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Correlation decay in certain soft billiards

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Abstract: Motivated by the 2D finite horizon periodic Lorentz gas, soft planar billiard systems with axis-symmetric potentials are studied in this paper. Since Sinai's celebrated discovery that elastic collisions of a point particle with strictly convex scatterers give rise to hyperbolic, and consequently, nice ergodic behaviour, several authors (most notably Sinai, Kubo, Knauf) have found potentials with analogous properties. These investigations concluded in the work of V. Donnay and C. Liverani who obtained general conditions for a 2-D rotationally symmetric potential to provide ergodic dynamics. Our main aim here is to understand when these potentials lead to stronger stochastic properties, in particular to exponential decay of correlations and central limit theorem. In the main argument we work with systems in general for which the rotation function satisfies certain conditions. One of these conditions has already been used by Donnay and Liverani to obtain hyperbolicity and ergodicity. What we prove is that if, in addition, the rotation function is regular in a reasonable sense, the rate of mixing is exponential, and, consequently the central limit theorem applies. Finally, we give examples of specific potentials that fit our assumptions. This way we give a full discussion in the case of constant potentials and show potentials with any kind of power law behaviour at the origin for which the correlations decay exponentially.

1. Introduction

Consider the planar motion of a point particle in a periodic array of strictly convex scatterers. Interaction with the scatterers is in the form of elastic collisions, otherwise motion is uniform. This dynamical system, the *planar Lorentz process* is a paradigm for strongly chaotic behaviour. Among other important properties ergodicity ([Si2,SCh]) and exponential decay of correlations ([Y,Ch]) have been proven for the corresponding billiard system.

In this paper we consider the following natural modification. The scatterers are no longer hard disks, the point particle may enter them. The particle moves according to some rotation symmetric potential which vanishes identically outside the disks.

Even the issue of these softened Lorentz processes has a large literature. Results point into two different directions. On the one hand, for quite general softening of the potential, the chaotic behaviour is no longer present. Stable periodic orbits and islands appear in the phase space. This is generally the case with smooth potentials, see [RT,Do2,Do1] and references therein.

However, in many cases, especially when the potential is not C^1, the chaotic behaviour persists. [1] The investigation of such soft billiards dates back to the pioneer works of Sinai ([Si1]) and Kubo et.al. ([Ku] and [KM]). There are two different approaches present in the literature to this hyperbolic case. On the one hand, under conditions on the derivatives (up to the second) of the potential the Hamiltonian flow turned out to be equivalent to a geodesic flow on a negatively curved manifold. This point of view is especially suitable for potentials with Coulomb type singularities, see [Kn1] on details.

The approach we follow is to study dynamics as a hyperbolic system with singularities. [M] and especially, [DL] – which is one of our main references – are written in the spirit of this principle. Actually, in most cases it is convenient to study the discrete time dynamics, a naturally defined Poincaré section map of the Hamiltonian flow – this is the track we are going to take.

Hyperbolicity of the system is mainly related to the properties of the so called *rotation function* that can be calculated from the potential. Being a bit technical its definition and relevant properties are

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[1] In [DL] there is a smooth potential example with ergodic behaviour, too, however it is unstable with respect to small perturbations like varying the full energy level.

discussed in the next section. Formulation of our main theorem (Theorem 1) is likewise left to the next section as it is in terms of the rotation function. Nevertheless, it might be useful to point out that

- In case the rotation function (and the billiard configuration) satisfies some hyperbolicity condition (see Definition 2), the soft billiard system is hyperbolic and ergodic. Although a little bit otherwise stated, this fact was proved in [DL]. The condition is, essentially, necessary for ergodicity (note however Remark 1).
- In this paper we concentrate on decay of correlations. If – in addition to those needed for hyperbolicity – the rotation function satisfies further regularity conditions (see Definition 3), the rate of mixing is proved to be exponential.

In most of the paper we think of the rotation function as being fixed with the desired properties. It is only section 5 when we turn to some specific potentials. Nevertheless, two technical conditions supposed to hold throughout the paper are:

- in order to be able to define a rotation function at all, we introduce $h(r) = r^2(1-2V(r))$ (cf. section 5) and require $h'(r) > 0$ for all but finitely many r (this condition ensures the lack of trapping zones, cf. [DL]):
- The scattering occurs on rotation symmetric potentials of finite range – that is, potential for every scatterer is concentrated on a circle and depends only on the distance from the center. (Note this is the case in our references like [DL], too.)
- The horizon is finite (i.e. the maximum time between two enterings of consecutive potential disks is uniformly bounded above for any trajectory).

Proof of our main theorem is based on our second main reference, on [Ch]. In this paper, by implementing the techniques of L. S. Young from [Y], N. Chernov showed that given any hyperbolic system with singularities for which one can show the validity of certain technical properies, correlations decay exponentially fast.

What we perform below is the proof of these technical properties for our ‘soft’ billiard system. Even though the existence of invariant cone fields is established in [DL], the *uniformity* of hyperbolicity (subsection 3.2) needs detailed investigation. An even more important new difficulty that we have to overcome is the treatment of quantities connected to the second derivative of the dynamics, especially while traveling through the potential. An analysis finer than before – in this sense – of the evolution of fronts is needed. This applies especially to the self-contained proof of curvature and distorsion bounds (subsection 3.3).

It is a key aspect of our method that arguments related to expansion and distortion can be carried out by considering motion inside and outside the potential disks separately. Actually, our choice of the outgoing phase space and the euclidean metric (see section 2) is related to this point of view and not to the tradition of [Ch]. (Using the Euclidean metric with the phase space of *incoming* particles instead of outgoing, our distortion bounds would no longer hold.) The splitting of motion into 'potential' and 'free' intervals is, however, slightly restrictive. Namely, certain soft billiard systems that seem ergodic and exponentially mixing are covered neither in this paper nor in [DL] (see also section 6, especially Remark 1).

The paper is organized as follows. In section 2 the dynamical system along with the rotation function and its properties are defined, and our main theorem is formulated. Section 3 gives a detailed geometric analysis of the system. After fixing notations and establishing some basic properties in subsection 3.1, subsection 3.2 is mainly concerned with uniform hyperbolicity (Proposition 1) and related issues. In subsection 3.3 important regularity properties of unstable manifolds are shown. Specifically, curvature bounds, distorsion bounds and absolute continuity of the holonomy map are proven (Propositions 2, 3 and 4). As a final bit of the general proof in section 4 we investigate the growth of unstable manifolds. The fact that expansion prevails the harmful effect of singularities is quantificated in the growth formulas of Proposition 5. As a conclusion we refer to Theorem 2.1 from [Ch]: a hyperbolic system with singularities for which Propositions 1, 2, 3, 4 and 5 are valid enjoys exponential decay of correlations. Last but not least, preceding some concluding remarks, in section 5 we turn to the investigation of specific potentials: as corollaries of our main theorem certain soft billiard systems are shown to exhibit exponential decay of correlations.

Part of the results in this paper and a sketch of the proof can also be found in the proceedings paper [BáTó].

2. Definition and basic properties of the system

The phase space.

Consider finitely many disjoint circles of radius R on the unit two-dimensional flat torus $\mathbf{T}^2$. (Thinking of a periodic array of circular disks on the Euclidean plane $\mathbb{R}^2$ would not be very much different.) We require that the configuration has finite horizon: there is a certain constant $\tau_{\max}$ such that any straight segment longer than $\tau_{\max}$ on $\mathbb{R}^2$ intersects at least one of the scatterers.

Remark. As the circles are disjoint, the minimum distance between two scatterers is bigger than some positive constant $\tau_{\min}$.

Let the Hamiltonian motion of our point particle be described by a potential which is identically zero outside and is some rotation symmetric function $V(r)$ inside the circular scatterers (here r is the distance from the center of the scatterer). For simplicity we fix the mass and the full energy of our point particle as

$$m = 1, \qquad E = \frac{1}{2}.$$

This way the free flight velocity has unit length, $|v| = 1$ (in other words $v \in \mathbf{S}^1$ where $\mathbf{S}^1$ is the unit circle in $\mathbb{R}^2$).

We assume (cf. Definition 2 and the remarks following it) that the Hamiltonian flow restricted to this surface of constant full energy is ergodic (with respect to Liouville measure). Equivalently one can say that the map corresponding to the naturally defined Poincaré section of the flow (see below) is ergodic. Our aim is *to study the rate of mixing for* this *map.*

Following tradition we work with the Poincaré section of outgoing velocities (particles that have just left one of the scatterers).

Notation

Denote by M the Poincaré section of outgoing particles. Sometimes we will also use the notation $M_+ = M$ to stress that this is the outgoing phase space, to avoid confusion.

The phase points are the boundary points of the scatterers, equipped with unit velocities pointing outwards. The phase space M is a finite union of cylinders (each corresponding to one of the circular scatterers). Coordinates for the cylinders are:

Notations

s denotes the arclength parameter along the scatterer (starting from a point arbitrarily fixed), describing position of the outgoing particle.

φ denotes the collision angle, the angle that the outgoing velocity makes with the normal vector of the scatterer in the point s. Clearly $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The position can be equivalently described by another angle parameter $\Theta \in [0, 2\pi]$, for which $s = R\Theta$ (here R is the radius of the scatterer).

Note that M defined this way is a (finite union of) Riemannian manifold(s).

Notation

Let

$$|dx|_e = \sqrt{ds^2 + d\varphi^2} \tag{2.1}$$

denote the Riemannian metric on M, which will be referred to as the Euclidean metric (e-metric).

Later on we will introduce another auxiliary metric quantity very common in the billiard literature, the p-metric.

As to dynamics, let T denote the first return map onto M.

Notation for the Lebesgue measure on M is m, i.e. $dm = ds\, d\varphi$. Furthermore, given a curve γ in M we denote the Lebesgue measure on γ with m_γ (this is simply the length on γ).

Denote by μ the natural invariant probability measure on M. μ is absolutely continuous w.r.t Lebesgue, and the density is of the form

$$d\mu = \text{ const. } \cos(\varphi)\, dm = \text{ const. } \cos(\varphi)\, ds\, d\varphi. \tag{2.2}$$

It is this latter measure for which T is assumed to be ergodic and K-mixing and this is the one we work with as well.

Remark. In a completely similar manner we could consider the Poincaré section M_- of incoming particles. The two coordinates would be the point of income and the angle the incoming velocity makes with the (opposite) normal vector. However, in some key steps of the proof – eg. the distorsion bounds of subsubsection 3.3.2 – we heavily use that our phase space is the outgoing, and *not* the incoming Poincaré section.

With slight abuse of notation we often refer to the incoming Poincaré coordinates with the same symbols s and φ. That should cause no confusion.

Rotation function, its basic properties and formulation of the main theorem. To describe the first return map T we decouple the motion into two parts: free flight among the scatterers and flight in the potential of the scatterers. Free flight can be treated completely analogously to the billiard case. The particle leaves one of the scatterers in the point s_0 with velocity φ_0 and reaches some other scatterer in point s (or equivalently, Θ) with unit incoming velocity that makes an angle φ with the (opposite) normal vector $n(s)$ at the point of income. After some inter-potential motion the particle leaves the circle in some point $s_1 = (R\Theta_1)$ with outgoing velocity specified by φ_1. Out of symmetry reasons $\varphi_1 = \varphi$, thus the only nontrivial quantity is the angle difference $\Delta\Theta = \Theta_1 - \Theta$. Again out of symmetry reasons $\Delta\Theta$ depends only on the angle φ.

The role in the map T played by the potential is completely described by the function $\Delta\Theta(\varphi)$.

Definition 1. *From here on we will refer to this function $\Delta\Theta(\varphi)$ as* **the rotation function**.

Being mainly interested in the differential aspects of T we introduce one more

Notation

$$\kappa(\varphi) = \frac{d\Delta\Theta(\varphi)}{d\varphi}.$$

Below two imporpant properties are defined in terms of which our main theorem is formulated.

Definition 2. *The soft billiard system satisfies property H in case*

1. *there is some positive constant c such that $|2+\kappa(\varphi)| > c$ for all φ;*
2. *the configuration of scatterers is such that the distance of any two circles is bounded below by $\tau_{\min}$ where*

$$\tau_{\min} \gneqq \max_{\varphi} \left\{ -2R\kappa(\varphi) \frac{\cos\varphi}{2+\kappa(\varphi)} \right\}.$$

Remarks.

- Although a bit otherwise formulated, it was essentially proven in [DL] that soft billiard systems with property H are hyperbolic and ergodic. The mechanism of hyperbolicity is briefly explained in Section 3.2.
- Note that in case $\kappa > 0$ or $\kappa < -2$ for all φ, the lower bound for $\tau_{\min}$ turns out to be negative. Thus the second assumption is only restrictive in the opposite case, and the closer κ may get to -2 from above the more restrictive it is.
- In case there is some φ for which $0 > \kappa > -2$, a positive lower bound on the free path is to be assumed. Thus a planar periodic configuration of circles is needed that has finite horizon and (a possibly great) given $\tau_{\min}$ simultaneously. At first sight it seems questionable whether such configurations exist at all, nevertheless, as proven in [BöTa], this happens with positive probability in a random construction.

Definition 3. *The rotation function is termed regular in case the following properties hold.*

1. $\Delta\Theta(\varphi)$ *is piecewise uniformly Hölder continuous. I.e. there are constants* $C < \infty$ *and* $\alpha > 0$*, and furthermore,* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *can be partitioned into finitely many intervals, such that for any* φ_1 *and* φ_2 *(from the interiour of one of the intervals):*

$$|\Delta\Theta(\varphi_1) - \Delta\Theta(\varphi_2)| \leq C|\varphi_1 - \varphi_2|^{\alpha}.$$

2. $\Delta\Theta(\varphi)$ *is a piecewise* C^2 *function of* φ *on the closed interval* $[-\frac{\pi}{2}, \frac{\pi}{2}]$*, in the above sense. (Note, however, that* κ*, in contrast to* $\Delta\Theta$*, can happen to have no finite one-sided limits at discontinuity points.)*
3. *There is some finite constant* C *such that*

$$|\kappa'(\varphi)| \leq C|\kappa^3(\varphi)|$$

where $\kappa'(\varphi)$ *is the derivative of* κ *with respect to* φ*.*

4. *For the final property consider any discontinuity point* φ_0 *where* $\kappa(\varphi)$ *(in contrast to* $\Delta\Theta(\varphi)$*) has no finite limit from the left. Of course, in case there is no finite limit from the right, the analogous property is similarly assumed.*

Restricted to some interval $[\varphi_0 - \epsilon, \varphi_0)$; $\omega(\varphi) = \frac{2+\kappa(\varphi)}{\cos\varphi}$ *is a monotonic function of* φ.

Remark. Note that in case κ is C^1 (or piecewise C^1 with boundedness of itself and of κ') regularity is automatic. In case the asymptotics of κ near some discontinuity is some power law $(\varphi_0 - \varphi)^{-\xi}$ (with $\xi > 0$), regularity means $\frac{1}{2} \leq \xi < 1$.

We need two more definitions for the statement of our theorem:

Definition 4. *Consider a phase space* M *with a dynamics* T *and a* T*-invariant probability measure* μ. *We say that the dynamical system* (M, T, μ) *has exponential decay of correlations (EDC) , if for every* $f, g : M \to \mathbb{R}$ *Hölder-continuous pair of functions there exist constants* $C < \infty$ *and* $a > 0$ *such that for every* $n \in \mathbb{N}$

$$\left| \int_M f(x) g(T^n x) d\mu(x) - \int_M f(x) d\mu(x) \int_M g(T^n x) d\mu(x) \right| \leq C e^{-an}.$$

Definition 5. *We say that* (M, T, μ) *satisfies central limit theorem (CLT) (for Hölder continuous functions) if for everz* $\eta > 0$ *and every Hölder-continuous function* $f : M \to \mathbb{R}$ *with* $\int f \, d\mu = 0$, *there exists a* $\sigma_f \geq 0$ *such that*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \overset{\text{distr}}{\longrightarrow} \mathcal{N}(0, \sigma_f)$$

where $\mathcal{N}(0, \sigma_f)$ *is the Gaussian distribution with variance* σ_f^2.

Now we are ready to formulate our main theorem.

Theorem 1. *Suppose that the soft billiard system* (M, T, μ) *satisfies property H and the rotation function is regular. Suppose furthermore that there are no corner points and the horizon is finite* ($0 < \tau_{min}$, $\tau_{max} < \infty$).

Then, dynamics enjoys, in addition to ergodicity and hyperbolicity, exponential decay of correlations and the central limit theorem for Hölder-continuous functions.

Proof. Ingredients for the proof are in sections 3 and 4. Actually, following tradition (eg. [Ch]) we modify the dynamical system in several steps (Conventions 1 and 2.) We will use a phase space $\bar{M}$, which is the original M cut into (countably many) connected components by singularities and so called ‘secondary singularities'. We will also use a higher iterate of the dynamics $T_1 = T^{m_0}$ with some m_0 to be found later.

It is the modified dynamical system $(\bar{M}, T_1, \mu)$ for which the conditions for EDC and LCT given in [Ch] are checked. Precisely, EDC and CLT for $(\bar{M}, T_1, \mu)$ are the consequence of Propositions 1, 2, 3, 4 and 5 and Theorem 2.1 from [Ch].

Exponential decay of correlations and the central limit theorem for (M, T, μ) follow easily from EDC and CLT for $(\bar{M}, T_1, \mu)$.

Now we turn to the details of the above proof.

Some conventions.

Constants that depend only on the map T itself (like $\tau_{\min}, \tau\max, \ldots$) will be called *global constants.*

Positive and finite global constants, whose value is otherwise not important, will be often denoted by just c or C (typically c for lower bounds and C for upper). That is, in two different lines of the same section, C can mean two different numbers.

Two quantities f and g defined on (the tangent bundle of) M (or on some subset like the unstable cone field, see subsection 3.1) will be called *equivalent* ($f \sim g$) if there are some global positive constants c and C such that $cf \le g \le Cf$.

2.1. Singularities.. Just like in billiards the dynamics T is not smooth at certain one-codimensional submanifolds (curves) of M. Consider the set of tangential reflections:

$$S_0 = \left\{ \; (s, \varphi) \in M \mid \varphi = \pm\frac{\pi}{2} \; \right\}.$$

Actually $S_0 = \partial M$ (the boundary of the phase space). It is not difficult to see that T is not continuous at $S_1 = T^{-1}S_0$, i.e. at the preimages of tangential reflections. However, additional singularities appear at

$$Z_0 = \{ \; (s, \varphi) \in M \mid \varphi = \varphi_0 \; \}$$

in case φ_0 is some discontinuity point for $\Delta\Theta(\varphi)$, $\kappa(\varphi)$ or $\kappa'(\varphi)$). In such a case we will consider the phase space as if it were cut into two regions, more precisely Z_0 is treated as part of the boundary. As κ is not differentiable at Z_0, T is not C^1 at the preimage of this set, at $Z_1 = T^{-1}Z_0$.

Furthermore we introduce the notations

$$S^{(n)} = S_1 \cup T^{-1}S_1 \cup \cdots \cup T^{-n+1}S_1$$

and $Z^{(n)}$, analogously. The n-th iterate of the dynamics is not smooth precisely at $Z^{(n)} \cup S^{(n)}$.

The geometrical structure of $Z^{(n)}$ is much similar to that of $S^{(n)}$. Indeed, one can think of Z_1 as the set of those trajectories that would touch tangentially a smaller disk (one of radius $R\sin(|\varphi_0|)$) at the next collision. The following properties of the singularity set are of crucial importance:

- $Z^{(n)} \cup S^{(n)}$ is a finite union of C^2 curves.
- *Continuation property.* Each endpoint, x_0, of every unextendable smooth curve $\gamma \subset Z^{(n)} \cup S^{(n)}$, lies either on the extended boundary $Z_0 \cup S_0$ or on another smooth curve $\gamma' \subset Z^{(n)} \cup S^{(n)}$ that itself does not terminate at x_0.
- *Complexity property.* Let us denote by K_n the complexity of $Z^{(n)} \cup S^{(n)}$, i.e. the maximal number of smooth curves in $Z^{(n)} \cup S^{(n)}$ that intersect or terminate at any point of $Z^{(n)} \cup S^{(n)}$. K_n grows sub-exponentially with n.

For the proof of these properties in the billiard setting see the literature, especially [ChY], our case is analogous.

One more similarity with 'hard' billiards is that for technical reasons later on we will introduce countably many secondary singularities parallel to the lines of $S^{(n)}$. Such secondary singularities are to be introduced parallel to $Z^{(n)}$ as well in case $|\kappa|$ is unbounded as $\varphi \to \varphi_0$, at least from one side. We will turn back to this question in subsubsection 3.2.6.

3. Fronts, u-manifolds and unstable manifolds

3.1. u-manifolds and their geometric properties. Fronts and their geometric description.

Our most important tools in describing hyperbolicity – local orthogonal manifolds or simply *fronts* – we inherit from billiard theory. A front $\mathcal{W}$ is defined in the flow phase space rather than in the Poincaré section.

Definition 6. *Take a smooth 1-codim submanifold E of the whole configuration space, and add the unit normal vector $v(q)$ of this submanifold at every point q as a velocity, continuously. Consequently, at every point the velocity points to the same side of the submanifold E. The set*

$$\mathcal{W} = \{(q, v(q)) | q \in E\} \subset \mathcal{M}, \tag{3.1}$$

where $v : E \to \mathbf{S}^1$ is continuous (smooth) and $v \perp E$ at every point of E, is called a front.

Analysis of the time evolution of fronts is the key to almost all the geometric properties of the system that we need. For this reason, we first discuss time evolution of an arbitrary front. Later subsections will deal with special cases.

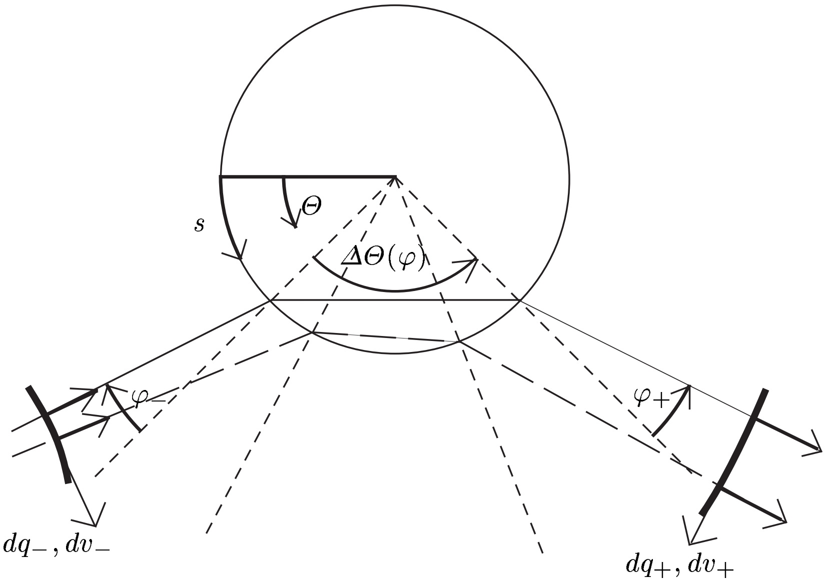


Fig. 1. conventions for notation and signs for fronts

Consider a front with a reference point just before reaching a scatterer, and another 'perturbed' point nearby. With the notations introduced before (see also Figure 1), the perturbation bringing the reference trajectory into the perturbed one is (dq_-, dv_-) just before collision, $(ds_-, d\varphi_-)$ in the incoming Poincaré section, $(ds_+, d\varphi_+)$ in the outgoing Poincaré section, (dq_+, dv_+) just after collision, and (dq'_-, dv'_-) just before the next collision. The evolution of the perturbations is:

$$\begin{aligned} ds_- &= \frac{dq_-}{\cos\varphi_-} \\ d\Theta_- &= \frac{ds_-}{R} \\ d\varphi_- &= dv_- + d\Theta_- \\ d\Theta_+ &= d\Theta_- + \kappa d\varphi_- \\ d\varphi := d\varphi_+ &= d\varphi_- \\ ds_+ &= R d\Theta_+ \\ dq_+ &= -\cos\varphi ds_+ \\ dv_+ &= -d\Theta_+ - d\varphi_+ \end{aligned} \tag{3.2}$$

while crossing the potential. For the evolution equations of free flight, we introduce the

Notation

$\tau = \tau(x)$ will denote the length of free flight of the particle before reaching the next scatterer.

So, during free flight we have

$$\begin{aligned} dq'_- &= dq_+ + \tau dv_+ \\ dv'_- &= dv_+ . \end{aligned} \tag{3.3}$$

Note that the angles of incidence and reflection are measured in different directions – in order to keep them equal, as they traditionally are, – but dq_- and dq_+ (just like dv_- and dv_+) are measured in the same direction, unlike usually in billiards.

Based on these, we can find out about the evolution of the derivative $B = \frac{dv}{dq}$.

Notations

B will denote the derivative of the unit normal vector (velocity) $v(q)$ of a front: $dv = Bdq$ for tangent vectors (dq, dv) *of the front.*

$m = \frac{d\varphi}{ds}$ will denote the slope of the (trace of the) front in the Poincaré section.

B is nothing else than the curvature of the submanifold E. Yet we will prefer to call it second fundamental form (SFF), in order to avoid confusion with other curvatures that are coming up. The term 'form' refers to higher dimensional cases when B is a symmetric operator.

(3.2) gives

$$\begin{aligned} m_- &= \cos\varphi B_- + \frac{1}{R} \\ \frac{1}{m_+} &= \frac{1}{m_-} + R\kappa \\ \cos\varphi B_+ &= m_+ + \frac{1}{R} \end{aligned} \tag{3.4}$$

while crossing the potential, which can be summarized in

$$B_+ = \frac{2+\kappa(\varphi)+(1+\kappa(\varphi))R\cos\varphi B_-}{R\cos\varphi(1+\kappa(\varphi)+\kappa(\varphi)R\cos\varphi B_-)}, \tag{3.5}$$

and (3.3) gives

$$\frac{1}{B'_-} = \frac{1}{B_+} + \tau \tag{3.6}$$

during free flight.

Notations

$$\lambda_1 := \frac{dq_+}{dq_-}, \tag{3.7}$$

$$\lambda_2 := \frac{dq'_-}{dq_+}, \tag{3.8}$$

$$\lambda := \lambda_1\lambda_2.$$

These are exactly the expansion factors along the front, for the respective 'pieces' of the dynamics. (They are also expansion factors in the Poincaré section, but in the p-metric to be introduced later.) We have

$$\lambda_1 = 1+\kappa+\kappa R\cos\varphi B_- = 1+\kappa R m_- = \frac{m_-}{m_+}, \tag{3.9}$$

$$\lambda_2 = 1+\tau B_+ = \frac{B_+}{B'_-}. \tag{3.10}$$

To study decay of correlations, we need one more derivative.

Notation

$D = \frac{dB}{dq}$.

This is exactly the *curvature* of the *front* as of a *subset of the flow phase space* (and not as of a subset of the configuration space – unlike B, cf. (3.1)).

To study the evolution of D we need to consider two small pieces of the front, one around the reference point, and one around the perturbed one. Let the change in the SFF be

$$dB_{--} = D_- dq.$$

before scattering, and

$$dB_{++} = D_+ dq.$$

after scattering. dB_{--} is *not* the difference of SFF-s at the points of incidence, because the perturbed point has to travel another $d\tau_- = \tan\varphi_- dq_-$ to reach the scatterer ($d\tau$ can be negative), which changes its SFF according to the rules (3.6)of free flight. Taking that into account, we have

$$dB_- = dB_{--} - B_-^2 d\tau_- = dB_{--} - B_-^2 \tan\varphi_- dq_-. \tag{3.11}$$

Similarly, for the fronts leaving the potential,

$$dB_{++} = dB_+ - B_+^2 d\tau_+ = dB_+ - B_+^2 \tan\varphi_+ dq_+. \tag{3.12}$$

(Note our convention on the signs of dq_-, dq_+, φ_- and φ_+.)

To follow the evolution of curvature we introduce

Notations

$D_1 = \frac{dB_-}{dq_-}(\neq D_-)$, $K_- = \frac{dm_-}{ds_-}$, $K_+ = \frac{dm_+}{ds_+}$, $D_2 = \frac{dB_+}{dq_+}$ and $\eta(\varphi) = \frac{d\kappa(\varphi)}{d\varphi}$.

With these we get from (3.2), (3.4), (3.7), (3.9), (3.11) and (3.12)

$$\begin{aligned} D_1 &= D_- - \tan\varphi B_-^2 \\ K_- &= \cos^2\varphi D_1 - \sin\varphi B_- m_- \\ K_+ &= \frac{1}{\lambda_1^3} K_1 - R\left(\frac{m_-}{\lambda_1}\right)^3 \eta \\ \cos^2\varphi D_2 &= -K_+ - \sin\varphi B_+ m_+ \\ D_+ &= D_2 - \tan\varphi B_+^2 \end{aligned} \tag{3.13}$$

while crossing the potential, and, from (3.3), (3.6) and (3.8)

$$D'_- = \frac{1}{\lambda_2^3} D_+ \tag{3.14}$$

during free flight.

3.2. Invariance of convex fronts, u-fronts and u-manifolds.. In [DL] it is shown – although not explicitely stated in this integrated form – that if Property H (defined in Definition 2) is satisfied, then convex fronts with suitably small SFF-s (the upper bound may be ∞) either remain convex, or focus before reaching the next scatterer, and become convex again, with suitably small SFF. This property is called the 'invariance of convex fronts'. In the present work we also require (see Theorem 1) that τ be bounded from below by some $\tau_{\min} > 0$ even in the case when [DL] did not (the 'no corner points' assumption), and an upper bound $\tau_{\max}$ (the 'finite horizon' assumption). In order to establish estimates that we will need later, we must repeat some steps of the argument in [DL]. We omit details of the calculations, these can be done by the reader or can be found in the above paper.

Notations

$$\begin{aligned} \tau_1 &= \max\left\{0, \max_{\varphi}\left\{-R\kappa(\varphi)\frac{\cos\varphi}{2+\kappa(\varphi)}\right\}\right\}, \\ B^* &= \frac{1}{\tau_1} \quad (\infty \text{ if } \tau_1 = 0). \end{aligned} \tag{3.15}$$

From (3.4) we get that that if $0 < B_- < B^*$ then either $m_+ > 0$ and thus $B_+ > \frac{1}{R}$ or $B_+ < -B^*$. This – by (3.6) – implies that $c < B'_- < B^{**}$ with some global constants $c > 0$ and $B^{**} < B^*$, assuming that $\tau_{min} > 2\tau_1$, which is exactly Property H. All in all,

$$c < B_- < B^{**} \text{ implies } c < B'_- < B^{**}. \tag{3.16}$$

This motivates our

Definition 7. *A u-front is a front with $c < B_- < B^{**}$. A u-manifold is the trace of a u-front on the Poincaré phase space.*

and

Definition 8. *An s-front is a front with $c < -B_+ < B^{**}$. An s-manifold is the trace of an s-front on the Poincaré phase space.*

As we have seen, u-manifolds remain u-manifolds under time evolution. s-fronts are exactly the u-fronts of the inverse dynamics.

The aim of this subsection is to show important properties of u-fronts and u-manifolds, which are stronger than those shown for an arbitrary front in the previous subsection. In subsection 3.3 we further restrict to the case of unstable manifolds, which are special kinds of u-manifolds.

3.2.1. Expansion estimates along u-fronts. First we work out estimates for the expansion along a front from one moment of incidence to the next. We will use these estimates later to estimate expansion of our dynamics T in our outgoing Poincaré phase space M.

Consider a u-front with the earlier notations. We start with an easy observation we will often use: from (3.4) and (3.16) we get $\frac{1}{R} < m_- < \frac{1}{R} + B^{**}$, which implies

$$m_- \sim 1. \tag{3.17}$$

To get the order of magnitude for the expansion factor λ, put the formulas in (3.5) and (3.9) together, and get that

$$\frac{R\cos\varphi B_+\lambda_1}{2+\kappa(\varphi)} = 1 + (1+\kappa(\varphi))RB_-\frac{\cos\varphi}{2+\kappa(\varphi)}.$$

The right hand side is trivially bounded from above since B_- is bounded, and so is $\frac{1+\kappa(\varphi)}{2+\kappa(\varphi)} = 1 - \frac{1}{2+\kappa(\varphi)}$. On the other hand,

- It is greater than 1 if $\frac{1+\kappa(\varphi)}{2+\kappa(\varphi)} > 0$.
- If $\frac{1+\kappa(\varphi)}{2+\kappa(\varphi)} \leq 0$ (that is, $-2 < \kappa(\varphi) \leq -1$), then

$$1 + (1+\kappa(\varphi))RB_-\frac{\cos\varphi}{2+\kappa(\varphi)} \geq 1 + (1+\kappa(\varphi))R\frac{\cos\varphi}{2+\kappa(\varphi)}B^* \geq$$

$$\geq 1 + (1+\kappa(\varphi))R\frac{\cos\varphi}{2+\kappa(\varphi)}\,\frac{2+\kappa(\varphi)}{-R\kappa(\varphi)\cos\varphi} = \frac{-1}{\kappa(\varphi)} \geq \frac{1}{2}.$$

All in all, using $\lambda_2 = \frac{B_+}{B'_-} \sim B_+$ (see (3.10) and (3.16)) we have

$$\lambda \sim B_+\lambda_1 \sim \frac{2+\kappa(\varphi)}{\cos\varphi}, \tag{3.18}$$

which is one of our key estimates. Notice that the right hand side cannot be too small due to Property H (Definition 2).

We can also get the order of magnitude for λ_1 and λ_2 separately: (3.9) and (3.17) gives

$$|\lambda_1|\sqrt{1+m_+^2} = \sqrt{\lambda_1^2 + m_-^2} \sim |2+\kappa(\varphi)|. \tag{3.19}$$

(The last equivalence is true because both sides are bounded away from zero, and can only be big when they grow linearly with κ.) Notice that λ_1 can be very small (even zero), and can even change signs while $2+\kappa(\varphi)$ remains positive. Of course, m_+ has to be infinity (and change signs) simultaneously.

Putting (3.18) and (3.19) together, we get

$$\frac{|\lambda_2|}{\sqrt{1+m_+^2}} \sim \frac{1}{\cos\varphi}. \tag{3.20}$$

This last line can be rewritten as

$$1 \sim \frac{|\lambda_2|\cos\varphi}{\sqrt{1+m_+^2}} \sim \frac{|B_+|\cos\varphi}{\sqrt{1+m_+^2}} = \frac{|m_+ + \frac{1}{R}|}{\sqrt{1+m_+^2}},$$

which implies that there is a global constant c such that

$$\left| m_+ + \frac{1}{R} \right| > c. \tag{3.21}$$

3.2.2. Expansivity. To obtain hyperbolicity, we must see that u-manifolds are expanded by the dynamics. In the first round we prove a Lemma about the expansion on u-fronts from collision to collision.

Lemma 1. *There exists a global constant $\Lambda > 1$, such that for every u-front, $|\lambda| \geq \Lambda$.*

Proof. Besides $\tau > 0$ and $B_- > 0$ we will use that $\tau \geq 2\tau_1 + d$ where $d := \tau_{min} - 2\tau_1 > 0$, and $\tau_1 \geq -R\kappa(\varphi)\frac{\cos\varphi}{2+\kappa(\varphi)}$ for every φ (see Definition 2 and (3.15)). Altogether:

$$\tau \geq d - 2R\kappa(\varphi)\frac{\cos\varphi}{2+\kappa(\varphi)}. \tag{3.22}$$

We will also use from (3.15) and Definition 7 that

$$0 < B_- \leq \frac{2+\kappa(\varphi)}{-R\kappa(\varphi)\cos\varphi} \tag{3.23}$$

whenever the right hand side is positive, which is the $-2 < \kappa(\varphi) < 0$ case. Now we start by putting together (3.9), (3.10) and (3.5) to get

$$\lambda = (1+\kappa(\varphi)Rm_-)(1+\tau B_+) = 1+\kappa(\varphi)+\kappa(\varphi)R\cos\varphi B_- + \tau\left(\frac{2+\kappa(\varphi)}{R\cos\varphi} + (1+\kappa(\varphi))B_-\right)$$

We estimate this taking care of the signs of the particular terms.

- If $\kappa(\varphi) \le -2-\delta$, then
$$\lambda \le 1+\kappa(\varphi) \le -1-\delta.$$
- If $-2+\delta \le \kappa(\varphi) \le -1$ then both coefficients of B_- are negative, so we can use (3.23) to estimate the right hand side from below. In the next step we find the coefficient of τ positive, so we can use (3.22). What we get is
$$\begin{aligned}\lambda &\ge 1+\kappa(\varphi)+\kappa(\varphi)R\cos\varphi\frac{2+\kappa(\varphi)}{-R\kappa(\varphi)\cos\varphi} + \\ &\quad +\tau\left(\frac{2+\kappa(\varphi)}{R\cos\varphi} + (1+\kappa(\varphi))\frac{2+\kappa(\varphi)}{-R\kappa(\varphi)\cos\varphi}\right) = \\ &= -1 \,+\, \tau\,\frac{2+\kappa(\varphi)}{-\kappa(\varphi)R\cos\varphi} \\ &\ge -1+\left(d-2R\kappa(\varphi)\frac{\cos\varphi}{2+\kappa(\varphi)}\right)\frac{2+\kappa(\varphi)}{-\kappa(\varphi)R\cos\varphi} \\ &\ge 1+\frac{d\delta}{2R}.\end{aligned}$$
- If $-1 \le \kappa(\varphi) \le 0$, then the coefficient of τ is positive, so we first use (3.22) to estimate the right hand side from below. In the next step we find one coefficient of B_- positive, so we just use $B_- > 0$, and one coefficient of B_- negative, so we can use (3.23). What we get is
$$\begin{aligned}\lambda &\ge 1+\kappa(\varphi)+\kappa(\varphi)R\cos\varphi B_- + \\ &\quad +\left(d-2R\kappa(\varphi)\frac{\cos\varphi}{2+\kappa(\varphi)}\right)\left(\frac{2+\kappa(\varphi)}{R\cos\varphi} + (1+\kappa(\varphi))B_-\right) = \\ &= 1+d\frac{2+\kappa(\varphi)}{R\cos\varphi} + d(1+\kappa(\varphi))B_- - \kappa(\varphi) - \frac{\kappa^2(\varphi)R\cos\varphi}{2+\kappa(\varphi)}B_- \\ &\ge 1\,+\,d\frac{2+\kappa(\varphi)}{R\cos\varphi}\,-\,\kappa(\varphi)\,-\,\frac{\kappa^2(\varphi)R\cos\varphi}{2+\kappa(\varphi)}\,\frac{2+\kappa(\varphi)}{-R\kappa(\varphi)\cos\varphi} \\ &\ge 1+\frac{d}{R}.\end{aligned}$$
- If $0 < \kappa(\varphi)$, then
$$\lambda \ge 1+\frac{2d}{R}.$$

3.2.3. Transversality.

Lemma 2. *We will see that u- and s-manifolds are uniformly transversal. I.e. there is some global constant* $\alpha_0 > 0$ *such that given any two tangent vectors (in the outgoing Poincaré phase space)* dx_s *and* dx_u *of an s- and a u-manifold, respectively, we have*

$$\sphericalangle(dx_u, dx_s) > \alpha_0.$$

Proof. To see this, use Definition 8 and (3.4) to get $m_+^s \sim -1$ for the slope of any s-manifold. This way, it is enough to see that the slopes of u- and s-manifolds are bounded away, that is, $|m_+^u - m_+^s| > c$. To get this, use – again – Definition 8, Definition 7, the estimates before them and (3.4) to get

$$-\frac{1}{R} > m_+^s > -\frac{1}{R} - \cos\varphi B^{**}$$

$$m_+^u > 0 \text{ or } m_+^u < -\frac{1}{R} - \cos\varphi B^* \tag{3.24}$$

so either $m_+^u - m_+^s > \frac{1}{R}$ or $m_+^s - m_+^u > \cos\varphi(B^* - B^{**})$. This implies the statement when $\cos\varphi$ is not too small. However, when $\cos\varphi$ is small, we have to use the estimate (3.21) and (3.24) to see also that

$$m_+^u > 0 \text{ or } m_+^u < -\frac{1}{R} - c$$

which completes the proof.

3.2.4. Hyperbolicity. In what follows we will consider time evolution of vectors tangent to u-manifolds. Notation both in the incoming and the outgoing phase space will be of the type $dx = (ds, d\varphi)$. In addition to the e-metric (2.1) we will use one more metric quantity, the p-metric:

$$|dx|_p = |ds|\cos(\varphi).$$

The p-metric measures distances along the corresponding u-front. It is degenerate on the whole tangent bundle. However, when restricted a u-manifold *in the incoming phase space*, by (3.17) we have:

$$|dx|_p \sim |dx|_e \cos(\varphi).$$

According to Lemma 1, u-vectors are expanded uniformly *(from collision to collision, that is, in the incoming phase space)* in the p-metric:

$$|DT|_p = \lambda \geq \Lambda > 1$$

To obtain expansion in the e-metric and the outgoing phase space, we look at the n-th iterate of the outgoing phase space dynamics the following way:

- switch to p-metric
- reach the next scatterer
- do $n-1$ steps in the incoming phase space
- cross the potential
- switch back to Euclidean metric.

This way we get

$$|DT^n\, dx|_e = \frac{\sqrt{1+m^2_{+(n)}}}{\cos\varphi_{(n)}} \lambda_{1(n)} \lambda_{(n-1)} \lambda_{(n-2)} \dots \lambda_{(1)} \lambda_2 \frac{\cos\varphi}{\sqrt{1+m^2_{+}}} |dx|_e$$

where symbols with ()-ed subscripts mean values at the appropriate iterate of the phase point. Using (3.19), (3.20) and Lemma 1 we get

$$|DT^n\, dx|_e \sim \lambda_{(n-1)} \lambda_{(n-2)} \dots \lambda_{(1)} \frac{2+\kappa(\varphi_{(n)})}{\cos\varphi_{(n)}} |dx|_e. \tag{3.25}$$

This way we have

$$|DT^n\, dx|_e > c_1 \Lambda^n |dx|_e \tag{3.26}$$

with some global constant c_1. Again, this is for u-vectors in the outgoing phase space.

The transversality of s- and u- vectors, stated in Proposition 1 implies that the product of (length) expansion factors for s- and u- vectors is equivalent to the n-step (Lebesgue) volume expansion factor. Using (2.2), and the T-invariance of μ, we get that if dx is a u-vector and dy is an s-vector, then

$$\frac{|DT^n\, dx|_e}{|dx|_e} \, \frac{|DT^n\, dy|_e}{|dy|_e} \sim \frac{\cos\varphi}{\cos\varphi_{(n)}}.$$

Combinig this with (3.25) we get

$$|DT^n\, dy|_e \sim \frac{\cos\varphi}{2+\kappa(\varphi_{(n)})} \frac{1}{\lambda_{(n-1)} \lambda_{(n-2)} \dots \lambda_{(1)}} |dy|_e,$$

which implies

$$|DT^n\,dy|_e < \frac{C_1}{\Lambda^n}|dx|_e \tag{3.27}$$

with some global constant C_1. Again, this is for s-vectors in the outgoing phase space.

Convention 1 *We choose a positive integer* m_0 *the following way. First take* m_1 *such that* $c_1\Lambda^{m_1} > 1$ *and* $\frac{C_1}{\Lambda^{m_1}} < 1$. *This way any enough high power of the dynamics,* T^m *with* $m > m_1$ *is uniformly expanding along u-manifolds and uniformly contracting along s-manifolds with* $\Lambda_1 = \Lambda^{m-m_1}$. *Now recall the notion and the basic properties of complexity* K_n *from subsection 2.1. As* K_n *grows subexponentially we may choose* m_2 *for which we have* $K_m < \Lambda^{m-m_1}$ *whenever* $m > m_2$. *We fix* $m_0 = \min(m_1, m_2) + 1$.

The advantage of this choice is that the iterate $T_1 = T^{m_0}$ *is uniformly hyperbolic (see the proposition to come) with constant* Λ_1 *for which* $\Lambda_1 > K_{m_0} + 1$. *This later fact we only use in section 4.*

Let us summarize what we have seen so far from the hyperbolic properties in the following

Proposition 1. *There exist two families of cones* $C_s(x)$ *and* $C_u(x)$ *– called stable and unstable cones – in the tangent space of* M *such that*

$$DT(C_u(x)) \subset C_u(Tx) \text{ and } C_s(Tx) \subset DT(C_s(x)).$$

The stable/unstable cone is uniformly contracting/expanding:

$$|DT_1^{-1}(dx)| \geq \Lambda_1|dx| \qquad \forall dx \in C_s(x),$$

$$|DT_1(dx)| \geq \Lambda_1|dx| \qquad \forall dx \in C_u(x).$$

Furthermore, the two cone fields are uniformly transversal in the sense above.

Vectors of the stable/unstable cone are often called s- and u-vectors.

Proof. The two cones are formed by the tangent vectors of s- and u-manifolds, respectively. Invariance is the implication (3.16), recalling Definition 7 and 8. Expansion and contraction are (3.26), (3.27) and Convention 1. Transversality is Lemma 2.

3.2.5. Alignment. We need to investigate the relative position of u-manifolds and singularities in order to find out how much of a u-manifold can be 'close' to a singularity Our aim is to prove the following

Lemma 3. *Take any smooth component Z of $T^{-k}Z_0$ with $k \geq 0$, where*

$$Z_0 = \{\ (s, \varphi) \in M \mid \varphi = \varphi_0\ \}$$

with any $\varphi_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Given some small positive δ let us denote the δ-neighborhood of Z by $Z^{[\delta]}$. There are global constants $C < \infty$ and $\alpha > 0$ such that for any u-manifold W we have

$$m_W\left(Z^{[\delta]} \cap W\right) \leq C\delta^{\alpha}, \tag{3.28}$$

where m_W is the Lebesgue measure – the length – on the u-manifold W.

Proof. If $k > 0$, then Z is an s-manifold, and is transversal to our u-manifold W according to Lemma 2, so the statement hold even with $\alpha = 1$.

So take $k = 0$, then Z is described by $m_Z = 0$. If $\kappa(\varphi)$ remains bounded near φ_0, then for our u-manifold W,

$$\frac{1}{m_+} = \frac{1}{m_-} + R\kappa(\varphi)$$

is bounded (see (3.4) and (3.17)), so the two curves are transversal again, we can choose $\alpha = 1$.

The interesting case is $k = 0$, $\kappa(\varphi) \to \infty$ as $\varphi \to \varphi_0$. In this case (3.17) ensures that $\frac{1}{m_-}$ is negligible – say, less than ε portion – compared to $R\kappa(\varphi)$. This – through (3.4) and the definition $m_+ = \frac{d\varphi_+}{ds_+} = \frac{d\varphi}{ds}$ – implies that for u-manifolds

$$(1-\varepsilon)R\kappa(\varphi) \leq \frac{ds}{d\varphi} \leq (1+\varepsilon)R\kappa(\varphi)$$

Integrating this with respedt to φ and using the definition of $\kappa(\varphi)$, we get

$$(1-\varepsilon)R(\mathit{\Delta\Theta}(\varphi) - \mathit{\Delta\Theta}(\bar{\varphi}) \leq s - \bar{s} \leq (1+\varepsilon)R(\mathit{\Delta\Theta}(\varphi) - \mathit{\Delta\Theta}(\bar{\varphi}))$$

which means that, close (enough) to a $\kappa(\varphi) \to \infty$ singularity, a u-manifold is (arbitrarily) similar to the graph of the rotation function $\mathit{\Delta\Theta}(\varphi)$. Now the Hölder-continuity of $\mathit{\Delta\Theta}(\varphi)$ required in the regularity condition (Definition 3) implies the statement of the Lemma.

3.2.6. Homogeneity strips, secondary singularities and homogeneous u-manifolds. **Notation**

$$\omega(\varphi) := \frac{2+\kappa(\varphi)}{\cos\varphi} \tag{3.29}$$

We will see that expansion in the e-metric is unbounded as $|\omega(\varphi)| \to \infty$. This certainly happens in the vicinity of $\pm\frac{\pi}{2}$, nevertheless, there can exist other discontinuity values φ_0 with the same property. Big expansion comes together with big variations of expansion (i.e. distorsion) rates along u-manifolds. For that reason we need to partition the phase space into homogeneity layers in which $\omega(\varphi)$ is nearly constant. We fix a large integer k_0 (to be specified in section 4) and define for $k > k_0$ the I-strips as

$$I_k = \left\{ (s,\varphi) \mid k^2 \le |\omega(\varphi)| < (k+1)^2 \right\} \tag{3.30}$$

Recall from Definition 3 that whenever $\lim_{\varphi\to\varphi_0} |\omega(\varphi)| = \infty$, there exists an interval $[\varphi_0-\epsilon, \varphi_0)$ restricted to which $|\omega(\varphi)|$ is a monotonic function of φ. We partition a subinterval of this interval into I-strips, thus k_0 is chosen accordingly large. In case there are several discontinuity points of $\omega(\varphi)$ (with unbounded one-sided limits) we may construct further I-strips, $I_k^{(s)}$, analogously. Here the index s labels the finitely many discontinuities of this kind.

Furthermore take $I_0^{(u)}$; $u = 1 \cdots U$ where the index u labels the finitely many connected components of the complement of all the above layers (that is, the ‘remaining part’ of the phase space).

We will use the notations Γ_0 for the countably many boundary components of I-strips.

Convention 2 *From now on, Γ_0 – just like S_0 and Z_0 before – is considered as part of the boundary of the phase space. That is, we will use a modified phase space $\bar{M}$, whose connected components are the homogeneity strips I_k (and $I_0^{(u)}$).*

In complete analogy with primary singularities we introduce furthermore the notations Γ_1 and $\Gamma^{(n)}$ for the corresponding preimages. The geometric properties of these secondary singularity lines are analogous to those of primary ones (for example, (3.28) applies).

Definition 9. *We will say that a u-manifold is homogeneous whenever it is contained in one of the homogeneity strips I_k (or $I_0^{(u)}$).*

In sections 3.3.2 and 4 we will be concerned with u-manifolds that remain homogeneous for several steps of the dynamics.

3.3. Regularity properties of unstable manifolds.

Definition 10. *An unstable manifold is a u-manifold for which all past iterates are u-manifolds as well.*

Analogously, a stable manifold is an s-manifold for which all future iterates are s-manifolds as well.

From the theory of hyperbolic systems (see [Ch] and references therein) we know that there is a unique unextendeble unstable (and similarly a unique unextendable stable) manifold through ($\mu-$)almost every point of $\bar{M}$. Thus it makes sense to talk about the (un)stable manifold through the point.

We will also refer to unstable manifolds as 'local unstable manifolds' (LUMs), stressing the fact that they are (and all their past iterates as well are) contained in some homogeneity layer I_k. (Remember that our phase space ends on the boundary of I_k, so I_{k+1} is already another connected component.)

In this subsection we deal with properties of unstable manifolds which are stronger than those proved before for arbitrary u-manifolds in subsection 3.2.

3.3.1. Curvature bounds. In what follows we obtain bounds on unstable manifolds that will guarantee that their curvature is uniformly bounded from above.

First we look at u-fronts as submanifolds of the flow phase space.

Putting the formulas in (3.13) and (3.14) together, we get

$$D'_- = -\frac{D_-}{\lambda^3} + \frac{2\sin\varphi B_-^2}{\lambda^3\cos\varphi} - \frac{2\sin\varphi B'^2_-}{\lambda_2\cos\varphi} + \frac{\sin\varphi B_-}{R\lambda^3\cos^2\varphi} + \frac{\sin\varphi B'_-}{R\lambda_2^2\cos^2\varphi} - Rm_-^3\frac{\eta}{\lambda^3\cos^2\varphi}$$

Our key estimate (3.20) implies

$$\cos\varphi|\lambda_2| \sim \sqrt{1+m_+^2}$$

which is bounded from below. So, in the above sum, terms number 2,3,4 and five are all bounded in absolute value. The last term is bounded due to our assumption

$$\left|\frac{\eta(\varphi)}{(2+\kappa(\varphi))^3}\right| < C.$$

As a consequence, we have

$$|D'_-| \le \frac{|D_-|}{\Lambda^3} + C_2, \tag{3.31}$$

with some global constant C_2 and can state

Lemma 4. *There is a global constant $\hat{D}$ such that for almost any point of the phase space, the front corresponding to the LUM has*

$$|D_-| \leq \hat{D}$$

Proof. Choose $\hat{D} = \frac{C_2\Lambda^3}{\Lambda^3-1}$. Now suppose indirectly that there is a set $H \subset M$ of positive measure, for the points of which $|D_-| > \hat{D} + \varepsilon$. Then (3.31) implies that there is a $c(\varepsilon) > 0$ such that $|D_-| > \hat{D} + \varepsilon + c$ on $T^{-1}H$. This implies that $|D_-| > \hat{D} + \varepsilon + 2c$ on $T^{-2}H$, and so on: $|D_-| > \hat{D} + \varepsilon + kc$ on $T^{-k}H$ for all $k > 0$. But the $T^{-k}H$-s are all sets of equal positive measure, which contradicts the finiteness of the phase space.

As a consequence, we can give curvature bounds for local unstable manifolds in the incoming and outgoing phase spaces. Since an unstable manifold in the Poincaré section is the graph of a function $\varphi = \varphi(s)$, its curvature is given by

$$g = \frac{\varphi''(s)}{\sqrt{1+(\varphi'(s))^2}^3} = \frac{K}{\sqrt{1+m^2}^3}.$$

We have reached

Proposition 2. *There is a global constant C such that for almost any point of the phase space, the front corresponding to the LUM has*

$$|g_+| \leq C.$$

Proof. It can be read from (3.13) that

$$|K_-| < C, \tag{3.32}$$

thus

$$|g_-| < C.$$

To find out about g_+, we write

$$g_+ = \frac{K_+}{\sqrt{1+m_+^2}^3} = \left(\frac{m_+}{\sqrt{1+m_+^2}}\right)^3 \frac{K_-}{m_-^3} - R\frac{\eta}{\sqrt{1+(\frac{1}{m_-}+R\kappa(\varphi))^2}^3}.$$

This is also bounded in absolute value due to our assumption

$$\left| \frac{\eta(\varphi)}{(2+\kappa(\varphi))^3} \right| < C$$

(see Definition 3).

3.3.2. Distorsion bounds. Length of a u-manifold W is expanded by T^n locally with a factor

$$J_{W,n}(x) = \frac{|DT^n dx|_e}{|dx|_e},$$

where dx is the vector tangent to the curve of W at x. The aim of this subsubsection is to prove

Proposition 3. *Let W be an unstable manifold on which T^n is smooth. Assume that $W_i = T^i W$ is a homogeneous unstable manifold for each $1 \leq i \leq n$. Then for all $x, \bar{x} \in W$*

$$|\ln J_{W,n}(x) - \ln J_{W,n}(\bar{x})| \leq C \left[\mathrm{dist}_{W_n}(T^n x, T^n \bar{x})\right]^{\frac{1}{5}}.$$

Proof. Note that $J_{W,n}(x) = \prod_{i=0}^{n-1} J_{W_i,1}(T^i x)$. Hence, it is enough to prove the Lemma for $n = 1$, because $\mathrm{dist}(T^i x, T^i \bar{x})$ grows uniformly exponentially in i due to (3.26). So we put $n = 1$.

Denote $x' = Tx$ and, we will use a ' to denote quantities related to the point x'.

Recall from section 2 that the expansion factor is easily calculated in the p-metric. To obtain $J := J_{W,1}(x)$ we transform $|dx|_e$ to $|dx|_p$, take the p-expansion factor from (3.7) and (3.8) and transform back. This way:

$$J = \frac{\sqrt{1+m'^2}}{\cos\varphi'} \lambda_1' \, \lambda_2 \, \frac{\cos\varphi}{\sqrt{1+m^2}}.$$

In order to calculate the change in the logarithm of J as we move from x to $\bar{x}$, it is best to write it with the help of (3.29) in the form

$$J = \omega(\varphi') J_1' J_2 \tag{3.33}$$

with

$$J_1 = \frac{\sqrt{1+m_+^2}}{2+\kappa(\varphi)} \lambda_1$$

and

$$J_2 = \frac{\cos\varphi}{\sqrt{1+m_+^2}} \lambda_2.$$

(3.19) and (3.20) imply

$$|J_1| \sim |J_2| \sim 1. \tag{3.34}$$

The change in logarithm of the three terms can be calculated independently, moreover, J_1 and J_2 are expected to change moderately, while $\omega(\varphi')$ can be kept under good control, because it depends only on φ'. The three terms are investigated in three sublemmas. Thus Proposition 3 is the direct consequence of the three Sublemmas 1, 2 and 3. Of course, the first and third (concerning J_1 and $\omega(\varphi)$) have to be applied with $'$-es. When applying Sublemma 3, we use the trivial fact $|\varphi - \bar{\varphi}| \le \mathrm{dist}(x, \bar{x})$.

In the arguments below, as usual, quantites with neither $+$ nor $-$ in their index are ment to have a $+$, that is, in the outgoing phase space.

Sublemma 1 *There exists a global constant C such that when a perturbation of size dx is performed on the base point, we have*

$$|d\ln J_1| \le C|dx|.$$

Proof. In many estimates, we will use – without further mention – that m_- and K_- are bounded (see (3.17) and (3.32)).

With the help of (3.9) we choose the form

$$J_1 = \frac{\sqrt{(1+\kappa(\varphi)Rm_-)^2 + m_-^2}}{2+\kappa(\varphi)}.$$

When calculating the differential, we use

$$d\kappa(\varphi) = \eta(\varphi)d\varphi = \frac{\eta(\varphi)m_+}{\sqrt{1+m_+^2}}dx$$

and

$$dm_- = K_- ds_- = K_- \frac{ds_+}{\lambda_1} = \frac{K_-}{\lambda_1}\frac{dx}{\sqrt{1+m_+^2}} = \frac{K_-}{\sqrt{\lambda_1^2 + m_-^2}}dx$$

Calculating the differential, we get

$$\begin{aligned} d\ln J_1 = & \frac{m_- + \kappa(\varphi)R + \kappa^2(\varphi)R^2 m_-}{((1+\kappa(\varphi)Rm_-)^2 + m_-^2)^{3/2}} K_- dx + \\ & + \frac{2Rm_- - 1 - m_-^2 + (2Rm_- - 1)Rm_-\kappa(\varphi)}{(2+\kappa(\varphi))((1+\kappa(\varphi)Rm_-)^2 + m_-^2)} \frac{\eta(\varphi)m_+}{\sqrt{1+m_+^2}}dx. \end{aligned}$$

The coefficient of dx in the first term is obviously bounded since the denominator is one degree higher in $\kappa(\varphi)$ and is bounded away from zero. In the second term, we use (3.9) and (3.19) to get

$$\left|\frac{m_+}{\sqrt{1+m_+^2}}\right| = \left|\frac{m_-}{\sqrt{1+m_+^2}\lambda_1}\right| \sim \left|\frac{1}{2+\kappa(\varphi)}\right| \tag{3.35}$$

so, looking again at the degrees of polynomials (in κ) in the numerator and denominator of the second term, we have

$$\frac{2Rm_- - 1 - m_-^2 + (2Rm_- - 1)Rm_-\kappa(\varphi)}{(2+\kappa(\varphi))(1+\kappa(\varphi)Rm_-)^2 + m_-^2)}\,\frac{\eta(\varphi)m_+}{\sqrt{1+m_+^2}} \leq C\left|\frac{\eta(\varphi)}{(2+\kappa(\varphi))^3}\right| \leq C_1.$$

Sublemma 2 *There exists a global constant C such that when a perturbation of size dx is performed on the base point, we have*

$$|d\ln J_2| \leq C|dx'|$$

(note the ' on the right hand side).

Proof. With the help of (3.4) and (3.10) we choose the form

$$J_2 = \frac{\cos\varphi + \tau m_+ + \frac{\tau}{R}}{\sqrt{1+m_+^2}}.$$

When calculating the differential, we use

$$d\varphi = \frac{m_+}{\sqrt{1+m_+^2}}dx$$

and

$$dm_+ = K_+ ds_+ = K_+\frac{dx}{\sqrt{1+m_+^2}} = (1+m_+^2)g_+dx.$$

This way we get

$$d\ln J_2 = \frac{-\sin\varphi m_+}{\cos\varphi\lambda_2\sqrt{1+m_+^2}}dx + B'_-d\tau - \frac{m_+(\frac{\tau}{R}+\cos\varphi)-\tau}{\cos\varphi\lambda_2}g_+dx.$$

Due to (3.20), the coefficient of dx in the first term is equivalent to $\frac{-\sin\varphi m_+}{1+m_+^2}$, and in the third term to $-\frac{m_+(\frac{\tau}{R}+\cos\varphi)-\tau}{\sqrt{1+m_+^2}}g_+$, both of which are bounded (cf. (3.35)).

We finish by estimating dx and $d\tau$ with dx'. First,

$$dx' = Jdx \sim \left|\frac{2+\kappa(\varphi')}{\cos\varphi'}\right|dx \geq cdx. \tag{3.36}$$

Second, the triangle inequality implies $|d\tau| \leq |ds| + |ds'_-|$. On the one hand, (3.36) implies $|ds| \leq |dx| \leq C|dx'|$. On the other hand (3.19) implies,

$$dx' = \sqrt{1+m_+'^2}|\lambda_1'|ds'_- \sim |2+\kappa(\varphi')|ds'_- \geq c ds'_-.$$

These give

$$|d\tau| \leq C|dx'|.$$

Sublemma 3 *There exists a global constant C such that if $x=(s,\varphi)$ and $\bar{x}=(\bar{s},\bar{\varphi})$ are in the same homogeneity layer*

$$I_k = \left\{ (s,\varphi) \mid k^2 \leq |\omega(\varphi)| < (k+1)^2 \right\},$$

then

$$|\ln|\omega(\varphi)| - \ln|\omega(\bar{\varphi})|| \leq C_{25}|\varphi-\bar{\varphi}|^{1/5}.$$

Proof. We use the notation $\omega'(\varphi) = \frac{d}{d\varphi}\omega(\varphi)$. It is easy to see that the regularity of $\kappa(\varphi)$ implies

$$\left|\frac{\omega'(\varphi)}{\omega^3(\varphi)}\right| \leq C.$$

That is, everywhere inside J_k,

$$\left|\frac{d|\ln\omega(\varphi)|}{d\varphi}\right| = \left|\frac{\omega'(\varphi)}{\omega(\varphi)}\right| \leq C|\omega(\varphi)|^2 \leq 2Ck^4.$$

This, together with the obvious $k^2 \leq |\omega(\varphi)|, |\omega(\bar{\varphi})| < (k+1)^2$, implies

$$|\ln|\omega(\varphi)| - \ln|\omega(\bar{\varphi})|| \leq \min\left\{2Ck^4|\varphi-\bar{\varphi}|, \ln(k+1)^2 - \ln k^2\right\} \leq \min\left\{2Ck^4|\varphi-\bar{\varphi}|, \frac{2}{k}\right\}.$$

It is easy to check that for every k and every ξ

$$\min\left\{2Ck^4|\xi|, \frac{2}{k}\right\} \leq 2C^{1/5}\xi^{1/5},$$

which completes the proof.

After proving that the expansion factors vary nicely between nearby points on the same u-manifold, we now investigate their behaviour at points of different u-manifolds that lie on the same s-manifold. This is the absolute continuity property. Just like it was with the distorsion bounds, it is important to consider homogeneous manifolds.

We introduce the simplified notation $J_k^u(x)$ and $J_k^s(x)$ for the k-step length expansion factor at x along the unstable and the stable manifold, respectively.

Proposition 4. *Let W_s be a small s-manifold, $x, \bar{x} \in W_s$, and $W_u, \bar{W}_u$ two u-manifolds crossing W_s at x and $\bar{x}$, respectively. Assume that T^k is smooth on W_s and $T^i W_s$ is a homogeneous s-manifold for each $0 \leq i \leq k$. Then*

$$|\ln J_k^u(x) - \ln J_k^u(\bar{x})| \leq C$$

where C is a global constant.

Proof. We have bounds on the change in expansion as we move along unstable manifolds. In order to have such bounds as we move along stable manifolds, we wish to use the fact that stable manifolds are turned into unstable ones when we revert time. However, this time reflection symmetry is not complete: we always work in the outgoing Poincaré section, and reverting time turns this into the incoming one. To deal with the problem, we introduce the map P which is the dynamics through the potential, and which maps from the incoming to the outgoing Poincaré section. That is,

$$P((s_-, \varphi)) := (s_+, \varphi) = (s_- + R\delta\Theta(\varphi), \varphi).$$

We can see from (3.4) that if $dx_- = (ds_-, d\varphi)$ is a tangent vector of the incoming phase space, then

$$|DP(dx_-)|_e = \sqrt{1 + m_+^2}\,|\lambda_1| \frac{1}{\sqrt{1 + m_-^2}} |dx_-|_e.$$

Denote by $\nu(x)$ the expansion factor of DP along the unstable manifold at x, that is $\nu(x) = \frac{|DP(dx_-)|_e}{|dx_-|_e}$ where dx is an unstable vector at x. We can use (3.19) and (3.17) to get

$$\nu(x) \sim |2 + \kappa(\varphi)|.$$

We also introduce the 'turn back' operator, which we will denote by a '$-$' sign: this turns incoming phase points into outgoing phase points which corresponds to reverting the velocity. '$-$' is almost the identity function from M_- to M_+, only the collision angle is reverted (see our sign convention in figure 1):

$$- \; : \; M_- \to M_+$$

$$-(s, \varphi_-) := (s_+, \varphi_+) = (s_-, -\varphi_-).$$

With these notations, if $x = P(y)$, the time reflection symmetry implies

$$J^s_k(x) = \frac{\nu(-x)}{J^u_k(-T^k y)\nu(-T^k x)} \sim \frac{1}{J^u_k(-T^k y)} \frac{|2+\kappa(\varphi)|}{|2+\kappa(\varphi_k)|} \tag{3.37}$$

The transversality of stable and unstable vectors, stated in Proposition 1 implies that $J^u_k(x)J^s_k(x)$ is equivalent to the k-step (Lebesgue) volume expansion factor. Using (2.2), and the T-invariance of μ, we get

$$J^u_k(x)J^s_k(x) \sim \frac{\cos\varphi}{\cos\varphi_k} \tag{3.38}$$

Putting together (3.37) and (3.38) we get

$$J^u_k(x) \sim J^u_k(-T^k y)\frac{2+\kappa(\varphi_k)}{2+\kappa(\varphi)}\frac{\cos\varphi}{\cos\varphi_k} = J^u_k(-T^k y)\frac{\omega(\varphi_k)}{\omega(\varphi)}$$

The same is true for $\bar{x} = P(\bar{y})$, so we have

$$|\ln J^u_k(x) - \ln J^u_k(\bar{x})| \leq |\ln J^u_k(-T^k y) - \ln J^u_k(-T^k \bar{y})| + \left|\ln\left|\frac{\omega(\varphi_k)}{\omega(\bar{\varphi}_k)}\right|\right| + \left|\ln\left|\frac{\omega(\varphi)}{\omega(\bar{\varphi})}\right|\right| + C.$$

To see the boundedness of the first term of the right hand size we can apply Proposition (3), because $-T^k y$ and $-T^k \bar{y}$ are on the same local *un*stable manifold. The second and third term is bounded because W_s and $T^k W_s$ are homogeneous, see Section 3.2.6. Now the proof of Proposition 4 is complete.

4. Growth properties of unstable manifolds

This last section is concerned with the growth properties of LUMs. Our aim is to show that LUMs 'grow large and round, on the avarage'. This is expressed in the formulas of Proposition 5 below.

Recall Convention 1. Throughout the section we use the higher iterate of the dynamics, $T_1 = T^{m_0}$. This has singularity set (secondary and primary) $\Xi = \Gamma^{(m_0)}$. For the higher iterates of T_1 the singularity set is $\Xi^{(n)} = \Xi \cup T_1{}^{-1}\Xi \cup \cdots \cup T^{-n+1}\Xi$.

δ_0-LUM's. To formulate and prove further important conditions on growth of LUMs we need to recall several notions and notations from [Ch]. Let $\delta_0 > 0$. We call W a δ_0-LUM if it is a LUM and $\operatorname{diam} W \leq \delta_0$. For an open subset $V \subset W$ and $x \in V$ denote by $V(x)$ the connected component of V containing the point x. Let $n \geq 0$. We call an open subset $V \subset W$ a (δ_0, n)-subset if $V \cap (\Xi^{(n)}) = \emptyset$ (i.e., the map T_1^n

is smooth and homogeneous on V) and $\operatorname{diam} T_1^n V(x) \le \delta_0$ for every $x \in V$. Note that $T_1^n V$ is then a union of δ_0-LUM's. Define a function $r_{V,n}$ on V by

$$r_{V,n}(x) = d_{T_1^n V(x)}(T_1^n x, \partial T_1^n V(x)).$$

Note that $r_{V,n}(x)$ is the radius of the largest open ball in $T_1^n V(x)$ centered at $T_1^n x$. In particular, $r_{W,0}(x) = d_W(x, \partial W)$.

One further notation we introduce is $\mathcal{U}_\delta$ (for any $\delta > 0$), the δ-neighborhood of the closed set $\Xi \cup S_0 \cup Z_0$.

The aim of this section is to prove the Proposition below.

Proposition 5. *There are constants $\alpha_0 \in (0,1)$ and $\beta_0, D_0, \eta, \chi, \zeta > 0$ with the following property. For any sufficiently small $\delta_0, \delta > 0$ and any δ_0-LUM W there is an open $(\delta_0, 0)$-subset $V_\delta^0 \subset W \cap \mathcal{U}_\delta$ and an open $(\delta_0, 1)$-subset $V_\delta^1 \subset W \setminus \mathcal{U}_\delta$ (one of these may be empty) such that $m_W(W \setminus (V_\delta^0 \cup V_\delta^1)) = 0$ and that $\forall \varepsilon > 0$*

$$m_W(r_{V_\delta^1,1} < \varepsilon) \le \alpha_0 \Lambda_1 \cdot m_W(r_{W,0} < \varepsilon/\Lambda_1) + \varepsilon \beta_0 \delta_0^{-1} m_W(W), \tag{4.1}$$

$$m_W(r_{V_\delta^0,0} < \varepsilon) \le D_0 \delta^{-\eta} m_W(r_{W,0} < \varepsilon) \tag{4.2}$$

and

$$m_W(V_\delta^0) \le D_0\, m_W(r_{W,0} < \zeta \delta^\chi). \tag{4.3}$$

Proof of this Proposition goes along the lines of the arguments from [Ch]. First let us consider

Accumulation of singularity lines. There are two sources of accumulation of the components of the set Ξ that can cut LUM's into arbitrary many pieces.

First, the set Γ_1 consists of countably many curves stretching approximately parallel to some curves in S_1 (or Z_1) and approaching them. So, each set $T^{-1} I_k$ and $k \ne 0$, is a narrow strip with curvilinear boundaries. The expansion of unstable fibers in these strips can be estimated using (3.33), (3.34) and (3.30). More precisely, let $W \subset T^{-1} I_k$ be a LUM, for some $k \ne 0$. Then the expansion factor, $J^u(x)$, on W satisfies

$$J^u(x) \sim \omega(\varphi) \sim k^2 \qquad \forall x \in W. \tag{4.4}$$

Second, there might be multiple intersections of the curves in $S_1 \cup Z_1$. Recall K_n, the complextity of $S^{(n)} \cup Z^{(n)}$ and it is properties from subsection 2.1. Specifically important for us is the choice of the higher iterate $T_1 = T^{m_0}$ with its relevant properties, see Convention 1.

Indexing system. Before proving the proposition we introduce a handy indexing system, cf. [Ch]. Let $\delta_0 > 0$ and W be a δ_0-LUM. If δ_0 is small enough, then W crosses at most K_{m_0} curves of the set $S^{(m_0)} \cup Z^{(m_0)}$, so the set $W \setminus (S^{(m_0)} \cup Z^{(m_0)})$ consists of at most $K_{m_0} + 1$ connected curves, let us call them $W_1, \dots, W_p$ with $p \leq K_{m_0} + 1$.

On each W_j the map T_1 (as a map on M) is smooth, but any W_j may be cut into arbitrary many (countably many) pieces by other curves in Ξ, which are the preimages of the boundaries of I_k. Let $\Delta \subset W$ be a connected component of the set $W \setminus \Xi$. It can be identified with the $(m_0 + 1)$-tuple $(k_1, \dots, k_{m_0}; j)$ such that $\Delta \subset W_j$ and $T^i \Delta \subset I_{k_i}$ for $1 \leq i \leq m_0$. Note that this identification is almost unique. Indeed, given j, $(T^i \Delta \subset) T^i W_j$ is contained in a strip of the phase space that lies between two horizontal lines: two components of $S_0 \cup Z_0$. It might happen that expansion factors diverge – and consequently, homogeneity strips have been constructed – at both sides of the strip. Thus given the index k_i, we have $T^i \Delta \subset I_{k_i}$, where I_{k_i} can be the k_ith layer from one of the two homogeneity structures. In such a case we use the following convention; the homogeneity layers at the 'upper' and 'lower' ends of the phase space strip (corresponding to j) are labelled by odd and even numbers, respectively. This way the indexing system is made unique and (4.4) remains true.

All in all, we will write $\Delta = \Delta(k_1, \dots, k_{m_0}; j)$. Of course, some strings $(k_1, \dots, k_{m_0}; j)$ may not correspond to any piece of W, for such strings $\Delta(k_1, \dots, k_{m_0}; j) = \emptyset$.

Denote by $J_1^u(x) = J^u(x) \cdots J^u(T^{m_0 - 1} x)$ the expansion factor of unstable vectors under DT_1. Let $|\Delta| = m_\Delta(\Delta)$ be the Euclidean length of a LUM Δ. We record two important facts:

(a) For every point $x \in \Delta(k_1, \dots, k_{m_0}; j)$ we have

$$J_1^u(x) \geq L_{k_1, \dots, k_{m_0}} := \max \left\{ \Lambda_1, \, C_{20} \prod_{k_i \neq 0} k_i^2 \right\}$$

where C_{20} is some positive global constant. This follows from (4.4).

(b) For each $\varDelta(k_1, \dots, k_{m_0}; j)$ we have

$$|\varDelta(k_1, \dots, k_{m_0}; j)| \le M_{k_1,\dots,k_{m_0}} := \min\left\{ |W|,\, C_{21} \prod_{k_i \neq 0} k_i^{-2} \right\}$$

where $C_{21} = C_{20}^{-1}|W|_{\max}$ and $|W|_{\max}$ is the maximal length of LUMs in M. This follows from the previous fact.

Next, put

$$\theta_0 := 2 \sum_{k=k_0}^{\infty} k^{-2} \le 4/k_0$$

and let us turn to the proof of our growth formulas.

Let W be a δ_0-LUM and $\delta > 0$ be small. For each connected component $\varDelta \subset W \setminus \varXi$ put $\varDelta^0 = \varDelta \cap \mathcal{U}_\delta$ and $\varDelta^1 = \mathrm{int}(\varDelta \setminus \mathcal{U}_\delta)$ (recall $\mathcal{U}_\delta$ is the δ-neighborhood of $\varXi \cup S_0 \cup Z_0$). Due to the Continuation property (cf. subsection 2.1) and to Alignment (cf. subsubsection 3.2.5), the set $\varDelta^0$ consists of two subintervals adjacent to the endpoints of $\varDelta$ (they may overlap and cover $\varDelta$, of course). The set $\varDelta^1$ is either empty or a subinterval of $\varDelta$. We put $W^1 = \cup_{\varDelta \subset W \setminus \varXi} \varDelta^1$.

Proof of (4.1). For each $\varDelta^1$ the set $T_1(\varDelta_1 \cap \{r_{W^1,1} < \varepsilon\})$ is the union of two subintervals of $T_1\varDelta^1$ of length ε adjacent to the endpoint of $T_1\varDelta^1$. Using the above indexing system we get

$$\begin{aligned} m_W(r_{W^1,1} < \varepsilon) &\le \sum_{k_1,\dots,k_{m_0},j} 2\varepsilon L^{-1}_{k_1,\dots,k_{m_0}} \\ &\le 2\varepsilon p \left[\varLambda_1^{-1} + {C_{20}}^{-1}(\theta_0 + \theta_0^2 + \cdots + \theta_0^{m_0})\right] \\ &\le 2\varepsilon(K_{m_0} + 1)\left(\varLambda_1^{-1} + {C_{20}}^{-1} m_0 \theta_0\right). \end{aligned}$$

We now assume that k_0 is large enough so that

$$\alpha_0 := (K_{m_0} + 1)(\varLambda_1^{-1} + {C_{20}}^{-1} m_0 \theta_0) < 1$$

and thus get

$$m_W(r_{W^1,1} < \varepsilon) \le \min\{|W|, 2\alpha_0\varepsilon\}.$$

The first term on the right hand side of (4.1) is equal to

$$\alpha_0 \varLambda_1 \min\{|W|, 2\varepsilon/\varLambda_1\} = \min\{\alpha_0\varLambda_1|W|, 2\alpha_0\varepsilon\}.$$

Since $\alpha_0 \Lambda_1 > 1$, we get

$$m_W(r_{W^1,1} < \varepsilon) \leq \alpha_0 \Lambda_1 \cdot m_W\left(r_{W,0} < \varepsilon/\Lambda_1\right). \tag{4.5}$$

Next, to obtain an open $(\delta_0, 1)$-subset V_δ^1 of W^1, one needs to further subdivide the intervals $\Delta^1 \subset W$ such that $|T_1 \Delta^1| > \delta_0$. Each such LUM $T_1 \Delta^1$ we divide into s_Δ equal subintervals of length $\leq \delta_0$, with $s_\Delta \leq |T_1 \Delta^1|/\delta_0$. If $|T_1 \Delta^1| < \delta_0$, then we set $s_\Delta = 0$ and leave Δ^1 unchanged. Then the union of the preimages under T_1 of the above intervals will make V_δ^1. Now we must estimate the measure of the ε-neighborhood of the additional endpoints of the subintervals of $T_1 \Delta^1$. This gives

$$\begin{aligned} m_W(r_{V_\delta^1,1} < \varepsilon) - m_W(r_{W^1,1} < \varepsilon) &\leq \sum_{\Delta \subset W \setminus \Xi} 2 s_\Delta \varepsilon |C_{22} \Delta^1| / |T_1 \Delta^1| \\ &\leq \sum_{\Delta \subset W \setminus \Xi} 2 C_{22} \varepsilon |\Delta^1| / \delta_0 \\ &\leq 2 C_{22} \varepsilon \delta_0^{-1} |W|. \end{aligned}$$

Here $C_{22} = \exp(\text{const} \cdot |W|_{\max}^{\frac{1}{5}})$ is an upper bound on distorsions on LUM's, see Proposition 3. Combining the above bound with (4.5) completes the proof of (4.1) with $\beta_0 = 2C_{22}$.

We now *prove (4.2)*. It is enough to consider $\varepsilon < |W|/2$, so that the right hand side of (4.2) equals $2D_0 \delta^{-\eta} \varepsilon$. We can put $V_\delta^0 = W \setminus \overline{V_\delta^1}$. Then the left hand side of (4.2) does not exceed $2J_\delta \varepsilon$, where J_δ is the number of nonempty connected components of the set $\overline{V_\delta^0}$, which is at most the number of connected components of $W \setminus \Xi$ of length $> 2\delta$. Hence, clearly $J_\delta \leq |W|/\delta \leq \delta_0/\delta$. This proves (4.2) with $\eta = 1$.

Finally, we *prove the inequality (4.3)*. Again, let Δ be a connected component of $W \setminus \Xi$ and Δ^0, Δ^1 be defined as above, with the set Δ^0 consisting of two subintervals adjacent to the endpoints of Δ. By (3.28) – and the analogous property for the secondary singularities, see subsubsections 3.2.5 and 3.2.6 – each of these subintervals has length smaller than $C\delta^\alpha$.

Now, the right hand side of (4.3) equals $D_0 \min\{|W|, 2\zeta\delta^\chi\}$. So, it is enough to show that $m_W(V_\delta^0) \leq B\delta^\chi$ for some $B, \chi > 0$. We have

$$\begin{aligned} m_W(V_\delta^0) &\leq \sum_{\Delta \subset W \setminus \Xi} \min\{2C\delta^\alpha, |\Delta|\} \\ &\leq \sum_{k_1, \ldots, k_{m_0}, j} \min\{2C\delta^{alpha}, M_{k_1, \ldots, k_{m_0}}\} \end{aligned}$$

$$\leq \text{const} \cdot \delta^\alpha + \text{const} \cdot \sum_{k_1,\dots,k_{m_0}}{}^{*} \min \left\{ \delta^\alpha, \prod_{k_i \neq 0} k_i^{-2} \right\}$$

where $\sum^*$ is taken over m_0-tuples that contain at least one nonzero index $k_i \neq 0$. The following Lemma – Lemma 7.2 from [Ch], which was proved in the Appendix of that paper – completes the proof of (4.3) with $\chi = \frac{\alpha}{2m_0}$.

Lemma 5. *Let* $\epsilon > 0$ *and* $m \geq 1$. *Then*

$$\sum_{k_1,\dots,k_m \geq 2} \min\left\{\epsilon, (k_1 \cdots k_m)^{-2}\right\} \leq B(m) \cdot \epsilon^{1/2m}.$$

With the help of this Lemma Proposition 5, and consequently, Theorem 1 is proved. □. □.

5. Specific potentials

In this section we would like to show that, as important corollaries of Theorem 1, exponential decay of correlations can be established for certain specific potentials. To prove such corollaries we need to calculate the rotation function $\Delta\Theta(\varphi)$ from the potential $V(r)$.

As to the detailed description of the Hamiltonian flow in a circularly symmetric potential, we refer to the literature, e.g. [DL] and references therein. Most important is that besides the full energy there is an additional integral of motion, the angular momentum l, that can be calculated for a specific trajectory as

$$l = R \sin \varphi$$

where φ is the collision angle at income. For brevity of notation it is worth introducing the function

$$h(r) = (1 - 2V(r))r^2.$$

By the presence of the angular momentum motion is completely integrable and is described by the pair of differential equations (recall our convention that the full energy is $E = \frac{1}{2}$):

$$\dot{r}^2 = r^{-2}(h(r) - l^2)$$

$$r^2 \dot{\Theta} = l.$$

Combining these we get

$$\frac{d\Theta}{dr} = \pm \frac{l}{r\sqrt{h(r) - l^2}} \tag{5.1}$$

where the sign depends on whether r is increasing or decreasing. More precisely, there is a minimum radius

$$\hat{r} = \hat{r}(\varphi) : \qquad h(\hat{r}) = l^2 = R^2 \sin^2 \varphi,$$

down to which r decreases (with negative sign in (5.1)) and from which r increases (with positive sign in (5.1)). This results in

$$\Delta\Theta(\varphi) = 2 \int_{\hat{r}}^{R} \frac{l}{r\sqrt{h(r) - l^2}} dr. \tag{5.2}$$

For a generic potential, the dependence of (5.2) on φ is rather implicit: φ is present both in the integrand (via l) and in the limits (via $\hat{r}$). One possible strategy to follow is to obtain some even more complicated formulas for the derivatives in the general case, and based on those perform estimates that guarrantee the desired dynamical properties. This is possible as long as only hyperbolicity and ergodicity is treated – like in [DL] – and thus only the first derivative, $\kappa(\varphi) = \Delta\Theta'(\varphi)$ is needed. However, for rate of mixing you need one more derivative, $\kappa'(\varphi) = \Delta\Theta''(\varphi)$, cf. Definition 3. Finding good sufficient conditions on the potential $V(r)$ that guarrantee the regularity of κ seems to be a very hard task, if possible at all. Thus we have chosen instead to investigate some specific cases where $\Delta\Theta$ is directly computable from (5.2). Of course, this way we could handle a much narrower class of potentials than [DL], nevertheless, the established dynamical property is stronger.

Corollary 1. *Consider the case of a constant potential,*

$$V(r) = V_0 \quad \text{for any} \quad r \in [0, R).$$

Correlations decay with an exponential rate in case

- $V_0 > 0$ *and the configuration is arbitrary,*
- $V_0 < 0$ *and the configuration is such that* $\tau_{\min} > \frac{2R}{\sqrt{1-2V_0-1}}$.

Remarks. Actually, the analysis of this constant potential case from the point of ergodicity dates back to the late eighties, to [Kn2] and [Ba]. Rate of mixing is, to our knowledge, discussed for the first time. For potential values $V_0 > \frac{1}{2}$ the particle cannot enter the disks, the system is equivalent to the traditional dispersing billiard, thus we consider the opposite case, $V_0 < \frac{1}{2}$.

Proof. Let us introduce the quantity

$$\nu = \sqrt{1 - 2V_0} \tag{5.3}$$

which is less or greater than 1 depending on the sign of V_0. Let us consider the case of positive V_0 first and introduce furthermore the angle φ_0 for which:

$$\nu = \sin \varphi_0 .$$

In case $|\varphi| > \varphi_0$, $|l|$ is greater than the maximum value $h(r)$ can take, which indicates that the particle has too large angular momentum to enter the potential, thus $\varDelta\varTheta = 0$. In the opposite case of $|\varphi| < \varphi_0$ it is easy to obtain $\hat{r} = \frac{R|\sin\varphi|}{\nu}$ and perform the integration of (5.2). All in all

$$\begin{aligned} \varDelta\varTheta(\varphi) = 2 \arccos\left(\frac{\sin\varphi}{\nu}\right) & \quad \text{if } |\varphi| < \varphi_0, \\ 0 \qquad\qquad & \quad \text{if } |\varphi| > \varphi_0. \end{aligned}$$

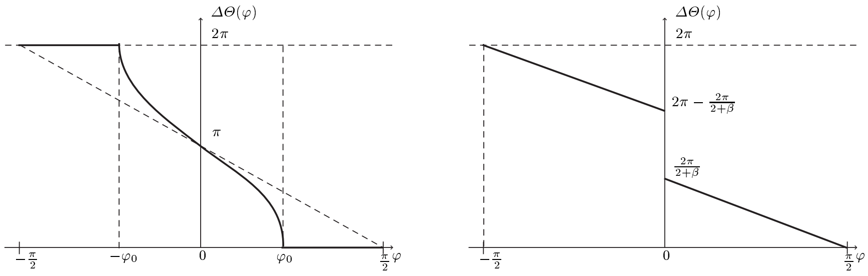
On the one hand, whatever a configuration we have, the system satisfies property H (cf. Definition 2), as either $\kappa = 0$ or $\kappa \leq \frac{-2}{\nu} < -2$. On the other hand, κ is a piecewise C^1 function of φ and it behaves as $(\varphi_0 - \varphi)^{-\frac{1}{2}}$ near the discontinuity point φ_0. Thus κ is regular (cf. Definition 3 and the remarks following it). This means that the first statement of our Corollary follows from Theorem 1.

Now let us turn to the case of $V_0 < 0$ (i.e. $\nu > 1$). It is even simpler to calculate the rotation function (5.2):

$$\varDelta\varTheta(\varphi) = 2 \arccos\left(\frac{\sin\varphi}{\nu}\right)$$

for all φ. As $\nu > 1$, this is a C^2 function on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, thus κ is definitely regular. As to property H, we have $0 > \kappa \geq -\frac{2}{\nu}$ where the minimum is obtained at $\varphi = 0$. Thus the assumption on the configuration from Definition 2 reads as $\tau_{\min} > \frac{2R}{\nu - 1}$ and the second statement of the Corollary follows from Theorem 1.

Remark. Note that motion in the constant potential is equivalent to the problem of diffraction form geometric optics. More precisely, we can think of the disks as if they were made of a material optically different from their neighborhood, where the relative diffraction coefficient is ν from (5.3). In case the disks are optically less dense than their neighborhood (i.e. $\nu < 1$, $V_0 > 0$), we may observe the phenomenon of complete reflection that corresponds to the limiting angle φ_0.



(a) $V(r) = V_0(> 0)$

(b) $V(r) = \frac{1}{2}\left(1 - \left(\frac{r}{R}\right)^{\beta}\right)$ $(\beta > 0)$

Fig. 2. rotation function for two examples

Corollary 2. *Given constants* $A > 0$ *and* $\beta > -2$*, consider the potential*

$$V(r) = A\left(1 - \left(\frac{r}{R}\right)^{\beta}\right).$$

Correlations decay at an exponential rate in case:

- $A = \frac{1}{2}$, $0 > \beta (> -2)$ *and the configuration is arbitrary,*
- $A = \frac{1}{2}$, $\beta > 0$ *and the configuration is such that* $\tau_{\min} > \frac{2R}{\beta}$.

Remark. Note that according to our construction the chosen value for the constant A, $A = \frac{1}{2}$ is exactly the full energy. If we had a different value for A, the integration in (5.2) would be much more complicated. In other words, Corollary 2, in contrast to Corollary 1 is unstable with respect to variations of the full energy (see also the discussion below, following the proof). Nevertheless it is nice to have at least one potential with exponential mixing for any kind of power law behaviour (if $\beta \leq -2$, a positive measure set of trajectories is pulled into the center of the disk, cf. [DL]).

Proof. By starightforward calculation

$$h(r) = \frac{r^{2+\beta}}{R^\beta}; \quad \text{and} \quad \hat{r} = R\,|\sin\varphi|^{\frac{2}{2+\beta}}.$$

Then it is not hard to integrate in (5.2):

$$\varDelta\varTheta(\varphi) = \frac{4}{2+\beta}\left(\frac{\pi}{2} - \varphi\right)$$

for all $\varphi \neq 0$. Thus $\varDelta\varTheta$ is piecewise linear (in the general case with one discontinuity of the first kind at $\varphi = 0$) and thus

$$\kappa = -\frac{4}{2+\beta}$$

identically. Regularity (in terms of Definition 3) is automatic.

Let us consider the attracting potentials, $\beta < 0$ first. In such a case the potential has a singularity at the center of the disk, resulting in the discontinuity at $\varphi = 0$. [2] Nevertheless, $\kappa < -2$, thus property H (cf. Definition 2) and consequently the first statement of the Corollary follows.

Now if $\beta > 0$, as $A = \frac{1}{2}$, the 'top' of the potential is equal to the energy. As a consequence, for the initial value $\varphi = 0$ the flow is not uniquely defined, resulting in the discontinuity for the rotation function. However, in accordance with Definition 2, property H is satisfied if $\tau_{\min} > \frac{2R}{\beta}$. Thus the second statement of the corollary holds.

Discussion.

As already mentioned, Corollary 2 is much sensitive to the convention $E = \frac{1}{2}$. Though very difficult to calculate, it is interesting to guess what happens if one perturbs the constant A (or equivalently, the full energy level).

Let us consider the case $\beta > 0$ first. With A either increased or decreased from the value $\frac{1}{2}$, the physical reason for the discontinuity at $\varphi = 0$ disappears and we expect smooth rotation functions. By continuity of the potential at R $\varDelta\varTheta(\frac{\pi}{2}) = 0$ seems also reasonable. As to the initial value $\varphi = 0$ let us have a look at the case $A < \frac{1}{2}$ first. There is no reason for the trajectory to deviate in direction: it slows

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[2] However, in case $\beta = -2(1 - \frac{1}{n})$, the left and right limits coincide, this correponds to the possibility of regularizing the flow, cf. [DL] and [Kn1].

down, reaches the center and then speeds up following a linear track. Thus $\varDelta\varTheta(0) = \pi$. This altogether implies on basis of Lagrange's mean value theorem that there definitely exists at least one $\varphi \in (0, \frac{\pi}{2})$ for which $\kappa(\varphi) = -2$. In such a case, however, stable periodic orbits tend to appear and the system is most likely not even ergodic, cf. [Do1]. One can suspect that a typical repelling potential which has a maximum less than the total energy, leads to non-ergodic soft billiards in a similar fashion.

In the opposite case of $A > \frac{1}{2}$ the behaviour of trajectories in the vicinity of $\varphi = 0$ is completely different. As the top of the potential is higher then the full energy, the particle cannot 'climb' it thus it should 'turn back'. We expect $\varDelta\varTheta(0) = 0$ and a smooth rotation function with $\kappa > -2$ for all φ. That would mean ergodicity and possibly exponential mixing in case of a suitable configuration (cf. Definition 2). All in all, ergodic and statistical behaviour is much sensitive to perturbation of the full energy level.

In case of $\beta < 0$ it is no so easy to guess. Nevertheless, we can say something rather surprising in one particular case that indicates similar sensitivity. Choose $\beta = -1$ and $A = 1$. It is not difficult to obtain $h(r) = 2r - r^2$. The integral in (5.2) is a bit more complicated now, nevertheless, its is possible to evaluate:

$$\varDelta\varTheta(\varphi) = 2\pi - 2\varphi \tag{5.4}$$

which means $\kappa = -2$ identically. This corresponds to the least ergodic behaviour we can have. It is straightforward to obtain that an identically zero potential ($V(r) = 0$ for all r) would result in $\varDelta\varTheta(\varphi) = \pi - 2\varphi$. Thus by (5.4) in this particular case of $A = 1, \beta = -1$ trajectories evolve as if they passed on freely and were reflected when leaving the disck.

Thus if $\beta = -1$, we may have exponential mixing ($A = \frac{1}{2}$) and stability ($A = 1$). As to other values of A it is worth mentioning that ergodicity follows from [DL] in case $A < \frac{1}{2}$.

6. Outlook

In this last section we list several possible interesting directions of future research.

1. As to the possibly most direct challenge, we conjecture that there exist rapidly mixing potentials for which the condition $|\kappa+2| > c$ (i.e. property H from Definition 2) is not satisfied for nearly tangential

trajectories. Thus these systems are not covered by Theorem 1, even more, at least to our knowledge, there is no result in the literature on the ergodicity or hyperbolicity of such soft billiards either. Thus we make the following

Remark 1. Note that it is possible that κ tends to -2 as $\varphi \to \frac{\pi}{2}$, nevertheless, $|\frac{2+\kappa}{\cos\varphi}| > c$ and the system can be hyperbolic (possibly ergodic or exponentially mixing). We will turn back to this question in a separate paper.

The difficulty with the treatment of this case is, as already mentioned in section 1, that the separate investigation of motion inside and outside the disks seems not to work at several arguments.

2. Further exciting open questions seem even more difficult. One natural direction of generalization is of course the higher dimensional case. As to softenings of multidimensional dispersing billiards (motivated eg. by the three dimensional Lorentz process with spherical scatterers) we are not aware of any mathematical result. Even hyperbolicity and ergodicity seem difficult, no to mention decay of correlations, especially in view of the recently observed pathological behaviour of singularity manifolds in multi-dimensional billiards (see [BChSzT]).
3. Another direction of future research, motivated mainly by applications to physics, could be the further investigation of those systems for which rapid mixing is already established. For example, as mathematical evidence on the existence of diffusion and other transport coefficents is given, it would be interesting to understand the dependence of these on certain parameters like the full energy level.
4. Last but not least, in contrast to the generality of Theorem 1, it is striking how narrow the class of specific potentials is for which we could apply the result in section 5. It would be desirable to establish – at least numerically – our reasonable regularity properties for as wide a class of potentials as possible.

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