Do Bosons Condense in a Homogeneous Magnetic Field ?

Philippe Briet[1], Horia D. Cornean[2], and Valentin A. Zagrebnov[3]

It has been known since the rigorous result by Angelescu and Corciovei [A-C] that the answer to the question in the title is negative for the Perfect Bose Gas (PBG). The main result of the present paper is that the answer could become positive if the bosons are simultaneously embedded in a periodic external potential. We show that it is true for PBG, as well as for the Bose gas with a mean-field repulsive particle interaction.

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1 Introduction

Consider a system of boson atoms confined by a nonhomogeneous magnetic field at temperature $T \geq 0$, and subjected to a cooling laser radiation. Nowadays this is a standard picture of the so-called magneto-optical traps. But the bosons in a magnetic field were studied even much earlier, back in the sixties, when experimentalists were exploring for artificial systems manifesting the Bose-Einstein Condensation (BEC). One of the first of them was the atomic Hydrogen, where atoms were prevented from recombining into molecules due to the presence of a strong homogeneous magnetic field. Since it was argued that this system remained a gas up to the temperature of absolute zero and since this gas was very diluted, it was expected to manifest at low temperatures a BEC similar to that in the PBG. But the rôle of magnetic field in

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[1] Université de Toulon et du Var, and Centre de Physique Théorique-CNRS, Campus de Luminy, Case 907 13288 Marseille cedex 9, France; e-mail: briet@univ-tln.fr

[2] Institut for Matematiske Fag, Aalborg Universitet, Fredrik Bajers Vej 7G, 9220 Aalborg, Danmark; H.C. is partially supported by MaPhySto – A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation; e-mail: cornean@math.auc.dk

[3] Université de la Méditerranée, and Centre de Physique Théorique-CNRS, Campus de Luminy, Case 907 13288 Marseille cedex 9, France; e-mail: zagrebnov@cpt.univ-mrs.fr

these experiments was reduced to preparation and trapping neutral boson atoms for further cooling.

The question in the title and the present paper are motivated by another problem. Namely, how thermodynamic properties of (charged) quantum gases will change in the presence of a magnetic field in the approximation when we neglect particle interaction? The first rigorous result in this direction was due to [A-C], who studied both Bose and Fermi perfect gases. One of their conclusion is a sort of "no-go" theorem forbidding BEC of the PBG in dimension $d = 3$. Since at the same time their result allows the BEC at higher dimensions ($d > 4$), it is clear that the impact of the homogeneous magnetic field reduces to a modification of the one-particle density of states at the bottom of the spectrum.

The aim of the present paper is to find an external ("electric") potential which is able to restore BEC in $d = 3$. Motivated by recent experiments with optical lattices (see e.g. [BeS-M] and references therein) we construct a class of periodic potentials with this property.

Now we come to our mathematical model. Denote by $\Lambda_1 \in \mathbb{R}^d$ an open, convex and simply connected domain with smooth boundary $\partial\Lambda_1$, containing the origin of coordinates; here $1 \leq d \leq 3$. The box which traps our system is given by ($L > 1$)

$$\Lambda_L := \{\mathbf{x} \in \mathbb{R}^d,\ \mathbf{x}/L \in \Lambda_1\}. \tag{1.1}$$

In this paper we consider continuous $\mathbb{Z}^d-$periodic external potentials V (i.e. $\boldsymbol{\gamma} \in \mathbb{Z}^d, V(\mathbf{x}+\boldsymbol{\gamma}) = V(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$), V_L denotes the restriction of V to Λ_L. If $d = 3$ we also consider a magnetic vector potential of the form:

$$\mathbf{a}(\mathbf{x}) = \omega \mathbf{a}_0(\mathbf{x}), \quad \omega \geq 0 \tag{1.2}$$

where either one of the two types of gauge: symmetric (transverse) $\mathbf{a}_0(\mathbf{x}) = 1/2(-x_2, x_1, 0)$ or Landau $\mathbf{a}_1(\mathbf{x}) = (0, x_1, 0)$ will be used; in both cases this generates a unit magnetic field "parallel to the third direction". Now let

$$h_L = h_L(\omega) = (-i\nabla - \mathbf{a})^2 + V_L, \tag{1.3}$$

be the one particle Hamiltonian defined on $L^2(\Lambda_L)$ with Dirichlet boundary conditions (DBC) on $\partial\Lambda_L$. Then h_L has purely discrete spectrum [R-SIV], we denote the set of eigenvalues (counting multiplicities and in increasing order) by $\{\lambda_j\}_{j\geq 1}$ and by $\{u_j\}_{j\geq 1}$ the corresponding set of eigenfunctions. We denote by h_∞ the unique self-adjoint extension of the operator $(-i\nabla - \mathbf{a})^2 + V$

defined on $C_0^\infty(\mathbb{R}^d)$ [R-SII]. Because of the magnetic field, the nature of the spectrum of h_∞ is not known in general; but since by our assumptions h_∞ is bounded from below and commutes with the (magnetic) translations, then h_∞ has no discrete spectrum. Let us denote by $E_0 := \inf \sigma(h_\infty)$. Moreover, due to standard arguments involving the min-max principle, for all $L > 1$ we have $E_0 \leq \lambda_1$ [R-SIV].

We first consider a perfect Bose gas (PBG) confined in the volume Λ_L, each particle of the gas interacts with the background potential V_L and the external magnetic field. The case of an imperfect gas with a Mean-Field (MF) type of particle potential will be considered in Section 5. In the grand-canonical ensemble, the pressure of a perfect gas at inverse temperature $\beta > 0$ and chemical potential $\mu < E_0$ is given by the well known expression, see e.g. [Hu] and also Section 5:

$$p_L(\beta, z) := -\frac{1}{\beta|\Lambda_L|}\sum_{j\geq 1}\ln(1 - ze^{-\beta\lambda_j}), \tag{1.4}$$

where $\{\lambda_j\}_{j\geq 1}$ is the set of eigenvalues of the one particle Hamiltonian (1.3); z is the fugacity $z := e^{\beta\mu}$. The density of the gas is:

$$\rho_L(\beta, z) := \beta z\frac{\partial p_L}{\partial z}(\beta, z) = \frac{1}{|\Lambda_L|}\sum_{j\geq 1}\frac{ze^{-\beta\lambda_j}}{1 - ze^{-\beta\lambda_j}}. \tag{1.5}$$

Since the semigroup $e^{-\beta h_L}$ generated by h_L is trace class, i.e. $\sum_{j\geq 1} e^{-\beta\lambda_j} < \infty$ [S 1], the series in (1.4) and (1.5) are absolutely convergent. It is known that under our assumptions the thermodynamic limit ($L \to \infty$) of the pressure p_L and of the particle density ρ_L exist [A-C] and we are now interested in the behavior of $\rho_\infty(\beta, z) := \lim_{L\to\infty} \rho_L(\beta, z)$ near the critical value $z_c = e^{\beta E_0}, \beta > 0$ since this determines whether the Bose-Einstein condensation takes place for our system [Hu], [Z-U-K].

Let $P_I(h_L)$ be the spectral projection of the operator h_L for a Borel set $I \subset \mathbb{R}$. Denoting by $N_L(\lambda) := |\Lambda_L|^{-1}\mathrm{Tr}\ (P_{(-\infty,\lambda)}(h_L))$ the counting function of eigenstates of h_L (the number of eigenstates of h_L for eigenvalues less than λ), we have:

$$\rho_L(\beta, z) = -\int_{E_0}^{\infty}\left[\partial_\lambda \frac{ze^{\beta\lambda}}{1 - ze^{\beta\lambda}}\right]\frac{N_L(\lambda)}{|\Lambda_L|}d\lambda. \tag{1.6}$$

Recall that the integrated density of states for h_∞, denoted by $n_\infty(\lambda)$ is defined as a weak limit:

$$n_\infty(\lambda) = \lim_{L\to\infty} \frac{N_L(\lambda)}{|\Lambda_L|} \tag{1.7}$$

on the space of continuous functions $C_0([E_0, \infty))$, see e.g. [P-F].

Moreover, let χ_{Λ_L} be the characteristic function of Λ_L and $P_I(h_\infty)$ be the spectral projection of h_∞ for a Borel set $I \subset \mathbb{R}$. Then under even more general conditions than ours, for any $\lambda \in \mathbb{R} \setminus \sigma_p(h_\infty)$, the pointwise limit

$$\tilde{n}_\infty(\lambda) := \lim_{L\to\infty} \frac{\mathrm{Tr}(\chi_{\Lambda_L} P_{(-\infty,\lambda)}(h_\infty)\chi_{\Lambda_L})}{|\Lambda_L|} \tag{1.8}$$

exists, is continuous and coincides with $n_\infty(\lambda)$ (see e.g. [B-S], [H-S] and [D-I-M]).

Notice that by (1.6) and (1.7), the density $\rho_L(\beta, z)$ admits for $z < z_c$ a thermodynamic limit of the form:

$$\rho_\infty(\beta, z) = -\int_{E_0}^{\infty} \left[\partial_\lambda \frac{z\ e^{-\beta\lambda}}{1 - z\ e^{-\beta\lambda}}\right] n_\infty(\lambda) d\lambda. \tag{1.9}$$

We easily see from (1.9) that the limit density $\rho_\infty(\beta, z = e^{\beta\mu})$, increases with μ and decreases with β. Moreover, $\rho_\infty(\beta, \cdot)$ has an analytic extension to the domain $\mathbb{C} \setminus [z_c, \infty)$.

Definition 1.1. *The homogeneous Bose gas manifests the Bose-Einstein condensation (BEC) if for every* $\beta > 0$*, it admits a finite critical density* $\rho_c(\beta)$*, where*

$$\rho_c(\beta) := \lim_{\mu \nearrow E_0} \rho_\infty(\beta, z = e^{\beta\mu}) < \infty. \tag{1.10}$$

Corespondingly, the critical temperature $1/\beta_c(\rho)$ *for a given density* ρ *is defined as the unique solution of the equation* $\rho = \rho_c(\beta)$*, i.e.*

$$\rho = \rho_c(\beta_c(\rho)).$$

For the "free" PBG, when $\omega = 0$ and $V = 0$, the integrated density of states is known explicitely $n_\infty(\lambda) = [d(2\sqrt{\pi})^d \Gamma(d/2)]^{-1}\lambda^{d/2}$, see e.g. [R-S IV]. Hence, by (1.9) one gets $\rho_c(\beta) < \infty$ for $d > 2$. This implies the BEC of the perfect gas for these dimensions.

On the other hand, we know from [A-C] that for a perfect Bose gas, the BEC does not exist (i.e. $\rho_c(\beta) = \infty$) in the presence of a homogeneous magnetic field ($\omega \neq 0$, $V = 0$). We will see that this is related to the fact that

$$n_\infty(\lambda) = B_{\omega,d} \cdot (\lambda - E_0(\omega))^{d/2-1} + o((\lambda - E_0(\omega))^{d/2-1})$$

for $\lambda \searrow E_0(\omega)$. Hence, integral (1.9) diverges for $z = z_c$.

In what follows, we will show that by adding a certain external periodic potential, we can restore the BEC in our system with magnetic field. In particular, we prove the following main theorem:

Theorem 1.2. *Consider a three dimensional perfect Bose gas in a homogeneous magnetic field, where the one-particle Hamiltonian is given by* $h_{0,L} = (-i\nabla - \mathbf{a})^2$ *on* $L^2(\Lambda_L)$ *with DBC on* $\partial\Lambda_L$*; here* $\mathbf{a} = \omega\mathbf{a}_0$, $\mathbf{a}_0(\mathbf{x}) := 1/2(-x_2, x_1, 0)$ *and* $\omega > 0$.

Then there exist $\mathbb{Z}^3-$*periodic and continuous external potentials* V*, such that the perturbed system described by the one-particle Hamiltonian* $h_L = h_{0,L} + V_L$ *on* $L^2(\Lambda_L)$*, manifests the BEC.*

This statement remains true by switching on a mean-field particle interaction.

The proof of the Theorem 1.2 is actually based on the following remark. Since the diamagnetic inequality (see [S 2]) yields $h_\infty \geq -\Delta + \min(V)$, by the Weyl estimate we get a polynomial upper bound $\sim \lambda^{d/2}$ for $n_\infty(\lambda)$ at infinity. Therefore, the only factor which can decide whether the limit in (1.10) is finite or not is the behavior of $n_\infty(\lambda)$ near the bottom E_0 of the spectrum $\sigma(h_\infty)$. Indeed, one can easily see that a sufficient condition for having a finite critical density is the estimate:

$$n_\infty(\lambda) \leq const \cdot (\lambda - E_0)^{1+\alpha}, \quad \lambda \in (E_0, E_0 + \epsilon) \tag{1.11}$$

for some $\alpha > 0$ and finite $\epsilon > 0$. On the contrary, a sufficient condition for an infinite critical density (or zero critical temperature) is the estimate

$$n_\infty(\lambda) \geq const \cdot (\lambda - E_0), \quad \lambda \in (E_0, E_0 + \epsilon) \tag{1.12}$$

for some finite $\epsilon > 0$.

Remark 1.3. *More generally, a necessary and sufficient condition for having a finite critical density for every* $\beta > 0$ *is the following estimate:*

$$\int_{E_0}^{E_0+1} \frac{n_\infty(\lambda)}{(\lambda - E_0)^2} d\lambda < \infty. \tag{1.13}$$

This condition also implies that $\rho_c : (0, \infty) \to (0, \infty)$ *is smooth and invertible (one shows that* ρ_c *is decreasing and onto).*

By the virtue of Definition 1.1, an equivalent way of defining BEC is imposing that the critical temperature $\beta_c(\rho)^{-1}$ *exists and is positive for every density. If the critical density is infinite, we set* $\beta_c(\rho) = \infty$*.*

We close this section by giving some technical points which are important in this paper. Let $f \in C_0^\infty(\mathbb{R})$, we have, (see [D-I-M]):

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}\ [\chi_{\Lambda_L} f(h_\infty) \chi_{\Lambda_L}] = -\int_{\mathbb{R}} f'(t) n_\infty(t) dt. \tag{1.14}$$

Moreover, we will show in Appendix 1 that $f(h_\infty)$ is an integral operator with a smooth integral kernel $f_{h_\infty}(\mathbf{x}, \mathbf{x}')$. Since h_∞ commutes with the magnetic translations (see Appendix 2, (5.36) for their definition), then:

$$\forall \boldsymbol{\gamma} \in \mathbb{Z}^d, \quad f_{h_\infty}(\mathbf{x} + \boldsymbol{\gamma}, \mathbf{x} + \boldsymbol{\gamma}) = f_{h_\infty}(\mathbf{x}, \mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d. \tag{1.15}$$

Therefore

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}\ [\chi_{\Lambda_L} f(h_\infty) \chi_{\Lambda_L}] = -\int_{\mathbb{R}} f'(t) n_\infty(t) dt = \frac{1}{|\Omega|} \int_\Omega f_{h_\infty}(\mathbf{x}, \mathbf{x}) d\mathbf{x}, \tag{1.16}$$

where $\Omega := (-1/2, 1/2)^d$ is the elementary cell, see Section 2.

Assume that the operator h_∞ is (magnetic) translation invariant in some subpace $\mathbb{R}^{d'}$ of $\mathbb{R}^d, d' < d$. For all $\mathbf{x} \in \mathbb{R}^d$ we write $\mathbf{x} = (\overline{x}, x)$, where $\overline{x}$ is the component of $\mathbf{x}$ in the subpace $\mathbb{R}^{d'}$. The kernel's diagonal of f_{h_∞} then is $\overline{x}$ independent. Thus (1.16) reads as

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathrm{Tr}\ [\chi_{\Lambda_L} f(h_\infty) \chi_{\Lambda_L}] = \frac{1}{|\Xi|} \int_\Xi f_{h_\infty}((\overline{0}, x); (\overline{0}, x)) dx. \tag{1.17}$$

where now $\Xi \subset \mathbb{R}^{d-d'}$ is the elementary cell in the subspace orthogonal to $R^{d'}$.

Our paper is organized as follows: Section 2 is devoted to a discussion on various results concerning the BEC for a perfect Bose gas in the presence of periodic external potential without magnetic field. Most of the facts given in this section are known but they are instructive for the rest of the paper. In Section 3 we discuss the stability of the BEC after an external magnetic field is switched on; we will also prove there the first part of Theorem 1.2. The results of Section 3 are applied in Section 4 where we study the imperfect Bose gas, in the case of a mean-field type interaction. We collect in Section 5 technical Appendices indinspensable for proofs placed in Section 3.

2 BEC for a Bose gas in a periodic external potential

In this section we assume that each particle of the Bose gas interacts with a continuous, $\mathbb{Z}^d$- periodic potential V; without loss of generality we will choose $\min(V) = 0$. The one particle Hamiltonian $h_L = -\Delta + V$ is then a self-adjoint operator on $L^2(\Lambda_L)$ (with DBC on $\partial\Lambda_L$) as well as the infinite-volume Hamiltonian $h_\infty = -\Delta + V$ on $L^2(\mathbb{R}^d)$.

We now apply the standard Floquet theory for periodic operators (see [R-S IV]). Let $\Omega^* = 2\pi\Omega = (-\pi, \pi)^d \subset \mathbb{R}^d$ be the elementary cell of the lattice dual to $\mathbb{Z}^d$, which is generated by translations of the cell $\Omega = (-1/2, 1/2)^d$. Define a unitary operator:

$$U : L^2(\mathbb{R}^d) \mapsto \int_{\Omega^*}^{\oplus} L^2(\Omega) d\mathbf{k}, \quad (Uf)(\mathbf{k}, \underline{x}) := \sum_{\boldsymbol{\gamma} \in \mathbb{Z}^d} \frac{1}{(2\pi)^{d/2}} e^{-i\mathbf{k}\cdot(\underline{x}+\boldsymbol{\gamma})} f(\underline{x} + \boldsymbol{\gamma}),$$

where $\mathbf{k} \in \Omega^*$ and $\underline{x} \in \Omega$. Then the unitary transformation of h_∞ is decomposable into the direct integral: $U h_\infty U^* = \int_{\Omega^*} h(\mathbf{k}) d\mathbf{k}$. Here the fiber Hamiltonians:

$$h(\mathbf{k}) = (-i\nabla + \mathbf{k})^2 + V, \quad \mathbf{k} \in \Omega^* \tag{2.1}$$

live in $L^2(\Omega)$ with periodic boundary conditions. They have purely discrete spectrum which accumulates at infinity; for a given $\mathbf{k} \in \Omega^*$, we denote the set of corresponding eigenvalues by $\{\lambda_j(\mathbf{k})\}_{j\geq 1}$.

An important ingredient for us is a result due to Kirsch and Simon [K-S] about the band structure of $\sigma(h_\infty)$:

Proposition 2.1. *Assume that the potential V is $\mathbb{Z}^d$-periodic, continuous and* $\min(V) = 0$. *Consider the operator $h_\infty = -\Delta + V$ on $L^2(\mathbb{R}^d)$ and let*

$\{\lambda_j(\mathbf{k})\}_{j\geq 1}$ be the eigenvalues of the fiber Hamiltonians $h(\mathbf{k}), \mathbf{k} \in \Omega^*$, defined in (2.1). Denoting by $E_0 = \inf \sigma(H_\infty) \geq 0$, we have:

(i). *$\lambda_1(\mathbf{k})$ has E_0 as a nondegenerate minimum at $\mathbf{k} = \mathbf{0}$ i.e.*

$$\min_{\mathbf{k}\in\Omega^*} \lambda_1(\mathbf{k}) = \lambda_1(\mathbf{0}) = E_0, \quad \lambda_1(\mathbf{k}) = E_0 + Q(\mathbf{k},\mathbf{k}) + o(|\mathbf{k}|^2), \text{for } \mathbf{k} \to \mathbf{0}, \tag{2.2}$$

where Q is a positive quadratic form on $\mathbb{R}^d$;

(ii). *E_0 is isolated from the rest of the spectrum $\sigma(h_\infty)$ i.e.*

$$\inf_{j\geq 2}\{\min_{\mathbf{k}\in\Omega^*} \lambda_j(\mathbf{k})\} - E_0 =: \lambda_0 > 0. \tag{2.3}$$

□

The eigenvalues of the quadratic form Q are related to the so-called effective masses in the corresponding directions.

The main result of this section is contained in the next proposition:

Proposition 2.2. *Under the same assumptions as in Proposition 2.1, the critical density defined as the limit in (1.10) is infinite for $d = 1$ or $d = 2$. If $d = 3$, the critical density (1.10) is finite and the PBG manifests the BEC.*

Proof. We show that if $d \in \{1, 2\}$ then (1.12) holds while if $d = 3$ then (1.11) holds. Let $f \in C_0^\infty(\mathbb{R})$. Since the kernel $f_{h_\infty}(\underline{x}, \underline{x}')$ is jointly continuous and decay polynomialy with respect to the variable $\underline{x} - \underline{x}'$ (see Appendix 1), the operator $f(h(\mathbf{k}))$ admits an integral kernel which due to the fiber decomposition has for $\underline{x}$ and $\underline{x}' \in \Omega$ the representation:

$$f_{h(\mathbf{k})}(\underline{x}, \underline{x}') = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^d} f_{h_\infty}(\underline{x} + \boldsymbol{\gamma}, \underline{x}')\, e^{-i\mathbf{k}\cdot(\underline{x}-\underline{x}'+\boldsymbol{\gamma})}. \tag{2.4}$$

We will see in the next section an extension of this formula for the more general magnetic case (cf. (3.17)). Now (2.4) implies that $f_{h(\mathbf{k})}(\underline{x}, \underline{x}')$ is jointly continuous in $\underline{x}$ and $\underline{x}'$. On the other hand $f(h(\mathbf{k}))$ is a finite rank operator and due to the smoothness property evoked above, its trace equals the integral of its kernel's diagonal:

$$\mathrm{Tr} f(h(\mathbf{k})) = \sum_{j\geq 1} f(\lambda_j(\mathbf{k})) = \int_\Omega f_{h(\mathbf{k})}(\underline{x}, \underline{x}) d\underline{x} = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^d} \int_\Omega f_{h_\infty}(\underline{x}+\boldsymbol{\gamma}, \underline{x})\, e^{-i\mathbf{k}\cdot\boldsymbol{\gamma}} d\underline{x}. \tag{2.5}$$

Then integrating (2.5) with respect to the $\mathbf{k}$ variable we have:

$$\sum_{j\geq 1}\frac{1}{(2\pi)^d}\int_{\Omega^*} f(\lambda_j(\mathbf{k}))d\mathbf{k} = \int_{\Omega} f_{h_\infty}(\underline{x},\underline{x})d\underline{x} = -\int_{\mathbb{R}} f'(t)n_\infty(t)dt, \qquad (2.6)$$

where the second equality comes from (1.16). Take a weakly converging sequence $f_n(t) \to \chi_{[E_0-0,\lambda]}(t)$, $n \to \infty$, and $\lambda \leq \max_{\mathbf{k}\in\Omega^*}\lambda_1(\mathbf{k})$. Then (2.6) and Proposition 2.1 imply:

$$n_\infty(\lambda) = \frac{1}{(2\pi)^d}\int_{\Omega^*}\chi_{[E_0-0,\lambda]}(\lambda_1(\mathbf{k}))d\mathbf{k}, \qquad (2.7)$$

where χ_I denotes the indicator of the set $I \subset \mathbb{R}$. Now, Proposition (2.1) (i) and a change of variables in (2.7) give:

$$n_\infty(\lambda) = A_d(\lambda - E_0)^{d/2} + o((\lambda - E_0)^{d/2}), \qquad (2.8)$$

for $\lambda \searrow E_0$. Notice that this is exactly the same behavior as in the "free" case (i.e. $V = 0$), and the proposition is proven. □

3 BEC for a Bose gas in presence of a constant magnetic field

In this section we prove the first part of our main Theorem 1.2. As it has been mentioned before, we are motivated by the work of Angelescu-Corcovei [A-C] who showed that for a free, three-dimensional Bose gas, the critical density is infinite in the presence of a constant magnetic field, i.e. BEC disappears. The mechanism of that is described in Section 1: by creating the Landau levels, the magnetic fields leads to increasing of the integrated density of states at the bottom of the spectrum $\sigma(h_0(\omega \neq 0))$. We only consider the case of dimension $d = 3$, for which the BEC in the PBG holds when $\mathbf{a} = 0$ even for a periodic external potential by Proposition (2.2). We first give an example of a periodic potential V where the BEC is distroyed by *any* constant magnetic field. Then we show that this is not always the case, i.e. we prove the first part of Theorem 1.2.

3.1 Instability of BEC in the presence of a magnetic field

We start with a simple case where the continuous external potential $V(\mathbf{x}) = v(x_1)$ i.e. it only depends on one variable and v is $\mathbb{Z}$-periodic. Throughout this section, we use the Landau gauge $\mathbf{a}_1(\mathbf{x}) = (0, x_1, 0) \in \mathbb{R}^3$. Notice that the choice of a particular gauge is irrelevant since the density of states is gauge invariant. Under these conditions, the "bulk" Hamiltonian is:

$$h_\infty = (-i\nabla - \omega \mathbf{a}_1)^2 + v = -\partial_{x_1}^2 + v(x_1) + (-i\partial_{x_2} - \omega x_1)^2 - \partial_{x_3}^2, \tag{3.1}$$

acting on $L^2(\mathbb{R}^3)$, where $\omega \geq 0$.

Proposition 3.1. *Consider a perfect Bose gas described by the one particle Hamiltonian h_L defined as the restriction of the operator (3.1) to $L^2(\Lambda_L)$ with DBC. Then for every $\omega > 0$, the critical density is infinite, i.e. the BEC is destroyed.*

Before coming to the proof of Proposition 3.1, we need some technical results. We will often write a vector $\mathbf{u} \in \mathbb{R}^3$ as $\mathbf{u} = (u_1, \tilde{u})$ with $\tilde{u} = (u_2, u_3) \in \mathbb{R}^2$. Decompose $L^2(\mathbb{R}^3)$ with the help of the partial Fourier transform with respect to x_2 and x_3:

$$U : L^2(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^2} L^2(\mathbb{R}) d\tilde{k},\ U = \int_{\mathbb{R}^2} U_{\tilde{k}} d\tilde{k},\ (U_{\tilde{k}} f)(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{x}} f(t, \tilde{x}) d\tilde{x}. \tag{3.2}$$

Then

$$U h_\infty U^* = \int_{\mathbb{R}^2} h(\tilde{k}) d\tilde{k}, \quad h(\tilde{k}) = -\frac{d^2}{dt^2} + (\omega t - k_2)^2 + v(t) + k_3^2. \tag{3.3}$$

If $z \in \mathbb{C} \setminus \mathbb{R}$ denote by $(h_\infty - z)^{-1}(x_1, \tilde{x}; x_1', \tilde{x}')$ and $[h(\tilde{k}) - z]^{-1}(t, t')$ the integral kernels of the corresponding operators. We are interested here in the analog of (2.4).

Lemma 3.2. *If $f \in C_0^\infty(\mathbb{R})$, then $f(h_\infty)$ admits a smooth integral kernel $f_{h_\infty}(t, \tilde{y}; t', \tilde{y}')$ and we have the representation*

$$f_{h(\tilde{k})}(t, t') = \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}} f_{h_\infty}(t, \tilde{y}; t', \mathbf{0}) d\tilde{y}, \tag{3.4}$$

for the kernel of the operator in (3.3) where the integral is absolutely convergent.

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Then $f(h_\infty)$ admits a smooth integral kernel which decays faster than any polynomial in $\overline{y}$ for t and t' fixed [S 1], [G-Kl] (see also Appendix 1). Moreover, by standard arguments [S 1], it is enough to prove (3.4) for the resolvent operator. Let $g \in C_0^\infty(\mathbb{R}^3)$. Then we have:

$$\begin{aligned} & [U_{\tilde{k}}(h_\infty - z)^{-1}U^* g](t) \qquad (3.5) \\ = & \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} d\tilde{x}' \int_{\mathbb{R}^2} d\tilde{x} \frac{e^{-i\tilde{k}\tilde{x}}}{2\pi} (h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}') \int_{\mathbb{R}^2} d\tilde{k}' \frac{e^{i\tilde{k}'\tilde{x}'}}{2\pi} g(t', \tilde{k}'). \end{aligned}$$

The above integral makes sense because $(h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}')$ decays exponentially in $|\tilde{x} - \tilde{x}'|$ for t and t' fixed (see [G-Kl]). Since h_∞ commutes with translations in both directions x_2 and x_3, we get:

$$(h_\infty - z)^{-1}(t, \tilde{x}; t', \tilde{x}') = (h_\infty - z)^{-1}(t, \tilde{x} - \tilde{x}'; t', \mathbf{0}).$$

Then the integrals in (3.5) take the form

$$\begin{aligned} & [U_{\tilde{k}}(h_\infty - z)^{-1}U^* g](t) \\ = & \int_{\mathbb{R}} dt' \left\{ \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}} (h_\infty - z)^{-1}(t, \tilde{y}; t', \mathbf{0}) d\tilde{y} \right\} g(t', \tilde{k}). \qquad (3.6) \end{aligned}$$

By virtue of (3.2) and (3.3), this yields the equality

$$[h(\tilde{k}) - z]^{-1}(t, t') = \int_{\mathbb{R}^2} e^{-i\tilde{k}\tilde{y}} (h_\infty - z)^{-1}(t, \tilde{y}; t', \mathbf{0}) d\tilde{y}, \qquad (3.7)$$

which has to be understood as equality between smooth functions outside the diagonal $t = t'$. ☐

Proof of Proposition 3.1. By the conditions on the external potential $v(t)$, the fiber operator $h(\tilde{k})$ has purely discrete spectrum for any $\tilde{k} \in \mathbb{R}^2$.

We denote by $\{\lambda_n(k_2)\}_{n \geq 1}$ the nondegenerate eigenvalues of the operator $h(k_2, 0) = -d^2/t^2 + (\omega t - k_2)^2 + v(t)$, and by $\{\psi_n(\cdot, k_2)\}_{n \geq 1}$ the corresponding eigenfunctions. Let $f \in C_0^\infty(\mathbb{R})$. Then we have

$$f_{h(\tilde{k})}(t, t') = \sum_{n \geq 1} f(\lambda_n(k_2) + k_3^2) \psi_n(t, k_2) \overline{\psi}_n(t', k_2). \qquad (3.8)$$

Here the sum over n is finite, since f has compact support, but $\lim_{n \to \infty} \lambda_n(k_2) = \infty$ uniformly in $k_2 \in \mathbb{R}$. Hence, $f_{h(\tilde{k})}$ is a finite-rank operator. This can be explicitely seen from the fact that the fiber operator $h(\tilde{k})$ in (3.3) is unitarily equivalent to the operator:

$$-\frac{d^2}{dt^2} + \omega^2 t^2 + v(t + k_2/\omega) + k_3^2,$$

which is a harmonic oscilator plus a bounded perturbation. Moreover, this representation makes evident that $\mathbb{Z}-$periodicity of v implies $\mathbb{Z}_\omega-$periodicity of $\lambda_n(k_2)$ for all $n \geq 1$.

Notice that we are only interested in what happens near the bottom of the spectrum, $E_0 = \inf \sigma(h_\infty) = \inf_{k_2} \lambda_1(k_2)$. Because of the non-degeneracy of the eigenvalues $\{\lambda_n(k_2)\}_{n\geq 1}$, E_0 is isolated from the other bands, i.e from $\mathrm{Ran}(\lambda_n)$ with $n \geq 2$. Applying usual arguments (see [R-S IV]), ψ_1 can be chosen positive. Hence, if f is supported close enough to E_0, by (3.8) we obtain:

$$f_{h(\tilde{k})}(t,t) = f(\lambda_1(k_2) + k_3^2)\ \psi_1^2(t, k_2).$$

Then taking the Fourier transform in (3.4) we get

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} f_{h(\tilde{k})}(t,t) d\tilde{k} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(\lambda_1(k_2) + k_3^2)\ \psi_1^2(t, k_2) d\tilde{k} = f_{h_\infty}(t, \mathbf{0}; t, \mathbf{0}). \tag{3.9}$$

Note that the second integral in (3.9) converges, since by standard methods we can prove that ψ_1 is sharply localized near k_2/ω and that it has a gaussian decay of the form $const\ e^{-\alpha(t-k_2/\omega)^2}$ (see [Ag]). Now using (1.16), (1.17) for λ close to E_0, we eventually get (in a way similar to (2.7)) that (3.9) implies

$$n_\infty(\lambda) = \frac{1}{4\pi^2}\int_{\Xi}\int_{\mathbb{R}^2} \chi_{[E_0 - 0, \lambda]}(\lambda_1(k_2) + k_3^2)\ \psi_1^2(t, k_2) dt\ d\tilde{k}, \tag{3.10}$$

where $\Xi = [-1/2, 1/2]$.

In the particular case when $v = 0$, we have $E_0 = \omega = \lambda_1(k_2)$. Fix λ between the first two Landau levels: $\lambda \in (\omega, 3\omega)$. Then by integrating with respect to k_2 we obtain

$$\int_{\mathbb{R}} \psi_1^2(t, k_2) dk_2 = \omega,$$

thus $n_\infty(\lambda) = \frac{\omega}{2\pi^2}(\lambda - \omega)^{1/2}$ for the pure magnetic case. Notice the "bad" exponent 1/2 which makes the critical density to diverge, see (1.12).

Now we show that even if $v \neq 0$, the integrated density of states still behaves like in (1.12). First, the general theory insures that $\lambda_1(k_2)$ is a real analytic function of k_2 [R-S IV]. If it is constant, then we are essentially back to the case $v = 0$, since one has a lower bound for n_∞ of the form:

$$n_\infty(\lambda) \geq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} \chi_{[E_0-0,\lambda]}(E_0 + k_3^2)dk_3\right) \inf_{-1/2\leq t\leq 1/2}\int_{\mathbb{R}} \psi_1^2(t, k_2)dk_2, \quad (3.11)$$

for every $\lambda \in (E_0, E_0+\epsilon)$. Since $\psi_1^2(t, k_2)$ is jointly smooth in both arguments and positive, the mapping

$$[-1/2, 1/2] \ni t \to \int_{\mathbb{R}} \psi_1^2(t, k_2)dk_2 \in \mathbb{R}$$

has a positive minimum, so $n_\infty(\lambda) \geq a(\lambda-E_0)^{1/2}+o((\lambda-E_0)^{1/2})$ for $\lambda \searrow E_0$ and for some $a > 0$, i.e. we get back to case (1.12).

Let $\lambda_1(k_2)$ be not constant. Since it is a real analytic function and $\mathbb{Z}_\omega$-periodic, there exists a finite set of points $\{\xi_1, ..., \xi_N\} \subset [0, \omega)$, where λ_1 takes its minimal value E_0. Let $\lambda_1(\xi_j) = E_0$, $j \in \{1, \dots, N\}$. Then there exists a positive integer $n_j \geq 1$ and a constant C_j so that for k_2 close to ξ_j

$$\lambda_1(k_1) \sim E_0 + C_j(k_2 - \xi_j)^{2n_j}.$$

To get a lower bound for $n_\infty(\lambda)$ we may take the integral (3.10) with respect to $\tilde{k}$ over compact domains around the minima of the function $\lambda_1(k_2)$. In fact, for λ close to E_0 we can bound n_∞ from below by taking into account just one of those minima:

$$n_\infty(\lambda) \geq const\left(\int_{-1/2}^{1/2} \psi_1^2(t, \xi_j)dt\right) \cdot \int_{\mathbb{R}^2} \chi_{[E_0,\lambda]}(E_0 + \delta C_j(k_2 - \xi_j)^{2n_j} + k_3^2)\, d\tilde{k}, \quad (3.12)$$

for some $\delta > 1$. This leads to

$$n_\infty(\lambda) \geq const(\lambda - E_0)^{\frac{1}{2}+\frac{1}{2n_j}},$$

which clearly implies (1.12) for $\lambda - E_0$ small enough. Therefore, the proposition is proven. □

Remark 3.3. *Proposition 3.1 can be easily extended to* $\mathbb{Z}^2-$*periodic potentials* $v = v(x_1, x_3)$. *In this case, a similar analysis shows that the corresponding Hamiltonian* h_∞ *is unitarily equivalent to the operator* $\int_{\mathbb{R}\times(-\pi,\pi)} h(\tilde{k})d\tilde{k}$, *where now*

$$h(\tilde{k}) = -\partial_t^2 + (\omega t - k_2)^2 + (-i\partial_s + k_3)^2 + v(t, s) \quad (3.13)$$

on $L^2(\mathbb{R} \times (-1/2, 1/2))$ with periodic boundary conditions on $\mathbb{R} \times \{\pm 1/2\}$. Notice that for every $\tilde{k} \in \mathbb{R} \times (-\pi, \pi)$, the fiber Hamiltonnian $h(\tilde{k})$ has a compact resolvent which is positivity improving (see [R-S IV]). Let $\{\lambda(\tilde{k})\}_{n\geq 1}$ denote the set of eigenvalues of $h(\tilde{k})$ and $\{\psi_n(\cdot, \tilde{k})\}_{n\geq 1}$ be the corresponding eigenvectors. Then $\lambda_1(\tilde{k})$ is continuous and nondegenerate for any $\tilde{k}$. Let E_0 be the minimal value of $\lambda_1(\tilde{k})$. Then there exists a point $(\xi, \zeta) \in \mathbb{R} \times (-\pi, \pi)$, $(n, m) \in \mathbb{N}^2$ and $(C, D) \in \mathbb{R}^2$ two non-negative constants such that in the neighborhood of (ξ, ζ) we have the expansion

$$\lambda_1(\tilde{k}) = E_0 + C(k_2 - \xi)^{2n} + D(k_3 - \zeta)^{2m} + o((k_2 - \xi)^{2n} + (k_3 - \zeta)^{2m}). \quad (3.14)$$

Notice that by a standard Thomas' argument (see [R-S IV]) concerning the k_3 variable one concludes that λ_1 cannot be constant in k_3, which implies $D > 0$ and $m \geq 1$.

If we choose λ close to E_0, formula (3.10) now takes the form

$$n_\infty(\lambda) = \frac{1}{2\pi} \int_{\Xi^2} dsdt \int_{\mathbb{R}\times(-\pi,\pi)} d\tilde{k}\ \chi_{[E_0-0,\lambda]}(\lambda_1(\tilde{k}))\ \psi_1^2(s, t, \tilde{k}).$$

Then the rest of the reasoning follows the same lines as above. For $C = 0$ in (3.14), we use the argument as the one for (3.11), while for $C > 0$ we take the estimate (3.12). This gives

$$n_\infty(\lambda) \geq const(\lambda - E_0)^{\frac{1}{2n}+\frac{1}{2m}},$$

for $\lambda \searrow E_0$, which implies (1.12), even for non-degenerate minimum $n = m = 1$ in (3.14).

3.2 An example of finite critical density for non-zero uniform magnetic field

The previous subsection showed that the Bose condensate can be destroyed by turning on a no matter how weak constant magnetic field. Here we want to show that this is not always true. Let dimension $d = 3$. We choose in this subsection the gauge $\mathbf{a}_0(x_1, x_2) = 1/2(-x_2, x_1, 0)$ and we construct an external periodic potential, which depends on all three variables such that the critical density becomes finite.

We assume that the external potential has the following form:

$$V_\epsilon(\mathbf{x}) = \epsilon \cdot [v_1(x_1) + v_2(x_2)] + v_3(x_3), \quad (3.15)$$

where $\epsilon > 0$ and small, each of the functions $\{v_j\}_{j=1}^3$ is smooth $\mathbb{Z}-$periodic potential, and we also suppose that neither one of v_1 and v_2 is constant.

Take the magnetic field intensity $\omega = 2\pi$, then the "bulk" Hamiltonian can be written as

$$h_\infty = (-i\nabla_{\mathbf{x}} - 2\pi\mathbf{a}_0(x_1, x_2))^2 + V_\epsilon = h_\epsilon \otimes \mathbf{1} + \mathbf{1} \otimes h_3, \tag{3.16}$$

where the operator $h_\epsilon = (-i\nabla - 2\pi\mathbf{a}_0)^2 + \epsilon(v_1 + v_2)$ lives in $L^2(\mathbb{R}^2)$ while the operator $h_3 = -d^2/dx_3^2 + v_3$ lives in $L^2(\mathbb{R})$.

First, let us introduce some notation. We write an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} = (\overline{x}, x_3)$ where $\overline{x} := (x_1, x_2)$. We often use the notation $\Xi = (-1/2, 1/2)$. The elementary cell is $\Omega = \Xi^3$ and the one of the dual lattice Ω^* is given by

$$\Omega^* := \{2\pi\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \Xi^3\} = 2\pi\Xi^3.$$

According to Appendix 2, the operator h_∞ is unitarily equivalent to $\int_{\Xi^3}^{\oplus} h(\boldsymbol{\xi})d\boldsymbol{\xi}$, where the fiber operator can be further written as

$$h(\boldsymbol{\xi}) = h_\epsilon(\overline{\xi}) \otimes \mathbf{1} + \mathbf{1} \otimes h_3(\xi_3).$$

Here the operators $h_\epsilon(\overline{\xi}) = h_0(\overline{\xi}) + \epsilon V(\overline{x})$, $h_0(\overline{\xi}) := [-i\nabla_{\overline{x}} - \mathbf{a}(\overline{x}) + \mathbf{k}(\overline{\xi})]^2$ live in $L^2(\Xi^2)$ with "magnetic" periodic boundary conditions (see Appendix 2 for definition), and $h_3(\xi_3) = (-id/dx_3 + \xi_3)^2 + v_3(x_3)$ in $L^2(\Xi)$.

Recall that $\mathbf{a}(\overline{x}) = 2\pi\mathbf{a}_0(\overline{x})$ and $\mathbf{k}(\overline{\xi}) = 2\pi(\mathbf{e}_1\xi_1 + \mathbf{e}_2\xi_2)$. If f is a $C_0^\infty(\mathbb{R})$-function then the integral kernel of $f(h(\boldsymbol{\xi}))$ is given by

$$\begin{aligned} f_{h(\boldsymbol{\xi})}(\overline{x}, x_3; \overline{x}', x_3') &= \sum_{\overline{\gamma}\in\mathbb{Z}^2} \sum_{\gamma_3\in\mathbb{Z}} e^{-i\phi(\overline{x},\overline{\gamma}) - ib(\overline{\gamma}) - 2\pi i\overline{\xi}\cdot(\overline{x}+\overline{\gamma}-\overline{x}')} e^{-2\pi i\xi_3(x_3+\gamma_3-x_3')} \\ &\times \; f_{h_\infty}(\overline{x} + \overline{\gamma}, x_3 + \gamma_3; \overline{x}', x_3'), \end{aligned} \tag{3.17}$$

for every $\mathbf{x}, \mathbf{x}' \in \Omega$, where $\phi(\overline{x}, \overline{\gamma}) = \pi(x_2 n - x_1 m)$ and $b(\overline{\gamma}) = \pi mn$ for $\overline{\gamma} = m\mathbf{e}_1 + n\mathbf{e}_2$, see Appendix 2. Notice that the third coordinate is not influenced by the magnetic field, while the first two coordinates are essentially treated in the Appendix 2; see for instance (5.47). Then by integrating the trace of $f_{h(\boldsymbol{\xi})}$ with respect to $\boldsymbol{\xi}$ we obtain

$$\int_{\Xi^3} \mathrm{Tr}\, f_{h(\boldsymbol{\xi})} d\boldsymbol{\xi} = \int_{\Xi^3}\int_\Omega f_{h(\boldsymbol{\xi})}(\mathbf{x}, \mathbf{x}) d\mathbf{x} d\boldsymbol{\xi} = \int_\Omega f_{h_\infty}(\mathbf{x}, \mathbf{x}) d\mathbf{x}. \tag{3.18}$$

Now, since we put the magnetic flux through Ξ^2 to be exactly 2π, all eigenvalues of $h_\epsilon(\overline{\xi})$ are simple and belong to an interval of width of order ϵ around the old Landau levels due to the analysis of Appendix 2. Denote them by (see also (5.54))

$$\{2\pi(2n+1)+\epsilon a_n(\epsilon,\overline{\xi}), \quad \overline{\xi}\in\Xi^2\}_{n\geq 0} \tag{3.19}$$

and by $l_m(\xi_3)$, $m\geq 1$ the eigenvalues of $h_3(\xi_3)$. Then the spectrum of h_∞ is given by the closure of the range of the function

$$2\pi(2n+1)+\epsilon a_n(\epsilon,\overline{\xi})+l_m(\xi_3), \quad \boldsymbol{\xi}=(\overline{\xi},\xi_3)\in\Xi^3, \quad n\geq 0, m\geq 1.$$

Notice that by Proposition 2.1 we know that the branch l_1 reaches its minimum at zero and $l_1(\xi_3)\sim l_1(0)+C\xi_3^2$ in its neighborhood. Then the bottom of the spectrum $\sigma(h_\infty)$ is equal to

$$E_0=2\pi+\epsilon\min_{\overline{\xi}\in\Xi^2}a_0(\epsilon,\overline{\xi})+l_1(0).$$

Moreover, E_0 is isolated from the other bands if ϵ is small enough, as it is in the nonmagnetic case. Thus, we can repeat our arguments leading to the estimate of the integral in (2.7) for $\lambda\searrow E_0$ and we get for the integrated density of states in this limit exponent $3/2$, provided the minimum of $a_0(\epsilon,\cdot)$ is nondegenerate. (See Lemma 3.4 for the definition of a nondegenerate minimum for $\epsilon=0$).

This would prove the first part of Theorem 1.2. Thus we continue by the following statement.

Lemma 3.4. *Assume $a_0(0,\cdot)$ has a nondegenerate (local) minimum at $\overline{\xi}_0\in\Xi^2$, i.e. there exists a symmetric and positive matrix $Q\in\mathcal{M}_2(\mathbb{R})$ such that for small $|\overline{\xi}-\overline{\xi}_0|$ we have*

$$a_0(0,\overline{\xi})=a_0(0,\overline{\xi}_0)+\langle\overline{\xi}-\overline{\xi}_0,Q(\overline{\xi}-\overline{\xi}_0)\rangle+\mathcal{O}(|\overline{\xi}-\overline{\xi}_0|^3). \tag{3.20}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{C}^2$.

Then for ϵ small enough, there exists $\overline{\xi}_\epsilon$ close to $\overline{\xi}_0$ such that the functional $a_0(\epsilon,\cdot)$ has $\overline{\xi}_\epsilon$ as nondegenerate minimum.

Proof. By assumption, we have $\overline{\xi}_0\in\Xi^2$ so that $(\nabla_\xi a_0)(0,\overline{\xi}_0)=\mathbf{0}$ while the differential at $\overline{\xi}_0$ given by $[D_\xi(\nabla_\xi a_0)](0,\overline{\xi}_0)=Q$ is invertible. Due to the smoothness properties of $a_0(\epsilon,\overline{\xi})$ with respect to all its variables, one can easily verify the hypotheses of the Implicit Function Theorem, which finishes the proof. $\square$

Remark 3.5. . *Assume that $a_0(0,\cdot)$ has a finite number of nondegenerate local minima, only. Then due to the joint analyticity of a_0 with respect to $(\epsilon,\overline{\xi})$, there exists $\epsilon_0>0$ small enough such that for $|\epsilon|<\epsilon_0$, the absolute minimum of $a_0(\epsilon,\cdot)$ will also be nondegenerate and there are no other critical points than the ones given by Lemma 3.4.*

Therefore, the only thing which remains to be studied is the behavior of $a_0(0,\cdot)$ near its absolute minimum. But (see Appendix 2), we have already got a fairly explicit expression for it in (5.55), where we have now to put $V(x_1,x_1)=v_1(x_1)+v_2(x_1)$, $\omega=2\pi$ and $n=0$. We use the notation

$$(\hat{v})_\gamma=\int_{-1/2}^{1/2}e^{-2\pi i\gamma x}v(x)dx,\quad \gamma\in\mathbb{Z},$$

for the discrete Fourier transform, to obtain, by virtue of (5.55), that

$$\begin{aligned}
a_0(0,\overline{\xi}) &= b_{0,1}(\overline{\xi})+b_{0,2}(\overline{\xi}), \qquad (3.21)\\
b_{0,1}(\overline{\xi})=b_{0,1}(\xi_2) &= \sum_{\gamma_2\in\mathbb{Z}}e^{-2\pi i\xi_2\gamma_2}\,e^{-\pi\gamma_2^2/2}(\hat{v}_1)_{\gamma_2},\\
b_{0,2}(\overline{\xi})=b_{0,2}(\xi_1) &= \sum_{\gamma_1\in\mathbb{Z}}e^{2\pi i\xi_1\gamma_1}\,e^{-\pi\gamma_1^2/2}(\hat{v}_2)_{\gamma_1}.
\end{aligned}$$

It is easy to see that by a judicios choice of v_1 and v_2 we can create any profile we want for the functions $b_{0,1}$ and $b_{0,2}$. In particular, we can make the local minima nondegenerate. Indeed, choose two nonconstant functions $p,q:(-1/2,1/2)\mapsto\mathbb{R}$ which admit C^∞-extensions to $\mathbb{R}$; assume they have nondegenerate absolute minima correspondingly at $\xi_{0,p}$ and $\xi_{0,q}$ in the interval $(-1/2,1/2)$. Denote by

$$\tilde{p}_s(x)=\sum_{k=-s}^{s}e^{2\pi ixk}(\hat{p})_k,\quad s>1,x\in(-1/2,1/2)$$

the approximation of p by its first $2s+1$ Fourier components. Since p is smooth, then for $s=M$ large enough the approximation $\tilde{p}_M$ will have a nondegenerate absolute minimum at $\xi_{M,p}$ close to $\xi_{0,p}$. Define

$$v_1(x):=\sum_{k=-M}^{M}e^{2\pi ixk}e^{\pi k^2/2}(\hat{p})_k,\ x\in(-1/2,1/2).$$

Then by (3.21) it follows that $b_{0,1}(\xi_2) = \tilde{p}_M(-\xi_2)$. Thus, $b_{0,1}$ has a nondegenerate absolute minimum at $-\xi_{M,p}$. Similar line of reasoning involving the function q implies the same conclusion for $b_{0,2}$. This finishes the proof of the first part of Theorem 1.2.

Remark 3.6. *By inspection of this line of reasoning, one finds that there is no need to restrict the potential periodic in directions* (x_1, x_2) *to a separable form, see (3.15). In fact. our proof goes through verbatim for* $\mathbb{Z}^2$*-periodic potential*

$$\epsilon \cdot [v_1(x_1) + v_2(x_2) + \delta \cdot v(x_1, x_2)]$$

for δ *small enough.*

4 Imperfect Bose gas: mean-field interaction

To discuss whether the BEC found in the two previous sections survives the switching on of a particle interaction, we consider here the simplest version of it, known as the Mean-Field (MF) interaction, see e.g. [Z-B].

To this end we need the second quantized form of the interacting gas Hamiltonian in the boson Fock space $\mathcal{F}^B(L^2(\Lambda_L))$:

$$H_{\Lambda_L}(\mu) := H_{\Lambda_L} - \mu N_{\Lambda_L} = T_{\Lambda_L} - \mu N_{\Lambda_L} + U_{\Lambda_L}. \tag{4.1}$$

Here

$$T_{\Lambda_L} := \int_{\Lambda_L} \mathrm{d}\mathbf{x} a^*(\mathbf{x}) h_L a(\mathbf{x}) \tag{4.2}$$

is the kinetic-energy part with one-particle operator h_L defined by (1.3) and

$$U_{\Lambda_L} := \frac{1}{2} \int_{(\Lambda_L)^2} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ a^*(\mathbf{x}) a^*(\mathbf{y}) v(\mathbf{x} - \mathbf{y}) a(\mathbf{y}) a(\mathbf{x}), \tag{4.3}$$

is the interaction defined by the two-body potential $v(\mathbf{x}-\mathbf{y})$, where $a^*(\mathbf{x}), a(\mathbf{x})$ are usual *boson-field* operators,

$$N_{\Lambda_L} := \int_{\Lambda_L} \mathrm{d}\mathbf{x}\, a^*(\mathbf{x}) a(\mathbf{x}) \tag{4.4}$$

is the particle-number operator, and μ is the chemical potential.

To ensure the existence of the thermodynamics of the Bose gas (4.1) for all parameters (β, μ) of the grand-canonical ensemble, it used to suppose that the interaction $v(\mathbf{x}-\mathbf{y})$ is superstable [Ru]. For example, let the pair interaction potential $v(\mathbf{x}) = v(-\mathbf{x})$ be a real, non-negative continuous function from $L^1(\mathbb{R}^d)$. Since $v \in L^1(\mathbb{R}^d)$, the Fourier transform $\hat{v}(\mathbf{q})$ exists, and

$$\hat{v}(\mathbf{0}) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, v(\mathbf{x}) > 0 \quad \text{with} \quad \hat{v}(\mathbf{0}) \geq \hat{v}(\mathbf{q}), \quad \mathbf{q} \in \mathbb{R}^d. \tag{4.5}$$

It is known [Ru] that the corresponding interaction is superstable, i.e. the n-body potential satisfies the inequality

$$\sum_{1 \leq i < j \leq n} v(\mathbf{x}_i - \mathbf{x}_j) \geq \frac{A}{2|\Lambda_L|} n^2 - Bn \tag{4.6}$$

for some constants $A > 0$, $B \geq 0$, for all $n \in \mathbb{N}$, $\mathbf{x}_i, \mathbf{x}_j \in \Lambda_L$ and L large enough which implies that the thermodynamic potentials exist for all values of the chemical potential μ.

To introduce the MF interaction consider the scaled potential :

$$v_\lambda(x) := \lambda^d v(\lambda x), \lambda \geq 0. \tag{4.7}$$

Denote by

$$p_{\Lambda_L}\left[H^\lambda_{\Lambda_L}\right](\beta, \mu) := \frac{1}{\beta|\Lambda_L|} \ln Tr_{\mathcal{F}^B(L^2(\Lambda_L))} e^{-\beta(H^\lambda_{\Lambda_L} - \mu N_{\Lambda_L})} \tag{4.8}$$

the grand-canonical pressure defined by the Hamiltonian $H^\lambda_{\Lambda_L}$ with the two-body interaction $v_\lambda(x)$. Then the limit

$$\lim_{\lambda \to 0} \lim_{L \to \infty} p_{\Lambda_L}\left[H^\lambda_{\Lambda_L}\right](\beta, \mu) = p^{vdW}(\beta, \mu), \tag{4.9}$$

exists and it is known as the *van der Waals limit* [deSm-Z]. If one chooses the scaled two-body potential (4.7) in the form

$$v_L(\mathbf{x}) := g|\Lambda_L|^{-1}, \tag{4.10}$$

Then the limit

$$\lim_{L \to \infty} p_{\Lambda_L}\left[H^{\lambda_L}_{\Lambda_L}\right](\beta, \mu) = p^{MF}(\beta, \mu), \tag{4.11}$$

exists and it is known as the *Mean-Field limit* [deSm-Z]. Notice that by virtue of (4.4) the interaction (4.3) in this case takes the form :

$$U^{MF}_{\Lambda_L} = \frac{1}{2} \int_{(\Lambda_L)^2} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\, a^*(\mathbf{x}) a^*(\mathbf{y}) v_L(\mathbf{x}-\mathbf{y}) a(\mathbf{y}) a(\mathbf{x}) = \frac{1}{2} \frac{g}{|\Lambda_L|} N_{\Lambda_L}(N_{\Lambda_L} - I), \tag{4.12}$$

i.e., the corresponding Hamiltonian in the Fock space $\mathcal{F}^B(L^2(\Lambda_L))$ is defined by $H_{\Lambda_L}^{MF} := T_{\Lambda_L} + U_{\Lambda_L}^{MF}$.

Since the spectrum of the one-particle kinetic-energy operator is such that $\inf \sigma(h_\infty) = E_0$, see Section 1, thermodynamic behaviour of the boson gas (4.1) is E_0-dependent.

Lemma 4.1 (Thermodynamic Functions). *The grand-canonical pressure $p^{MF,E_0}(\beta,\mu)$ (4.11) of the M-F boson gas (4.1) exists for all $\beta \geq 0, \mu \in \mathbb{R}$ and is given by the Legendre transformation:*

$$p^{MF,E_0}(\beta,\mu) = \sup_{\rho\geq 0} \left(\mu\rho - f^{MF,E_0}(\beta,\rho)\right), \tag{4.13}$$

where the canonical free-energy density $f^{MF,E_0}(\beta,\rho)$ at inverse temperature β and density ρ is given by

$$f^{MF,E_0}(\beta,\rho) = f^{PBG,E_0}(\beta,\rho) + \frac{g}{2}\rho^2. \tag{4.14}$$

Here $f^{PBG,E_0}(\beta,\rho)$ is the free-energy density of the PBG, corresponding to (4.2).

Proof. The grand-canonical thermodynamic pressure of the PBG (4.2) is given by the limit

$$p^{PBG,E_0}(\beta,\mu) = \lim_{L\to\infty} p_{\Lambda_L}^{PBG,E_0}(\beta,\mu) = \lim_{L\to\infty} \frac{1}{\beta|\Lambda_L|} Tr_{\mathcal{F}^B(L^2(\Lambda_L))} e^{-\beta(T_{\Lambda_L} - \mu N_{\Lambda_L})}, \tag{4.15}$$

which implies that in order to be well defined, the chemical potential μ must be bounded from above: $\mu < E_0$, see Section 1. On the other hand, one has

$$p_{\Lambda_L}^{MF,E_0}(\beta,\mu) = \frac{1}{\beta|\Lambda_L|} \ln \sum_{N=0}^{\infty} e^{\beta(\mu N - gN(N-1)/2|\Lambda_L|)} Tr_{\mathcal{F}_N^B(L^2(\Lambda_L))} e^{-\beta T_{\Lambda_L}^{(N)}}, \tag{4.16}$$

where $T_{\Lambda_L}^{(N)}$ is a restriction of the kinetic-energy operator (4.3) on the N-particle sector $\mathcal{F}_N^B(L^2(\Lambda_L))$ of the Fock space $\mathcal{F}^B(L^2(\Lambda_L))$, $N = 0, 1, 2, \ldots$. Put

$$\begin{aligned} &f^{PBG,E_0=0}(\beta,\rho) := f^{PBG}(\beta,\rho), \\ &p^{PBG,E_0=0}(\beta,\mu) := p^{PBG}(\beta,\mu). \end{aligned} \tag{4.17}$$

Since the canonical free-energy density $f^{PBG,E_0}(\beta,\rho)$, is the Legendre transformation of $p^{PBG,E_0}(\beta,\mu)$, by definitions (4.17) one gets:

$$\begin{aligned} f^{PBG,E_0}(\beta,\rho) &= \sup_{\mu\leq E_0}\left(\rho\mu - p^{PBG,E_0}(\beta,\mu)\right) \\ = \sup_{\mu\leq E_0}&\left(\rho(\mu-E_0) - p^{PBG}(\beta,\mu-E_0) + E_0\rho\right) \\ &= f^{PBG}(\beta,\rho) + E_0\rho. \end{aligned} \tag{4.18}$$

The free-energy density of the mean-field model (4.12) at temperature β and density $\rho = N/|\Lambda_L|$ is defined by

$$f_{\Lambda_L}[H^{MF}_{\Lambda_L}](\beta,\rho) = -\frac{1}{\beta|\Lambda_L|}\ln Tr_{\mathcal{F}^B_N(L^2(\Lambda_L))}e^{-\beta H^{MF(N)}_{\Lambda_L}}, \tag{4.19}$$

where $Tr_{\mathcal{F}^B_N(L^2(\Lambda_L))}(\cdot)$ denotes the trace over the Hilbert space $\mathcal{F}^B_N(L^2(\Lambda_L))$ of symmetrized functions for $N = \rho|\Lambda_L|$ bosons. Since $\mathcal{F}^B_N(L^2(\Lambda_L))$ is the proper space of the particle-number operator N_{Λ_L} with the proper value N, the mean-field interaction term on this space is constant. Thus, we immediately find (4.14) in the thermodynamic limit:

$$\lim_{L\to\infty} f_{\Lambda_L}[H^{MF}_{\Lambda_L}](\beta,\rho) = \lim_{L\to\infty} f_{\Lambda_L}[T_{\Lambda_L}](\beta,\rho) + \frac{g}{2}\rho^2. \tag{4.20}$$

By (4.16) the pressure of the mean-field gas is well-defined for all $\mu \in \mathbb{R}$, and it is again the Legendre transform of $f^{MF}(\beta,\rho)$, yielding formula (4.13). □

Corollary 4.2. (Pressure of the M-F Bose Gas)
The grand-canonical pressure of a mean-field Bose Gas (4.13) is given by

$$p^{MF,E_0}(\beta,\mu) = \begin{cases} \mu\overline{\rho}(\beta,\mu) - f^{MF,E_0}(\beta,\overline{\rho}(\beta,\mu)), & \mathit{for}\ \mu \leq E_0 + g\rho_c(\beta); \\ (\mu-E_0)^2/2g + p^{PBG}(\beta,0), & \mathit{for}\ \mu > E_0 + g\rho_c(\beta), \end{cases} \tag{4.21}$$

where $\overline{\rho}(\beta,\mu))$ is a unique solution of the chemical potential equation

$$\mu = \partial_\rho f^{MF,E_0}(\beta,\rho) = \partial_\rho f^{PBG}(\beta,\rho) + E_0 + g\rho. \tag{4.22}$$

Here $\rho^{PBG,E_0}(\beta,\mu) = \rho^{PBG}(\beta,\mu-E_0)$ is the total density of the Perfect Bose gas and $\rho^{PBG,E_0}(\beta,E_0) \equiv \rho_c(\beta)$, defined by (1.10).

Theorem 4.3. *For the mean-field Bose gas (4.12), one gets the following expressions for particle densities in the thermodynamic limit. The total grand-canonical density is given by*

$$\rho^{MF,E_0}(\beta,\mu) = \begin{cases} \overline{\rho}(\beta,\mu)), & \text{for} \quad \mu \leq E_0 + g\rho_c(\beta), \\ (\mu - E_0)/g, & \text{for} \quad \mu > E_0 + g\rho_c(\beta). \end{cases} \tag{4.23}$$

The condensate density is given by

$$\rho_0^{MF,E_0}(\beta,\mu) = \begin{cases} 0, & \text{for} \quad \mu \leq E_0 + g\rho_c(\beta), \\ (\mu - E_0)/g - \rho_c(\beta), & \text{for} \quad \mu > E_0 + g\rho_c(\beta). \end{cases} \tag{4.24}$$

Proof: Since the total grand-canonical density of the mean-field Bose gas is defined by thermodynamic relation $\rho^{MF,E_0}(\beta,\mu) = \partial_\mu p^{MF,E_0}(\beta,\mu)$, the part (4.23) of our theorem follows directly from (4.21) and (4.22).

The part (4.24) is a more delicate matter. It is based on the strong equivalence of ensembles for the mean-field Bose gas and the fact that expectations in the canonical ensemble coincide with those for the PBG. This implies [vdB-L-deSm],[P-Z] that the particle density in the states with the energies higher than some $\delta > 0$ is equal to

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{\{j:\,\lambda_j > E_0+\delta\}} \langle a^*(u_j)a(u_j)\rangle_{\Lambda_L}(\beta,\mu) = \int_{E_0+\delta}^{\infty} \frac{dn_\infty(\lambda)}{e^{\beta(\lambda-\mu+g\rho^{MF,E_0}(\beta,\mu))} - 1}, \tag{4.25}$$

where $a^*(u_j) = \int_{\Lambda_L} d\mathbf{x} u_j(\mathbf{x}) a^*(\mathbf{x}) = (a(u_j))^*$ for the eigenvectors $\{u_j(x)\}_{j\geq 1}$ of the operator h_L (1.3). By virtue of (4.23) we get from (4.25) that

$$\lim_{\delta\to 0}\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{\{j:\,\lambda_j > E_0+\delta\}} \langle a^*(u_j)a(u_j)\rangle_{\Lambda_L}(\beta,\mu) = \rho_c(\beta) < \infty \tag{4.26}$$

for $\mu > E_0 + g\rho_c(\beta)$. Since the total particle density

$$\rho^{MF,E_0}(\beta,\mu) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{j\geq 0} \langle a^*(u_j)a(u_j)\rangle_{\Lambda_L}(\beta,\mu) \tag{4.27}$$

is equal to (4.23), the limit (4.26) proves the Bose-Einstein condensation (4.24) in the following form:

$$\rho_0^{MF,E_0}(\beta,\mu) = \lim_{\delta\to 0}\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{\{j:\,\lambda_j \leq E_0+\delta\}} \langle a^*(u_j)a(u_j)\rangle_{\Lambda_L}(\beta,\mu). \tag{4.28}$$

□

5 Appendices

5.1 Appendix 1: Smoothness and decay for the integral kernel of f_{h_∞}

For simplicity we consider her only the dimension $d = 3$, and the operator $h_\infty = (-i\nabla - \mathbf{a})^2 + V$ as in Introduction. Let $f \in C_0^\infty(\mathbb{R})$. We write

$$f(h_\infty) = \exp(-h_\infty) \cdot \tilde{f}(h_\infty) \cdot \exp(-h_\infty), \quad \tilde{f}(t) = e^{2t} f(t). \tag{5.29}$$

It is well-known (see e.g. the arguments via Feynman-Kac-Ito formula in [S 2]) that $\exp(-h_\infty)$ admits an integral kernel, denoted by $e^{-h_\infty}(\mathbf{x}, \mathbf{x}')$, which obeys the so-called diamagnetic inequality

$$\left| e^{-h_\infty}(\mathbf{x}, \mathbf{x}') \right| \leq e^{-\min(V)} \cdot \frac{1}{(4\pi)^{3/2}} e^{-|\mathbf{x}-\mathbf{x}'|^2/4}.$$

Moreover, let $K \in \mathbb{R}^3$ be a compact set, and $\alpha_1, \alpha_2 \in \mathbb{N}^3$. Under the smoothness conditions we assumed for $\mathbf{a}$ and V, the semigroup kernel obeys (see [S 2])

$$\left| \partial_{\mathbf{x}}^{\alpha_1} \partial_{\mathbf{x}'}^{\alpha_2} e^{-h_\infty}(\mathbf{x}, \mathbf{x}') \right| \leq C_1 \cdot e^{-C_2|\mathbf{x}-\mathbf{x}'|} < \infty, \quad \mathbf{x}' \in \mathbb{R}^3, \ \mathbf{x} \in K, \tag{5.30}$$

where C_1 and C_2 are constants which may depend on α's and K.

By virtue of the previous estimate, we can then write down the kernel of $f(h_\infty)$ as

$$f_{h_\infty}(\mathbf{x}, \mathbf{x}') = \int_{\mathbb{R}^3} e^{-h_\infty}(\mathbf{x}, \mathbf{y}) \left[\tilde{f}(h_\infty) e^{-h_\infty}(\cdot, \mathbf{x}') \right] (\mathbf{y}) d\mathbf{y}. \tag{5.31}$$

By the Cauchy-Schwarz inequality with respect to the $\mathbf{y}$ variable in (5.31), we get that the above kernel belongs to $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Since this is also true for $\tilde{f}$, we can then rewrite (5.31) as

$$f_{h_\infty}(\mathbf{x}, \mathbf{x}') = \int_{\mathbb{R}^6} e^{-h_\infty}(\mathbf{x}, \mathbf{y}) \cdot \tilde{f}_{h_\infty}(\mathbf{y}, \mathbf{y}') \cdot e^{-h_\infty}(\mathbf{y}', \mathbf{x}') d\mathbf{y} d\mathbf{y}'.$$

This together with (5.30) allow us to conclude that $f_{h_\infty}(\cdot, \cdot) \in C^\infty(\mathbb{R}^6)$.

Finally, regarding the decay of the above kernel, we recall a result of Germinet and Klein [G-Kl], which adapted to our setting assures such that for every $N \geq 1$, there exists a positive constant $C_{N,f}$ so that

$$|f_{h_\infty}(\mathbf{x}, \mathbf{x}')| \leq C_{N,f} \cdot (1 + |\mathbf{x} - \mathbf{x}'|)^{-N}, \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^3.$$

5.2 Appendix 2: Magnetic field in two dimensions and banding of Landau levels

Consider a 2-dimensional particle subjected to a constant magnetic field $\mathbf{B} = (0, 0, \omega)$, which is orthogonal to the plane $\mathbb{R}^2$, where the particle is allowed to move. We use here the transverse (symmetric) gauge i.e. $\mathbf{a}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \wedge \mathbf{x} = \omega \mathbf{a}_0(\mathbf{x}) = \omega/2(-x_2, x_1)$. Therefore, it is a two dimensional restriction of the model we consider in Section 3.2.

Take $\mathbf{e}_1$ and $\mathbf{e}_2$ as elements of the standard orthonormal basis in $\mathbb{R}^2$ and consider the lattice $\mathbb{Z}^2$. Let $\Xi = (-1/2, 1/2)$, the we denote the elementary cell in $\mathbb{R}^2$ by $\Xi \times \Xi$.

We denote the dual lattice by $(\mathbb{Z}^2)^*$, and define the dual elementary cell by

$$(\Xi^2)^* := \{\mathbf{k}(\xi) := \xi_1 \mathbf{k}_1 + \xi_2 \mathbf{k}_2,\ \xi = (\xi_1, \xi_2) \in \Xi^2\} = 2\pi \Xi^2,$$

where $\mathbf{k}_{1,2} = 2\pi \mathbf{e}_{1,2}$.

We suppose here that the magnetic field satisfies the "rationality condition", i.e. there exists $N \in \mathbb{N}^*$ such that the magnetic flux through Ξ^2 is:

$$\mathbf{B} \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) = \omega |\mathbf{e}_1 \wedge \mathbf{e}_2| = 2\pi N\ . \tag{5.32}$$

It is well-known (see e.g. [B-E-S], [J-P] and references therein) that for the particle restricted to the plane $\mathbb{R}^2$, the "free" magnetic Hamiltonian $h_0 = (-i\nabla - \mathbf{a})^2$ has only pure point spectrum (Landau levels), which is given by the set $\sigma(h_0) = \{(2n+1)\omega\ : n \in \{0, 1, \dots\}\}$. Let $V \in C^0(\mathbb{R}^2)$ be a $\mathbb{Z}^2$-periodic external potential. For $\epsilon \geq 0$ the perturbed Hamiltonian

$$h_\epsilon = h_0 + \epsilon V$$

acts on $L^2(\mathbb{R}^2)$. We now are interested in two questions: first, to justify the representation (3.17) and second, to investigate the nature of the spectrum of h_ϵ.

For $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^2$, define the "magnetic phase"

$$\phi(\mathbf{x}, \mathbf{y}) := -\frac{1}{2}\, \mathbf{B} \cdot (\mathbf{x} \wedge \mathbf{y}) \tag{5.33}$$

and recall that its main property is:

$$\exp[-i\phi(\mathbf{x}, \mathbf{y})](-i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x})) \exp[i\phi(\mathbf{x}, \mathbf{y})] = -i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x} - \mathbf{y})\ . \tag{5.34}$$

For all $\mathbf{e} = m\mathbf{e}_1 + n\mathbf{e}_2 \in \mathbb{Z}^2$, define $b(\mathbf{e}) := \pi N mn$. Let $\mathbf{f} = p\mathbf{e}_1 + q\mathbf{e}_2 \in \mathbb{Z}^2$. Then by virtue of (5.32)

$$\phi(\mathbf{e}, \mathbf{f}) = -\frac{1}{2}\,\mathbf{B} \cdot (\mathbf{e} \wedge \mathbf{f}) = \pi N(pn - qm)\ . \tag{5.35}$$

These imply that $b(\mathbf{e}) + b(\mathbf{f}) - b(\mathbf{e}+\mathbf{f}) - \phi(\mathbf{e}, \mathbf{f}) \in 2\pi\mathbb{Z}$, and that the modified magnetic translations

$$(T_{\mathbf{e}}\psi)(\mathbf{x}) := \exp\left[i\phi(\mathbf{x}, \mathbf{e}) + ib(\mathbf{e})\right]\psi(\mathbf{x} - \mathbf{e}), \quad \mathbf{e} \in \Gamma, \quad \psi \in L^2(\mathbb{R}^2) \tag{5.36}$$

form an abelian group, i.e. $T_{\mathbf{e}}T_{\mathbf{f}} = T_{\mathbf{e}+\mathbf{f}}$, which commutes with the perturbed Hamiltonian h_ϵ, $\epsilon \geq 0$. This means that in the case of "rational" magnetic fields some sort of the Floquet banding of Landau levels should exist.

Now we explicitely decompose the operator h_ϵ into a direct fiber integral. This will be done first for h_0 (in fact for its resolvent) and then for h_ϵ. Define a direct fiber integral of $L^2(\Xi^2)$-spaces, $\mathcal{H} := \int_{\Xi^2}^{\oplus} L^2(\Xi^2)\, d\xi$, together with the unitary operator $U : L^2(\mathbb{R}^2) \mapsto \mathcal{H}$ whose action on smooth and compactly supported functions is:

$$(U\psi)(\xi, \underline{x}) = \sum_{\mathbf{e}\in\mathbb{Z}^2} \exp\left[-i\mathbf{k}(\xi)\cdot\mathbf{e} - i\phi(\underline{x}, \mathbf{e}) - ib(\mathbf{e})\right]\psi(\underline{x} + \mathbf{e}), \tag{5.37}$$

here $\underline{x}$ denotes the position variable $(x_1, x_2) \in \Xi^2$. Formula (5.37) is then extended by continuity on $L^2(\mathbb{R}^2)$. Its adjoint reads as ($\mathbf{e} \in \mathbb{Z}^2$):

$$(U^*\psi)(\underline{x} + \mathbf{e}) = \int_{\Xi^2} d\xi' \exp\left[i\mathbf{k}(\xi')\cdot\mathbf{e} + i\phi(\underline{x}, \mathbf{e}) + ib(\mathbf{e})\right]\psi(\xi', \underline{x}). \tag{5.38}$$

It is known (see [J-P], [C-N] and references therein) that for z from the resolvent set $\rho(h_0)$, the resolvent $(h_0 - z)^{-1}$ admits the following integral kernel $K_0(\mathbf{x}, \mathbf{x}'; z)$:

$$\begin{aligned} K_0(\mathbf{x}, \mathbf{x}'; z) &= e^{i\phi(\mathbf{x},\mathbf{x}')}\, G_0(\mathbf{x}, \mathbf{x}'; z) \\ &\equiv \frac{\gamma(\alpha)}{4\,\pi} e^{i\phi(\mathbf{x},\mathbf{x}')}\, e^{-\psi(\mathbf{x},\mathbf{x}')}\mathcal{F}(\alpha, 1; 2\,\psi(\mathbf{x}, \mathbf{x}')) \end{aligned} \tag{5.39}$$

where $\psi(\mathbf{x}, \mathbf{x}') = \omega|\mathbf{x} - \mathbf{x}'|^2/4$, $\alpha = -\left(z/\omega - 1\right)/2 \neq -1, -2, \ldots$, γ is the Euler function, and $\mathcal{F}(\alpha, \beta; \zeta)$ is the confluent hyper-geometric function [Ab-St].

Take any $g \in C_0^\infty(\mathbb{R}^2)$. Since

$$\big(T_{\mathbf{e}}(h_0 - z)^{-1} g\big)(\mathbf{x}) = \big((h_0 - z)^{-1} T_{\mathbf{e}} g\big)(\mathbf{x}) \tag{5.40}$$

for any $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{e} \in \mathbb{Z}^2$, one has:

$$K_0(\mathbf{x}, \mathbf{x}' + \mathbf{e}; z) \exp\left(i\phi(\mathbf{x}', \mathbf{e})\right) = \exp\left(i\phi(\mathbf{x}, \mathbf{e})\right) K_0(\mathbf{x} - \mathbf{e}, \mathbf{x}'; z) \tag{5.41}$$

for any $\mathbf{x}' \in \mathbb{R}^2$ and for each $\mathbf{e}' \in \mathbb{Z}^2$ one has

$$K_0(\mathbf{x}, \underline{x}' + \mathbf{e}'; z) \exp\left(i\phi(\underline{x}', \mathbf{e}')\right) = \exp\left(i\phi(\mathbf{x}, \mathbf{e}')\right) K_0(\mathbf{x} - \mathbf{e}', \underline{x}'; z), \tag{5.42}$$

for any $\underline{x}' \in \Xi^2$ (notice that $\phi(\mathbf{x}, \mathbf{x}) = 0$). Take a smooth $g \in \mathcal{H}$. Then by (5.38) and (5.42), we get

$$\begin{aligned} \{[(h_0 - z)^{-1}] U^* g\}(\mathbf{x}) &= \sum_{\mathbf{e}' \in \mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \int_{\Xi^2} d\xi' e^{i\mathbf{k}(\xi') \cdot \mathbf{e}' + i\phi(\mathbf{x}, \mathbf{e}') + ib(\mathbf{e}')} \\ &\times \; K_0(\mathbf{x} - \mathbf{e}', \underline{x}'; z) g(\xi', \underline{x}') \,. \end{aligned} \tag{5.43}$$

Then with the help of (5.37) in the above equation, the expression for $\{U[(h_0 - z)^{-1}] U^* g\}(\xi, \underline{x})$ reads as:

$$\begin{aligned} &\sum_{\mathbf{e}, \mathbf{e}' \in \mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \exp\left[-i\phi(\underline{x}, \mathbf{e}) - ib(\mathbf{e}) + i\phi(\underline{x} + \mathbf{e}, \mathbf{e}') + ib(\mathbf{e}')\right] \qquad (5.44) \\ \times \;\; &K_0(\underline{x} + \mathbf{e} - \mathbf{e}', \underline{x}'; z) \exp\left[-i\mathbf{k}(\xi) \cdot \mathbf{e}\right] \int_{\Xi^2} d\xi' \exp\left[i\mathbf{k}(\xi') \cdot \mathbf{e}'\right] g(\xi', \underline{x}') \,. \end{aligned}$$

Changing the summation over the variable $\mathbf{e}$ to $\mathbf{f} = \mathbf{e} - \mathbf{e}'$, one gets for (5.43) that:

$$\begin{aligned} &\sum_{\mathbf{e}' \in \mathbb{Z}^2} \sum_{\mathbf{f} \in \mathbb{Z}^2} \int_{\Xi^2} d\underline{x}' \exp\left[-i\phi(\underline{x}, \mathbf{f}) - ib(\mathbf{e}' + \mathbf{f}) + i\phi(\mathbf{f}, \mathbf{e}') + ib(\mathbf{e}')\right] \\ \times \;\; &K_0(\underline{x} + \mathbf{f}, \underline{x}'; z) \exp\left[-i\mathbf{k}(\xi) \cdot \mathbf{f}\right] \exp\left[-i\mathbf{k}(\xi) \cdot \mathbf{e}'\right] \\ \times \;\; &\int_{\Xi^2} d\xi' \exp\left[i\mathbf{k}(\xi') \cdot \mathbf{e}'\right] g(\xi', \underline{x}'). \end{aligned} \tag{5.45}$$

Since by the magnetic flux rationality (5.32) one has $-b(\mathbf{e}' + \mathbf{f}) + \phi(\mathbf{f}, \mathbf{e}') + b(\mathbf{e}') + b(\mathbf{f}) \in 2\pi\mathbb{Z}$, and since

$$\sum_{\mathbf{e}' \in \mathbb{Z}^2} \exp\left[-i\mathbf{k}(\xi) \cdot \mathbf{e}'\right] \int_{\Xi^2} d\xi' \exp\left[i\mathbf{k}(\xi') \cdot \mathbf{e}'\right] g(\xi', \underline{x}') = g(\xi, \underline{x}'), \tag{5.46}$$

we conclude by (5.45) that the resolvent $(h_0 - z)^{-1}$ is is decomposable, and its fibers are integral operators on $L^2(\Xi^2)$ with kernels:

$$K_0(\xi; \underline{x}, \underline{x}'; z) = \sum_{\mathbf{f} \in \mathbb{Z}^2} \exp\left[-i\phi(\underline{x}, \mathbf{f}) - ib(\mathbf{f}) - i\mathbf{k}(\xi) \cdot \mathbf{f}\right] K_0(\underline{x} + \mathbf{f}, \underline{x}'; z). \quad (5.47)$$

In order to obtain decomposition of the operator h_0 into fibers $h_0(\xi)$ which have a common domain (i.e. independent of ξ), one has to "rotate" $\mathcal{H}$ with the unitary operator $\mathcal{V}$ defined on each fiber by the multiplication $\mathcal{V}(\xi, \underline{x}) = \exp\left[-i\mathbf{k}(\xi) \cdot \underline{x}\right]$. Let $\mathcal{U} := \mathcal{V}U$. Then $\mathcal{U}(h_0 - z)^{-1}\mathcal{U}^* = \int_{\Xi^2} d\xi [h_0(\xi) - z]^{-1}$ and

$$[h_0(\xi) - z]^{-1}(\underline{x}, \underline{x}') = \exp\left[-i\mathbf{k}(\xi) \cdot \underline{x}\right] K_0(\xi; \underline{x}, \underline{x}'; z) \exp\left[i\mathbf{k}(\xi) \cdot \underline{x}'\right]. \quad (5.48)$$

One can see that the range $[h_0(\xi) - z]^{-1} C_0^\infty(\Xi^2)$ is contained in the restriction to $\overline{\Xi^2}$ of all $C^\infty(\mathbb{R}^2)$-functions ψ with the property that both $\psi(\mathbf{x})$ and $[-i\nabla_{\mathbf{x}} - \mathbf{a}(\mathbf{x})]\psi(\mathbf{x})$ are invariant with respect to the modified magnetic translations (5.36). We say that the functions with such property verify "magnetic" periodic boundary conditions. Denote this restriction by D. Then the operator $h_0(\xi) := [-i\nabla_{\underline{x}} - \mathbf{a}(\underline{x}) + \mathbf{k}(\xi)]^2$ with "magnetic" periodic boundary conditions is essentially self-adjoint on D. Now, if $\epsilon > 0$, everything remains true for h_ϵ, whose fibers are defined as operator sum:

$$h_\epsilon(\xi) = h_0(\xi) + \epsilon V(\underline{x}), \quad \underline{x} \in \Xi^2. \quad (5.49)$$

Then from (5.48) and (5.49) we derive (3.17).

As it is well-known (see e.g. [B-E-S]), the orthogonal projectors of h_0 corresponding to the n-th Landau eigenvalue $\omega(2n+1)$ are integral operators, with the kernel (see also (5.39) for notations):

$$P_{0,n}(\mathbf{x}, \mathbf{x}') = \frac{\omega}{2\pi} e^{i\phi(\mathbf{x},\mathbf{x}')}\, e^{-\psi(\mathbf{x},\mathbf{x}')} \mathcal{L}_{n+1}(2\psi(\mathbf{x}, \mathbf{x}')), \quad (5.50)$$

where $\mathcal{L}_m(\zeta)$ is the m-th Laguerre polynomial, with $\mathcal{L}_m(0) = 1$, for any $m \geq 1$.

Then for each fiber $h_0(\xi)$ we have $h_0(\xi) = \sum_{n \geq 0} \omega(2n+1) P_{0,n}(\xi)$, where similar to (5.47) the "free" fiber projectors have the kernels:

$$P_{0,n}(\xi; \underline{x}, \underline{x}') = \sum_{\mathbf{f} \in \mathbb{Z}^2} \exp\left[-i\phi(\underline{x}, \mathbf{f}) - ib(\mathbf{f}) - i\mathbf{k}(\xi) \cdot (\underline{x} + \mathbf{f} - \underline{x}')\right] P_{0,n}(\underline{x} + \mathbf{f}, \underline{x}'; z). \quad (5.51)$$

Notice that the rank r of $P_{0,n}(\xi)$ can be easily obtained from:

$$\begin{aligned} r &= \int_{\Xi^2} d\underline{x}\; P_{0,n}(\xi;\underline{x},\underline{x}) \\ &= \frac{\omega}{2\pi}\sum_{\mathbf{f}\in\mathbb{Z}^2}\int_{\Xi^2} d\underline{x}\; \exp\left[-2i\phi(\underline{x},\mathbf{f})-ib(\mathbf{f})-i\mathbf{k}(\xi)\cdot\mathbf{f}\right] \\ &\times\; \exp\left[-\omega\mathbf{f}^2/4\right]\mathcal{L}_{n+1}(\omega\mathbf{f}^2/2). \end{aligned} \tag{5.52}$$

Since $\mathbf{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 \in \mathbb{Z}^2$ and $\underline{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \Xi^2$, then $-2i\phi(\underline{x},\mathbf{f}) = 2i\pi N(f_1x_2 - f_2x_1)$. Therefore, the integral

$$\int_{\Xi^2} d\underline{x}\; \exp\left[-2i\phi(\underline{x},\mathbf{f})\right] \tag{5.53}$$

is zero except for $\mathbf{f} = 0$, when it is equal to $|\Xi^2| = |\mathbf{e}_1\wedge\mathbf{e}_2|$. Then by virtue of (5.32) we get $r = N$, for any $n \geq 0$. Then if $\epsilon > 0$ and small enough, by analytic perturbation theory one obtains that $h_\epsilon(\xi)$ has in the neighborhood of each Landau level $\omega(2n+1)$ exactly N discrete eigenvalues $\{\lambda_j^{(n)}(\epsilon,\xi)\}_{j=1}^N$. If $P_{\epsilon,n}(\xi)$ is the projector corresponding to each group of eigenvalues $\{\lambda_j^{(n)}(\epsilon,\xi)\}_{j=1}^N$ and if $\mathcal{S}_{\epsilon,n}(\xi)$ is the intertwining unitary:

$$\mathcal{S}_{\epsilon,n}(\xi)P_{\epsilon,n}(\xi) = P_{0,n}(\xi)\mathcal{S}_{\epsilon,n}(\xi)\ ,$$

then after rotation by $\mathcal{S}_{\epsilon,n}(\xi)$ of the "reduced" perturbed Hamiltonian given by $P_{\epsilon,n}(\xi)h_\epsilon(\xi)P_{\epsilon,n}(\xi)$, one obtains that its eigenvalues

$$\{\lambda_j^{(n)}(\epsilon,\xi) =: \omega(2n+1) + \epsilon a_{n,j}(\epsilon,\xi)\}_{j=1}^N \tag{5.54}$$

are localized (up to an error of the order ϵ^2 and uniformly in ξ) in the neighborhood of the eigenvalues of the operator

$$L_{n,\epsilon}(\xi) := \omega(2n+1)P_{0,n}(\xi) + \epsilon P_{0,n}(\xi)VP_{0,n}(\xi)\ .$$

In the particular case of Section 3.2, when $N = 1$ (or $\omega = 2\pi$), one obtains that the operator $L_{n,\epsilon}(\xi)$ has only one eigenvalue which differs from $2\pi(2n+1)$ by $\epsilon a_{n,N=1}(\epsilon = 0,\xi)$, where (see (3.19))

$$\begin{aligned} a_n(\epsilon=0,\xi) &:= a_{n,N=1}(\epsilon=0,\xi) = \int_{\Xi^2} d\underline{x}\; V(\underline{x})P_{0,n}(\xi;\underline{x},\underline{x}) \\ &= \sum_{\mathbf{f}\in\mathbb{Z}^2}\exp\left[-ib(\mathbf{f})-i\mathbf{k}(\xi)\cdot\mathbf{f}\right]\,\exp\left[-\pi\mathbf{f}^2/2\right]\mathcal{L}_{n+1}(\pi\mathbf{f}^2) \\ &\times \int_{\Xi^2} d\underline{x}\; V(\underline{x})\exp\left[-2i\phi(\underline{x},\mathbf{f})\right]. \end{aligned} \tag{5.55}$$

Denote with

$$\hat{v}_{f_1,f_2} := \int_{\Xi^2} dx_1 \, dx_2 \, e^{-2\pi i(f_1 x_1 + f_2 x_2)} V(x_1, x_2)$$

the Fourier components of V $(f_1, f_2 \in \mathbb{Z})$. Since $\omega = 2\pi N$ and $N = 1$ then

$$\int_{\Xi^2} d\underline{x} \, V(\underline{x}) \, \exp\left[-2i\phi(\underline{x}, \mathbf{f})\right] = \hat{v}_{f_2,-f_1} \, .$$

By virtue of (5.55) we get that

$$\int_{\Xi^2} d\xi \, |\nabla_\xi a_n(\epsilon = 0, \xi)|^2 = \sum_{\mathbf{f} \in \mathbb{Z}^2} \mathbf{f}^2 \, \exp\left[-\pi \mathbf{f}^2/2\right] \mathcal{L}^2_{n+1}(\pi \mathbf{f}^2) \, |\hat{v}_{f_2,-f_1}|^2 \, . \quad (5.56)$$

If the above quantity is nonzero, then by analyticity of $a_n(\epsilon = 0, \xi)$ one obtains that this function is not a constant. For example, if $n = 0$, then $\mathcal{L}_1(\zeta) = 1$ and (5.56) imply that any nonconstant potential ϵV transforms (at least for small ϵ) the unperturbed ($\epsilon = 0$) Landau fundamental state into a simple, absolutely continuous spectral band.

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