Quantization of edge currents for continuous magnetic operators

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Abstract

For a magnetic Hamiltonian on a half-plane given as the sum of the Landau operator with Dirichlet boundary conditions and a random potential, a quantization theorem for the edge currents is proven. This shows that the concept of edge channels also makes sense in presence of disorder. Moreover, Gaussian bounds on the heat kernel and its covariant derivatives are obtained.

1 Introduction

Topological quantization of edge currents has been proven rigorously only for discrete magnetic Schrödinger operators [23, 18, 14]. The purpose of this work is prove similar results also for continuous Schrödinger operators. In order to describe the main result, H_L denotes the Landau operator on $L^2(\mathbb{R}^2)$ and V is a differentiable potential given as sum of a periodic and a random part. Let $\hat{H}$ denote the restriction of H_L+V to the half-plane with Dirichlet boundary conditions and let $J_1 = \imath[\hat{H}, X_1]$ be the current operator along the boundary. Suppose that the interval Δ is a gap of H_L+V (but this is then *not* a gap of $\hat{H}$) and that $G:\mathbb{R}\to[0,1]$ is a decreasing differentiable function equal to 1 to the left of Δ and 0 to its right. Hence its derivative G' is negative and supported by Δ. Furthermore let χ be a smooth, positive, and compactly supported function on $\mathbb{R}$ with unit integral, namely $\int \chi = 1$. Under these conditions, it is shown that $\chi(X_1)J_1G'(\hat{H})\chi(X_1)$ is a traceclass operator and that

$$\mathbf{E}\ \mathbf{Tr}\left(\chi(X_1)J_1G'(\hat{H})\chi(X_1)\right) \;=\; \frac{1}{2\pi}\,\mathrm{Ind}\;,$$

where $\mathbf{E}$ denotes the disorder average and $\mathrm{Ind}\in\mathbf{Z}$ is the index of a certain Fredholm operator which depends on Δ, but not on the choice of the functions G and χ. The index is also equal to the non-commutative winding number of the unitary operator $\mathcal{U} = \exp(-2\pi\imath\, G(\hat{H}))$.

The quantity $\mathbf{E}\ \mathbf{Tr}\left(\chi(X_1)J_1G'(\hat{H})\chi(X_1)\right)$ can physically be interpreted as the conductivity of the edge. Indeed, $-G'(\hat{H}) \geq 0$ is a density matrix of edge states in Δ which is normalized because $-\int G' = 1$. The operator $\chi(X_1)J_1G'(\hat{H})\chi(X_1)$ gives the corresponding current along the boundary and within a in strip of unit width which is perpendicular to the boundary. According to the above, its averaged trace is quantized. Another interpretation, fully developed in [23], is obtained when the smooth function $G'(\hat{H})$ approximates $\frac{1}{|\Delta'|}\chi_{\Delta'}(\hat{H})$ where $\chi_{\Delta'}$ is the indicator function of the interval $\Delta' \subset \Delta$. The boundaries of Δ' are thought to be the local Fermi levels at the upper and lower boundary of a bar-like sample. Then the above quantity measures the net edge current, namely the sum of the current along the upper boundary and the (reversed) one on the lower boundary. This net edge current is hence quantized. Moreover the above index is equal to the bulk conductivity as given by the Kubo formula as long as the Fermi level lies in Δ (the proof of this fact is deferred to a forthcoming work [19]). Hence both edge and bulk currents in the bar are quantized with the same Fredholm index, a fact of crucial importance for the quantum Hall effect (see [23, 18] and references therein).

Let us briefly discuss the hypothesis. The gap condition on (the density of states of) H should be satisfied in clean high mobility samples. However, the quantization of the edge currents probably also holds under a weaker dynamical localization condition, just as the quantization of the Kubo-Chern formula does [6, 1]. The Dirichlet boundary conditions could also be replaced by a soft edge modeled by a confining potential. Technical modifications would mainly be needed in Section 6.

Compared to [18], the main technical difficulties en route concern proving Gaussian bounds on the heat kernels as well as their (covariant) derivatives. Even though this is known for the heat kernels themselves [24, 13, 7], to our knowledge the derivatives have not been controlled as explicitely. The proof of the quantization of edge currents is based on an index theorem for covariant families of unitaries (Section 9).

2 Magnetic Hamiltonians

Let $\vec{X} = (X_1, X_2)$ be the position operator on $L^2(\mathbb{R}^2)$ and $\vec{\partial} = (\partial_1, \partial_2)$ the associated partial derivatives. Then $\vec{X}$ and $\imath\vec{\partial}$ are self-adjoint with common core $C_c^\infty(\mathbb{R}^2)$, the smooth functions with compact support. Setting $\gamma = \frac{qB}{\hbar}$, the Landau operator in the Landau gauge is then given by

$$H_L \;=\; \frac{\hbar^2}{2m}\left(\imath\partial_1 - \gamma\, X_2\right)^2 + \frac{\hbar^2}{2m}\left(\imath\partial_2\right)^2 \;\; .$$

In order to simplify notations, units are chosen such that $\hbar^2/m = 1$. With the following operators (all with common core $C_c^\infty(\mathbb{R}^2)$)

$$D_1 \;=\; \imath\partial_1 - \gamma X_2 \;, \qquad D_2 \;=\; \imath\partial_2 \;, \qquad K_1 \;=\; \imath\partial_1 \;, \qquad K_2 \;=\; \imath\partial_2 - \gamma X_1 \;,$$

it can be written as $H_L = \frac{1}{2}(D_1^2 + D_2^2)$. One readily verifies that both $D_{1,2}$ commute with both $K_{1,2}$. Hence the Landau operator has a large symmetry group and its spectrum is infinitely degenerated. The $D_{1,2}$ and $K_{1,2}$ are respectively the generators of the Landau translations $T(\vec{\xi}\,)$ and the magnetic translation operators $U(\vec{\xi}\,)$ defined by (for $\psi \in L^2(\mathbb{R}^2)$ and $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$)

$$\begin{array}{rclrcl} (U(\vec{\xi}\,)\psi)(\vec{x}\,) & = & \hat{\Phi}(\vec{\xi}, \vec{x} - \vec{\xi}\,)\, \psi(\vec{x} - \vec{\xi}\,) \;, & \hat{\Phi}(\vec{\xi}, \vec{x}\,) & = & e^{-\imath\gamma\xi_2 x_1} \;, \\ (T(\vec{\xi}\,)\psi)(\vec{x}\,) & = & \Phi(\vec{\xi}, \vec{x} - \vec{\xi}\,)\, \psi(\vec{x} - \vec{\xi}\,) \;, & \Phi(\vec{\xi}, \vec{x}\,) & = & e^{-\imath\gamma\xi_1 x_2} \;. \end{array}$$

It can be easily verified that the following relations hold

$$U(\vec{\xi}\,)U(\vec{\eta}\,) \;=\; \hat{\Phi}(\vec{\xi}, \vec{\eta}\,)U(\vec{\xi} + \vec{\eta}\,) \;, \qquad T(\vec{\xi}\,)T(\vec{\eta}\,) \;=\; \Phi(\vec{\xi}, \vec{\eta}\,)T(\vec{\xi} + \vec{\eta}\,) \;.$$

The aim is now to add a potential to H_L having possibly a periodic and a disordered component. Let the set Ω of configurations of the potential be compact and let $\mathbb{R}^2$ act homeomorphically on it. This action will simply be denoted by $\omega \mapsto \vec{x} \cdot \omega$, $\omega \in \Omega$. The set Ω is often called the hull. Furthermore let $\mathbf{P}$ be an invariant and ergodic probability measure on the hull. Given a measurable positive function $V \in L^\infty(\Omega, \mathbf{P})$, a family of bounded multiplication operators V_ω on $L^2(\mathbb{R}^2)$ is defined by

$$(V_\omega \psi)(\vec{x}) \;=\; V(-\vec{x} \cdot \omega)\, \psi(\vec{x}) \;.$$

The family of Hamiltonians studied here is now $H_\omega = H_L + V_\omega$. Note that H_ω transforms covariantly with respect to the magnetic translations:

$$U(\vec{\xi}\,)H_\omega U(\vec{\xi}\,)^* \;=\; H_{\vec{\xi}\cdot\omega} \;, \qquad \vec{\xi} \in \mathbb{R}^2 \;. \tag{1}$$

By functional calculus, any (continuous) function of the Hamiltonian is also covariant.

This work will mainly focus on the analysis of magnetic operators with an infinite boundary. Such a boundary could be modeled by a confining potential, but in this work Dirichlet boundary conditions are chosen. Hence let $H_{\omega,s}$ be the operator H_ω restricted to the domain $\{\vec{x} \in \mathbb{R}^2 \,|\, x_2 > -s\}$ and with Dirichlet boundary conditions. The operator $\hat{H}$ from the introduction was meant to be of that type. $H_{\omega,s}$ now satisfies the following covariance relation:

$$U(\vec{\xi}\,)H_{\omega,s} U(\vec{\xi}\,)^* \;=\; H_{\vec{\xi}\cdot\omega, s+\xi_2} \;, \qquad \vec{\xi} \in \mathbb{R}^2 \;. \tag{2}$$

This relation can be made even more similar to (1) if one introduces the new (non-compact) hull $\hat{\Omega} = \Omega \times (\mathbb{R} \cup \infty)$ and furnishes it with the $\mathbb{R}^2$-action $\vec{\xi} \cdot (\omega, s) = (\vec{\xi} \cdot \omega, s + \xi_2)$ where $\infty + x_2 = \infty$. As $s = \infty$ is left invariant under the shift, the point $\omega \in \Omega$ (without boundary conditions) can be identified with the point $(\omega, \infty) \in \hat{\Omega}$ (boundary conditions pushed to ∞). In other words, equation (2) incorporates (1).

Example: For sake of concreteness, let us construct such a potential explicitely. Set $\Omega = \mathbb{R}^2/\mathbf{Z}^2 \times [-\lambda, \lambda]^{\times \mathbf{Z}^2}$ and let $\mathbf{P}$ be the product measure of the Lebesgue measures with i.i.d. measures on the λ-components. If $\omega = (\vec{x}_0, (\lambda_\omega(\vec{n}))_{\vec{n}\in\mathbf{Z}^2})$, then the action is given by $\vec{x} \cdot \omega = (\vec{x} + \vec{x}_0, (\lambda_\omega(\vec{n} + [\vec{x} + \vec{x}_0]))_{\vec{n}\in\mathbf{Z}^2})$ where $[\vec{x}]$ denotes the integer parts of $\vec{x}$. Suppose now given two positive functions $w, v \in L^\infty(\mathbb{R}^2)$ where w is periodic with unit cell $[0,1]^2$ and v vanishes on the boundary and outside of the unit cell. Then set $V(\omega) = w(\vec{x}_0) + \lambda_\omega(0)\, v(\vec{x}_0)$. The associated multiplication operator is the sum of the periodic potential and disordered potential of the following type:

$$V_\omega(\vec{x}) \;=\; w(\vec{x}+\vec{x}_0) + \sum_{\vec{n}\in\mathbf{Z}^2} \lambda_\omega(\vec{n})\, v(\vec{x}+\vec{x}_0+\vec{n}) \;.$$

It is well-known that the periodic potential splits the Landau bands, each giving Harper-like spectra, and that the disordered potential leads to localization (*e.g.* [9, 16], and references therein). $\diamond$

3 Analysis of covariant families of integral operators

Equation (2), incorporating (1), is a covariance relation for the Hamiltonians $(H_{\hat\omega})_{\hat\omega\in\hat\Omega}$. By functional calculus it leads to a covariance relation also for functions of these operators. As will be proven in Sections 4 and 5 below, certain functions of the Hamiltonians $H_{\hat\omega}$ will actually be bounded integral operators on $L^2(\mathbb{R}^2)$. Therefore this section is concerned with the set $\mathcal{A}$ of weakly continuous families $A=(A_{\hat\omega})_{\hat\omega\in\hat\Omega}$ of bounded integral operators on $L^2(\mathbb{R}^2)$ for which the covariance relation

$$U(\vec{\xi}\,)A_{\hat\omega}U(\vec{\xi}\,)^* \;=\; A_{\vec{\xi}\cdot\hat\omega}\;, \qquad \vec{\xi}\in\mathbb{R}^2\;, \tag{3}$$

holds and

$$\|A\|_\infty \;=\; \operatorname*{ess\,sup}_{\hat\omega}\,\|A_{\hat\omega}\| \;<\; \infty\;,$$

where the essential supremum is taken with respect to the product of the probability measure $\mathbf{P}$ with the Lebesgue measure. As already mentioned above, $\Omega\times\{\infty\}$ is an $\mathbb{R}^2$-invariant compact subspace of $\hat\Omega$ so that $(A_{\omega,\infty})_{\omega\in\Omega}$ forms also a weakly continuous covariant family of bounded integral operators on $L^2(\mathbb{R}^2)$. The set of these families is denoted by $\mathcal{A}_\infty$ and in order to simplify notations A_ω will be written for $A_{\omega,\infty}$.

Pointwise linear combinations and products of such families form new families, hence $\mathcal{A}$ is actually an algebra, a generalized convolution algebra. In fact, various crossed product algebras are naturally associated to covariant families of integral operators (smooth and C*-algebras, as well as a von Neumann algebra [12, 5]), but this point of view will not be developed here. Let us only mention that the restriction of a family $(A_{\hat\omega})_{\hat\omega\in\hat\Omega}$ to values $s=\infty$ yields an algebra homomorphism $\mathcal{A}\to\mathcal{A}_\infty$ which plays an important role in [19]. In the following, the standard algebraic structures like derivations, covariant derivatives and various traces for covariant operator families are introduced.

The integral kernel of $A_{\hat\omega}$ will be denoted by $\langle\vec{x}\,|A_{\hat\omega}|\vec{y}\,\rangle$. Continuity and differentiability properties of these kernels in $\vec{x}$ and $\vec{y}$, as well as estimates on the decay in $\vec{x}-\vec{y}$ will be studied in the next section. These properties transpose again to sums and products. Using the covariance relation and the Cauchy-Schwarz inequality, one establishes that

$$\|A\|_\infty \;\leq\; \int_{\mathbb{R}} d\vec{x}\;\operatorname*{ess\,sup}_{\hat\omega}\left|\langle\vec{x}\,|A_{\hat\omega}|\vec{0}\,\rangle\right|\;. \tag{4}$$

Given $A \in \mathcal{A}$, new elements $\nabla_j A \in \mathcal{A}$, $j = 1, 2$, are defined by

$$(\nabla_j A)_{\hat{\omega}} \;=\; \imath[X_j, A_{\hat{\omega}}] \;, \tag{5}$$

as long as the r.h.s. are again bounded integral operators. By (4), this can be assured for through decay properties of the kernels $\langle \vec{x}\,|A_{\hat{\omega}}|\vec{y}\,\rangle$ in $|\vec{x} - \vec{y}\,|$. ∇_j is a derivation, i.e. it satisfies the Leibniz rule $\nabla_j(AB) = \nabla_j(A)B + A\nabla_j(B)$. Furthermore, $D_j A \in \mathcal{A}$ has integral kernel

$$\langle \vec{x}\,|D_j A_{\hat{\omega}}|\vec{y}\,\rangle \;=\; \left(\imath\partial_{x_j} - \delta_{j,1}\gamma x_2\right) \langle \vec{x}\,|A_{\hat{\omega}}|\vec{y}\,\rangle \;, \tag{6}$$

again provided the r.h.s. is the integral kernel of a bounded operator. If a covariant family $A = (A_{\omega,s})_{\omega,s}$ is actually independent of ω, then the covariance relation implies that the integral kernel $\langle \vec{x}\,|A_{\omega,s}|\vec{y}\,\rangle$ depends only on $x_1 - y_1, x_2, y_2$ and that of $\langle \vec{x}\,|A_{\omega,\infty}|\vec{y}\,\rangle$ only on $\vec{x} - \vec{y}$. As a consequence, for such A

$$[D_1, A_{\omega,s}] \;=\; \imath\gamma\nabla_2 A_{\omega,s} \;, \qquad [D_2, A_{\omega,\infty}] \;=\; 0 \;. \tag{7}$$

Let χ be a positive compactly supported function on $\mathbb{R}$ satisfying $\int dx\,\chi(x) = 1$. For $j = 1, 2$, let us set $\chi_j(\vec{x}) = \chi(x_j)$ and consider χ_j also as a multiplication operator on $L^2(\mathbb{R}^2)$. Let $\|T\|_1$ denote the (Schatten) traceclass norm of an operator T on $L^2(\mathbb{R}^2)$. Whenever $\|\chi_1 A_{\hat{\omega}}\|_1$ is integrable w.r.t. $\mathbf{P}$, the family $A \in \mathcal{A}$ will be called $\hat{\mathcal{T}}$-traceclass. Whenever $\|\chi_1\chi_2 A_{\omega,\infty}\|_1$ is integrable w.r.t. $\mathbf{P}$, the family $A \in \mathcal{A}_\infty$ is called $\mathcal{T}$-traceclass. For traceclass families, one can set

$$\hat{\mathcal{T}}(A) \;=\; \int d\mathbf{P}(\omega)\;\mathbf{Tr}(\chi_1 A_{\omega,s}) \;, \qquad \mathcal{T}(A) \;=\; \int d\mathbf{P}(\omega)\;\mathbf{Tr}(\chi_1\chi_2 A_{\omega,\infty}) \;,$$

where $\mathbf{Tr}$ is the usual trace on $L^2(\mathbb{R}^2)$. In order to write out more explicit formulas and see that the definition of $\hat{\mathcal{T}}$ is independent of the choice of s, recall that if T is a traceclass integral operator on $L^2(\mathbb{R}^2)$ with jointly continuous integral kernel, then $\mathbf{Tr}(T) = \int d\vec{x}\,\langle \vec{x}\,|T|\vec{x}\,\rangle$ (jointly continuous means that $(\vec{x}, \vec{y}) \mapsto \langle \vec{x}\,|T|\vec{y}\,\rangle$ is continuous; references herefore are given, *e.g.*, in [3] where it is also shown that the same formula holds if the integral kernel has a finite number of isolated point singularities). Using the covariance relation (3) and the invariance of $\mathbf{P}$,

$$\hat{\mathcal{T}}(A) \;=\; \int d\mathbf{P}(\omega) \int ds\;\langle \vec{0}\,|A_{\omega,s}|\vec{0}\,\rangle \;, \qquad \mathcal{T}(A) \;=\; \int d\mathbf{P}(\omega)\;\langle \vec{0}\,|A_{\omega,\infty}|\vec{0}\,\rangle \;. \tag{8}$$

This shows that A is $\hat{\mathcal{T}}$-traceclass (respectively $\mathcal{T}$-traceclass) if the integral kernels of $|A|$ are jointly continuous and integrable in the 2-direction (respectively, jointly continuous and uniformly bounded).

Lemma 1 *$\hat{\mathcal{T}}$ and $\mathcal{T}$ are traces on $\mathcal{A}$. This means in the case of $\hat{\mathcal{T}}$ that for $A, B \in \mathcal{A}$ with $\hat{\mathcal{T}}$-traceclass B:*
(i) $\hat{\mathcal{T}}(AB) = \hat{\mathcal{T}}(BA)$.
(ii) $\hat{\mathcal{T}}(AB) \leq \|A\|_\infty \hat{\mathcal{T}}(|B|)$.
(iii) $\hat{\mathcal{T}}(|A + B|) \leq \hat{\mathcal{T}}(|A|) + \hat{\mathcal{T}}(|B|)$.
Similar relations hold for $\mathcal{T}$.

Proof: Because of the translation invariance of $\mathbf{P}$, (i) can immediately be deduced from the definition of $\hat{\mathcal{T}}$. In order to prove (ii), one can use the polar decomposition $B = U|B|$ where the unitary $U = (U_{\hat\omega})_{\omega\in\hat\Omega}$ is easily seen to satisfy the covariance relation, just as the positive operator $|B|$. Then

$$\hat{\mathcal{T}}(AB) \;=\; \int d\mathbf{P}(\omega)\,\mathbf{Tr}(A_{\hat\omega}U_{\hat\omega}|B_{\hat\omega}|\chi_1) \;\leq\; \int d\mathbf{P}(\omega)\,\|A_{\hat\omega}U_{\hat\omega}\|\;\mathbf{Tr}(|B_{\hat\omega}|\chi_1) \;\leq\; \|A\|_\infty\,\hat{\mathcal{T}}(|B|)\;.$$

For the proof of (iii), set $|A+B| = U(A+B)$ by polar decomposition. Then $\hat{\mathcal{T}}(|A+B|) = \hat{\mathcal{T}}(UA) + \hat{\mathcal{T}}(UB)$ which allows to conclude by (ii). □

It follows from the covariance relation and the Birkhoff theorem that $\mathcal{T}$ is the trace per unit volume [5], while $\hat{\mathcal{T}}$ is the (disorder averaged) trace per unit volume in the 1-direction combined with the usual trace in the 2-direction. Finally let us remark that the traces are invariant w.r.t. the derivations:

$$\hat{\mathcal{T}}(\nabla_1 A) \;=\; 0\;, \qquad \mathcal{T}(\nabla_j A) \;=\; 0\;, \quad j=1,2\;,$$

as long as A has jointly continuous integral kernel and $\nabla_1 A$ is $\hat{\mathcal{T}}$-traceclass (resp. $\nabla_j A$ is $\mathcal{T}$-traceclass). Under these hypothesis, this can directly be verified from the expressions (8).

4 Integralkernels associated to the planar Hamiltonian

In this section, Hamiltonians without boundary conditions are considered. Hence $s = \infty$ and ω stands for (ω, ∞). Following Davies [13, Sec. 3.4], the functional calculus of the Hamiltonian $H_\omega = H_L + V_\omega$ will be done via the complex heat kernel:

$$F(H_\omega) \;=\; \int_{-\infty}^{\infty} dt\,\tilde{F}(t)\,e^{-H_\omega(1+\imath t)}\;, \tag{9}$$

where

$$\tilde{F}(t) \;=\; 2\pi\int_{-\infty}^{\infty} dE\,e^{\imath Et}\,e^{E}\,F(E)\;.$$

For compactly supported differentiable functions $F \in C_c^k(\mathbb{R})$, one has the standard Fourier estimates $|\tilde{F}(t)| \leq c_k(1+|t|^{k-1})^{-1}$. Such an estimate may also hold for functions with infinite support, but we do not intend here to give the most general formulation.

Proposition 1 *Let $V_\omega \in L^\infty(\mathbb{R}^2)$ and $F \in C_c^k(\mathbb{R})$ with $k > 2$. Then $F(H_\omega)$ is an integral operator the integral kernel of which satisfies uniformly in ω and for any $\delta > 0$*

$$|\langle\vec{x}|F(H_\omega)|\vec{y}\rangle| \;\leq\; \frac{c_\delta}{1+|\vec{x}-\vec{y}|^{k-2-\delta}}\;.$$

Proof: As the following estimates are pointwise in ω, the index will be suppressed. Let us begin with the integral kernel of $e^{-H_L z}$ explicitly using Mehler's formula for the (shifted) harmonic oscillator $h(k) = \frac{1}{2}(-\partial_{x_2}^2 + \gamma^2(X_2 + \frac{k}{\gamma})^2)$ in the 2-direction ($\Re e(z) > 0$):

$$\begin{aligned} \langle \vec{x}|e^{-H_L z}|\vec{y}\rangle &= \int \frac{dk}{2\pi}\, e^{\imath k(x_1-y_1)} \langle x_2|e^{-h(k)z}|y_2\rangle \\ &= \frac{\gamma}{4\pi}\,\frac{1}{\sinh(\frac{\gamma}{2}z)}\, e^{-\frac{\gamma}{4}\coth(\frac{\gamma}{2}z)|\vec{x}-\vec{y}|^2}\, e^{-\imath\frac{\gamma}{2}(x_1-y_1)(x_2-y_2)}\, \Phi(\vec{x}-\vec{y},\vec{y}) \,. \end{aligned} \tag{10}$$

In order to obtain upper bounds for the integral kernel let us use the following elementary inequalities ($\Re e(z) > 0$):

$$\frac{1}{|\sinh(z)|} \le \frac{1}{\Re e(z)}\,, \qquad \Re e(\coth(z)) \ \ge\ \Re e(z^{-1})\,. \tag{11}$$

They lead directly to the following estimate:

$$\big|\langle \vec{x}|e^{-H_L z}|\vec{y}\rangle\big| \ \le\ \frac{1}{2\pi}\,\frac{1}{\Re e(z)}\, e^{-\frac{1}{2}|\vec{x}-\vec{y}|^2\,\Re e(z^{-1})}\,. \tag{12}$$

Now set $V_+ = \|V\|_\infty$ and $\tilde{V} = V - V_+$ so that $\tilde{H} = H_L + \tilde{V}$ has a negative potential $\tilde{V}$. Furthermore Duhamel's formula reads

$$\langle \vec{x}|e^{-\tilde{H}z}|\vec{y}\rangle \ =\ \langle \vec{x}|e^{-H_L z}|\vec{y}\rangle - z\int_0^1 dq \int_{\mathbb{R}^2} d\vec{r}\, \langle \vec{x}\,|e^{-(1-q)H_L z}|\vec{r}\rangle\, \tilde{V}(\vec{r})\, \langle \vec{r}\,|e^{-q\tilde{H}z}|\vec{y}\rangle\,. \tag{13}$$

Using this iteratively, one obtains the Dyson series for $z = t > 0$ which is estimated term by term using (12)

$$\big|\langle \vec{x}|e^{-Ht}|\vec{y}\rangle\big| \ \le\ e^{-tV_+} \sum_{n\ge 0} \frac{V_+^n}{(2\pi)^n} \left(\prod_{l=1}^{n} \int_{\mathbb{R}^2} d\vec{r}_l \int_0^{q_{l+1}} dq_l\right) \left(\prod_{l=0}^{n-1} \frac{e^{-\frac{|\vec{r}_{l-1}-\vec{r}_l|^2}{2(q_l-q_{l-1})t}}}{q_l - q_{l-1}}\right) \frac{e^{-\frac{|\vec{r}_n-\vec{y}|^2}{2q_n t}}}{q_n t}\,, \tag{14}$$

where $q_{n+1} = 1$ and $\vec{r}_0 = \vec{x}$ in each term. A short calculation using rotation invariance shows, for $\Re e(a) > 0$ and $\Re e(b) > 0$,

$$\int_{\mathbb{R}^2} d\vec{r}\, e^{-\frac{|\vec{x}-\vec{r}\,|^2}{a}}\, e^{-\frac{|\vec{r}-\vec{y}\,|^2}{b}} \ =\ \pi\,\frac{ab}{a+b}\, e^{-\frac{|\vec{x}-\vec{y}\,|^2}{a+b}}\,. \tag{15}$$

Applying this n times in the nth order term of the Dyson series shows

$$\big|\langle \vec{x}|e^{-Ht}|\vec{y}\rangle\big| \ \le\ \frac{1}{2\pi\, t}\, e^{-\frac{1}{2t}|\vec{x}-\vec{y}\,|^2}\,. \tag{16}$$

Now the arguments of Lemma 3.4.6 and Theorem 3.4.8 of [13] imply that, for $\Re e(z) > 0$,

$$|\langle\vec{x}|e^{-zH}|\vec{y}\,\rangle| \;\leq\; \frac{c}{|\Re e(z)|}\,e^{-\frac{1}{4}|\vec{x}-\vec{y}\,|^2\,\Re e(z^{-1})}\;. \tag{17}$$

As in Theorem 3.4.9 of [13], one therefore has

$$|\langle\vec{x}|F(H_\omega)|\vec{y}\,\rangle| \;\leq\; \int dt\;\frac{c_k}{1+|t|^{k-1}}\;\exp\left(-\frac{1}{4}\,\frac{|\vec{x}-\vec{y}\,|^2}{1+t^2}\right)\;,$$

so that the inequality $e^{-r}\leq c_\beta/(1+r)^\beta$ for $r,\beta>0$ leads to

$$|\langle\vec{x}|F(H_\omega)|\vec{y}\,\rangle| \;\leq\; \int dt\;\frac{c_k\,c_\beta}{1+|t|^{k-1}}\,(1+t^2)^\beta\,\frac{1}{(1+\frac{1}{4}|\vec{x}-\vec{y}\,|^2)^\beta}\;.$$

Hence the t-integral is bounded as long as $2\beta<k-2$ which concludes the proof. □

As an aside be mentioned that there are various other ways to get estimates on the integral kernel of the semigroup e^{-tH}. One is a Combes-Thomas-like argument which will be used in Section 5. Another is to simply apply the diamagnetic inequality [10, Theo. 1.13], which reads $|e^{-tH}\phi(\vec{x})|\leq e^{t\Delta/2}|\phi|(\vec{x})$ for positive V and any $\phi\in L^2(\mathbb{R}^2)$ and $t>0$ where $\Delta=\partial_1^2+\partial_2^2$ is the two-dimensional Laplacian. As is moreover known (consult *e.g.* [7]) that the integral kernels $\langle\vec{x}|e^{-tH}|\vec{y}\,\rangle$ are jointly continuous, one also deduces the pointwise estimate (16) because the r.h.s. of (16) is precisely the integral kernel of $e^{t\Delta/2}$. Here the above Dyson series argument was used because the same technique will be used to derive estimates on the covariant derivatives of the integral kernels.

Proposition 2 *Let $V_\omega\in L^\infty(\mathbb{R}^2)$ and $F\in C_c^k(\mathbb{R})$ with $k>6$. Then $D_jF(H_\omega)$ is an integral operator satisfying for any $\delta>0$*

$$|\langle\vec{x}|D_jF(H_\omega)|\vec{y}\rangle| \;\leq\; \frac{c_\delta}{1+|\vec{x}-\vec{y}\,|^{k-6-\delta}}\;.$$

Suppose that $\partial_jV_\omega\in L^\infty(\mathbb{R}^2)$ and $F\in C_c^k(\mathbb{R})$ with $k>10$. Then $D_jD_iF(H_\omega)$ is an integral operator satisfying for any $\delta>0$

$$|\langle\vec{x}|D_jD_iF(H_\omega)|\vec{y}\rangle| \;\leq\; \frac{c_\delta}{1+|\vec{x}-\vec{y}\,|^{k-10-\delta}}\;.$$

Proof: Again the index ω will be suppressed. One has

$$D_je^{-zH} \;=\; D_je^{-zH_L}-z\int_0^1 dq\;D_je^{-z(1-q)H_L}\;V\,e^{-zqH}\;. \tag{18}$$

Hence estimates on the covariant derivatives of the Landau Hamiltonian will be needed. Using $|\coth(z)|\leq\frac{e^{-\Re e(z)}}{\Re e(z)}+1$ and the inequalities (11) (from now on c denotes varying constants and $\Re e(z)\geq 0$)

$$\begin{aligned}\big|(\imath\partial_{x_j}-\gamma\delta_{j,1}x_2)\,\langle\vec{x}|e^{-zH_L}|\vec{y}\,\rangle\big| \;&\le\; \frac{c}{\Re e(z)}\,|\vec{x}-\vec{y}\,|\,\left(1+\frac{1}{\Re e(z)}\right)\,e^{-\frac{1}{4}|\vec{x}-\vec{y}\,|^2\,\Re e(z^{-1})} \\ &\le\; \frac{c}{\Re e(z)^2\sqrt{\Re e(z^{-1})}}\,\left(\Re e(z)+1\right)\,e^{-\frac{1}{8}|\vec{x}-\vec{y}\,|^2\,\Re e(z^{-1})}\;,\end{aligned}\tag{19}$$

where in the second step $ae^{-2a^2}\le e^{-a^2}$ for $a>0$ was used. Let now I_1 denote the integral kernel of the second contribution in (18). Using (17) and then again (15), one gets

$$\begin{aligned}|I_1| \;&\le\; c\,\frac{\Re e(z)+1}{\Re e(z)^2\sqrt{\Re e(z^{-1})}}\,|z|\int_0^1 dq\int d\vec{r}\,\frac{1}{(1-q)^{\frac{3}{2}}}\,e^{-\frac{1}{8(1-q)}|\vec{x}-\vec{r}\,|^2\,\Re e(z^{-1})}\,\frac{1}{q\Re e(z)}\,e^{-\frac{1}{8q}|\vec{r}-\vec{y}\,|^2\,\Re e(z^{-1})} \\ &\le\; c\,\frac{\Re e(z)+1}{\Re e(z)^3\sqrt{\Re e(z^{-1})}}\,\frac{|z|}{\Re e(z^{-1})}\,e^{-\frac{1}{8}|\vec{x}-\vec{y}\,|^2\,\Re e(z^{-1})}\;.\end{aligned}\tag{20}$$

Hence,

$$\big|\langle\vec{x}|D_je^{-zH}|\vec{y}\,\rangle\big| \;\le\; c\,\frac{\Re e(z)+1}{\Re e(z)^2\sqrt{\Re e(z^{-1})}}\,\left(1+\frac{|z|}{\Re e(z)\Re e(z^{-1})}\right)\,e^{-\frac{1}{8}|\vec{x}-\vec{y}\,|^2\,\Re e(z^{-1})}\tag{21}$$

yielding

$$\big|\langle\vec{x}|D_je^{-(1+\imath t)H}|\vec{y}\,\rangle\big| \;\le\; c\,(1+t^2)^2e^{-\frac{|\vec{x}-\vec{y}\,|^2}{8(1+t^2)}}\;.\tag{22}$$

This implies just as in Proposition 1 that $D_jF(H)$ satisfies the stated bound.

To prove the second statement, let us use (7) which implies

$$\begin{aligned}D_jD_ie^{-zH} \;&=\; D_jD_ie^{-zH_L}+\delta_{i,1}\,\imath\gamma z\int_0^1 dq\,D_j\left(\nabla_2e^{-(1-q)zH_L}\right)Ve^{-qzH} \\ &\quad +\imath z\int_0^1 dq\;D_je^{-(1-q)zH_L}\partial_iVe^{-qzH}\;+\;z\int_0^1 dq\;D_je^{-(1-q)zH_L}VD_ie^{-qzH}\;.\end{aligned}$$

As in (19) one shows for the first term

$$\big|\langle\vec{x}|D_jD_ie^{-zH_L}|\vec{y}\,\rangle\big| \;\le\; c\,\frac{(\Re e(z)+1)^2}{\Re e(z)^3}\,\frac{1}{\Re e(z^{-1})}\,e^{-\frac{|\vec{x}-\vec{y}\,|^2}{16}\,\Re e(z^{-1})}\;.$$

For the second term, let us commute D_j and ∇_2. The integral kernel of the contribution $\int_0^1 dq[D_j,\nabla_2]e^{-(1-q)zH_L}Ve^{-qzH}$ satisfies a bound as (20). Let I_2 be the integral kernel of $\int_0^1 dq\nabla_2D_je^{-(1-q)zH_L}Ve^{-qzH}$. Using

$$\big|\langle\vec{x}|\nabla_2D_je^{-zH_L}|\vec{y}\,\rangle\big| \le c\,\frac{(\Re e(z)+1)^2}{\Re e(z)^2}\,|\vec{x}-\vec{y}\,|^2e^{-\frac{|\vec{x}-\vec{y}\,|^2}{4}\,\Re e(z^{-1})} \le \frac{c\,(\Re e(z)+1)^2}{\Re e(z)^2\Re e(z^{-1})}\,e^{-\frac{|\vec{x}-\vec{y}\,|^2}{16}\,\Re e(z^{-1})}\;,$$

and performing a similar calculation as in (20) one finds that it can be bounded by

$$|I_2| \;\leq\; c\,\frac{|z|\,(\Re e(z)+1)^2}{\Re e(z)^3 \Re e(z^{-1})^2}\; e^{-\frac{|\vec{x}-\vec{y}|^2}{16}\Re e(z^{-1})}$$

The integral kernel of the third contribution can be bounded as above, hence let us focus on the integral kernel I_4 of the forth contribution. Using (19) and (21), one gets by a similar calculation as in (20)

$$|I_4| \;\leq\; c\,\frac{(\Re e(z)+1)^2}{\Re e(z)^4 \Re e(z^{-1})}\left(1+\frac{|z|}{\Re(z)\Re e(z^{-1})}\right)\,\frac{|z|}{\Re e(z^{-1})}\; e^{-\frac{|\vec{x}-\vec{y}|^2}{16}\Re e(z^{-1})}\;.$$

Finally, the number k determining the decay of the integral kernel of $D_j D_i F(H_\omega)$ depends on the leading power in t of $\left|\langle\vec{x}|D_j D_i e^{-(1+\imath t)H}|\vec{y}\rangle\right|$. Comparing the above contributions one sees that this power is determined by I_4, and, setting $z = 1+\imath t$, one has

$$|I_4| \;\leq\; c\,(1+t^2)^4\, e^{-\frac{|\vec{x}-\vec{y}|^2}{16(1+t^2)}}\;.$$

As in Proposition 1, the statement of the proposition follows. □

Let us remark that Proposition 2 implies in particular that the integral kernel of $F(H_\omega)$ is twice differentiable. In dimension 2, the same argument goes through for $D_1^2 D_2^2\, F(H)$, but not for $D_j^3\, F(H)$. In higher dimension, more regularity can be obtained.

5 Integralkernels of operators on the half-plane

The aim of this section is to show that Proposition 1 and the part of Proposition 2 concerning covariant derivatives in the 1-direction remain essentially valid for the operators $H_{\omega,s}$ on the half-plane. This is done by proving estimates like (12) and (19) for the kernel of the semigroup generated by the Landau operator $\hat{H}_L$ (and its covariant derivative) with Dirichlet boundary conditions at $s = 0$. Covariance then implies that these estimates also hold for arbitrary $s < \infty$ and the perturbative arguments based on the Dyson series expansion can be directly transposed to obtain a power law decay of the integral kernels of functions of the Hamiltonian on the half-plane.

Proposition 3 *For* $\Re e(z) > 0$ *and* $n = 0, 1, 2$. *Then*

$$\Big|\,\langle\vec{x}\,|D_1^n\, e^{-z\hat{H}_L}|\vec{y}\rangle\,\Big| \;\leq\; c\,\frac{1+|z|^{n+1}}{\Re e(z)^{\frac{n}{2}+1}}\exp\left(-\frac{|\vec{x}-\vec{y}|^2}{10}\,\Re e(z^{-1})\right)\;.$$

Furthermore, $\langle\vec{x}\,|D_1^n e^{-z\hat{H}_L}|\vec{y}\rangle$ *is continuous in* $\vec{x},\vec{y}$ *for* $n = 0, 1$.

This in particular implies that the integral kernel of $e^{-tH_{\omega,s}}$ is continuous so that the diamagnetic inequality implies

$$\left|\langle\vec{x}|e^{-tH_{\omega,s}}|\vec{y}\rangle\right| \;\leq\; \langle\vec{x}|e^{t\Delta_s}|\vec{y}\rangle \;=\; \frac{1}{4\pi t}\,e^{-|\vec{x}-\vec{y}|^2/t}\left(1-e^{-2(x_2+s)(y_2+s)/t}\right)\;\chi(x_2\geq -s)\,\chi(y_2\geq -s)\;. \tag{23}$$

This also shows how the integral kernels of functions of $H_{\omega,s}$ vanish near $x_2=-s$ or $y_2=-s$.

For the proof of Proposition 3, the semigroup of $\hat{H}_L$ is calculated via Fourier transform just as in (10):

$$\langle\vec{x}\,|e^{-t\hat{H}_L}|\vec{y}\rangle \;=\; \int\frac{dk}{2\pi}\;e^{\imath k(x_1-y_1)}\;\langle x_2|e^{-t\hat{h}(k)}|y_2\rangle\;, \tag{24}$$

where $\hat{h}(k)=\frac{1}{2}(-\partial^2+\gamma^2(X+\frac{k}{\gamma})^2)$ with Dirichlet boundary conditions at the origin. As we did not succeed in calculating this kernel explicitly, recourse to more abstract analytical arguments is necessary. For a complex dilation argument on the heat kernel, the following will be needed:

Lemma 2 $k+\imath\kappa\in\mathbb{C}\mapsto e^{-t\hat{h}(k+\imath\kappa)}$ *is entire for all* $t>0$ *and the integral kernel satisfies*

$$\left|\langle x|e^{-t\hat{h}(k+\imath\kappa)}|y\rangle\right| \;\leq\; e^{\frac{1}{2}\,t\kappa^2}\;\langle x|e^{-th(k)}|y\rangle\;. \tag{25}$$

Proof: First let us show that X is relatively bounded w.r.t. $\hat{h}(0)$ with relative bound 0. Therefore let $|n\rangle$ denote the Hermite eigenfunctions of $h(0)$ and recall $X|n\rangle=(2\gamma)^{-1/2}(\sqrt{n+1}\,|n+1\rangle+\sqrt{n}\,|n-1\rangle)$. The odd Hermite functions $|2l+1\rangle$ form an eigenbasis of $\hat{h}(0)$ which is complete in $L^2(\mathbb{R}_+)$. Now let $\psi=\sum_{l\geq 0}a_l\,|2l+1\rangle$ so that $\|\hat{h}(k)\psi\|^2=\sum_{l\geq 0}|a_l|^2(2l+\frac{3}{2})^2$. As $\|X\psi\|^2\leq c\sum_{l\geq 0}|a_l|^2(2l+2)$, the relative bound estimates follow immediately. In conclusion, $\hat{h}(k+\imath\kappa)=\hat{h}(0)+\gamma X(k+\imath\kappa)+\frac{1}{2}(k+\imath\kappa)^2$ is automatically closed and [17, Thm. IX.2.6] implies the desired analyticity property.

In order to prove the estimate, let us cite a norm-convergent version of the Trotter product formula from [8]: Given two m-sectorial operators A,B on a given Hilbert space $\mathcal{H}$ satisfying that $(A+1)^{-1}$ is compact and $D(A)\cap D(B)$ is dense in $\mathcal{H}$,

$$e^{-t(A\dot{+}B)} \;=\; \lim_{n\to\infty}\left(e^{-\frac{t}{n}A}e^{-\frac{t}{n}B}\right)^n\;,$$

where the convergence is in the norm topology and $A\dot{+}B$ is the form sum. This will be applied for $A=\hat{h}(k)-\alpha W(k)$ and $B=\alpha W(k)-\imath\gamma\kappa(X+\frac{k}{\gamma})$ where $W(k)=\frac{\gamma^2}{2}(X+\frac{k}{\gamma})^2+1$. Indeed, A is a strictly positive selfadjoint operator with compact resolvent as long as $\alpha>0$ is small enough and B is m-sectorial. As $A\dot{+}B-\frac{1}{2}\kappa^2$ and $\hat{h}(k+\imath\kappa)$ coincide on the domain of the latter and the semigroups are bounded, one deduces

$$e^{-t\hat{h}(k+\imath\kappa)} \;=\; e^{\frac{1}{2}\,t\kappa^2}\;\lim_{n\to\infty}\left(e^{-\frac{t}{n}A}e^{-\frac{t}{n}B}\right)^n\;.$$

Setting $x=r_0$ and $y=r_n$, one can therefore bound as follows:

$$
\begin{aligned}
\left|\langle x|\left(e^{-\frac{t}{n}A}e^{-\frac{t}{n}B}\right)^n|y\rangle\right| &\leq \int_{\mathbb{R}_+^{n-1}} dr_1\cdots dr_{n-1}\prod_{j=1}^{n}\langle r_{j-1}|e^{-\frac{t}{n}(\hat{h}(k)-\alpha W(k))}|r_j\rangle\, e^{-\frac{t}{n}\alpha W(k)(r_j)} \\
&= \langle x|\left(e^{-\frac{t}{n}(\hat{h}(k)-\alpha W(k))}e^{-\frac{t}{n}\alpha W(k)}\right)^n|y\rangle \\
&= \langle x|e^{-t\hat{h}(k)}|y\rangle\ ,
\end{aligned}
$$

where the last equality follows from recomposing with the Trotter formula. To conclude, one just notes that the integral kernel of $e^{-t\hat{h}(k)}$ is bounded by that of $e^{-th(k)}$ (this follows easily, e.g., from the Feynman-Kac path-integral in which Dirichlet boundary conditions are incorporated by characteristic functions). □

Lemma 3 *For* $t>0$ *and* $n=0,1,2,$

$$
\left|\ D_1^n\,\langle\vec{x}\,|e^{-t\hat{H}_L}|\vec{y}\,\rangle\ \right|\ \leq\ c\,\frac{(1+t)^{n+\frac{1}{2}}}{t^{\frac{n}{2}+1}}\exp\left(-\frac{|\vec{x}-\vec{y}|^2}{2^{n+1}t}\right)\ .
$$

Proof: Applying D_1^n to equation (24) and multiplying it with $e^{\kappa(x_1-y_1)}$, $\kappa\in\mathbb{R}$ leads to

$$
e^{\kappa(x_1-y_1)}\ D_1^n\,\langle\vec{x}\,|e^{-t\hat{H}_L}|\vec{y}\,\rangle\ =\ \int_{\mathbb{R}}\frac{dk}{2\pi}\,(-k-\gamma x_2)^n\,e^{\imath(k-\imath\kappa)(x_1-y_1)}\langle x_2|e^{-t\hat{h}(k)}|y_2\rangle\ .
$$

Let us change variables $k-\imath\kappa\mapsto k$, then use analyticity (Lemma 2) and decay properties on the boundaries of a Cauchy contour in order to obtain:

$$
e^{\kappa(x_1-y_1)}\ D_1^n\,\langle\vec{x}\,|e^{-t\hat{H}_L}|\vec{y}\,\rangle\ =\ \int_{\mathbb{R}}\frac{dk}{2\pi}\,(-k-\imath\kappa-\gamma x_2)^n\,e^{\imath k(x_1-y_1)}\langle x_2|e^{-t\hat{h}(k+\imath\kappa)}|y_2\rangle\ . \qquad (26)
$$

Now the estimate (25) will be used, along with the fact $\langle x|e^{-th(k)}|y\rangle=\langle x+\frac{k}{\gamma}|e^{-th(0)}|y+\frac{k}{\gamma}\rangle$ and the following estimate for the Mehler kernel:

$$
\begin{aligned}
\langle x|e^{-th(0)}|y\rangle &= \sqrt{\frac{\gamma}{2\pi\sinh(\gamma t)}}\ \exp\left(-\frac{\gamma}{4}\coth\left(\frac{\gamma t}{2}\right)|x-y|^2-\frac{\gamma}{4}\tanh\left(\frac{\gamma t}{2}\right)|x+y|^2\right) \\
&\leq \sqrt{\frac{1}{2\pi t}}\ \exp\left(-\frac{1}{2t}|x-y|^2-\frac{\gamma}{4}\tanh\left(\frac{\gamma t}{2}\right)|x+y|^2\right)\ .
\end{aligned}
$$

Replacing this and substituting k for $k+\frac{\gamma}{2}(x_2+y_2)$, one obtains

$$
\left|\ e^{\kappa(x_1-y_1)}\ D_1^n\,\langle\vec{x}\,|e^{-t\hat{H}_L}|\vec{y}\,\rangle\ \right|\leq\int\frac{dk}{2\pi}\,\left|k+\frac{\gamma}{2}(x_2-y_2)+\imath\kappa\right|^n\frac{e^{\frac{1}{2}t\,\kappa^2}}{\sqrt{2\pi t}}\ e^{-\frac{|x_2-y_2|^2}{2t}-\frac{1}{\gamma}\tanh(\frac{\gamma t}{2})k^2}\ .
$$

Now let us choose $\kappa=\frac{(x_1-y_1)}{t}$ and integrate over k. Then

$$|\langle\vec{x}|e^{-t\hat{H}_L}|\vec{y}\rangle| \;\leq\; \left(\frac{\pi\gamma}{t\tanh\left(\frac{\gamma t}{2}\right)}\right)^{\frac{1}{2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} \;\leq\; c\,\frac{(1+t)^{\frac{1}{2}}}{t}\,e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}}\;,$$

because $\coth(t) < \frac{1+t}{t}$. Using $\int dk\,|k-b|\,e^{-ak^2} \leq a^{-1} + \sqrt{\frac{\pi}{a}}\,|b|$ for $a > 0$ it follows that

$$\begin{aligned}
|D_1\langle\vec{x}|e^{-t\hat{H}_L}|\vec{y}\rangle| \;&\leq\; \left(\frac{\gamma^{\frac{1}{2}}}{\left(\tanh\left(\frac{\gamma t}{2}\right)\right)^{\frac{1}{2}}} + \frac{\sqrt{\pi}|x_1-y_1|}{t} + \gamma\frac{\sqrt{\pi}|x_2-y_2|}{2}\right)\frac{\gamma^{\frac{1}{2}}}{\left(t\tanh\left(\frac{\gamma t}{2}\right)\right)^{\frac{1}{2}}}e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} \\
&\leq\; c\,\frac{(1+t)^{\frac{3}{2}}}{t^{\frac{3}{2}}}\,e^{-\frac{|\vec{x}-\vec{y}|^2}{4t}}
\end{aligned}$$

where for the second bound $xe^{-2x^2} \leq e^{-x^2}$ was used. The last estimate ($n = 2$) is obtained similarly upon using $\int dk(k-b)^2e^{-ak^2} = (b^2+2a^{-1})\sqrt{\frac{\pi}{a}}$. □

Proof of Proposition 3 (This argument follows closely [13, Thm. 3.4.8] and is hence kept sketchy). Let us set $K(z,\vec{x},\vec{y}) = \langle\vec{x}\,|D^n\,e^{-z\hat{H}_L}|\vec{y}\,\rangle$. If $z = t + \imath s$, one has

$$|K(z,\vec{x},\vec{y}\,)| \;\leq\; \left\|D^n\,e^{-z\hat{H}_L}\right\|_{\infty,1} \;\leq\; \left\|D^n\,e^{-\frac{t}{2}\hat{H}_L}\right\|_{\infty,2}\left\|e^{-\frac{t}{2}\hat{H}_L}\right\|_{2,1} \;=\; \left\|D^n\,e^{-\frac{t}{2}\hat{H}_L}\right\|_{\infty,2}\left\|e^{-\frac{t}{2}\hat{H}_L}\right\|_{\infty,2}.$$

Since $\|A\|_{\infty,2}^2 \leq \sup_{\vec{x}}\int_{\mathbb{R}\times\mathbb{R}_+} d\vec{y}\,|\langle\vec{x}|A|\vec{y}\rangle|^2$, Lemma 3 implies

$$\left\|D^n\,e^{-\frac{t}{2}\hat{H}_L}\right\|_{\infty,2} \leq c\,\frac{(1+t)^{n+\frac{1}{2}}}{t^{\frac{n+1}{2}}}$$

so that

$$|K(z,\vec{x},\vec{y}\,)| \;\leq\; c\,f(t)\;, \qquad f(z) \;=\; \frac{1+z^{n+1}}{z^{\frac{n}{2}+1}}\;.$$

Now for $0 \leq \gamma < \frac{\pi}{2}$, let $D = \{z\,|\,0 \leq \arg(z) \leq \gamma\;,\;|z| \geq 1\}$ and set

$$g(z) \;=\; \frac{1}{f(z^{-1})}\,K(z^{-1},\vec{x},\vec{y}\,)\;\exp\left(\frac{1}{8}\,|\vec{x}-\vec{y}\,|^2e^{\imath(\frac{\pi}{2}-\gamma)}\,\frac{z}{\sin(\gamma)}\right)\;.$$

The hypothesis of the Phragmen-Lindelöf Theorem can be verified, showing that $|g(z)| \leq c\,\cos(\gamma)^{-\frac{n+1}{2}}$ for $z \in D$. Applying this also to $\overline{z}$ and choosing $\gamma = \frac{\pi}{2}(1-\epsilon) + \epsilon|\arg(z)|$ for some $\epsilon < 1$ allows to conclude the first statement of the Proposition.

In order to prove continuity in $\vec{x},\vec{y}$ of $\langle\vec{x}\,|D_1^ne^{-z\hat{H}_L}|\vec{y}\,\rangle$ for $n = 0, 1$, one may follow the same strategy as above to obtain a bound on $\langle\vec{x}\,|D_1^n\,D_2e^{-z\hat{H}_L}|\vec{y}\,\rangle$. This involves calculating the derivative of the half-sided Mehler kernel with Duhamel's formula,

$$\partial_x\,\langle x|e^{-t\hat{h}(k)}|y\rangle \;=\; \int_0^1 dq\int_0^\infty dr\;\partial_x\,\langle x|e^{-(1-q)t\hat{h}(0)}|r\rangle\;\gamma kr\;\langle r|e^{-qt\hat{h}(k)}|y\rangle\;e^{-\frac{1}{2}(1-q)tk^2}\;,$$

which can be done exactly as the kernel of $e^{-t\hat{h}(0)}$ is known explicitly by the reflection principle. One then replaces in (26), carries out the k-integral and uses

$$\int_{\mathbb{R}} dr\; r^p e^{-\frac{(r-d)^2}{(1-q)t}} e^{-\frac{r^2}{qt}} \;\leq\; c\, e^{-\frac{d^2}{t}} \,(1-q)^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} (1+t^{\frac{p}{2}}) \;,$$

to bound the r-integral. Application of the inequality $xe^{-2x^2} \leq e^{-x^2}$ allows to obtain an expression which is integrable in q at 0 and 1. This yields an estimate similar to but more cumbersome than the ones in Lemma 3. Since only the continuity result is needed here, further details are left out. □

6 Comparing integral kernels

For a given function F, one can compare the integral kernels of $F(H_{\omega,s})$ and $F(H_{\omega,\infty})$ and estimate the difference in particular for arguments which are far from the boundary at $x_2 = -s$. Therefore let us construct the semigroup of $H_{\omega,s}$ by means of the reflection principle. The reflection $S_s : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ at the line $x_2 = -s$ is defined by $(S_s\psi)(x_1, x_2) = \psi(x_1, -x_2 - 2s)$. Let Π_2^s be the indicator function on the half-plane $x_2 \geq -s$. Note that $S_s H_L S_s$ is the Landau operator with reversed magnetic field. Now set

$$\tilde{H}_{\omega,s} \;=\; \Pi_2^s H_\omega + (1-\Pi_2^s) S_s H_\omega S_s$$

with core $C^\infty_{c,s}(\mathbb{R}^2)$, given by the functions in $\psi \in C^\infty_c(\mathbb{R}^2)$ satisfying the antisymmetry relation $S_s\psi = -\psi$. These functions vanish on the boundary $x_2 = -s$. By construction, $S_s\tilde{H}_{\omega,s}S_s = \tilde{H}_{\omega,s}$ and therefore $C^\infty_{c,s}(\mathbb{R}^2)$ is left invariant. Moreover, if ψ is a smooth compactly supported function in the domain of $\tilde{H}_{\omega,s}$ then $\tilde{H}_{\omega,s}\psi = \Pi_2^s H_{\omega,s}(1-S)\psi$ so that for $\Re e(z) > 0$

$$e^{-zH_{\omega,s}} \;=\; \Pi_2^s\, e^{-z\tilde{H}_{\omega,s}}\,(1-S_s)\Pi_2^s \;. \tag{27}$$

Furthermore let $\phi \in C^\infty(\mathbb{R})$ be monotonously increasing, $\phi(-\infty) = 0$, $\phi(\infty) = 1$ and $\operatorname{supp}(\phi') \subset [0,1]$ and set $\phi_s(x) = \phi(s+x)$. The following result is similar to the discrete case [18, 14].

Theorem 1 *Let $V_\omega \in L^\infty(\mathbb{R}^2)$ and $F \in C^k_c$, $k > 6$ and $s < \infty$. Then $F(H_{\omega,s})$ is an integral operator which can be decomposed as*

$$F(H_{\omega,s}) \;=\; \phi_s\, F(H_\omega) + K_{\omega,s} \;,$$

where $K_{\omega,s}$ form a covariant family of integral operators the kernels of which satisfy for any $\delta > 0$

$$|\langle \vec{x}|K_{\omega,s}|\vec{y}\rangle| \;\leq\; \frac{c_\delta}{1+|x_2+s|^{k-6-\delta}+|y_2+s|^{k-6-\delta}} \;. \tag{28}$$

Proof: Again we set $s = 0$, drop the indices ω and s and denote the half-plane operator by $\hat{H}$, the one on the plane by H. Furthermore set:

$$\tilde{H} \;=\; H + P \;, \qquad P \;=\; (1 - \Pi_2)\left(-2\gamma\, X_2\, D_1 + 2\gamma^2 X_2^2 - V + SVS\right)(1 - \Pi_2)\;,$$

One easily verifies the arguments of Section 5 which imply that also the integral kernel of $e^{-z\tilde{H}}$ satisfies the estimates of Proposition 3. Using (27) and Duhamel's formula, one gets the following operator identity on $L^2(\mathbb{R} \times \mathbb{R}_+)$:

$$e^{-z\hat{H}} \;=\; \Pi_2 e^{-zH}\Pi_2 + z\int_0^1 dq\; \Pi_2 e^{-(1-q)z\tilde{H}} P e^{-qzH}\Pi_2 - \Pi_2\, e^{-z\tilde{H}} S \Pi_2\;.$$

Replacing this into (9), the first term gives rise to $\Pi_2 F(H)\Pi_2$, which can easily be replaced by $\phi_0 F(H)\phi_0$ up to an error satisfying (28). The third term leads to $\Pi_2 F(\tilde{H})S\Pi_2$, which according to Proposition 1 (holding also for $\tilde{H}$) can directly be seen to satisfy (28), even with Π_2 replaced by ϕ_0.

Now let us consider the second contribution to $e^{-z\hat{H}}$ and denote it $I(z)$. In order to estimate it, it will be used that the kernel of $e^{-z\tilde{H}}$ satisfies the estimate (17) following from (23). Then, using the particular form of P and the estimate (21), one first obtains

$$\left|\langle\vec{r}|Pe^{-q(1+\imath t)H}|\vec{y}\rangle\right| \;\leq\; c\left(\frac{|r_2|}{q^{3/2}}\,(1+t^2)^2 \,+\, \frac{r_2^2}{q} \,+\, \frac{|V|}{q}\right) e^{-\frac{|\vec{r}-\vec{y}|^2}{8q(1+t^2)}}\;.$$

Due to (17), one can bound

$$\begin{aligned}
|\langle\vec{x}|I(z)|\vec{y}\rangle| \;&\leq\; c\int_0^1 dq\int_{r_2\leq 0} d\vec{r}\,\frac{1}{1-q}\; e^{-\frac{|\vec{x}-\vec{r}|^2}{8(1-q)(1+t^2)}}\left(\frac{|r_2|}{q^{3/2}}\,(1+t^2)^2 \,+\, \frac{r_2^2}{q} \,+\, \frac{|V|}{q}\right) e^{-\frac{|\vec{r}-\vec{y}|^2}{8q(1+t^2)}} \\
&\leq\; c\int_0^1 dq\int_0^\infty dr_2\,\frac{q^{1/2}}{(1-q)^{1/2}}\left(\frac{|r_2|}{q^{3/2}}\,(1+t^2)^2 \,+\, \frac{r_2^2}{q} \,+\, \frac{|V|}{q}\right) e^{-\frac{|x_2+r_2|^2}{8q(1+t^2)}-\frac{|r_2+y_2|^2}{8(1-q)(1+t^2)}} \\
&\leq\; c\,(1+t^2)^2 e^{-\frac{x_2^2+y_2^2}{8(1+t^2)}}\;,
\end{aligned}$$

where in the second step the integral over r_1 was carried out and the resulting Gaussian factor $e^{-\frac{|x_1-y_1|^2}{8(1+t^2)}}$ simply bounded by 1, and the third follows from the estimate $|x_2 - r_2|^2 + |r_2 - y_2|^2 \geq x_2^2 + y_2^2 + 2r_2^2$, followed by another Gaussian integration (then over all $r_2 \in \mathbb{R}$). Just as in the proof of Proposition 2 the desired bound on the contribution to $F(\hat{H})$ follows. □

7 Traceclass estimates

To begin with, $\mathcal{T}$-traceclass properties on compactly supported smooth functions of the planar Hamiltonians are examined. Proposition 2 implies the continuity of the integral kernel of $D_j F(H)$ so that one obtains the following:

Corollary 1 *Let* $\partial_j V_\omega \in L^\infty(\mathbb{R}^2)$*,* $j = 1, 2$*, and* $F \in C_c^\infty(\mathbb{R})$*. Then* $F(H) \in \mathcal{A}_\infty$ *and* $D_j F(H) \in \mathcal{A}_\infty$ *are* $\mathcal{T}$*-traceclass. Their* $\mathcal{T}$*-trace can be calculated by* (8)*.*

This result allows to transpose the formalism developed in [6, 22] to prove the Kubo formula for tight-binding Schrödinger operators also to continuous Schrödinger operators. For the definition and evaluation of the edge currents, the following $\hat{\mathcal{T}}$-traceclass estimates will be important.

Corollary 2 *Let* $\Delta \subset \mathbb{R}$ *be a gap of* $H_{\omega,\infty}$ *and* $F : \mathbb{R} \to \mathbb{R}_+$ *be a smooth positive function supported by* Δ*. Suppose* $\partial_j V_\omega \in L^\infty(\mathbb{R}^2)$*,* $j = 1, 2$*. Then* $F(H) \in \mathcal{A}$ *and* $D_1 F(H) \in \mathcal{A}$ *are* $\hat{\mathcal{T}}$*-traceclass. Their trace can be calculated by* (8)*.*

Proof: If Δ is a gap of $H_{\omega,\infty}$, then $F(H_{\omega,\infty}) = 0$ so that the first term in Theorem 1 vanishes and the second term is $K = F(H)$. As $F(H) \geq 0$, one calculate $\hat{\mathcal{T}}(F(H))$ directly using the integral kernels which satisfy the estimate of Theorem 1. This immediately implies that $\hat{\mathcal{T}}(F(H)) < \infty$. As F is positive, $D_1 F(H) = (D_1 F(H)^{1/2}) F(H)^{1/2}$. As $F(H)^{1/2}$ is $\hat{\mathcal{T}}$-traceclass by the above argument, so is $D_1 F(H)$. $\Box$

The next result does not allude to properties of the Hamiltonian, but rather gives a general property of $\hat{\mathcal{T}}$-traceclass operators. Therefore let $|\nabla_j|$ be new operations on $\mathcal{A}$ defined by $\langle \vec{x} \,|\, (|\nabla_j| A)_{\hat{\omega}} |\vec{y}\,\rangle = |x_j - y_j| \langle \vec{x}\,| A_{\hat{\omega}} |\vec{y}\,\rangle$. Whether $|\nabla_j| A \in \mathcal{A}$ can, for example, easily be deduced from (4) if the integral kernel of A decays off the diagonal. Furthermore let us introduce the function $\Sigma(\vec{x}\,) = \text{sign}(x_1)$ and denote the associated multiplication operator also by Σ.

Proposition 4 *Suppose that* $A \in \mathcal{A}$ *is* $\hat{\mathcal{T}}$*-traceclass and that the integral kernels are jointly continuous. Moreover let* $|\nabla_1| A \in \mathcal{A}$*. Then for any* $s < \infty$*, the operators* $[\Sigma, A_{\omega,s}]$ *are Hilbert-Schmidt and the square of their Hilbert-Schmidt norm is* $\mathbf{P}$*-integrable.*

Proof: It follows from the hypothesis and the ideal property that $(|\nabla_1| A^*) A$ is $\hat{\mathcal{T}}$-traceclass and therefore

$$\hat{\mathcal{T}}((|\nabla_1| A)^* A) \;=\; \int_{\mathbb{R}} ds \int_{\Omega} d\mathbf{P}(\omega) \int d\vec{y}\; |y_1|\, |\langle \vec{y}\,| A_{\omega,s} |0\rangle|^2$$

is finite. Replacing the identity

$$|y_1| \;=\; \frac{1}{2} \int dx_1 \;\left(1 - \Sigma(\vec{y} + \vec{x}\,)\Sigma(\vec{x}\,)\right) \;,$$

and using the covariance relation one obtains

$$\begin{aligned}
\hat{\mathcal{T}}((|\nabla_1| A)^* A) &= \frac{1}{2} \int d\mathbf{P}(\omega) \int d\vec{x} \int d\vec{y}\, (1 - \Sigma(\vec{y})\Sigma(\vec{x}))\, |\langle \vec{y}\,| A_{-\vec{x}\cdot\omega,s} |\vec{x}\,\rangle|^2 \\
&= \frac{1}{4} \int d\mathbf{P}(\omega) \int d\vec{x}\; \langle \vec{x}\,| [\Sigma, A_{-\vec{x}\cdot\omega,s}]^* [\Sigma, A_{-\vec{x}\cdot\omega,s}] |\vec{x}\,\rangle \\
&= \frac{1}{4} \int d\mathbf{P}(\omega)\; \mathbf{Tr}\left(|[\Sigma, A_{\omega,s}]|^2\right) \;,
\end{aligned}$$

where in the last step the integrations were exchanged and the translation invariance of **P** was used. This implies the claim because of the weak continuity of $A_{\omega,s}$ in ω. $\square$

8 Currents

By the Heisenberg equations of motion the current operators are given by

$$J_j \;=\; \frac{d}{dt}X_j(t)\Big|_{t=0} \;=\; \imath[H_{\hat{\omega}},X_j] \;=\; -2D_j\;, \qquad j=1,2\;. \tag{29}$$

Accessible in experiment is the expectation value of the current w.r.t. to a given one-particle density matrix ρ. The current density in the bulk is then calculated in the planar model using the trace per unit volume $\mathcal{T}$. The following result implies that no bulk current flows at equilibrium and absence of electric field, that is, if the density matrix is a function of the Hamiltonian such as the Fermi-Dirac function. This result was already given in [6], but only with a very sketchy proof.

Proposition 5 *Let $F\in C^k_c(\mathbb{R})$ with $k>5$. Then*

$$\mathcal{T}(J_jF(H)) \;=\; 0\;.$$

Proof: Let us begin by noting that ∇_j-invariance of $\mathcal{T}$ and Duhamel's formula imply that for $\Re e(z)>0$

$$0 \;=\; \mathcal{T}(\nabla_j e^{-zH}) \;=\; z\,\mathcal{T}\left(\int_0^1 dq\, e^{-(1-q)zH}(\nabla_jH)e^{-qzH}\right)\;.$$

Since e^{-qzH} is $\mathcal{T}$-traceclass only for $q>0$, the integral $\int_0^1 = \int_{\frac{1}{2}}^1+\int_0^{\frac{1}{2}}$ is split. This allows to use cyclicity in order to obtain

$$0 \;=\; z\,\mathcal{T}\left((\nabla_jH)e^{-zH}\right)\;.$$

Finally the representation by a norm convergent Riemann integral (9) can be used to conclude

$$\mathcal{T}(J_j\,F(H)) \;=\; -2\int_{\mathbb{R}} dt\,\tilde{F}(t)\;\mathcal{T}\left((\nabla_jH)e^{-(1+\imath t)H}\right) \;=\; 0\;,$$

where the trace $\mathcal{T}$ and the sum defining the Riemann integral over t could be exchanged because $e^{-(1+\imath t)H}$ is $\mathcal{T}$-traceclass for any t due to the results of Section 4. $\square$

For a system with a boundary in the 1-direction, an edge current flows along the infinite boundary of the half-plane. However, this current only flows in the vicinity of the boundary so that the trace per unit volume $\mathcal{T}$ of $J_1\rho$ vanishes. In fact, the physical current density along the boundary is rather obtained by taking the trace per unit volume in the 1-direction followed by the usual trace in the 2-direction, an operation precisely given by $\hat{\mathcal{T}}$. Corollary 2 implies that the following definition of the edge current is also mathematically sound as long as F is positive and supported by a gap of $H_{\omega,\infty}$:

$$j^e(F) \;=\; \hat{\mathcal{T}}(J_1\, F(H)) \;. \tag{30}$$

One might erroneously believe that analogous to Proposition 5 one has $\hat{\mathcal{T}}(J_1F(H))=0$ at least if F is supported by a gap of H. In fact, the proof of Proposition 5 does not carry over because the semigroup is *not* $\hat{\mathcal{T}}$-traceclass. What the (finite) value of $\hat{\mathcal{T}}(J_1F(H))$ is, will be analyzed in the next sections.

At this point let us comment on what happens if the spectrum of H_ω does not have a gap. Then $F(H_{\omega,s}) = \phi_sF(H_{\omega,\infty}) + K_s$ where $\phi_sF(H_{\omega,\infty})$ is definitely not $\hat{\mathcal{T}}$-traceclass and Theorem 1 implies that K_s is a boundary operator, although it does not directly imply that K_s is moreover $\hat{\mathcal{T}}$-traceclass because it may not have a definite sign. In order to make nevertheless sense of the edge current in this situation, one can regularize the expression and rather define the edge current by

$$j^e(F) \;=\; \lim_{S\to\infty}\int_{-S}^{S} ds\;\int d\mathbf{P}(\omega)\;\langle\vec{0}\,|J_1F(H_{\omega,s})|\vec{0}\,\rangle\;.$$

Due to Proposition 5, one then sees that the contribution coming from $\phi_sF(H_{\omega,\infty})$ vanishes for every finite S. Hence, assuming that the remainder K_s is actually $\hat{\mathcal{T}}$-traceclass, one then obtains $j^e(F)=\hat{\mathcal{T}}(J_1K_s)<\infty$, hence a reasonable definition.

9 Winding numbers

On $\mathcal{A}\times\mathcal{A}$ consider the densely defined bilinear map

$$\xi(A,B) \;=\; \imath\,\hat{\mathcal{T}}(A\nabla_1 B)\;, \tag{31}$$

If A is $\hat{\mathcal{T}}$-traceclass and $\nabla_1 B\in\mathcal{A}$ (or vice versa) then (A,B) belongs to the domain of definition of ξ denoted $\mathcal{D}(\xi)$.

Lemma 4 *ξ is a 1-cocycle, namely it satisfies whenever $(A,B),(B,C)(C,A)\in\mathcal{D}(\xi)$ have jointly continuous integral kernels:*
(i) *Cyclicity: $\xi(A,B)=-\xi(B,A)$.*
(ii) *Closedness under the Hochschild operator: $\xi(AB,C)-\xi(A,BC)+\xi(CA,B)=0$.*

Proof: This follows from a short algebraic calculation using the Leibniz rule for ∇_1 and the ∇_1-invariance of $\hat{\mathcal{T}}$ holding under the stated hypothesis. □

By general principles [12] (see also [18, 19]), 1-cocycles can be paired with unitaries. The pairing in the present context stems from a Fredholm module so that it leads to an index theorem. Let Π_1 be the indicator function on the half-space with positive first coordinate, i.e. $\Pi_1=\frac{1}{2}(\Sigma+1)$. The projection from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}_+\times\mathbb{R})$ is also denoted by Π_1.

Theorem 2 *Let* $\mathcal{U}$ *be a unitary such that* $\mathcal{U}-1 \in \mathcal{A}$ *is* $\hat{\mathcal{T}}$*-traceclass and has jointly continuous integral kernel. Furthermore let* $\nabla_1 \mathcal{U} \in \mathcal{A}$ *and* $|\nabla_1|\mathcal{U} \in \mathcal{A}$*. Then for fixed* $s < \infty$ *and* $\omega \in \Omega$*,* $\Pi_1\,\mathcal{U}_{\hat{\omega}}\Pi_1$ *is a Fredholm operator on* $L^2(\mathbb{R}_+\times\mathbb{R})$*. If* $\vec{\xi} \in \mathbb{R}^2 \mapsto \mathcal{U}_{\vec{\xi}\cdot\omega,s}$ *is moreover norm-continuous, the corresponding index* Ind *is* $\mathbf{P}$*-almost surely independent of* ω*, always independent of* s*, and given by*

$$\text{Ind} \;=\; -\,\xi(\mathcal{U}^* - 1, \mathcal{U}) \;.$$

Proof: By Proposition 4, the conditions imply that $[\Sigma, \mathcal{U}_{\hat{\omega}}]$ is Hilbert-Schmidt. From the algebraic identity:

$$\Pi_1 A_{\hat{\omega}} B_{\hat{\omega}} \Pi_1 - \Pi_1 A_{\hat{\omega}} \Pi_1 B_{\hat{\omega}} \Pi_1 \;=\; -\frac{1}{4}\,\Pi_1\,[\Sigma, A_{\hat{\omega}}]\,[\Sigma, B_{\hat{\omega}}]\;, \qquad A, B \in \mathcal{A}\;, \tag{32}$$

follows that $\Pi_1 - \Pi_1\mathcal{U}_{\hat{\omega}}\Pi_1\mathcal{U}^*_{\hat{\omega}}\Pi_1$ and $\Pi_1 - \Pi_1\mathcal{U}^*_{\hat{\omega}}\Pi_1\mathcal{U}_{\hat{\omega}}\Pi_1$ are traceclass. By Fedosov's formula (*e.g.* [12, 6, 18]), $\Pi_1\mathcal{U}_{\hat{\omega}}\Pi_1$ is a Fredholm operator on $L^2(\mathbb{R}_+\times\mathbb{R})$ whose index is given by

$$\text{Ind}_{\hat{\omega}} \;=\; \mathbf{Tr}\,(\Pi_1 - \Pi_1\,\mathcal{U}^*_{\hat{\omega}}\Pi_1\,\mathcal{U}_{\hat{\omega}}\Pi_1) - \mathbf{Tr}\,(\Pi_1 - \Pi_1\,\mathcal{U}_{\hat{\omega}}\Pi_1\,\mathcal{U}^*_{\hat{\omega}}\Pi_1)\;.$$

By hypothesis, $\vec{\xi} \in \mathbb{R}^2 \mapsto \Pi_1\mathcal{U}_{\vec{\xi}\cdot\omega,s}\Pi_1$ is a norm-continuous family of Fredholm operators so that by homotopy invariance and ergodicity of $\mathbf{P}$ their Fredholm index is $\mathbf{P}$-almost surely constant. The identity $\Pi_1\,\mathcal{U}_{\omega,s+\xi_2}\Pi_1 = U(0,\xi_2)^*\Pi_1\,\mathcal{U}_{(0,\xi_2)\cdot\omega,s}\Pi_1 U(0,\xi_2)$ implies that $s \in \mathbb{R} \mapsto \Pi_1\,\mathcal{U}_{\omega,s}\Pi_1$ is norm continuous and therefore $\text{Ind}_{\omega,s}$ constant in s.

The almost sure index Ind is, for any $s \in \mathbb{R}$, given by

$$\text{Ind} \;=\; \int d\mathbf{P}(\omega)\;\text{Ind}_{\omega,s} \;=\; -\,\eta_s(\mathcal{U}^* - 1, \mathcal{U})\;,$$

where $\eta_s(A,B) = \int d\mathbf{P}(\omega)\,\eta_{\omega,s}(A,B)$ with

$$\eta_{\hat{\omega}}(A,B) \;=\; \mathbf{Tr}\,(\Pi_1 B_{\hat{\omega}} A_{\hat{\omega}}\Pi_1 - \Pi_1 B_{\hat{\omega}}\Pi_1 A_{\hat{\omega}}\Pi_1) - \mathbf{Tr}\,(\Pi_1 A_{\hat{\omega}} B_{\hat{\omega}}\Pi_1 - \Pi_1 A_{\hat{\omega}}\Pi_1 B_{\hat{\omega}}\Pi_1)\;.$$

Introduce next the 1-cocycle ζ_s

$$\zeta_s(A,B) \;=\; \int d\mathbf{P}(\omega)\;\zeta_{\omega,s}(A,B)\;, \qquad \zeta_{\hat{\omega}}(A,B) \;=\; \frac{1}{4}\,\mathbf{Tr}\,(\Sigma\,[\Sigma, A_{\omega,s}]\,[\Sigma, B_{\omega,s}])\;.$$

Then $\eta_{\hat{\omega}}(A,B) = \zeta_{\hat{\omega}}(A,B)$ because of identity (32) and the cyclicity property of $\zeta_{\hat{\omega}}$.

Using the invariance of $\mathbf{P}$ as well as the identity

$$\int dy_1\;\Sigma(\vec{y})\;\left(\Sigma(\vec{y}) - \Sigma(\vec{y}+\vec{x})\right)^2 \;=\; -4\,x_1\;,$$

one can verify as in the proof of Proposition 4 that $\xi(A,B) = \zeta_s(A,B)$ for all finite s. □

The above calculations follow closely [18, Sec. 4.2, 4.3]. However, there is one crucial difference. The invariance of the index in the 2-direction holds for all unitaries, while in the discrete case it was only true for unitaries in the image of the exponential map [18, Prop. 4.10].

The reason is that the exponential map is an isomorphism in the continuous case, namely it is Connes' Thom isomorphism [11]. Further explanations will be given in [19].

Note that $\mathcal{U}-1$ being $\hat{\mathcal{T}}$-traceclass implies that also $\mathcal{U}^k-1$ is $\hat{\mathcal{T}}$-traceclass for any $k\in\mathbf{Z}$. In fact, this follows from $\mathcal{U}^k-1=(\mathcal{U}-1)\sum_{l=0}^{k-1}\mathcal{U}^l$ and the fact that traceclass operators form an ideal. It is then elementary to verify that under the assumptions of the theorem

$$\eta_s((\mathcal{U}^*)^k-1,\mathcal{U}^k) \;=\; k\,\eta_s(\mathcal{U}^*-1,\mathcal{U})\;. \tag{33}$$

10 Quantization of edge currents

Let $\Delta=[E',E'']$ be in a gap of the spectrum of $H_{\omega,\infty}$. Let $G\in C^\infty(\mathbb{R})$ be a monotonously decreasing function with $G(-\infty)=1$, $G(\infty)=0$, and $\mathrm{supp}(G')\subset\Delta\backslash G^{-1}(\frac{1}{2})$. The support of a function is closed by definition and hence all derivatives of G vanish on the pre-image $G^{-1}(\frac{1}{2})$. Define via functional calculus the following unitary operator

$$\mathcal{U}(\Delta) \;=\; \exp(-2\pi\imath\, G(H))\;. \tag{34}$$

Theorem 3 *Suppose* $\partial_j V_\omega\in L^\infty(\mathbb{R}^2)$ *for* $j=1,2$. *Let* Δ *be in a gap of the spectrum of* $H_{\omega,\infty}$. *Then* $J_1G'(H)\in\mathcal{A}$ *and* $\mathcal{U}(\Delta)-1\in\mathcal{A}$ *are both* $\hat{\mathcal{T}}$*-traceclass,* $\nabla_1\mathcal{U}(\Delta)\in\mathcal{A}$ *and for* $\mathbf{P}$*-almost all* ω *and all* $s\in\mathbb{R}$,

$$-\,2\pi\,\hat{\mathcal{T}}(J_1G'(H)) \;=\; \imath\,\hat{\mathcal{T}}((\mathcal{U}(\Delta)^*-1)\,\nabla_1\mathcal{U}(\Delta)) \;=\; \mathrm{Ind}(\Pi_1\,\mathcal{U}_{\omega,s}(\Delta)\Pi_1)\;. \tag{35}$$

Proof: First the assumptions of Theorem 2 are established. As the function G is monotonously decreasing, G' is a negative smooth function supported by Δ so that Corollary 2 implies that $J_1G'(H)$ is $\hat{\mathcal{T}}$-traceclass. To prove the $\hat{\mathcal{T}}$-traceclass property of $\mathcal{U}(\Delta)-1$, let us write it as the linear combination of three positive and smooth functions of H. Indeed, $E\mapsto\sin_\pm(2\pi\,G(E))$ and $E\mapsto\cos(2\pi\,G(E))-1$ (where $g_\pm$ denotes the positive and negative parts of a real function g) are of this type since $\sin_\pm(2\pi\,G(E))$ vanishes with all its derivatives at $E\in G^{-1}(\frac{1}{2})$. That $\nabla_1\mathcal{U}(\Delta)\in\mathcal{A}$ and $|\nabla_1|\mathcal{U}(\Delta)\in\mathcal{A}$ follows directly from Proposition 1 for the half-plane operators combined with (4). Finally, by Duhamel's formula

$$\vec{\xi}\in\mathbb{R}^2\;\mapsto\; e^{-tH_{\vec{\xi}\cdot\omega,s}} \;=\; \int_0^1 dq\; e^{-(1-q)tH_{L,s}}\,V_\omega(.\,-\vec{\xi})\,e^{-qtH_{\vec{\xi}\cdot\omega,s}}\;,$$

so that the continuity of the potential implies the norm-continuity of the semigroups and via the norm-convergent functional calculus (9) also of $\vec{\xi}\in\mathbb{R}^2\;\mapsto\;\mathcal{U}(\Delta)_{\vec{\xi}\cdot\omega,s}$. In conclusion, the conditions of Theorem 2 are verified and only the first equality in (35) remains to be shown.

For that express $\mathcal{U}(\Delta)$ as exponential series and use the Leibniz rule to obtain

$$\hat{\mathcal{T}}((\mathcal{U}(\Delta)^*-1)\,\nabla_1\mathcal{U}(\Delta)) \;=\; \sum_{m=0}^{\infty}\frac{(-2\pi\imath)^m}{m!}\sum_{l=0}^{m-1}\hat{\mathcal{T}}\left((\mathcal{U}(\Delta)^*-1)\,G(H)^l\,\nabla_1G(H)\,G(H)^{m-l-1}\right)\;,$$

where the trace and the infinite sum could be exchanged because of the traceclass properties (note that also $\nabla_1 G(H) \in \mathcal{A}$). Due to cyclicity and the fact that $[\mathcal{U}(\Delta), G(H)] = 0$, each summand is now equal to $\hat{\mathcal{T}}((\mathcal{U}(\Delta)^* - 1)\, G(H)^{m-1}\, \nabla_1 G(H))$. Exchanging again sum and trace and summing the exponential up again, one gets

$$\hat{\mathcal{T}}((\mathcal{U}(\Delta)^* - 1)\, \nabla_1 \mathcal{U}(\Delta)) \;=\; -2\pi\imath\, \hat{\mathcal{T}}\,((1 - \mathcal{U}(\Delta))\, \nabla_1 G(H)) \;.$$

Repeating the same argument for $\mathcal{U}(\Delta)^k = \exp(-2\pi\imath\, k\, G(H))$ where $k \in \mathbf{Z}$ and using (33) more generally implies that, for $k \neq 0$,

$$\hat{\mathcal{T}}\,((1 - \mathcal{U}(\Delta))\, \nabla_1 G(H)) \;=\; \hat{\mathcal{T}}\,\big((1 - \mathcal{U}(\Delta)^k)\, \nabla_1 G(H)\big) \;.$$

Writing $G(E) = \int dt\, \tilde{G}(t)\, e^{-E(1+\imath t)}$ as in (9), the above r.h.s. is, for $k \neq 0$, equal to

$$-2\pi\imath \int dt\, \tilde{G}(t)\, (1 + \imath t) \int_0^1 dq\, \hat{\mathcal{T}}\,\big((1 - \mathcal{U}(\Delta)^k)\, e^{-(1-q)(1+\imath t)H} (\nabla_1 H) e^{-q(1+\imath t)H}\big) \;.$$

The integral over t is a norm convergent Riemann integral. One therefore finds using $G'(E) = -\int dt\, (1 + \imath t)\, \tilde{G}(t)\, e^{-E(1+\imath t)}$, for $k \neq 0$,

$$\hat{\mathcal{T}}((\mathcal{U}^*(\Delta) - 1)\, \nabla_1 \mathcal{U}(\Delta)) \;=\; 2\pi\imath\, \hat{\mathcal{T}}\,\big((\mathcal{U}(\Delta)^k - 1)\, (\nabla_1 H)\, G'(H)\big) \;,$$

while, for $k = 0$, the r.h.s. vanishes.

To conclude, let $\phi : [0, 1] \to \mathbb{R}$ be a differentiable function vanishing at the boundary points 0 and 1. Let its Fourier coefficients be denoted by $a_k = \int_0^1 dx\, e^{-2\pi\imath kx} \phi(x)$. Then $\sum_k a_k e^{2\pi\imath kx} = \phi(x)$ and, in particular, $\sum_k a_k = 0$. Hence

$$\begin{aligned}\left(\sum_{k\neq 0} a_k\right) \hat{\mathcal{T}}((\mathcal{U}^*(\Delta) - 1)\, \nabla_1 \mathcal{U}(\Delta)) \;&=\; 2\pi\imath \sum_k a_k\, \hat{\mathcal{T}}\,\big((\mathcal{U}(\Delta)^k - 1)\, (\nabla_1 H)\, G'(H)\big) \\ &=\; 2\pi\imath\, \hat{\mathcal{T}}(G'(H)\, \phi(G(H))\, (\nabla_1 H)) \;.\end{aligned}$$

Let now ϕ converge to the indicator function of $[0, 1]$. Then $a_0 \to 1$ and $\sum_{k\neq 0} a_k \to -1$, while $G'(H)\phi(G(H)) \to G'(H)$ (the Gibbs phenomenon is damped). As $J_1 = -\nabla_1 H$, this concludes the proof. □

11 Link to bulk Hall conductivity

The Chern character is a trilinear form ch defined by

$$\mathrm{ch}(A, B, C) \;=\; 2\pi\imath\, \mathcal{T}(A((\nabla_1 B)(\nabla_2 C) - (\nabla_2 B)(\nabla_1 C))) \;, \qquad A, B, C \in \mathcal{A}_\infty \;, \tag{36}$$

as long as the r.h.s. is well-defined. The following theorem is well-known [20, 4, 3].

Theorem 4 *Let $P \in \mathcal{A}_\infty$ be a projection with integral kernels satisfying* $\mathbf{P}$*-a.s. for some $\delta > 0$*

$$|\langle \vec{x}|P_\omega|\vec{y}\,\rangle| \;\leq\; \frac{c_\delta}{1+|\vec{x}-\vec{y}\,|^{2+\delta}}\ . \tag{37}$$

Then $ch(P, P, P)$ is well-defined and equal to an integer given as the index of a Fredholm operator.

The importance of this result stems from the fact that the bulk Hall conductivity of a gas of independent electrons described by H at zero-temperature, zero dissipation and with chemical potential μ is given by [2, 20, 4, 21, 3, 6, 1]

$$\sigma_b^\perp(\mu) \;=\; \frac{q^2}{h}\,\mathrm{ch}(P_\mu, P_\mu, P_\mu)\ ,$$

where $P_\mu = \chi_{(-\infty,\mu]}(H)$ is the family of associated Fermi projections. This fact can be deduced from Kubo's formula [6, 1] or the adiabatic Laughlin *Gedankenexperiment* [3].

As discussed in great detail in [6, 1] in the discrete setting, (37) is a dynamical localization condition on the spectral region in the vicinity of the Fermi level. For the purposes of the present article, however, we restrict ourselves to the situation where the Fermi level μ is in a gap of the spectrum of H. Then P_μ, defined with a characteristic function, can also be written as a smooth function of H for which the estimate (37) holds by Proposition 1. By homotopy of a Fredholm index, one then deduces:

Corollary 3 *Let the interval Δ be a gap of the spectrum of $H_{\omega,\infty}$. Then $\mu \in \Delta \mapsto \sigma_\perp^b(\mu)$ is constant and equal to an integer multiple of $\frac{q^2}{h}$.*

The following result, analogous to the discrete case [23, 18, 14], albeit based on Connes' Thom isomorphism and its dual in cyclic cohomology [11, 15] instead of the Pimsner-Voiculescu sequence and its dual, will be proven in [19]:

Theorem 5 *Let the interval Δ be a gap of $H_{\omega,\infty}$. Then $\sigma_\perp^b(\mu) = \mathrm{Ind}(\Pi\,\mathcal{U}_{\hat{\omega}}(\Delta)\Pi)$ for $\mu \in \Delta$.*

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