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Symbolic Dynamics for Transition Tori - II

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We use Conley index theory to approach the problem of the transition tori of Arnold which have transverse intersections. We prove that near any finite collection of transition tori, there exists a set invariant for a certain power of the symplectic diffeomorphism, on which the dynamics is semi-conjugate to a full shift. Our treatment relies on an earlier article of C. Robinson.

Key Words: Arnold diffusion, Conley index, symbolic dynamics, transition tori.

1. INTRODUCTION

This paper continues the study of the transition tori of Arnold with topological tools, started in [10]. The objective of this paper is to prove the existence of orbits that drift chaotically near a finite collection of KAM tori with transverse homoclinic and heteroclinic connections. The chaotic behavior is described in terms of symbolic dynamics.

The new ideas of this paper are that we remove the strong transversality condition on the tori, as required in [3], [10], and we prove the existence of true symbolic dynamics, rather than a Shadowing Lemma type of result.

The use of Conley index theory and correctly aligned windows which we pursue in greater detail replaces the traditional method of finding an orbit passing near a sequence of points and reduces the role of the symplectic geometry. The topological character of the construction takes the key role here, to the extent that we can replace the transversality condition on the stable and unstable manifolds of the tori by a topological substitute, referred as topological crossing. We will deal with this situation in a future paper.

In this section, M represents a n-dimensional symplectic manifold and f_μ a family of symplectic diffeomorphisms. The maps f_μ are the result of perturbing a symplectic diffeomorphism f_{μ_0}. About f_{μ_0} it is assumed that it has a partially hyperbolic fixed point p_{μ_0}. There exists $a_0 > 0$ such that f_μ has a partially hyperbolic fixed point p_μ, for all $|\mu - \mu_0| < a_0$. Let n_h be the dimension of the stable manifold $W^s(p_{\mu_0}, f_{\mu_0})$ and of the unstable manifold $W^u(p_{\mu_0}, f_{\mu_0})$, and let $2n_c$ be the dimension of the center manifold $W^c(p_{\mu_0}, f_{\mu_0})$. We also assume that f_{μ_0} satisfies a twist condition on the center manifold $W^c(p_{\mu_0}, f_{\mu_0})$. By the KAM theorem [1], there exists a Cantor set of n_c-dimensional tori $\{T_{\mu,\alpha}\}_{\alpha\in\mathcal{I}}$ in $W^c(p_\mu, f_\mu)$, invariant under f_μ, for all $|\mu - \mu_0| < a_1$ and some $0 < a_1 < a_0$. The map f_μ determines a quasi-periodic motion on each of these tori $T_{\mu,\alpha}$, with irrational rotations in each dimension which are independent of each other over integers. The size of the index set $\mathcal{I}$ can be taken uniform in μ. For details and for a discussion on a more general possible setting of the problem, the reader is invited to see [10].

Definition 1. A *family of transition tori* is a collection $\{T_{\mu,\alpha}\}_{\alpha\in\mathcal{I}_\mu}$ indexed by a set $\mathcal{I}_\mu \subseteq \mathcal{I}$ for a range of parameters $|\mu - \mu_0| < a_2$ for some $0 < a_2 \leq a_1$, such that $W^u(T_{\mu,\alpha}, f_\mu)$ and $W^s(T_{\mu,\beta}, f_\mu)$ have a transverse intersection at a point $q_{\mu,\alpha\beta}$, for all $\alpha, \beta \in \mathcal{I}_\mu$.

The usual way of obtaining transition tori is by perturbing a completely integrable Hamiltonian to measure the splitting of some stable and unstable manifolds. The usual procedure is by the Melnikov functions. If a Melnikov function has a nondegenerate zero, then there exists $0 < a_2 \leq a_1$ such that f_μ has a family of transition tori indexed on some $\mathcal{I}_\mu \subseteq \mathcal{I}$. The existence of transition tori is associated, in general, to the mechanism of drifting orbits from torus to torus known as Arnold diffusion. Xia shows in [11] that, in the planar three body problem, each pair of $W^u(T_{\mu,\alpha}, f_\mu)$ and $W^s(T_{\mu,\beta}, f_\mu)$ have a non-empty transverse intersection, and concludes the existence of Arnold diffusion. The fact that the invariant tori have stable and unstable manifolds that intersect transversally has also been proved in the five-body problem (see [7]), as a route to Arnold diffusion. Our main theorem describes a phenomenon similar to Arnold diffusion.

Theorem 2 (The Main Theorem). *Let $\{T_{\mu,\alpha}\}_{\alpha\in\mathcal{I}_\mu}$ be a family of transition tori. For a fixed μ with $|\mu-\mu_0| < a_2$, let $T_{\mu,\alpha_1}, \ldots, T_{\mu,\alpha_s}$ be an arbitrary, finite sub-collection of tori and $\epsilon > 0$. Then there exist a fixed power f_μ^k of f_μ (for some $k > 0$), a compact set S_μ^ϵ invariant for f_μ^k, with S_μ^ϵ*

contained in some ϵ-neighborhood of $\bigcup_{i=1,\dots,s} T_{\mu,\alpha_i}$*, and a semi-conjugacy* $\rho : S^\epsilon_\mu \to \Sigma$ *onto a full shift* Σ *on s-symbols. Moreover, the inverse image through* ρ *of each periodic orbit of the shift map* σ *contains a periodic orbit for* f^k_μ *in* S^ϵ_μ*, and the semi-conjugacy is stable under a small perturbation near the initial value of* μ*.*

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2. THE CONLEY INDEX AND CORRECTLY ALIGNED WINDOWS

Symbolic dynamics can be used to show the existence of complicated dynamics. A system can be shown to posses a subsystem modelled by symbolic dynamics whenever a set of boxes (or windows) can be found so that the image of any box crosses a subset of the boxes in a manner which is correctly aligned. The precise nature of the correctly aligned crossing can be expressed in the language of Conley index. In this section, we present some relevant aspects of the Conley index theory. See [5], [8] for further background on Conley index. We end this section with a discussion on topological crossing from the Conley index theory point of view.

Let f denote a diffeomorphism of a smooth n-dimensional manifold M. The image of a box N_1 under f is correctly aligned with the box N_2 provided (1) the 'ends' of N_1 come out the 'ends' of N_2 and (2) the image of N_1 crosses N_2 in some correct dimensional sense. In terms of the Conley index theory, these concepts are expressed in terms of compact pairs (N, L), where $L \subseteq N$ represents the 'ends' and both sets are compact. The nature of the crossing is expressed in terms of cohomology (or homology). We say that the image of the the compact pair (N_1, L_1) under f is *compatible* to the compact pair (N_2, L_2) if the following conditions are satisfied:

(i) $f(N_1 \setminus L_1) \subseteq N_2$
(ii) $f(L_1) \cap N_2 \subseteq L_2$.

See Figure 1.

In order to get a point whose orbit passes through a sequence of boxes (or windows) N_{i_j}, it is necessary that the image $f(N_{i_j})$ of one crosses the next $N_{i_{j+1}}$ in some consistent sense. We could use homology to define this sufficient condition as in [2], but cohomology is usually used in the Conley index theory because it works well with infinite intersections and it satisfies a strong excision property. Therefore we use cohomology in the sequel. For each box N_{i_j} of the sequence, the set $N_{i_j} \setminus f^{-1}\left(N_{i_{j+1}}\right)$ consists of the points which miss the next box, and so are the exit points. Below we will assume that there is a dimension k such that $H^k\left(N_{i_j}, N_{i_j} \setminus f^{-1}(N_{i_{j+1}})\right) = \mathbf{Q}$.

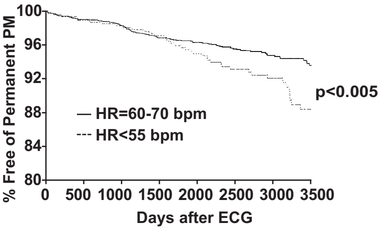

FIG. 1. Compatible compact pairs

Its generator α_{i_j} measures the k expanding directions in N_{i_j}. For the sequence of boxes to be correctly aligned, the pull back $f^*(\alpha_{i_j})$ of α_{i_j} must be nonzero in the expanding directions of $N_{i_{j-1}}$, i.e. an element of $H^k\left(N_{i_{j-1}}, N_{i_{j-1}} \setminus f^{-1}(N_{i_j})\right)$. This pull back is expressed in terms of the transfer map which we define next.

For the compact pairs (N_1, L_1) and (N_2, L_2) one can define a transfer map between their Alexander-Spanier cohomologies with rational coefficients as follows. The map

$$f : (N_1, L_1) \to (N_2 \cup f(N_1), L_2 \cup f(L_1))$$

induces a homomorphisms

$$\hat{f}^*_{N_1,N_2} : H^*(N_2 \cup f(N_1), L_2 \cup f(L_1)) \to H^*(N_1, L_1)$$

in cohomology, due to the aforementioned compatibility properties. Notice that the induced map on cohomology pulls back the elements. Since we do not assume that $f(N_1)$ is contained in N_2, we use the inclusion

$$i : (N_2, L_2) \to (N_2 \cup f(N_1), L_2 \cup f(L_1))$$

to induce an isomorphism

$$i^* : H^*(N_2 \cup f(N_1), L_2 \cup f(L_1)) \to H^*(N_2, L_2)$$

in cohomology by excision. The map

$$f^*_{N_1,N_2} = \hat{f}^*_{N_1,N_2} \circ (i^*)^{-1} : H^*(N_2, L_2) \to H^*(N_1, L_1)$$

is called the transfer map from (N_1, L_1) to (N_2, L_2). Here the cohomology generators of $H^*(N_1, L_1)$ and $H^*(N_2, L_2)$ describe the 'expanding' directions of these boxes. The transfer map $f^*_{N_1,N_2}$ describes how these 'expanding directions' match one another when f maps (N_1, L_1) across (N_2, L_2).

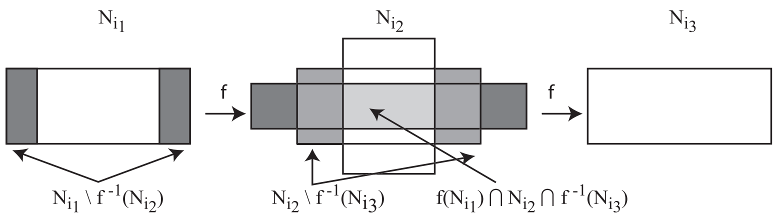

FIG. 2. Correctly aligned windows

Let $\{N_i\}_{i\in\mathcal{I}}$ be a (finite or infinite) collection of compact sets (or boxes, or windows). A matrix $A = (a_{ij})_{i,j\in\mathcal{I}}$ of entries zero or one is called a *transition matrix for the collection* provided

$$f(N_{i_1}) \cap N_{i_2} \cap f^{-1}(N_{i_3}) \subseteq \mathrm{int} N_{i_2} \tag{1}$$

for each $i_1, i_2, i_3 \in \mathcal{I}$ with $a_{i_1 i_2} = 1 = a_{i_2 i_3}$. See Figure 2.

Lemma 3. *Assume $\{N_i\}_{i\in\mathcal{I}}$ is a collection of compact sets with transition matrix A satisfying condition (1). Then the image of the compact pair $\big(N_{i_1}, N_{i_1} \setminus f^{-1}(intN_{i_2})\big)$ under f is compatible to the compact pair $\big(N_{i_2}, N_{i_2} \setminus f^{-1}(intN_{i_3})\big)$, for all $i_1, i_2, i_3 \in \mathcal{I}$ with $a_{i_1 i_2} = 1 = a_{i_2 i_3}$.*

This is proved in [5].

A collection of compact sets (windows) $\{N_i\}_{i\in\mathcal{I}}$ is said to be *k-dimensionally correctly aligned for the transition matrix A* provided they satisfy condition (1) and

$$H^l\left(N_{i_1}, N_{i_1} \setminus f^{-1}(\mathrm{int} N_{i_2})\right) = \begin{cases} \mathbf{Q} & \text{if } l = k, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

$$H^l\left(N_{i_2}, N_{i_2} \setminus f^{-1}(\mathrm{int} N_{i_3})\right) = \begin{cases} \mathbf{Q} & \text{if } l = k, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

and the transfer maps

$$f^k_{N_{i_1}, N_{i_2}} \neq 0 \tag{4}$$

for all $i_1, i_2, i_3 \in \mathcal{I}$ with $a_{i_1 i_2} = 1 = a_{i_2 i_3}$.

Conditions (2), (3) and (4) are a way of saying that the image of N_{i_1} crosses N_{i_2} in terms of the k-th cohomology, and thus the windows are correctly aligned.

The following theorem from [5] provides the existence of an invariant set with the correct symbolic dynamics. For a finite transition matrix A, the subshift of finite type Σ_A is the set of all strings of symbols $\underline{s} = (s_i)_{i\in\mathbf{Z}}$ such that $a_{s_i s_{i+1}} = 1$ for all i. The shift map $\sigma : \Sigma_A \to \Sigma_A$ is defined by $\underline{t} = \sigma(\underline{s})$ where $t_i = s_{i+1}$. See [9].

THEOREM 4. *Let* $\{N_i\}_{i=1,\ldots,s}$ *be a finite collection of compact sets such that* $N_i \cap N_j = \emptyset$ *or* $N_i = N_j$ *for any* $i, j = 1, \ldots, s$. *Assume that* $\{N_i\}_{i=1,\ldots,s}$ *is* k*-dimensionally correctly aligned for the transition matrix* A*, i.e. they satisfy conditions (1)-(4) from above. Then there exist a compact invariant set* $S \subseteq \bigcup_{i=1,\ldots,s} N_i$ *and a semi-conjugacy* $\rho : S \to \Sigma_s$ *onto the subshift of finite type on* s *symbols for the transition matrix* A. *Moreover, the inverse image of each periodic orbit of the shift map* σ *on* Σ_A *contains a periodic orbit of* f*, with the same principal period.*

In the proof of the Theorem 2, we will consider windows that are correctly aligned under different mappings (which are different powers of the same map). In order to deal with this situation, we need the following technical lemma (which proof is left as an exercise to the reader).

LEMMA 5. *Let* $\{N_i\}_{i=0,\ldots,5}$ *be compact windows and* $\{f_i\}_{i=1,\ldots,4}$ *be diffeomorphisms satisfying the following* k*-dimensionally correctly alignment conditions*

$$f_{i+1}(N_i) \cap N_{i+1} \cap f_{i+2}^{-1}(N_{i+2}) \subseteq intN_{i+1}, \text{ for all } i \in \{0, \ldots, 2\},$$
$$H^l\left(N_i, N_i \setminus f_{i+1}^{-1}(intN_{i+1})\right) = \begin{cases} \mathbf{Q} & \text{if } l = k, \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding transfer maps

$$(f_{i+1})^k_{N_i, N_{i+1}} \neq 0$$

for all $i \in \{0, \ldots, 4\}$. *Moreover, suppose that the following stronger condition is satisfied*

$$f_2(N_1) \cap f_3^{-1}(N_3) \subseteq int(N_2). \tag{5}$$

Let $g = f_3 \circ f_2$. *Then, the windows* N_0, N_1, N_3, N_4 *are* k*-dimensionally correctly aligned in the following sense*

$$\begin{aligned} & f_1(N_0) \cap N_1 \cap g^{-1}(N_3) \subseteq intN_1, \\ & g(N_1) \cap N_3 \cap f_4^{-1}(N_4) \subseteq intN_3, \\ & H^l\left(N_0, N_0 \setminus f_1^{-1}(intN_1)\right) = \\ & = H^l\left(N_1, N_1 \setminus g^{-1}(intN_3)\right) = \\ & = H^l\left(N_3, N_3 \setminus f_4^{-1}(intN_4)\right) = \begin{cases} \mathbf{Q} & \text{if } l = k, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and the corresponding transfer maps

$$(f_1)^k_{N_0,N_1} \neq 0,\ g^k_{N_1,N_3} \neq 0, \text{ and } (f_4^k)_{N_3,N_4} \neq 0.$$

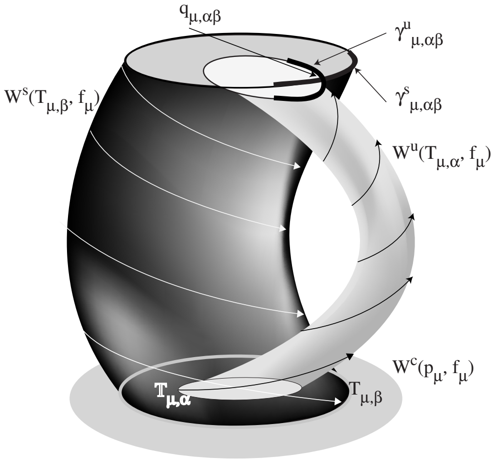

FIG. 3. Transverse intersections of the tori

3. PRELIMINARY RESULTS

Proposition 6. *Suppose that* $W^u(T_{\mu,\alpha}, f_\mu)$ *and* $W^s(T_{\mu,\beta}, f_\mu)$ *have a non-empty transverse intersection at a point* $q_{\mu,\alpha\beta}$. *Then*

(i) $W^u(T_{\mu,\alpha}, f_\mu)$ *and* $W^{cs}(p_\mu, f_\mu)$ *have non-empty transverse intersection within some neighborhood of* $q_{\mu,\alpha\beta}$ *in* M *and the intersection* $\gamma^u_{\mu,\alpha\beta}$ *of* $W^u(T_{\mu,\alpha}, f_\mu)$ *and* $W^{cs}(p_\mu, f_\mu)$ *is locally a* n_c*-dimensional submanifold containing* $q_{\mu,\alpha\beta}$. *Similarly,* $W^s(T_{\mu,\beta}, f_\mu)$ *and* $W^{cu}(p_\mu, f_\mu)$ *have non-empty transverse intersection within some neighborhood of* $q_{\mu,\alpha\beta}$ *in* M *and the intersection* $\gamma^s_{\mu,\alpha\beta}$ *of* $W^s(T_{\mu,\beta}, f_\mu)$ *and* $W^{cu}(p_\mu, f_\mu)$ *is locally a* n_c*-dimensional submanifold containing* $q_{\mu,\alpha\beta}$.

(ii) $\gamma^u_{\mu,\alpha\beta}$ *and* $W^s(T_{\mu,\beta}, f_\mu)$ *have a transverse intersection at* $q_{\mu,\alpha\beta}$ *in* $W^{cs}(p_\mu, f_\mu)$. *Similarly,* $\gamma^s_{\mu,\alpha\beta}$ *and* $W^u(T_{\mu,\alpha}, f_\mu)$ *have a transverse intersection at* $q_{\mu,\alpha\beta}$ *in* $W^{cs}(p_\mu, f_\mu)$.

(iii) $f^n_\mu(\gamma^u_{\mu,\alpha\beta})$ *and* $W^s(T_{\mu,\beta}, f_\mu)$ *have a transverse intersection at* $f^n(q_{\mu,\alpha\beta})$ *in* $W^{cs}(p_\mu, f_\mu)$, *for all* $n \geq 0$. *Similarly,* $f^{-n}_\mu(\gamma^s_{\mu,\alpha\beta})$ *and* $W^u(T_{\mu,\alpha}, f_\mu)$ *have a transverse intersection at* $f^{-n}(q_{\mu,\alpha\beta})$ *in* $W^{cs}(p_\mu, f_\mu)$, *for all* $n \geq 0$.

(iv) $\gamma^u_{\mu,\alpha\beta}$ *and* $\gamma^s_{\mu,\alpha\beta}$ *have a transverse intersection at* $q_{\mu,\alpha\beta}$ *in* $W^{cu}(p_\mu, f_\mu) \cap W^{cs}(p_\mu, f_\mu)$.

See Figure 3.

Proof. For economy, we will denote $q_{\mu,\alpha\beta}$ by q and p_μ by p. We will prove (i), (ii) and (iii) only for the unstable part, since the stable counterpart of the proof follows similarly. Assume that

$$T_qM = T_q\left(W^u(T_{\mu,\alpha}, f_\mu)\right) + T_q\left(W^s(T_{\mu,\beta}, f_\mu)\right).$$

Since $T_q\left(W^s(T_{\mu,\alpha}, f_\mu)\right) \subseteq T_q\left(W^{cs}(p, f_\mu)\right)$, we then have

$$T_qM = T_q\left(W^u(T_{\mu,\alpha}, f_\mu)\right) + T_q\left(W^{cs}(p, f_\mu)\right).$$

This implies that $W^u(T_{\mu,\alpha}, f_\mu)$ and $W^{cs}(p, f_\mu))$ have non-empty transverse intersection within some neighborhood of q. Hence

$$\gamma^u_{\alpha\beta} := W^u(T_{\mu,\alpha}, f_\mu) \cap W^{cs}(p, f_\mu)$$

is locally a n_c-dimensional submanifold containing q, which proves (i). We then derive

$$T_q\left(W^{cs}(p, f_\mu)\right) = T_q(W^u(T_{\mu,\alpha}, f_\mu)) \cap T_q\left(W^{cs}(p, f_\mu)\right) + T_q(W^s(T_{\mu,\beta}, f_\mu)),$$

hence

$$T_q\left(W^{cs}(p)\right) = T_q\gamma^u_{\alpha\beta} + T_q\left(W^s(T_{\mu,\beta}, f_\mu)\right),$$

which proves (ii). Statement (iii) follows from the invariance of $W^s(T_{\mu,\beta}, f_\mu)$ under forward iterations. For (iv), we first notice that $W^{cu}(p, f_\mu)$ and $W^{cs}(p, f_\mu)$ have a transverse intersection at $q_{\mu,\alpha\beta}$, hence $W^{cu}(p, f_\mu) \cap W^{cs}(p, f_\mu)$ is locally a $2n_c$-dimensional manifold containing $q_{\mu,\alpha\beta}$. Then, using the facts that $T_q\gamma^u_{\alpha\beta} \subseteq T_q\left(W^u(T_{\mu,\alpha}, f_\mu)\right)$, $T_q\gamma^s_{\alpha\beta} \subseteq T_q\left(W^s(T_{\mu,\beta}, f_\mu)\right)$, and $W^u(T_{\mu,\alpha}, f_\mu)$ and $W^s(T_{\mu,\beta}, f_\mu)$ are transverse at q and have complementary dimensions, we obtain

$$\begin{aligned} &T_q\left(W^{cs}(p, f_\mu)\right) \cap T_q\left(W^{cu}(p, f_\mu)\right) = \\ &= \left(T_q\gamma^u_{\alpha\beta} + T_q\left(W^s(T_{\mu,\beta}, f_\mu)\right)\right) \cap \left(T_q\gamma^s_{\alpha\beta} + T_q\left(W^u(T_{\mu,\alpha}, f_\mu)\right)\right) \\ &= T_q\gamma^u_{\alpha\beta} + T_q\gamma^s_{\alpha\beta}. \end{aligned}$$

∎

LEMMA 7. *Let $T_1, T_2, \ldots, T_s$ be a family of n-dimensional tori. For each $i = 1, \ldots, s$, let $\theta_i : T_i \to T_i$, $i = 1, \ldots, s$ be a translation by irrational angles ω_i^j ($j = 1, \ldots, n$) in each dimension, with all angular frequencies $\omega_i^1, \ldots, \omega_i^n$ independent over the integers. Assume that the angular frequency vectors $\Omega_i := (\omega_i^1, \ldots, \omega_i^n)$, $i = 1, \ldots, s$, are linearly independent over the integers. Let p_i, p_i' be a fixed pair of points on T_i, for each*

$i = 1, \dots, s$. Then, for every $\epsilon > 0$ and every integer $h_0 > 0$, there exists an integer $h > h_0$ such that $d(\theta_i^h p_i, p_i') < \epsilon$ for all $i = 1, \dots, s$.

Proof. Consider the ns-dimensional torus $T_1 \times \ldots \times T_s$ endowed with the product metric given by the supremum, and the map $\Theta : T_1 \times \ldots \times T_s \to T_1 \times \ldots \times T_s$ defined by $\Theta(x_1, \dots, x_s) = (\theta_1(x_1), \dots, \theta_s(x_s))$. This is a translation on the ns-dimensional torus, of irrational angles given by $(\Omega_1, \dots, \Omega_s)$. Since these vectors are linearly independent, the translation Θ is topologically transitive, so for a given $\epsilon > 0$ and $h_0 > 0$ there exists $h > h_0$ such that $\Theta^h(p_1, \dots, p_s) = \big(\theta_1^h(x_1), \dots, \theta_s^h(x_s)\big)$ is ϵ-close to $(p_1', \dots, p_s')$ with respect to the product metric. This proves our statement. ▌

4. PROOF OF THE MAIN THEOREM

Since we keep μ fixed, we simplify the notation by setting $f_\mu = f$, $p_\mu = p$, $T_{\mu,\alpha_i} = T_i$, $q_{\alpha_i \alpha_j} = q_{ij}$. Let $\gamma^u_{\alpha_i\alpha_j} = \gamma^u_{ij}$, $\gamma^s_{\alpha_i\alpha_j} = \gamma^s_{ij}$, $i, j = 1, \dots, s$, as in Proposition 6.

Thus, the following hold true for all $i, j = 1, \dots, s$:

- $W^u(T_i)$ and $W^s(T_j)$ are transverse at q_{ij} in M
- $f^n(\gamma^u_{ij})$ and $W^s(T_j)$ are transverse at $f^n(q_{ij})$ in $W^{cs}(p)$ for all $n \geq 0$
- $f^{-n}(\gamma^s_{ij})$ and $W^u(T_i)$ are transverse at $f^{-n}(q_{ij})$ in $W^{cu}(p)$ for all $n \geq 0$
- γ^u_{ij} and γ^s_{ij} are transverse at q_{ij} in $W^{cu}(p) \cap W^{cs}(p)$.

When n goes to infinity, $f^n(q_{ij})$ approaches T_j, and $f^n(\gamma^u_{ij})$ approaches $W^c(p)$ and remains transverse to $f^n(\gamma^s_{ij})$ in a $2n_c$-dimensional disk contained in $f^n\left(W^{cu}(p) \cap W^{cs}(p)\right)$. If γ^s_{ij} is transverse to the foliation of $W^s(T_j)$ by stable manifolds of points in T_j, then $f^n(\gamma^s_{ij})$ becomes C^1-close to some n_c-disk contained in T_j and depending on n (see Remark 8). However, if some tangent directions to γ^s_{ij} are also tangent to the foliation, then this does not occur. Nevertheless, we can always find a n_c-disk t^s_{ij} near γ^s_{ij}, contained in $W^s(T_j)$, such that $f^n(t^s_{ij})$ approaches C^1-smoothly a n_c-disk D^s_{ij} in T_j (where the disk depends on n). The condition on t^s_{ij} is that the n_c tangent directions along t^s_{ij} are transverse to the stable directions at q_{ij} in $W^s(T_j)$. Therefore, γ^u_{ij} and t^s_{ij} are transverse in some $2n_c$-disk that is transverse to the stable direction at q_{ij} in $W^{cs}(p)$. Under forward iteration, $f^n(\gamma^s_{ij})$ and $f^n(t^s_{ij})$ may tilt differently along the stable directions; all we require from t^s_{ij} is that it is initially contained in some small tubular neighborhood of γ^s_{ij} in $W^s(T_j)$. Intuitively, we think of t^s_{ij} as an approximation of $D^s_{ij} \subseteq T_j$, hence the existence of t^s_{ij} is a way of saying that the stable holonomy f^s along the stable leaves of $W^{cs}(p)$ maps γ^u_{ij} to a n_c-disk transverse to T_j in $W^c(p)$. Below we define t^s_{ij} by using a linear map L^s_{ij} to tilt γ^s_{ij} while leaving γ^u_{ij} fixed.

Similarly, there exists a n_c-disk t^u_{ij} in $W^u(T_i)$ near γ^u_{ij} such that $f^{-n}(t^u_{ij})$ approaches a moving n_c-disk D^u_{ij} in T_i.

Now we define a triple of correctly aligned windows near q_{ij}. Let N_{ij} be a compact set defined by the following properties, for all $i, j = 1, \dots, s$:

(i) N_{ij} is the diffeomorphic image of a rectangle

$$\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \overline{B^{n_h}}(0,1) \times \overline{B^{n_h}}(0,1)$$

under a local diffeomorphism ν_{ij}

(ii) $\gamma^u_{ij} = \nu_{ij}\left(\{0\} \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)$

(iii) $\gamma^s_{ij} = \nu_{ij}\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\} \times \{0\}\right)$

(iv) $t^u_{ij} = \nu_{ij}\left(L^u_{ij}\left(\{0\} \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)$ for some linear map $L^u_{ij} : \mathbf{R}^{2n_h+2n_c} \to \mathbf{R}^{2n_h+2n_c}$ with $\nu_{ij}\left(L^u_{ij}\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\} \times \{0\}\right)\right) = \gamma^s_{ij}$

(v) $t^s_{ij} = \nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\} \times \{0\}\right)\right)$ for some linear map $L^s_{ij} : \mathbf{R}^{2n_h+2n_c} \to \mathbf{R}^{2n_h+2n_c}$ with $\nu_{ij}\left(L^s_{ij}\left(\{0\} \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right) = \gamma^u_{ij}$

(vi) $W^u(T_i) \cap N_{ij} = \nu_{ij}\left(\{0\} \times \overline{B^{n_c}}(0,1) \times \{0\} \times \overline{B^{n_h}}(0,1)\right)$

(vii) $W^s(T_j) \cap N_{ij} = \nu_{ij}\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \overline{B^{n_h}}(0,1) \times \{0\}\right).$

Using the same homeomorphism which provides N_{ij}, we define

$$N^s_{ij} := \nu_{ij}\left(\overline{B^{n_c}}(0,1+\delta) \times \overline{B^{n_c}}(0,1-\delta) \times \overline{B^{n_h}}(0,1+\delta) \times \overline{B^{n_h}}(0,1-\delta)\right)$$

and

$$N^u_{ij} := \nu_{ij}\left(\overline{B^{n_c}}(0,1-\delta) \times \overline{B^{n_c}}(0,1+\delta) \times \overline{B^{n_h}}(0,1-\delta) \times \overline{B^{n_h}}(0,1+\delta)\right),$$

for some small $\delta > 0$. Since t^s_{ij} is close to γ^s_{ij} and t^u_{ij} is close to γ^u_{ij}, if $\delta > 0$ is small enough, then N^u_{ij}, N_{ij}, and N^s_{ij} are correctly aligned windows under the identity mapping. We have

$$\begin{aligned} N^s_{ij} \cap N_{ij} \cap N^u_{ij} = \\ &= \nu_{ij}\left(\overline{B^{n_c}}(0,1-\delta) \times \overline{B^{n_c}}(0,1-\delta) \times \overline{B^{n_h}}(0,1-\delta) \times \overline{B^{n_h}}(0,1-\delta)\right) \\ &\subseteq \mathrm{int} N_{ij}, \end{aligned}$$

$$H^l(N^s_{ij}, N^s_{ij} \setminus N_{ij}) = H^l(N_{ij}, N^u_{ij} \setminus N_{ij}) = \begin{cases} \mathbf{Q} & \text{if } l = n_c + n_h, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{id}^{n_h+n_c}_{N^s_{ij}, N^s_{ij}} = \mathrm{id}.$$

This implies that

$$f^{-l}(N^u_{ij}),\ N_{ij},\ f^l(N^s_{ij}) \tag{6}$$

are correctly aligned windows under f^l, for any integer $l > 0$.

Since the finite collection of points q_{ij}, $i, j = 1, \ldots, s$, is within a bounded distance from $W^c(p)$, we can choose N^u_{ij}, N_{ij} and N^s_{ij} such that there exists $l > 0$ for which $f^l(N^s_{ij})$ is contained in some $\epsilon/2$-neighborhood of T_i, and $f^{-l}(N^u_{ij})$ is contained in an $\epsilon/2$-neighborhood of T_j, for all $i, j = 1, \ldots, s$. We emphasize that we choose the same fixed power f^l for all tori T_i. This means that it will take us the power f^{2l} in order to be able to 'transit' from near the torus T_i to near the torus T_j (including the case when $i = j$).

Thus, if l is chosen large enough, each set $f^l(N^s_{ij})$ will stretch along the unstable direction and shrink along the stable direction at the same rate. The symplectic form ω is non-degenerate on the 'rectangle' $\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)$, hence f^l preserves its 'area' given in terms of the symplectic form. That is, when this 'rectangle' stretches along D^s_{ij} under positive iterations, due to the twist, it necessarily shrinks in the complementary directions. In addition, we have that the directions along the disk $f^l\left(\nu_{ij}\left(\{0\} \times \{0\} \times \overline{B^{n_h}}(0,1) \times \{0\}\right)\right)$ are close to the unstable directions along $W^u(T_j)$, the directions along the disk $f^l\left(\nu_{ij}\left(\{0\} \times \{0\} \times \{0\} \times \overline{B^{n_h}}(0,1)\right)\right)$ are close to the stable directions along $W^s(T_j)$, and $f^l(\gamma^u_{ij})$, $f^l(t^s_{ij})$ are contained in a $2n_c$-disk transverse to the stable and unstable directions, provided l is chosen large enough.

At this point, near each torus T_j we have a family of s windows $f^l(N^s_{ij})$ and another family of s windows $f^{-l}(N^u_{jk})$, where $i, k = 1, \ldots, s$. All points of the windows in the family $f^l(N^s_{ij})$ are within an $\epsilon/2$-distance away from the n_c-disks D^s_{ij}, while all points in the windows of the family $f^{-l}(N^u_{jk})$ are within an $\epsilon/2$-distance away from the n_c-disks D^u_{jk}.

Choose a family of n_c-disks D_j in T_j with $\mathrm{diam} D_j$ greater than any of the diameters $\mathrm{diam} f^l(D^s_{ij})$ and $\mathrm{diam} f^{-l}(D^u_{jk})$, $i, j, k = 1, \ldots, s$, but smaller than ϵ. By KAM theory, near T_j there exist a Cantor set of invariant tori for which the angular frequency vectors vary from tori to tori. There exist disks D'^s_{ij} and D'^u_{jk} that are C^1-close to D^s_{ij} and D^u_{jk} respectively, with $\mathrm{diam} f^l(D^s_{ij})$ and $\mathrm{diam} f^{-l}(D^u_{jk})$ smaller than $\mathrm{diam} D_j$, lying on tori with independent frequency vectors. By Lemma 7, there exists an arbitrarily large $h > 0$ such that f^h maps the center of each disk D'^s_{ij} $\epsilon/2$-close to the center of D_j and f^{-h} maps the center of each disk D'^u_{jk} $\epsilon/2$-close to the center of D_j.

There is a twist condition on $W^c(p)$ which makes every disk in $W^c(p)$ transverse to T_j shear along T_j. The twist condition is the same at each point of the tori since our system is the perturbation of an integrable system.

At this point we define a sequence of $n_c + n_h$-dimensionally correctly aligned windows $\{N_j\}_{j=1,\dots,s}$, of uniform size, lying near the transition tori. Each N_j is defined by the following properties, as $j = 1, \dots, s$:

(1) N_j is the diffeomorphic image of a rectangle $\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \overline{B^{n_h}}(0,1) \times \overline{B^{n_h}}(0,1)$ under a local diffeomorphism ν_j
(2) $\nu_j\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \overline{B^{n_h}}(0,1) \times \{0\}\right) \subseteq W^s(T_j)$
(3) $\nu_j\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\} \times \overline{B^{n_h}}(0,1)\right) \subseteq W^u(T_j)$
(4) $\nu_j\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)$ is contained in a $2n_c$-dimensional disk transverse to the stable and unstable directions
(5) all points of N_j are within a distance less than ϵ from D_j
(6) the sets N_j are mutually disjoint.

We claim that N^u_{ij}, N_j and N^s_{ij} can be chosen in such a way that

$$f^h(N^s_{ij}),\ N_j,\ f^{-h}(N^u_{jk}) \tag{7}$$

is a triple of correctly aligned windows relative to some fixed power f^h of f, with the order of the iterate h independent of the particular torus T_j.

Under f^h, the rectangle $f^l\left(\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)\right)$ twists in such a way that the image

$$f^l\left(\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \mathrm{bd}\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)\right)$$

comes out of

$$\nu_j\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)$$

under some appropriate coordinate change, provided that N^s_{ij} is a small enough set and $h > 0$ is large enough integer. So the boundary of $f^l(\gamma^u_{ij})$ comes out of $\nu_j\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times 0 \times 0\right)$ and stays away from $\nu_j\left(\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\} \times \{0\}\right)$ (which is an approximation of D_j). Hence the image of

$$f^l\left(\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \mathrm{bd}\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)\right)$$

under f^h stretches across

$$\nu_j\left(\overline{B^{n_c}}(0,1) \times \overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)$$

in a n_c-manner.

Now we take the image of both

$$f^{l+h}\left(\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1) \times \mathrm{bd}\overline{B^{n_c}}(0,1) \times \{0\} \times \{0\}\right)\right)\right)$$

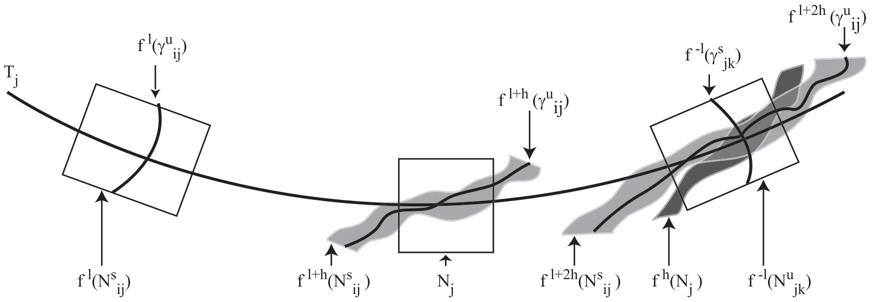

FIG. 4. Correctly aligned windows along the transition tori

and

$$\nu_j\left(\overline{B^{n_c}}(0,1)\times\overline{B^{n_c}}(0,1)\times\{0\}\times\{0\}\right)$$

under f^h. The image of $\nu_j\left(\overline{B^{n_c}}(0,1)\times\overline{B^{n_c}}(0,1)\times\{0\}\times\{0\}\right)$ is stretched across some section $f^{-l}\left(\nu^u_{ij}\left(\mathrm{bd}\overline{B^{n_c}}(0,1)\times\overline{B^{n_c}}(0,1)\times\{0\}\times\{0\}\right)\right)$, due to the twist map. The image of

$$f^{l+h}\left(\nu_{ij}\left(L^s_{ij}\left(\overline{B^{n_c}}(0,1)\times\mathrm{bd}\overline{B^{n_c}}(0,1)\times\{0\}\times\{0\}\right)\right)\right)$$

under f^h is stretched across the same section even more. If N^s_{ij} and N^u_{jk} are chosen small enough, these crossings are consistent for all $2n_c$-dimensional sections of the three windows. In both cases the crossing occurs in a n_c-dimensional manner, correctly aligned with the direction of crossing in the previous iteration, as shown in Figure 4.

The hyperbolic directions along $W^u(T_j)$ and $W^s(T_j)$ are all right, so we have

$$f^{l+h}(N^s_{ij})\cap N_j\cap f^{-l-h}N^u_{ij}\subseteq \mathrm{int}N_j,$$

$$\begin{aligned}&H^l(f^l(N^s_{ij}),f^l(N^s_{ij})\setminus f^{-h}(N_j))=\\&=H^l(N_j,N_j\setminus f^{-l-h}(N^u_{ij}))=\begin{cases}\mathbf{Q} & \text{if } l=n_c+n_h,\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and the corresponding transfer map

$$\mathrm{id}^{n_h+n_c}_{f^l(N^s_{ij}),N_j}\neq 0.$$

The latter equation is a consequence of the twist condition along the central manifold.

Note that all correct alignment of windows described so far satisfies the stronger condition (5). By Lemma 5, we have obtained a collection of compact sets $\{N_i\}_{i=1,\ldots,s}$ satisfying $N_i \cap N_j = \emptyset$ or $N_i = N_j$, and a power $f_\mu^k := f_\mu^{2l+h}$ of f_μ such that $\{N_i\}_{i=1,\ldots,s}$ is $n_c + n_h$-dimensionally correctly aligned. All cohomolgy groups equal $\mathbf{Q}$ in dimension $n_h + n_c$ and are trivial in any other dimension and all transfer maps are equal to the identity in dimension $n_h + n_c$. Theorem 4 implies that there exists a set $S_\mu^\epsilon \subseteq \bigcup_{i=1,\ldots,s} N_i$ invariant for f_μ^k such that the dynamics induced by f_μ^k on S_μ^ϵ is semi-conjugate to a full shift on s symbols and the inverse image of each periodic orbit for σ contains a periodic orbit for f_μ^k. Each N_i has been constructed within an ϵ-neighborhood of T_i, hence S_μ^ϵ is contained in some ϵ-neighborhood of $\bigcup_{i=1,\ldots,s} T_i$. Under a small perturbation $f_{\mu'}$ near f_μ all transversality relationships will be preserved. Moreover, if μ' is close enough to μ, the windows constructed above will remain correctly aligned under $f_{\mu'}$, hence there exists a compact set $S_{\mu'}^\epsilon$ on which $f_{\mu'}^k$ is semi-conjugate to a full shift on s symbols. ▮

Remark 8. In [3], A. Garcia considers a strong topological crossing condition of $W^u(T_i)$ and $W^s(T_j)$ at q_{ij}, requiring the existence of a point $z_j' \in T_j$ such that $W^s(z_j')$ is transverse to $W^{cu}(p)$ at q_{ij} (or, equivalently, the existence of a point $z_i \in T_i$ such that $W^u(z_i)$ is transverse to $W^{cs}(p)$ at q_{ij}). If this stronger requirement is satisfied, then we can take $t_{ij}^s = \gamma_{ij}^s$ and $t_{ij}^u = \gamma_{ij}^u$, so γ_{ij}^s approaches a n_c-disk in T_j under forward iterations and γ_{ij}^u approaches a n_c-disk in T_i under backwards iterations. The construction of correctly aligned windows follows easier, as in [10].

Remark 9. The Main Theorem does not prove Arnold diffusion *in stricto senso.* Usually, the term Arnold diffusion describes the existence of a set of bounded size of orbits that travel between KAM tori, as μ approaches μ_0. Arnold diffusion has been shown in several specific models, by using Melnikov analysis, for example in [6]. In the more general context we consider, we need to rely heavier on topological arguments. The difficulty of proving Arnold diffusion is related to the so called 'gap problem': as $\mu \to \mu_0$, the set of parameters for the KAM tori which maintain transverse intersections of their stable and unstable manifolds shrinks to zero size. Therefore the distance of diffusion shrinks to zero as $\mu \to \mu_0$. Our result shows the existence of orbits passing any near prescribed sequence of a fixed set of tori for a fixed μ, and for very small perturbations near μ. In order to construct the correctly aligned windows, it is sufficient to have transverse intersections of $W^u(T_i)$ and $W^s(T_j)$, but it might be possible to construct these windows even in the absence of transversality. One simple case in which we can construct correctly aligned windows is when the stable and unstable manifolds of the tori have topological crossing, rather than transverse intersection. The reader is invited to see [2] for the definition of topological crossing, but it seems that even topological crossing in a weaker sense is all it takes to build up correctly aligned windows. Topological crossing has the following advantage over transversality: one needs to check only the existence of a change of sign in the Melnikov function (rather than the existence of a simple zero) in order to conclude the presence of topological crossing. This type of approach will make the subject of a future paper.

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