Existence of Diffusion Orbits in *a priori* Unstable Hamiltonian Systems

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Abstract. Under open and dense conditions we show that Arnold diffusion orbits exist in *a priori* unstable and time-periodic Hamiltonian systems with two degrees of freedom.

1, Introduction and Results

By the KAM (Kolmogorov, Arnold and Moser) theory we know that there are many invariant tori in nearly integrable Hamiltonian systems with arbitrary n degrees of freedom. These tori are of n dimension and occupy a nearly full Lebesgue measure set in the phase space. As an important consequence, all orbits are stable in autonomous system with two degrees of freedom, or time-periodic system with one degree of freedom, in the sense that the actions do not change much along the orbits. However, the KAM theory dose not guarantee such stability when the system has three or more degrees of freedom for the autonomous case or when it has two or more degrees of freedom for the time-periodic case, simply because the KAM torus can not separate the phase space (or integral manifold) into two disconnected parts.

In his celebrated paper [Ar], Arnold constructed an example of nearly integrable Hamiltonian system, where some orbits are unstable. His example is a time periodic system with two degrees of freedom. In Arnold's example the perturbations are chosen so specifically that all hyperbolic invariant tori preserve in the perturbed system. Hence one can use so called Poincaré-Melnikov method to construct transition chain along which the action has substantial variation. However, in generic case the perturbed systems do not possess such a good property, some resonant gaps between invariant tori break up the transition chain, thus it seems unclear whether one can apply Arnold's method to find diffusion orbits. Despite of this technical difficulty, Arnold asked whether there is such a phenomenon for a "typical" small perturbation. After near four decades of study some remarkable generalizations of Arnold's result have been announced ([X1],[DLS],[Ma5]). A few years ago, Xia [X1] announced that Arnol'd diffusion exists in generic *a priori* unstable systems, recently Mather announced ([Ma5]) that, under so-called cusp residual condition, Arnold diffusion exists in *a priori* stable systems with two degrees of freedom in time-periodic case, or with three degrees of freedom in autonomous case.

In this paper, we study generic perturbations of *a priori* unstable Hamiltonian systems which have two degrees of freedom and are time-periodic, and give a complete proof of the existence of diffusion orbits. The approach of our proof is different from the approaches proposed by Mather and by Xia (cf. [Ma5] and [X2]). The starting point of our proof is based on the previous work of Mather ([Ma3], [Ma4]). With his profound insight, Mather opened a way to study Hamiltonian dynamics in higher dimensions. In [Ma3] Mather established the variational set-up of time-dependent positive definite Lagrangian systems and showed the existence of minimal measures. By exploiting the properties of barrier functions in [Ma4], he introduced the idea of C-equivalence and pointed out a possible way to construct connecting orbits. The difficulty to apply this method to interesting problem in higher dimensions is that we do not know the structures of related c-minimal orbit sets. In this paper we are succeeded in getting sufficient information of the topological structure of the relevant Mañé sets and in providing the proof of a theorem of connecting C-equivalent Mañé sets formulated by Mather in [Ma4]. Consequently, we are able to construct the diffusion orbits crossing the gaps. However, it appears unclear whether such C-equivalence can be established at the place where uncountably many whisker tori cluster together. Fortunately, this is the place where there is no big gap. Arnold's mechanism can be used here because a transition chain of whisker tori clearly exists in this case. Crucially relying on such geometric structure, we are able to establish local variational principle (cf. [Bs], [BCV]), the local minimum corresponds to some diffusion orbits crossing these whisker tori. It is the variational version of Arnold's mechanism. Our another work is to show that we can join the orbits constructed by C-equivalence smoothly with the orbits which realize the minimum of the local variational principle. In this way we do find some diffusion orbits in generic systems.

Given a Hamiltonian function $H(p,q,t)$ the Hamiltonian equation has the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1.1}$$

The Hamiltonian function studied here has the following form:

$$H(p,q,t) = f(p_1) + g(p_2,q_2) + P(p,q,t) \tag{1.2}$$

where $p=(p_1,p_2)\in\mathbb{R}^2$, $q=(q_1,q_2)\in\mathbb{T}^2$, $H\in C^r$ $(r\geq 3)$, P is a time-1-periodic small perturbation. We assume it satisfies following conditions:

1, $f+g$ is a convex function in p i.e. the Hessian matrix $\partial_{pp}(f+g)$ is positive definite, finite everywhere and has superlinear growth in p, $(f+g)/\|p\|\to\infty$ as $\|p\|\to\infty$;

2, it is *a priori* unstable in the sense that g has non-degenerate saddle critical point, i.e. $\partial_{p_2q_2}g^2 - \partial_{p_2p_2}g\partial_{q_2q_2}g > 0$ at (p_2,q_2). Without loss of generality, we assume $(p_2,q_2)=0$. As the Hamiltonian flow ϕ_g^t induced by g is integrable, it has two homoclinic loops on the cylinder $(p_2,q_2)\in\mathbb{R}\times\mathbb{T}$.

As $f+g$ is an integrable system and H is its small perturbation, every solution of H is well defined for $t\in\mathbb{R}$.

Let $\mathcal{B}_{\epsilon,K}$ denote a ball in the function space $C^r(\{(p,q)\in\mathbb{T}^2\times\mathbb{R}^2 : \|p\|\le K\}\to\mathbb{R})$, centered at the origin with radius of ϵ. Now we can state the theorem which was formulated by Arnold in [Ar]:

Theorem 1.1. *Let $A<B$ be two arbitrarily given numbers and assume H satisfies the above two conditions. There exist a small number $\epsilon>0$, a large number $K>0$ and an open and dense set in $B_{\epsilon,K}$ such that for each $P\in B_{\epsilon,K}$ there exists an orbit of the Hamiltonian flow which connects the region with $p_1<A$ to the region with $p_1>B$.*

We shall use variational argument to complete the proof. In the section 2, by using Legender transformation we follow Mather's work [Ma4] and put this problem into the Lagrangian formalism. The diffusion orbits are found by searching for the minimal action of the Lagrangian. Some properties such as upper semi-continuity of some set-valued functions are also proved in this section. In the section 3, we investigate the topological structure of some relevant Mañé sets. The section 4 is devoted to the study of the barrier function when the Aubry set contains a codimension one torus. In the section 5, by making use of the semi-continuity property shown in the section 2 we obtain the proof of a theorem of connecting C-equivalent Mañé sets, formulated by Mather in [Ma4]. Based on the understanding of the topological structure of the relevant Mañé sets shown in the section 3, we establish the C-equivalence among those relevant Mañé sets and use this C-equivalence to construct the diffusion orbits crossing resonant gaps. In virtue of the techniques developed in [BCV] and the analytic expression of the barrier function obtained in the section 4 we join the orbits constructed by C-equivalence smoothly with the orbits constructed via transition chain. Thus we obtain the diffusion orbits. In the section 6 we show the open and dense property.

2, Variational set-up

Roughly speaking, the diffusion orbits are constructed by connecting different c-minimal orbit sets, along which the Lagrange action takes its minimum. Therefore, we shall study the Lagrangian equation equivalent to the Hamiltonian equation (1.1):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)-\frac{\partial L}{\partial q}=0, \tag{2.1}$$

where the Lagrangian function $L(\dot q,q,t)$ is obtained from the Hamiltonian function (1.1) by using Legendre transformation $\mathcal{L}$: $(p,q,t)\to(\dot q,q,t)$ such that

$$L(\dot q,q,t)=\max_p\{\langle p,\dot q\rangle-H(p,q,t)\}. \tag{2.2}$$

where $\dot q=\dot q(p,q,t)$ is implicitly determined by $\dot q=\frac{\partial H}{\partial p}$. Since we study a nearly integrable system, the Lagrangian has the form of

$$L=L_0(q_2,\dot q)+L_1(q,\dot q,t)$$

where L_0 corresponds to $f+g$ through the Legender transformation.

Throughout this paper, we use ϕ^t to denote the Euler-Lagrange flow determined by L, use Φ^t to denote the Hamiltonian flow determined by H. To specify the Euler-Lagrange (Hamiltonian) flow determined by other functions we add the subscript, e.g. $\phi^t_{L_0}$, Φ^t_{f+g}, etc.

Clearly, the equation (2.1) corresponds to the critical point of the functional

$$A(\gamma) = \int L(\gamma, \dot\gamma, t)dt.$$

We can think that L is a function defined on $TM \times \mathbb{T}$ where $M = \mathbb{T}^2$. By the assumption on H we see that L satisfies the following conditions introduced by Mather [Ma3]:

Positive definiteness. For every $(q,t) \in M \times \mathbb{T}$, the Lagrangian function is strictly convex in velocity: the Hessian $L_{\dot q \dot q}$ is positive definite.

Superlinear growth. We suppose that L has fiber-wise superlinear growth: for every $(q,t) \in M \times \mathbb{T}$, we have $L/\|\dot q\| \to \infty$ as $\|\dot q\| \to \infty$.

Completeness. All solutions of the Lagrange equations are well defined for all $t \in \mathbb{R}$.

Under these conditions Mather established the theory of c-minimal measure and c-minimal orbits [Ma3, Ma4]. To introduce some basic results of Mather, let us observe the fact that the functional $\int Ldt$ has the same critical point as $\int (L - \eta_c)dt$ does if η_c is a closed 1-form on $M \times \mathbb{T}$, whose first de Rham co-homology class is c, i.e. $[\eta_c] = c$, in other words, their Lagrange equations are the same.

Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^1(I, M)$ is called a c-minimizer or a c-minimal curve if it minimizes the action among all curves $\xi \in C^1(I, M)$ which satisfy the same boundary conditions:

$$A_c(\gamma) = \min_{\substack{\xi(a)=\gamma(a)\\ \xi(b)=\gamma(b)}} \int_a^b (L - \eta_c)(d\xi(t), t)dt. \tag{2.3}$$

As we have the condition of completeness the minimizer must be a C^1-curve by Tonelli's theorem. Without the completeness the minimizer can only be an absolutely continuous curve ([BM]). If J is a non compact interval, the curve $\gamma \in C^1(J, M)$ is said a c-minimizer if $\gamma|_I$ is c-minimal for any compact interval $I \subset J$. An orbit $X(t)$ of ϕ^t is called c-minimizing if the curve $\pi \circ X$ is c-minimizing, where the operator π is the standard projection from tangent bundle to the underling manifold along the fibers, a point $(z, s) \in TM \times \mathbb{R}$ is c-minimizing if its orbit $\phi^t(z, s)$ is c-minimizing. We use $\tilde{\mathcal{G}}_L(c) \subset TM \times \mathbb{R}$ to denote the set of minimal orbits of $L - \eta_c$ (the c-minimal orbits of L). We shall drop the subscript L when it is clear which Lagrangian is under consideration. It is not necessary to assume the periodicity of L in t for the definition of $\tilde{\mathcal{G}}$. When it is periodic in t, $\tilde{\mathcal{G}}(c) \subset TM \times \mathbb{R}$ is a nonempty compact subset of $TM \times \mathbb{T}$, invariant for the Euler-Lagrange flow ϕ^t.

We can extend the definition of action along a C^1-curve to the action on a probability measure. Let $\mathfrak{M}$ be the set of Borel probability measures on $TM \times \mathbb{T}$. For each $\nu \in \mathfrak{M}$, the action $A_c(\nu)$ is defined as the following:

$$A_c(\nu) = \int (L - \eta_c) d\nu. \tag{2.4}$$

Mather has proved [Ma3] that for each first de Rham cohomology class c there is a probability measure μ which minimizes the actions over $\mathfrak{M}$

$$A_c(\mu) = \inf_{\nu \in \mathfrak{M}} \int (L - \eta_c) d\nu.$$

This μ is invariant to the Euler-Lagrange flow. We use $\tilde{\mathcal{M}}(c)$ to denote the support of the measure, use $\alpha(c) = A_c(\mu)$ to denote the minimum c-action. It defines a function α: $H^1(M, \mathbb{R}) \to \mathbb{R}$, usually called α-function. Its Legendre transformation β: $H_1(M, \mathbb{R}) \to \mathbb{R}$ is usually called β-function. Both functions are convex, finite everywhere and have super-linear growth [Ma3]. As $\tilde{\mathcal{M}}(c)$ is defined as the limit measure of c-minimal orbits, the following lemma is a straightforward result of topological dynamics:

Lemma 2.1. *For each co-homological class c and each positive number ϵ, there exists a positive number $T_0 = T_0(c, \epsilon)$, if $T \geq T_0$ and γ: $[0, T] \to M \times \mathbb{T}$ is a curve minimizing the action of $L - \eta_c$, $[\eta_c] = c$, then there is $t \in [0, T]$ such that*

$$d(d\gamma(t), \tilde{\mathcal{M}}(c)) \leq \epsilon.$$

Before to start the existence proof of diffusion orbits we need to introduce some more concepts and investigate some relevant properties, which shall be made use of below for our purpose.

We have defined the set $\tilde{\mathcal{M}}(c)$ and $\tilde{\mathcal{G}}(c)$. It is easy to see that $\tilde{\mathcal{M}}(c)$ is contained in the set $\tilde{\mathcal{G}}(c)$. Between the set $\tilde{\mathcal{G}}$ and set $\tilde{\mathcal{M}}$ we can also define so-called Aubry set $\tilde{\mathcal{A}}(c)$ and Mañé set $\tilde{\mathcal{N}}(c)$ as well as the limit point set $\tilde{\mathcal{L}}(c)$.

As all orbits are well defined on the whole $\mathbb{R}$, they have ω-limit set and α-limit set. Let $\tilde{\omega}(c)$ be the union of ω-limit points of c-minimal orbits $X(t) : [0, \infty) \to TM \times \mathbb{T}$, let $\tilde{\alpha}(c)$ be the union of α-limit points of c-minimal orbits $X(t) : (-\infty, 0] \to TM \times \mathbb{T}$. We call $\tilde{L}(c) = \tilde{\omega}(c) \cup \tilde{\alpha}(c)$ the limit set.

To define the Aubry set and the Mañé set let us define

$$h_c(x, x', t, t') = \min_{\substack{\gamma \in C^1([t,t'], M) \\ \gamma(t) = x, \gamma(t') = x'}} \int_t^{t'} (L - \eta_c)(d\gamma(s), s) ds - (t' - t)\alpha(c), \tag{2.5}$$

$$F_c(x, x', s, s') = \inf_{\substack{s = t \bmod 1 \\ s' = t' \bmod 1 \\ t' \geq t + 1}} h_c(x, x', t, t').$$

$$h_c(x,x') = h_c(x,x',0,1). \tag{2.6}$$

Let

$$h_c^n(x,x') = \min\Big\{ \sum_{i=0}^{n-1} h_c(m_i,m_{i-1}) : m_0 = x, m_n = x' \\ \text{and } m_i \in M \text{ for } 0 \le i \le n \Big\},$$

and let

$$h_c^\infty(x,x') = \liminf_{n\to\infty} h_c^n(x,x') \tag{2.7}$$

$$d_c(x,x') = h_c^\infty(x,x') + h_c^\infty(x',x) \tag{2.8}$$

Mather showed in [Ma4] that d_c is a pseudo-metric on the set $\{x \in M : h_c^\infty(x,x) = 0\}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-semi-static if

$$A_c(\gamma|_{[a,b]}) - \alpha(c)(b-a) = F_c(\gamma(a), \gamma(b), a \bmod 1, b \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-static if

$$A_c(\gamma|_{[a,b]}) - \alpha(c)(b-a) = -F_c(\gamma(b), \gamma(a), b \bmod 1, a \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$. An orbit $X(t) = (d\gamma(t), t \bmod 1)$ is called static (semi-static) if γ is static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of global c-semi-static orbits, the set $\tilde{\mathcal{A}}(c)$ is defined as the union of global c-static orbits, we call it Aubry set.

We use $\mathcal{M}(c)$, $\mathcal{L}(c)$, $\mathcal{A}(c)$, $\mathcal{N}(c)$ and $\mathcal{G}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{L}}(c)$, $\tilde{\mathcal{A}}(c)$, $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{G}}(c)$ from $TM \times \mathbb{T}$ to $M \times \mathbb{T}$ respectively. We have the following inclusions ([Be])

$$\tilde{\mathcal{M}}(c) \subseteq \tilde{\mathcal{L}}(c) \subseteq \tilde{\mathcal{A}}(c) \subseteq \tilde{\mathcal{N}}(c) \subseteq \tilde{\mathcal{G}}(c). \tag{2.9}$$

The set $\tilde{\mathcal{G}}(c)$ and $\tilde{\mathcal{N}}(c)$ have the good property of upper semi-continuity in c. To describe these minimal orbit sets, Mather introduced two kinds of barrier functions B_c and B_c^*, it is defined as follows

$$B_c(q) = h_c^\infty(q,q)$$

$$B_c^*(q) = \min\{h_c^\infty(\xi,q) + h_c^\infty(q,\eta) - h_c^\infty(\xi,\eta) : \forall\ \xi,\eta \in \pi(\mathcal{M}(c)|_{t=0})\}. \tag{2.10}$$

Clearly, we have $0 \le B_c^* \le B_c$. When $d_c(\xi,\eta) = 0$ for all $\xi,\eta \in \mathcal{M}(c)|_{t=0}$, then $B_c = B_c^*$ ([Ma4]). It is not hard to see that $\mathcal{A}(c)|_{t=0} = \{x \in M : B_c(x) = 0\}$. The following lemma is a modified version of the proposition 2.1 in [Be].

Lemma 2.2. *Let M be a compact, connected Riemanian manifold. Assume $L \in C^r(TM \times \mathbb{R}, \mathbb{R})$ $(r \geq 2)$ satisfies the positive definite, superlinear-growth and completeness conditions. Considered as the function of t, L is assumed periodic for $t \in (-\infty, 0]$ and for $t \in [1, \infty)$. Then the map $L \to \tilde{\mathcal{G}}_L \subset TM \times \mathbb{R}$ is upper semi-continuous. As an immediate consequence, $\tilde{\mathcal{G}}(c)$ is a non-empty compact set in $TM \times \mathbb{T}$ and the map $c \to \tilde{\mathcal{G}}(c)$ is upper semi-continuous if L is periodic in t.*

We can consider t is defined on $(\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim$, where $\sim$ is defined by identifying $\{0\} \in [0,1]$ with some point on one circle, and identifying $\{1\} \in [0,1]$ with some point on another circle. Let $U_k = \{(\zeta, q, t) : (q,t) \in M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim, \|\zeta\| \leq k, \}$, $\cup_{k=1}^{\infty} U_k = TM \times \mathbb{R}$. Let $L_i \in C^r(TM \times \mathbb{T}, \mathbb{R})$. We say L_i converges to L if for each $\epsilon > 0$ and each U_k there exists i_0 such that $\|L - L_i\|_{U_k} \leq \epsilon$ if $i \geq i_0$.

Proof: Since M is connected and compact, any two point x_1 $x_2 \in M$ can be connected by a geodesic. Let $\ell(x_1, x_2)$ be the length of the shortest geodesic connecting these two points, there is uniformly upper bound $K_1 > 0$ of $\ell(x_1, x_2)$ for all x_1, $x_2 \in M$. Let

$$K = \max_{\substack{(q,t) \in \mathbb{T}^2 \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim \\ \|\zeta\| \leq K_1}} L(q, \zeta, t).$$

Given time interval $[a,b]$ with $b-a \geq 1$, if we reparametrize the shortest geodesic $\gamma(s)$ by $\bar{\gamma}(t) = \gamma(\ell(x_1,x_2)(t-a)/(b-a))$, then $\bar{\gamma}(t)$ is a C^1-curve such that $\bar{\gamma}(a) = x_1$, $\bar{\gamma}(b) = x_2$. Clearly, the action of L along this curve is not bigger than $K(b-a)$. Obviously, there is a upper bound uniformly for all minimizing action of L' if they close to L on $\{\|\zeta\| \leq K_1\}$, still denoted by $K(b-a)$.

Since the super-linear growth is assumed, there are two constant C and D such that $L'(q, \dot{q}, t) \geq C\|\dot{q}\| - D$ for all $(q, \dot{q}, t) \in TM \times [a,b]$ and for all L' close to L. It follows that

$$\frac{\mathrm{dist}(\gamma(a), \gamma(b))}{b-a} \leq \frac{1}{b-a} \int_a^b \|d\gamma\| \leq \frac{(K+D)}{C} \tag{2.11}$$

if γ is a minimizer. As (2.11) holds for any $b - a \geq 1$, it implies that there must be some $\tau_i \in [a+i, a+i+1]$ $(i \in \mathbb{Z})$ such that $\|\dot{\gamma}(\tau_i)\| \leq C^{-1}(K+D)$. By the compactness of $M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim$ we see that there exists $K_2 > 0$ such that $\cup_{s \in [0,1]} \phi^s(\{q, \xi, t : (q,t) \in M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim, \|\xi\| \leq C^{-1}(K+D)\}) \subset \{q, \xi, t : (q,t) \in M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim, \|\xi\| \leq K_2\}$.

Let $L_i \in C^r(TM \times \mathbb{R}, \mathbb{R})$ be a sequence converging to L, let γ_i: $[a,b] \to M$ be the minimizer of L_i with $b - a \geq 1$. By the argument above, we see there exists some $U_k \supset \{\xi, q, t : (q,t) \in M \times \mathbb{R}, \|\xi\| \leq K_2\}$. So $\|(L(z,t) - L_i(z,t)\|_{U_k} \leq \epsilon_i$, where $\epsilon_i \to 0$ as $i \to \infty$. Thus

$$\int_a^b L(d\gamma_i(t), t) dt \leq (K + \epsilon_i)(b-a). \tag{2.12}$$

As all γ_i is a C^1-curve and the actions of L on each γ_i are bounded by [2.12], the set $\{\gamma_i\}_{i \in \mathbb{Z}_+}$ is compact in the C^0-topology (cf [Ma3]). Moreover this set is compact in the $C^1([a,b], M)$-topology as we have $\|\dot{\gamma}_i\| \leq K_2$ and as $\partial^2 L/\partial \dot{q}^2$ is positive definite so we can write the Lagrange equations in the form of $\ddot q = f(q,\dot q,t)$, which implies γ_i is bounded in C^2-topology.

Let γ: $[a,b]\to M$ be one of the accumulate point of this set. Clearly, γ: $[a,b]\to M$ is the minimizer of L and we have

$$A_c(\gamma)=\lim_{i\to\infty}\int_a^b L_i(d\gamma_i(t),t)dt.$$

We let $I_i=[-T_i,T_i]$ and let $T_i\to\infty$, there is a sequence of minimizers of L_i, γ_i: $I_i\to M$. By diagonal extraction argument we can find a subsequence of γ_i which converges C^1 uniformly on each compact set to a C^1-curve γ: $\mathbb{R}\to M$ which is the minimizer of L on any compact interval of $\mathbb{R}$. This proves the upper semi-continuity of $L\to\tilde{\mathcal{G}}_L$.

Given L periodic in t, we let $L_c=L-\eta_c$ where η_c is a closed one form such that $[\eta_c]=c$. η_c is a linear function in $\dot q$. If $c_i\to c$, we can choose a sequence of closed 1-form η_{c_i} such that $[\eta_{c_i}]=c_i$ and $|\eta_{c_i}-\eta_c|_{\|\dot q\|\le K_1}\to 0$. In this case $L_{c_i}\to L_c$ implies $c_i\to c$. Since the c-minimal orbits are independent of the choice of η_i, applying the argument above we obtain the upper semi-continuity $c\to\tilde{\mathcal{G}}(c)$. $\square$

In the application, the set $\tilde{\mathcal{G}}(c)$ seems too bigger to be used for the construction of connecting orbits in interesting problems. Mañé sets seem good candidates. In the time-periodic case, Mañé set can be a proper subset of $\tilde{\mathcal{G}}(c)$, $\tilde{\mathcal{N}}(c)\subsetneq\tilde{\mathcal{G}}(c)$. It is closely related to the problem whether the Lax-Oleinik semi-group converges or not, some example can be found in [FM]. To establish the connection between two Mañé sets we consider a modified Lagrangian

$$L_{\eta,\mu}=L-\eta-\mu$$

where η is a closed 1-form on M such that $[\eta]=c$, μ is a 1-form depending on t in the way that the restriction of μ on $\{t\le 0\}$ is 0, the restriction on $\{t\ge 1\}$ is a closed 1-form $\bar\mu$ on M with $[\bar\mu]=c'-c$. Let $m_0,m_1\in M$, we define

$$\begin{aligned}h_{\eta,\mu}^{T_0,T_1}(m_0,m_1)=&\inf_{\substack{\gamma(-T_0)=m_0\\ \gamma(T_1)=m_1}}\int_{-T_0}^{T_1}(L-\eta-\mu)(d\gamma(t),t)dt\\ &-T_0\alpha(c)-T_1\alpha(c').\end{aligned}\tag{2.13}$$

Clearly there is $m^*\in M$ and some constant C_μ, $C_{\eta,\mu}$, independent of T_0,T_1, such that

$$\begin{aligned}h_{\eta,\mu}^{T_0,T_1}(m_0,m_1)&\le h_\eta^{T_0}(m_0,m^*)+h_{\eta+\bar\mu}^{T_2}(m^*,m_2)+C_\mu\\ &\le C_{\eta,\mu}.\end{aligned}$$

Thus its limit infimum is bounded

$$\begin{aligned}h_{\eta,\mu}^{\infty}(m_0,m_1)=&\liminf_{T_0,T_1\to\infty}h_{\eta,\mu}^{T_0,T_1}(m_0,m_1)\\ \le&C_{\eta,\mu}.\end{aligned}\tag{2.14}$$

Let $\{T_0^i\}_{i\in\mathbb{Z}_+}$ and $\{T_1^i\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i \to \infty$ $(j=0,1)$ as $i\to\infty$ and the following limit exists

$$\lim_{i\to\infty} h_{\eta,\mu}^{T_0^i,T_1^i}(m_0,m_1) = h_{\eta,\mu}^{\infty}(m_0,m_1).$$

Let $\gamma_i(t,m_0,m_1)$: $[-T_0^i,T_1^i]\to M$ be the minimizer connecting m_0 and m_1

$$h_{\eta,\mu}^{T_0^i,T_1^i}(m_0,m_1) = \int_{-T_0^i}^{T_1^i}(L-\eta-\mu)(d\gamma_i(t),t)dt - T_0^i\alpha(c) - T_1^i\alpha(c').$$

From the proof of the lemma 2.2 we can see that for any compact interval $[a,b]$ there is some $I\in\mathbb{Z}_+$ such that the set $\{\gamma_i\}_{i\geq I}$ is pre-compact in $C^1([a,b],M)$.

Lemma 2.3. *Let γ: $\mathbb{R}\to M$ be an accumulation point of $\{\gamma_i\}$. If $s\geq 1$ then*

$$\begin{aligned} A_{L_{\eta,\mu}}(\gamma|[s,\tau]) = \inf_{\substack{\tau_1-\tau\in\mathbb{Z},\tau_1>s\\ \gamma^*(s)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} &\int_s^{\tau_1}(L-\eta-\mu)(d\gamma^*(t),t)dt \\ &-(\tau_1-\tau)\alpha(c'); \end{aligned} \tag{2.15a}$$

if $\tau\leq 0$ then

$$\begin{aligned} A_{L_{\eta,\mu}}(\gamma|[s,\tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},s_1<\tau\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau)=\gamma(\tau)}} &\int_{s_1}^{\tau}(L-\eta-\mu)(d\gamma^*(t),t)dt \\ &+(s_1-s)\alpha(c); \end{aligned} \tag{2.15b}$$

if $s\leq 0$ and $\tau\geq 1$ then

$$\begin{aligned} A_{L_{\eta,\mu}}(\gamma|[s,\tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},\tau_1-\tau\in\mathbb{Z}\\ s_1\leq 0,\tau_1\geq 1\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} &\int_{s_1}^{\tau_1}(L-\eta-\mu)(d\gamma^*(t),t)dt \\ &+(s_1-s)\alpha(c)-(\tau_1-\tau)\alpha(c'). \end{aligned} \tag{2.15c}$$

Proof: To show that let us suppose the contrary, for instance, (2.15b) does not hold. Thus there would exist $\Delta>0$, $s<\tau\leq 0$, $s_1<\tau\leq 0$, $s_1-s\in\mathbb{Z}$ and a curve γ^*: $[s_1,\tau]\to M$ with $\gamma^*(s_1)=\gamma(s_1)$, $\gamma^*(\tau)=\gamma(\tau)$ such that

$$A_{L_{\eta,\mu}}(\gamma|[s,\tau]) \geq \int_{s_1}^{\tau}(L-\eta-\mu)(d\gamma^*(t),t)dt + (s_1-s)\alpha(c)+\Delta.$$

Let $\epsilon=\frac{1}{4}\Delta$. By the definition of limit infimum there exist subsequences $T_j^{i_k}$ $(j=0,1,$ $k=0,1,2,\cdots)$ such that

$$T_0^{i_k}-T_0^{i_0}\geq s-s_1, \qquad \forall\ k\geq 1, \tag{2.16}$$

$$h_{\eta,\mu}^{T_0,T_1}(m_0,m_1) \geq h_{\eta,\mu}^{\infty}(m_0,m_1) - \epsilon, \qquad \forall \;\; T_0 \geq T_0^{i_0},\; T_1 \geq T_1^{i_0}, \tag{2.17}$$

and

$$|h_{\eta,\mu}^{T_0^{i_k},T_1^{i_k}}(m_0,m_1) - h_{\eta,\mu}^{\infty}(m_0,m_1)| < \epsilon, \qquad \forall \; k. \tag{2.18}$$

By taking subsequence further we can assume $\gamma_{i_k} \to \gamma$. In this case, we can choose sufficiently large k such that $\gamma_{i_k}(s)$ and $\gamma_{i_k}(\tau)$ are so close to $\gamma(s)$ and $\gamma(\tau)$ respectively that we can construct a curve $\gamma_{i_k}^*\colon [s_1,\tau] \to M$ which has the same endpoints as γ_{i_k}: $\gamma_i^*(s_1) = \gamma_i(s)$, $\gamma_i^*(\tau) = \gamma_i(\tau)$ and satisfies the following

$$A_{L_{\eta,\mu}}(\gamma_{i_k}|[s,\tau]) \geq \int_{s_1}^{\tau} (L-\eta-\mu)(d\gamma_{i_k}^*(t),t)dt + (s_1 - s)\alpha(c) + \frac{3}{4}\Delta. \tag{2.19}$$

Let $T_0' = T_0^{i_k} + (s - s_1)$, if we extend $\gamma_{i_k}^*$ to $\mathbb{R} \to M$ such that

$$\gamma_{i_k}^* = \begin{cases} \gamma_{i_k}(t - s_1 + s), & t \leq s_1, \\ \gamma_{i_k}^*(t), & s_1 \leq t \leq \tau, \\ \gamma_{i_k}(t), & t \geq \tau, \end{cases}$$

then we obtain from (2.18) and (2.19) that

$$\begin{aligned} h_{\eta,\mu}^{T_0',T_1^{i_k}}(m_0,m_1) \leq & A_{L_{\eta,\mu}}(\gamma_{i_k}^*|[-T_0',T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0'\alpha(c) \\ \leq & A_{L_{\eta,\mu}}(\gamma_{i_k}|[-T_0^{i_k},T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0^{i_k}\alpha(c) - \frac{3}{4}\Delta \\ \leq & h_{\eta,\mu}^{\infty}(m_0,m_1) - 2\epsilon. \end{aligned}$$

but this contradicts (2.17) since $T_0' \geq T_0^{i_0}$ and $T_1^{i_k} \geq T_1^{i_0}$, guaranteed by (2.16). (2.15a) and (2.15c) can be proved in the same way. $\square$

We define a set for $L_{\eta,\mu}$ somewhat similar to the definition of Mañé set

$$\tilde{\mathcal{N}}_{\eta,\mu} = \{d\gamma \in \tilde{\mathcal{G}}_{L_{\eta,\mu}} : \; (2.15a)\; (2.15b) \text{ and } (2.15c) \text{ hold } \}.$$

Lemma 2.4. *The map $(\eta,\mu) \to \tilde{\mathcal{N}}_{\eta,\mu}$ is upper semi-continuous. $\tilde{\mathcal{N}}_{\eta,0} = \tilde{\mathcal{N}}(c)$ if $[\eta] = c$, consequently, the map $c \to \tilde{\mathcal{N}}(c)$ is upper semi-continuous.*

Proof: Let $\eta_i \to \eta$ and $\mu_i \to \mu$, let $\gamma_i \in \tilde{\mathcal{N}}_{\eta_i,\mu_i}$ and let γ be an accumulation point of the set $\{\gamma_i \in \tilde{\mathcal{N}}_{\eta_i,\mu_i}\}_{i\in\mathbb{Z}^+}$. Clearly, $\gamma \in \tilde{\mathcal{N}}_{\eta,\mu}$. If $\gamma \notin \tilde{\mathcal{N}}_{\eta,\mu}$ there would be two point $\gamma(s)$,$\gamma(\tau) \in M$ such that one of the following three possible cases takes place. Either $\gamma(s)$ and $\gamma(\tau) \in M$ can be connected by another curve $\gamma^*\colon [s+n,\tau] \to M$ with smaller action

$$A_{\eta,\mu}(\gamma|[s,\tau]) < A_{\eta,\mu}(\gamma^*|[s+n,\tau]) - n\alpha(c)$$

in the case $\tau < 0$; or there would a curve $\gamma^*\colon [s,\tau+n] \to M$ such that

$$A_{\eta,\mu}(\gamma|[s,\tau]) < A_{\eta,\mu}(\gamma^*|[s,\tau+n]) - n\alpha(c')$$

in the case $s \geq 1$, or when $s \leq 0$ and $\tau \geq 1$ there would be a curve γ^*: $[s+n_1, \tau+n_2] \to M$ such that

$$A_{\eta,\mu}(\gamma|[s,\tau]) < A_{\eta,\mu}(\gamma^*|[s+n_1, \tau+n_2]) - n_1\alpha(c) - n_2\alpha(c')$$

where $s + n_1 \leq 0$, $\tau + n_2 \geq 1$. Since γ is an accumulation point of γ_i, for any small $\epsilon > 0$, there would be sufficiently large i such that $\|\gamma - \gamma_i\|_{C^1[s,t]} < \epsilon$, it follows that $\gamma_i \notin \tilde{\mathcal{N}}_{\eta_i,\mu_i}$ but that is absurd.

Let us consider the case that $\mu = 0$. In this case, $L - \eta$ is periodic in t. If some orbit $\gamma \in \tilde{\mathcal{N}}_{\eta,0}$: $\mathbb{R} \to M$ is not semi-static, then there exist $s < \tau \in \mathbb{R}$, $n \in \mathbb{Z}$, $\Delta > 0$ and a curve γ^*: $[s, \tau + n] \to M$ such that $\gamma^*(s) = \gamma(s)$, $\gamma^*(\tau + n) = \gamma(\tau)$ and

$$A_{\eta,0}(\gamma|[s,\tau]) \geq A_{\eta,0}(\gamma^*|[s, \tau + n]) - n\alpha(c) + \Delta.$$

We can extend γ^* to $[s_1, \tau_1 + n] \to M$ such that $s_1 \leq \min\{s, 0\}$, $\min\{\tau_1, \tau_1 + n\} \geq 1$, $\tau_1 \geq \tau$ and

$$\gamma^* = \begin{cases} \gamma(t), & s_1 \leq t \leq s, \\ \gamma^*(t), & s \leq t \leq \tau + n, \\ \gamma(t-n), & \tau + n \leq t \leq \tau_1 + n. \end{cases}$$

Since $L - \eta$ is periodic in t, we would have

$$A_{\eta,0}(\gamma|[s_1, \tau_1]) \geq A_{\eta,0}(\tau^*\gamma|[s_1, \tau_1 + n]) - n\alpha(c) + \Delta.$$

but this contradicts to (2.15c). $\square$

The upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ will be fully exploited to build the C-equivalence among some $\tilde{\mathcal{N}}(c)$, the construction of diffusion orbits in this paper depends crucially on this property. Towards that, we shall also make use of the Lipschitz property of the Aubry sets. Let π: $TM \times \mathbb{T} \to M \times \mathbb{T}$ be the projection along the fibers, Mather discovered the following (cf. [Ma3,4]):

Lemma 2.5. *π: $\tilde{\mathcal{A}}(c) \to M \times \mathbb{T}$ is injective. Its inverse (considered as a map from $\mathcal{A}(c) = \pi\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c)$) is Lipschitz, i.e. $\exists$ a constant C_L such that for any $x, y \in \mathcal{A}(c)$ we have*

$$\text{dist}(\pi^{-1}(x), \pi^{-1}(y)) \leq C_L \text{dist}(x, y).$$

The concept of of regular Lagrangian is useful for us in this paper. L is said to be c-regular if the following limit exists for all (x, x', s, s')

$$h_c(x, x', s, s') = \lim_{\substack{s=t \mod 1 \\ s'=t' \mod 1 \\ t'-t\to\infty}} F_c(x, x', t, t'). \tag{2.20}$$

Lemma 2.6. (Bernard 2002) *If $\tilde{\mathcal{M}}(c)$ is minimal in the sense of topological dynamics and if there exists a sequence γ_n of n-periodic curves such $A_c(\gamma_n) \to 0$ as $n \to \infty$, then L_c is regular, hence $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c)$.*

For the completeness sake, we shall present his proof in the appendix. Applying this lemma to the area-preserving twist map we have the following:

Corollary 2.7. *Let $\omega \in \mathbb{R}\backslash\mathbb{Q}$ be the rotation number and $c = \beta'(\omega)$, then L_c is regular and $\tilde{\mathcal{G}}(c) = \tilde{\mathcal{A}}(c)$.*

3, Structure of some c-minimal orbit sets

Our construction of connecting orbits between different c-minimal orbit sets exploit fully the upper-semi continuity of the set-valued function $c \to \tilde{\mathcal{N}}(c)$, and the structure of the relevant Mañé sets.

Let us consider the Hamiltonian flow Φ^t which is a small perturbation of Φ^t_{f+g}. Let Φ and Φ_{f+g} be their time-1-map. As the cylinder $\mathbb{T} \times \mathbb{R} \times \{(q_2, p_2) = (0,0)\} = \Sigma_0$ is the normally hyperbolic invariant manifold for Φ_{f+g} and the *a priori* unstable condition is assumed, it follows from the fundamental theorem of normally hyperbolic invariant manifold (cf. [HPS]) that there is $\epsilon = \epsilon(A, B) > 0$ such that if $\|P\|_{C^r} \le \epsilon$ on the region $\{|p| \le \max(|A|, |B|) + 1\}$ the map $\Phi^t|_{t=s+k}$ $(k \in \mathbb{Z})$ also has a C^{r-1} invariant manifold $\Sigma(s) \subset \mathbb{R}^2 \times \mathbb{T}^2$. This manifold is a small deformation of the manifold $\Sigma_0|_{\{|p_1| \le \max(|A|,|B|)+1\}}$, and is also normally hyperbolic and time-1-periodic. Let $\Sigma = \Sigma(0)$, it can be considered as the image of a map ψ: $\Sigma_0 \to \mathbb{R}^2 \times \mathbb{T}^2$, $\Sigma = \{p_1, q_1, p_2(p_1, q_1), q_2(p_1, q_1)\}$. This map induces a 2-form $\psi^*\omega$ on Σ_0

$$\psi^*\omega = \left(1 + \frac{\partial(p_2, q_2)}{\partial(p_1, q_1)}\right) dp_1 \wedge dq_1.$$

Since the second de Rham co-homology group of Σ_0 is trivial, by using Moser's argument on the isotopy of symplectic forms [Mo], we find that there exists a diffeomorphism ψ_1 on $\Sigma_0|_{\{|p_1| \le \max(|A|,|B|)+1\}}$ such that

$$(\psi \circ \psi_1)^*\omega = dp_1 \wedge dq_1. \tag{3.1}$$

Since Σ is invariant for Φ and $\Phi^*\omega = \omega$, we have

$$\Big((\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)\Big)^* dp_1 \wedge dq_1 = dp_1 \wedge dq_1$$

i.e. $(\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)$ preserves the standard area. Clearly, it is exact and twist since it is a small perturbation of Φ_f. In this sense, we say that the restriction of Φ on Σ is obviously area-preserving and twist. If $r > 4$ there are many invariant homotopically non-trivial curves, including many KAM curves. As it still remains open whether the invariant curves must be differentiable, we can only assume all these curves are of Lipschitz. Given $\rho \in \mathbb{R}$ there is an Aubry-Mather set with rotation number ρ, which is either an invariant circle, or a Denjoy set if $\rho \in \mathbb{R}\backslash\mathbb{Q}$, or periodic orbits if $\rho \in \mathbb{Q}$. Under the generic condition we can assume there is no homotopically non-trivial invariant curves with rational rotation number for $\Phi|_\Sigma$, there is only one minimal periodic orbit on Σ for each rational rotation number.

Let us consider the Legendre transformation $\mathcal{L}$. By abuse of terminology we continue to denote $\Sigma(s)$ and its image under the Legendre transformation by the same symbol. Let

$$\tilde{\Sigma} = \bigcup_{s \in \mathbb{T}} (\Sigma(s), s),$$

which has the normal hyperbolicity as well. Under the Legendre transformation those Aubry-Mather sets (invariant curves, Denjoy sets or minimal periodic orbits) on Σ correspond to the support of some c-minimal measures. Recall $H^1(M,\mathbb{R}) = \mathbb{R}^2$. We claim that each of these sets corresponds to an interval or a rectangle in $H^1(M,\mathbb{R})$, in other words, for all c in this interval (rectangle), the time-1-section of the support of the c-minimal measure is exactly this Aubry-Mather set.

Towards that goal, we introduce the coordinate transformation $(p_1, q_2, p_2, q_2) \to (p_1, q_2, p_2 + \zeta(q_2), q_2)$ where ζ is defined in the way such that

$$\frac{\partial g}{\partial p_2}(\zeta(q_2), q_2) = 0, \tag{3.2}$$

and let $g'(p_2, q_2) = g(p_2 + \zeta(q_2), q_2)$. By the assumption on g we now have

$$\frac{\partial^2 g'}{\partial q_2^2}(0,0) < 0, \qquad \frac{\partial^2 g'}{\partial p_2 \partial q_2}(0, q_2) = 0. \tag{3.3}$$

To simplify the notation we still use g to denote the function g'. Let L_0 be the Lagrangian obtained from $f + g$ by Legendre transformation, it has the form

$$L_0(q_2, \dot{q}) = \ell_1(\dot{q}_1) + \ell_2(q_2, \dot{q}_2)$$

where ℓ_1 and ℓ_2 are the Legendre transformation of f and g respectively. As g is a convex function in p_2, $\dot{q}_2 = \dot{q}_2(p_2, q_2) = \partial_{p_2} g(p_2, q_2)$, we find from (3.2) and the convexity of g that $\dot{q}_2(0, q_2) = 0$ and $\partial \dot{q}_2/\partial p_2 > 0$, thus ℓ_2 can be written in the form

$$\ell_2(q_2, \dot{q}_2) = V(q_2) + U(q_2, \dot{q}_2) \tag{3.4}$$

where $V(q_2) = -g(0, q_2)$, $U \geq 0$ is a convex function in $\dot{q}_2$ with super-linear growth, attains its minimum at $\dot{q}_2 = 0$ ($\forall q_2 \in \mathbb{T}$). By the assumption, V has a global minimum at $q_2 = 0$ which is non-degenerate.

Now let us consider the β function of L_0. Under the flow $\phi^t_{L_0}$ each invariant circle on Σ with rotation number ρ is the support of the minimal measure $\mu_{(\rho,0)}$ whose rotation vector is $(\rho, 0)$. If we consider the α function of L_0, it corresponds to the support of those c-minimal measures that $c \in \{c_1\} \times [c_2^-, c_2^+] \subset \mathbb{R}^2$ with $-\infty < c_2^- < 0 < c_2^+ < \infty$. We have $c_2^- < c_2^+$ since the β function of the twist map has corner at rational numbers. β is differentiable at some rational number p/q if and only there exists a homotopically non-trivial invariant curve of rotation number p/q, consists entirely of periodic orbits of period q ([Ba],[Ma2]). We obtain that $-\infty < c_2^-$ and $c_2^+ < \infty$, from the property that both α and β functions are finite everywhere and has super-linear growth.

Next, let us consider the α function of L. We use $c = (c_1, c_2) \in \mathbb{R}^2$ to denote the first de Rham cohomology class of M. For each $c \in \mathbb{R} \times (c_2^-, c_2^+)$ the action variable on each c-minimal orbit of L_0 takes value $(p_1, 0)$ which is independent of t. Let A^*, B^* be such numbers that for each $c \in [A^*, B^*] \times (c_2^-, c_2^+)$ the corresponding p_1 satisfies the condition

$$A - 1 \leq p_1 \leq B + 1.$$

In the following, we shall use $\bar{M} = \{(\bar{q}_1, \bar{q}_2) \in \mathbb{T} \times \mathbb{R}\}$ to denote a covering space of M. For a curve γ: $I \to M$, we use $\bar{\gamma}$ to denote its lift to this covering space.

Lemma 3.1. *There exists $\epsilon_0>0$, if $\|P\|_r\le\epsilon_0$ on the region $\{|p|\le\max(|A|,|B|)+1\}$, there is a strip $C=[A^*,B^*]\times[-c_2^*,c_2^*]\subset H^1(M,\mathbb{R})$ $(c_2^*>0)$, such that for each $c\in C$, the c-minimal orbit set $\tilde{\mathcal{G}}(c)\subset\tilde{\Sigma}$.*

Proof: Note the Lagrange flow of L_0 is integrable and is decoupled between two phase sub-space $(q_1,\dot q_1)$ and $(q_2,\dot q_2)$. Let us consider $\phi^t_{\ell_2}$, the second component of the flow $\phi^t_{L_0}$. It has two homoclinic loops Γ^+ and Γ^-, which can be thought as the graph of the functions $G^{\pm}(q_2)$, i.e. $\Gamma^{\pm}=\{q_2,G^{\pm}(q_2)\}$. The orbit dq_2^+ on Γ^+ encircles the cylinder $\mathbb{T}\times\mathbb{R}$ in counter clockwise direction ($\dot q_2>0$), the orbit dq_2^- on Γ^- encircles the cylinder in clockwise direction ($\dot q_2<0$). Clearly we have some positive numbers $C_A^{\pm}>0$, $C_K^{\pm}>0$ such that

$$\int_{-\infty}^{\infty}\ell_2(q_2^{\pm}(t),\dot q_2^{\pm}(t))dt=C_A^{\pm},\qquad \int_{-\infty}^{\infty}U(q_2^{\pm}(t),\dot q_2^{\pm}(t))dt=C_K^{\pm},$$

it is easy to see that

$$c_2^+=\frac{1}{2\pi}C_A^+,\qquad c_2^-=-\frac{1}{2\pi}C_A^-.$$

Let us single out one lift of these homoclinic loops $d\bar q_2^+$ and $d\bar q_2^-$ with $\bar q_2^+(-\infty)=0$, $\bar q_2^+(\infty)=2\pi$, $\bar q_2^-(-\infty)=2\pi$, $\bar q_2^-(\infty)=0$. For $a_1,a_2>0$, we assume $\bar q_2^+(0)=a_1$, $\bar q_2^-(0)=2\pi-a_2$. Let $T_+(a_1,a_2)$ and $T_-(a_1,a_2)$ be the numbers such that $\bar q_2^+(T_+(a_1,a_2))=2\pi-a_2$ and $\bar q_2^-(T_-(a_1,a_2))=a_1$. Clearly, $T_{\pm}(a_1,a_2)\to\infty$ as one of $a_i\to 0$. There exists $a_0>0$ for $a_i\le a_0$ $(i=1,2)$ we have

$$\int_0^{T_{\pm}(a_1,a_2)}\ell_2(q_2^{\pm}(t),\dot q_2^{\pm}(t))dt\ge\frac{9}{10}C_A^{\pm}. \tag{3.5}$$

Let $\Omega=\{(q_2,\dot q_2):G^-(q_2)\le\dot q_2\le G^+(q_2)\}$, it is a fish-shaped area and filled with periodic orbits of ℓ_2. The circles made up by these periodic orbits are contractible on the cylinder. For small $a_2>0$ there exists $a_1=a_1(a_2)>0$ such that if the periodic orbit $dq_2(t)$ intersects the line $\{q_2=a_1\}$ with $\dot q_2(0)\ge 0$ then it can intersect the line $q_2=2\pi-\frac{a_2}{2}$ also. Let $\bar a_2=\bar a_2(a_0)\le a_0$ be the biggest number such that $a_1(\bar a_2)\le a_0$. Let $T_+(a_0)>0$ be the smallest number such that $q_2(0)=a_1(\bar a_2)$ and $q_2(T_+(a_0))=2\pi-\bar a_2$. Since ℓ_2 is convex in $\dot q_2$, it is easy to see that $T_+(a_0)>T_+(a_1(\bar a_2),\bar a_2(a_0))$. Similarly, we can define $\bar a_1=\bar a_1(a_0)\le a_0$ and $T_-(a_0)$ so that $q_2(0)=2\pi-a_2(\bar a_1)$ and $q_2(T_-(\bar a_0))=\bar a_1$. Obviously, there exists $\bar a_0>0$ and $\tau>0$ such that for each $a_0\le\bar a_0$ and for all $T_+(a_1,a_2)-\tau\le T\le T_+(a_0)$ we have

$$\inf_{\substack{\bar\zeta(0)=a_1\\ \bar\zeta(T)=2\pi-a_2}}\int_0^T\ell_2(\zeta(t),\dot\zeta(t))dt\ge\frac{4}{5}C_A^+, \tag{3.6}$$

where $a_2=\bar a_2(a_0)$ and $a_1=a_1(\bar a_2)$. Similarly, for all $T_-(a_1,a_2)-\tau\le T\le T_-(a_0)$ we have

$$\inf_{\substack{\bar\zeta(0)=2\pi-a_2\\ \bar\zeta(T)=a_1}}\int_0^T\ell_2(\zeta(t),\dot\zeta(t))dt\ge\frac{4}{5}C_A^-, \tag{3.6'}$$

where $a_1 = \bar{a}_1(a_0)$ and $a_2 = a_2(\bar{a}_1)$. On the other hand, for each $a_1, a_2 > 0$ and each $T > 0$ there exists $\xi_T^{\pm}$: $[0,T] \to \mathbb{R}$ such that $\xi_T^+(0) = a_1$ and $\xi_T^+(T) = -a_2$, $\xi_T^-(0) = -a_2$ and $\xi_T^-(T) = a_1$, which minimizes the action

$$\begin{aligned} A_2(\xi_T^+) =& \int_0^T \ell_2(\xi_T^+(t), \dot{\xi}_T^+(t))dt \\ =& \inf_{\substack{\zeta^+(0)=a_1 \\ \zeta^+(T)=-a_2}} \int_0^T \ell_2(\zeta_T^+(t), \dot{\zeta}_T^+(t))dt. \\ A_2(\xi_T^-) =& \int_0^T \ell_2(\xi_T^-(t), \dot{\xi}_T^-(t))dt \\ =& \inf_{\substack{\zeta^-(0)=-a_2 \\ \zeta^-(T)=a_1}} \int_0^T \ell_2(\zeta_T^-(t), \dot{\zeta}_T^-(t))dt. \end{aligned}$$

It is easy to see that $dist(d\xi_T^{\pm}, \Gamma^{\pm}) \to 0$ as $T \to \infty$, so there exists $T_a > 0$, for each $T \geq T_a$ we have

$$\begin{aligned} A_2(\xi_T^{\pm}) \leq& 2\left(\int_{-\infty}^0 + \int_{T_{\pm}(a_1,a_2)}^{\infty}\right) \ell_2(q_2(t), \dot{q}_2(t))dt \\ \to& 0 \qquad\qquad (\text{as } a_1 \to 0,\ a_2 \to 0). \end{aligned} \tag{3.7}$$

It follows from (3.5) and (3.7) that there exists $0 < a_0^* \leq \bar{a}_0$, if $a_i \leq a_0^*$ and if $T'_+ \geq T_+(a_1, a_2) - \tau$, $T'_- \geq T_-(a_1, a_2) - \tau$ then

$$A_2(\xi_{T'_+}^+) \leq \frac{1}{5}C_A^+, \qquad A_2(\xi_{T'_-}^-) \leq \frac{1}{5}C_A^-. \tag{3.8}$$

Let us now consider those orbits of L of which the second component $d\gamma_2$ encircles around the phase cylinder $(q_2, \dot{q}_2) \in \mathbb{T} \times \mathbb{R}$ at least once. We assume $d\gamma_2$ goes around the cylinder in counter clock-wise direction. In this case, there is $t_0 \in \mathbb{R}$ such that $\bar{\gamma}_2(t_0) = a_1(\bar{a}_2(a_0^*))$ with $\bar{\gamma}_2(t_0 + t) > a_1(\bar{a}_2(a_0^*))$ for small $t > 0$. Let $t_1 = \min\{t \in \mathbb{R} : t > t_0, \bar{\gamma}_2(t_1) = 2\pi - \bar{a}_2(a_0^*)\}$. Based on the understanding on the phase portraits of ℓ_2 and the smooth dependence of solutions of ODE's on parameter, we see that for given $\delta > 0$ there exists $\epsilon > 0$ such that if $\|P\| \leq \epsilon$ then

$$t_1 - t_0 \leq T_+(a_0^*) + \tau, \tag{3.9}$$

and if the initial value is chosen so that $\dot{\bar{\gamma}}_2(t_0) \leq G^+(\bar{\gamma}_2(t_0)) + \delta$, we have further

$$t_1 - t_0 \geq T_+(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau. \tag{3.10}$$

Similarly, in the case of $d\gamma_2$ going around the cylinder in clockwise direction, there is t'_0, $t'_1 \in \mathbb{R}$ such that $\bar{\gamma}_2(t'_0) = 2\pi - a_2(\bar{a}_1(a_0^*))$ and $\bar{\gamma}_2(t'_1) = \bar{a}_1(a_0^*)$, if $\|P\| \leq \epsilon$ then

$$t_1 - t_0 \leq T_-(a_0^*) + \tau, \tag{3.9'}$$

and if the initial value is chosen so that $\dot{\bar{\gamma}}_2(t_0) \geq G^-(\bar{\gamma}_2(t_0)) - \delta$, we have further

$$t_1 - t_0 \geq T_-(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau. \tag{3.10'}$$

Let us consider the minimizer of $L(\dot{q}, q, t) - c_1\dot{q}_1$ in the covering space $\mathbb{T} \times \mathbb{R}$ for (q_1, q_2). Let ξ: $[0, T] \to \mathbb{T} \times \mathbb{R}$ be a curve which minimize the action:

$$\begin{aligned} A(\xi|[0,T]) &= \int_0^T (L(d\xi(t), t) - c_1\dot{\xi}_1(t))dt \\ &= \min_{\substack{\zeta(0)=\hat{q} \\ \zeta(T)=q^*}} \int_0^T (L(d\zeta(t), t) - c_1\dot{\zeta}_1(t))dt \end{aligned}$$

where $\hat{q}_2 = -a_2$ and $q_2^* = a_1$. Since there is a normally hyperbolic structure around $\tilde{\Sigma}|_{\{|p_1| \leq \max(|A|,|B|)+1\}}$, there is $T_+^* > 0$ such that

$$\dot{\xi}_2(T) < G^+(a_1) + \delta, \qquad \forall\ a_1, a_2 \leq a_0^* \text{ and } \ T \geq T_+^*. \tag{3.11}$$

Now we are on the stage to claim that the second component $d\gamma_2$ of each c-minimal orbit of L can not go around the cylinder twice for small $|c_2|$. If our claim is not true, there exist $t_0, T \in \mathbb{R}$ such that $\bar{\gamma}_2(t_0) = 0$, $\bar{\gamma}_2(t_0 + 2T) = 4\pi$. If we choose another curve $\gamma^*(t) = (\gamma_1(t), q_2^T(t/2))$ where $(q_2^T(t), \dot{q}_2^T(t))$ is an orbit of $\phi_{\ell_2}^t$ going around the cylinder with period T, then $\gamma(t_0) = \gamma^*(t_0)$ and $\gamma(t_0 + 2T) = \gamma^*(t_0 + 2T)$. Since L is a small perturbation of L_0, i.e.

$$L(q, \dot{q}, t) - \langle c, \dot{q}\rangle = \ell_1(\dot{q}_1) + \ell_2(q_2, \dot{q}_2) - \sum_{i=1}^{2} c_i\dot{q}_i + P(q, \dot{q}, t),$$

we can compare the action of L along these two curves. Toward that we consider $q_2^T(t)$, a T-periodic orbit of ℓ_2 with $\dot{q}_2^T > 0$, the action of ℓ_2 on $q_2^T(t)|[0, T]$ minimizes the action of ℓ_2 along all curves ξ: $[0, T] \to \mathbb{R}$ such that $\bar{\xi}(0) = 0$ and $\bar{\xi}(T) = 2\pi$, i.e.

$$\begin{aligned} A_2(q_2^T(t)|[0,T]) &= \int_0^T \{V(q_2^T(t)) + U(q_2^T(t), \dot{q}_2^T(t))\}dt \\ &= \inf_{\substack{\bar{\xi}(0)=0 \\ \bar{\xi}(T)=2\pi}} \int_0^T \{V(\xi(t)) + U(\xi(t), \dot{\xi}(t))\}dt. \end{aligned}$$

Let $\hat{q}_2^T(t) = q_2^T(t/2)$, then we have

$$A_2(q_2^T(t)|[0, 2T]) - A_2(\hat{q}_2^T(t)|[0, 2T]) \geq \frac{1}{2}\int_0^{2T} U(q_2^T(t), \dot{q}_2^T(t))dt,$$

where we have used the facts such as $U(0, q_2) \equiv 0$, $\partial_{\dot{q}_2} U(0, q_2) \equiv 0$ as well as the positive definiteness of $\partial^2_{\dot{q}_2} U$. Repeating this process by replacing T with $2T$, $4T$ and so on, note $A_2(\hat{q}_2^T|[0, 2T]) \geq A_2(q_2^{2T}|[0, 2T])$ we obtain

$$A_2(q_2^T(t)|[0, 2T]) - A_2(\hat{q}_2^T(t)|[0, 2T]) \geq \frac{1}{2} C_K^+.$$

It follows that

$$\begin{aligned}
A_c(\gamma|[t_0, t_0 + 2T]) - &A_c(\gamma^*|[t_0, t_0 + 2T]) \\
&\geq A_2(\gamma_2|[t_0, t_0 + 2T]) - A_2(\hat{q}_2^T|[t_0, t_0 + 2T]) - 4T\epsilon - 2\pi c_2 \\
&\geq A_2(q_2^T|[t_0, t_0 + 2T]) - A_2(\hat{q}_2^T|[t_0, t_0 + 2T]) - 4T\epsilon - 2\pi c_2 \\
&\geq \frac{1}{2} C_K^+ - 4T\epsilon - 2\pi c_2 \\
&> 0
\end{aligned}$$

if $T \leq T_+^* + T_+(a_0^*) + \tau$, $|c_2| \leq \frac{1}{8\pi} C_K^+$ and $\epsilon \leq \frac{1}{17} C_K^+ (T_+^* + T_+(a_0^*) + \tau)^{-1}$, but this contradicts to the minimality of γ.

Let us consider the case when $T \geq T_+^* + T_+(a_0^*)$. Since $d\gamma_2$ encircles around the cylinder, there exist t' and $T' \geq T$ such that $\bar{\gamma}_2(t') = -\bar{a}_2(a_0^*)$, $\bar{\gamma}_2(t' + T') = 2\pi - \bar{a}_2(a_0^*)$ and $\bar{\gamma}_2(t) < 2\pi - \bar{a}_2(a_0^*)$ for $t < t' + T'$. Let

$$\hat{t} = \max\{t \in (t', t' + T') : \ \bar{\gamma}_2(t) = a_1(\bar{a}_2(a_0^*)), \ \dot{\bar{\gamma}}_2(t) \geq 0\}.$$

Clearly, there is such a $\hat{t}$ which satisfies the following:

$$T_+(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau \leq t' + T' - \hat{t} \leq T_+(a_0^*) + \tau. \tag{3.12}$$

The second inequality is from (3.9) and the choice of this $\hat{t}$. To verify the first, let us suppose the contrary. If $t' + T' - \hat{t} < T_+(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau$, then $\hat{t} - t' > T_+^* + \tau$. In this case we obtain from (3.11) that $\dot{\bar{\gamma}}_2(\hat{t}) < G^+(\bar{\gamma}_2(\hat{t})) + \delta$. But this contradicts to the definition of $\hat{t}$ and δ, since $d\bar{\gamma}_2$ can not reach the line $q_2 = 2\pi - \bar{a}_2(a_0^*)$ within time $T_+(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau$, which verifies the first inequality of (3.12).

Let us construct a curve $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ in the way that its lift has the form

$$\bar{\zeta}_2(t) = \begin{cases} \bar{\gamma}_2(t) & t \leq \hat{t} \\ \xi_2(t) & \hat{t} \leq t \leq t' + T' \\ \bar{\gamma}_2(t) - 2\pi & t \geq t' + T' \end{cases} \tag{3.13}$$

and $\zeta_1(t) = \gamma_1(t)$, where ξ_2 is the extremal of the Lagrangian ℓ_2 with the boundary condition $\xi_2(\hat{t}) = a_1(\bar{a}_2(a_0^*))$ and $\xi_2(t' + T') = -\bar{a}_2(a_0^*)$. Thus, $\gamma = \zeta$ at $t = \hat{t}$ as well as at $t = t' + T'$. The difference between the action of $L - \eta_c$ along γ and ζ is

$$\begin{aligned}
A_c(\gamma) - A_c(\zeta) =& \int_{\hat{t}}^{t'+T'} \Big\{\ell_2(d\gamma_2(t)) - \ell_2(d\zeta_2(t))\Big\}dt \\
&+ \int_{\hat{t}}^{t'+T'} \{P(d\gamma(t),t) - P(d\zeta(t),t)\}dt - 2\pi c_2 \\
\geq& \frac{3}{5}C_A^+ - 2(T_+(a_0^*) + \tau)\epsilon - 2\pi c_2 \\
& \qquad\qquad\qquad\qquad \text{(by (3.6), (3.8) and (3.12))} \\
>&0
\end{aligned}$$

if $|c_2| \leq \frac{1}{10\pi}C_A^+$ and $\epsilon \leq \frac{1}{10}C_A^+(T_+(a_0^*)) + \tau)^{-1}$, but this contradicts the fact that γ is a c-minimal orbit.

Let $a^* = \min\{a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*), \bar{a}_1(a_0^*), a_2(\bar{a}_1(a_0^*))\}$, let $\epsilon > 0$ be suitably small such that

$$\min\{V(q_2) : |q_2 - 2k\pi| \geq a^*\} \geq 3\epsilon.$$

For each lift of γ we claim there is $k \in \mathbb{Z}$ such that $|\bar{\gamma}_2(t) - 2k\pi| \geq a_0^*$ holds for all $t \in \mathbb{R}$. Let us suppose the contrary, thus we have two alternatives:

1) For all $k \in \mathbb{Z}$ and all $t \in \mathbb{R}$ we have $|\bar{\gamma}_2(t) - 2k\pi| > a^*$, or
2) There exists $t_0 \in \mathbb{R}$ such that $|\bar{\gamma}_2(t) - 2k\pi| = a^*$ for some $k \in \mathbb{Z}$.

In the first case, we consider those lift of γ_2 which takes value between 0 and 2π and construct a curve $\zeta(t) = (\gamma_1(t), \zeta_1(t))$ in the way such that its lift has the form

$$\bar{\zeta}_2(t) = \begin{cases} \bar{\gamma}_2(t) & t \leq t_1 \\ (1 + t_1 - t)\bar{\gamma}_2(t) & t_1 \leq t \leq t_1 + 1 \\ 0 & t_1 + 1 \leq t \leq t_2 - 1 \\ (t_2 - t)\bar{\gamma}_2(t) & t_2 - 1 \leq t \leq t \\ \bar{\gamma}_2(t) & t \geq t_2. \end{cases}$$

Comparing the action of $L - \eta_c$ along γ with that along ζ, we find

$$\begin{aligned}
A_c(\gamma) - A_c(\zeta) =& \left(\int_{t_1}^{t_1+1} + \int_{t_2-1}^{t_2}\right) \Big\{\ell_2(d\gamma(t)) - \ell_2(d\zeta(t))\Big\}dt \\
&+ \int_{t_1}^{t_2} \Big\{P(d\gamma(t),t) - P(d\zeta(t),t)\Big\}dt \\
&+ \int_{t_1+1}^{t_2-1} \ell_2(d\gamma(t))dt \\
\geq& \left(\int_{t_1}^{t_1+1} + \int_{t_2-1}^{t_2}\right) \Big\{\ell_2(d\gamma(t)) - \ell_2(d\zeta(t))\Big\}dt \\
&+ \epsilon(t_2 - t_1 - 6) \\
>&0,
\end{aligned}$$

if $t_2 - t_1$ is sufficiently large. Thus the first alternative does not occur.

Let us consider the second possibility. By taking suitable lift of γ, we can assume $|\bar\gamma_2(t_0)| = a_0^*$. If there is some point $t \in \mathbb{R}$ such that $\dot\gamma_2(t) - G^+(\gamma_2(t)) > \delta$, then $d\gamma_2(t)$ shall encircle around the cylinder for at least twice, it has been shown impossible. Thus we have $\dot\gamma_2(t_0) - G^+(\gamma_2(t_0)) < \delta$. In this case, $d\gamma_2(t)$ keeps close to $\Gamma^\pm$ for $t_0 \le t \le t_0 + T_+(a_1(\bar a_2(a_0^*)), \bar a_2(a_0^*))$, or for $t_0 - T_+(a_1(\bar a_2(a_0^*)), \bar a_2(a_0^*)) \le t \le t_0$, or for $t_0 \le t \le t_0 + T_-(\bar a_1(a_0^*), a_2(\bar a_1(a_0^*)))$, or for $t_0 - T_-(\bar a_1(a_0^*), a_2(\bar a_1(a_0^*))) \le t \le t_0$. In each case, we can construct a curve as we did in (3.13) and find γ is not c-minimal, but this is absurd. Therefore, all c-minimal orbits should be in a^*-neighborhood of $\tilde\Sigma$. Since $\tilde{\mathcal{G}}(c)$ is invariant, by the normal hyperbolicity, it is contained in $\tilde\Sigma$. □

Although the structure of minimal measures is unclear in general case, we know very well the structures of those $\tilde{\mathcal{M}}(c) \subset \tilde\Sigma$ since the time-1-map Φ is an area-preserving twist map when it is restricted on Σ. Under the projection from $TM \times \mathbb{T}$ to $TM \times \{t = 0\}$, the support of those c-minimal measures are the image of those Anbry-Mather sets such as homotopically non-trivial invariant curves, Denjoy sets or minimal periodic orbits on Σ under the Legendre transformation $\mathcal{L}$. We use Γ to denote those Aubry-Mather sets on Σ in the Hamitonian formalism, let $\Gamma(t) = \Phi_H^t(\Gamma) \subset \Sigma(t)$, $\tilde\Gamma = \cup_{t\in\mathbb{T}}(\mathcal{L}(\Gamma(t)), t)$.

Before going on the study of some c-minimal measures, let us note a fact as follows:

Proposition 3.2. *Let $c', c^* \in H^1(M,\mathbb{R})$, μ' and μ^* be the corresponding minimal measures respectively. If $\langle c' - c^*, \rho(\mu')\rangle = \langle c' - c^*, \rho(\mu^*)\rangle = 0$, then $\alpha(c') = \alpha(c^*)$.*

Proof. By the definition of the α function we find that

$$\begin{aligned}
\alpha(c') &= \inf_{\nu\in\mathfrak{M}} \int (L - \eta_{c'}) d\nu \\
&= \int (L - \eta_{c'}) d\mu' \\
&= \int (L - \eta_{c^*}) d\mu' + \langle c^* - c', \rho(\mu')\rangle \\
&\ge \alpha(c^*).
\end{aligned}$$

In the same way we find that $\alpha(c^*) \ge \alpha(c')$. □

Lemma 3.3. *Assume $\tilde\Gamma \in \tilde{\mathcal{M}}(\bar c)$ for some $\bar c = (\bar c_1, \bar c_2) \in [A^*, B^*] \times [-c_2^*, c_2^*]$, then there is an interval $I = I(\bar c_1) = \{(c_1, c_2) \in H^1(M,\mathbb{R}) : c_1 = \bar c_1, a(c_1) \le c_2 \le b(c_1)\}$ with $-\infty < a(c_1) < 0 < b(c_1) < \infty$, such that $\tilde{\mathcal{M}}(c) = \tilde\Gamma$ for all $c \in \mathrm{Int} I$ and $\tilde{\mathcal{M}}(c) \supseteq \tilde\Gamma$ for $c \in \partial I$; if there is an invariant curve containing Γ we have further $\tilde{\mathcal{M}}(c) = \tilde\Gamma$ for all $c \in I$.*

Proof: Let $\bar\mu$ be a $\bar c$-minimal measure. We have shown in the lemma 3.1 that there can not be any support of $\bar\mu$ outside of $\tilde\Sigma$. From the well known theory for twist map it follows that $\bar\mu$ is uniquely ergodic, i.e. $\mathrm{supp}(\bar\mu) = \tilde\Gamma$, since there is a dense orbit in the Aubry-Mather set because we have assumed that there is only one minimal orbit when the rotation number is rational. Let $\phi^t(z,\theta) \in TM \times \mathbb{T}$ be the Lagrangian flow,

z_t be the TM component, $\hat{\eta} = dq_2$. For any invariant measure μ, if $\text{supp}(\mu) \subset \tilde{\Sigma}$, we have

$$\begin{aligned}\int \hat{\eta} d\mu =& \frac{1}{T}\int_0^T ds \int (\hat{\eta} \circ \phi^s) d\mu \\ =& \frac{1}{T}\int_0^T ds \int \langle \hat{\eta}, z_s \rangle d\mu(z) \\ \leq& \frac{1}{T}\int \Big| \int_0^T \langle \hat{\eta}, z_s \rangle ds \Big| d\mu(z) \\ \leq& \frac{2\pi}{T} \to 0\end{aligned}$$

as $T \to \infty$. Since $\int \hat{\eta} d\mu$ is independent of T, $\int \hat{\eta} d\mu = 0$. Therefore, it follows from the proposition 3.2 that $\alpha(\bar{c}) = \alpha(\hat{c})$ if both $\bar{c}$- and $\hat{c}$-minimal measures are on $\tilde{\Sigma}$ with $\bar{c} - \hat{c} = (0, c_2)$. As the β function for a twist map is strictly convex, $\tilde{\mathcal{M}}(\bar{c}) = \tilde{\mathcal{M}}(\hat{c})$. As the α function is convex and has super-linear growth, I is connected and $-\infty < a < 0 < b < \infty$. What remains to show is I is closed. If not, there is a sequence $c_k \to c$ such that $\tilde{\Gamma} \subset \tilde{\mathcal{M}}(c_k)$ and $\tilde{\Gamma} \not\subseteq \tilde{\mathcal{M}}(c)$, so there exists μ such that $A_c(\mu_{\tilde{\Gamma}}) > A_c(\mu)$, where $\mu_{\tilde{\Gamma}}$ is the invariant measure on $\tilde{\Gamma}$. Let k be sufficiently large so that c_k is sufficiently close to c,

$$A_{c_k}(\mu) = \int L d\mu - \langle \rho(\mu), c_k \rangle = A_c(\mu) - \langle \rho(\mu), c_k - c \rangle < A_c(\mu_{\tilde{\Gamma}}).$$

On the other hand, it follows from $\langle c - c_k, \rho(\mu_{\tilde{\Gamma}}) \rangle = 0$ that $A_c(\mu_{\tilde{\Gamma}}) = A_{c_k}(\mu_{\tilde{\Gamma}})$, thus we have $A_{c_k}(\mu_{\tilde{\Gamma}}) > A_{c_k}(\mu)$, but it contradicts to the fact that $\mu_{\tilde{\Gamma}}$ is c_k-minimal measure.

If there is another measure μ which can also minimize the c-action of L when $a(c_1) < c_2 < b(c_1)$, then $\langle dq_2, \mu \rangle = 0$. Indeed, for all $a(c_1) < c_2' < b(c_1)$ we have

$$\begin{aligned}\int \Big(L - c_1 \dot{q}_1 - c_2' \dot{q}_2 \Big) d\mu_\Gamma =& \int \Big(L - c_1 \dot{q}_1 - c_2 \dot{q}_2 \Big) d\mu_\Gamma \\ =& \int \Big(L - c_1 \dot{q}_1 - c_2 \dot{q}_2 \Big) d\mu \quad \text{(by assumption)} \\ =& \int \Big(L - c_1 \dot{q}_1 - c_2' \dot{q}_2 \Big) d\mu + (c_2' - c_2) \langle dq_2, \mu \rangle,\end{aligned}$$

thus we can choose c_2' in the way that $(c_2' - c_2)\langle dq_2, \mu \rangle > 0$ if $\langle dq_2, \mu \rangle \neq 0$. But this contradicts to the minimality of μ_Γ. Consequently, we always have

$$\int (L - c_1 \dot{q}_1) d\mu = \int (L - c_1 \dot{q}_1) d\mu_{\tilde{\Gamma}},$$

which is independent of the value c_2 takes between $a(c_1)$ and $b(c_1)$, it implies that $\mu = \mu_{\tilde{\Gamma}}$ as $\mu_{\tilde{\Gamma}}$ is the only c-minimal measure when $|c_2| \leq c_2^*$.

Let us consider the case when Γ is contained in an invariant curve and $c_2 \in \partial I$. In this case, the Aubry set $\tilde{\mathcal{A}}(c)$ contains a co-dimensional one torus. As any c-minimal curve $\gamma \subset \mathcal{A}(c)$ can not cross the 2-torus $\pi\tilde{\Gamma} \subset T^2 \times \mathbb{T}$, where π is the projection $TM \times \mathbb{T} \to M \times \mathbb{T}$, because the inverse map to the Aubry sets is of Lipschitz and $\pi\tilde{\Gamma}$ is a codimension 1 torus. Thus there exist $\delta > 0$ such that for any $T > 0$

$$-\delta \leq \left| \int_{-T}^{T} \dot{\gamma}_2(t)dt \right| \leq 2\pi + \delta.$$

So, if μ is also a c-minimal measure and $c' = (c_1, 0)$, then

$$\begin{aligned} A_{c'}(\mu_{\tilde{\Gamma}}) =& \int (L - c_1\dot{q}_1)d\mu_{\tilde{\Gamma}} \\ =& \int (L - c_1\dot{q}_1 - c_2\dot{q}_2)d\mu \quad \text{(by condition)} \\ =& \int (L - c_1\dot{q}_1)d\mu \quad \text{(by } \ (3.13)) \\ =& A_{c'}(\mu) \end{aligned}$$

it implies that the only minimal measure is $\mu_{\tilde{\Gamma}}$. $\square$

It follows from the lemma 3.3 we see that there is a strip $\mathcal{S} = \{(c_1, c_2) \in H^1(M, \mathbb{R}) : c_1 \in \mathbb{R},\ a(c_1) \leq c_2 \leq b(c_1),\ A^* < c_1 < B^*,\ -\infty < a(c_1) < 0 < b(c_1) < \infty\}$, such that if $c \in \text{int}\mathcal{S}$, the c-minimal measure is on $\tilde{\Sigma}$ and is uniquely ergodic. If $c \in \partial\mathcal{S}$ and $\Gamma \subset \tilde{\mathcal{M}}(c)$ is an invariant curve, the c-minimal measure is also uniquely ergodic. In these cases, we have $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$.

In the following, we use $I(c_1) = \{c = (c_1, c_2) : a(c_1) \leq c_2 \leq b(c_1)\}$ to denote the maximal interval in the following sense: for each $c' = (c_1, c_2')$ with $a(c_1) < c_2' < b(c_1)$, the c'-minimal measure has some $\tilde{\Gamma} \subset \tilde{\Sigma}$ as its support, this $\tilde{\Gamma}$ is not contained in the support of any c^*-minimal measure where $c^* = (c_1, c_2^*)$ with either $c_c^* < a(c_1)$ or $c_2^* > b(c_1)$.

Lemma 3.4. *Let $\tilde{\Gamma} \subset \tilde{\Sigma}$ be the support of some minimal measure for $\bar{c} \in I(c_1)$, we assume that it has dense orbit. Then $\tilde{\mathcal{N}}(c) \subset \tilde{\Sigma}$ for each $c \in intI(c_1) = \{(c_1, c_2) : a(c_1) < c_2 < b(c_1)\}$. If Γ is an invariant curve or a Denjoy set contained in an invariant curve, and if $c \in \partial I = \{(c_1, c_2) : c_2 = a(c_1)$ or $c_2 = b(c_1)\}$ we have further that $\tilde{\mathcal{N}}(c)$ consists of $\tilde{\Gamma}$ and the c-minimal orbits homoclinic to $\tilde{\Gamma}$.*

Proof. Let us consider a c-minimal orbit $d\gamma$ with $c \in intI(c_1)$ ($c \in I(c_1)$ if Γ is an invariant curve). If this orbit is not contained in $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$, then $d\gamma$ is semi-asymptotic to $\tilde{\Gamma}$ as $t \to \pm\infty$. We say an orbit is semi-asymptotic to an invariant set Γ as $t \to \infty$ if every invariant subset of its ω-limit set that is minimal in Birkhoff sense is contained in Γ. We use the argument in [Bo] to show it. Let N is a minimal (in Birkhoff sense) invariant subset of the ω-limit set of $d\gamma$, there exists a sequence

$t_k \to \infty$ such that $\mathrm{dist}(d\gamma(t_k), N) \to 0$. We claim that there is a sequence $T_k \to \infty$ such that

$$\limsup_{k\to\infty}\{\mathrm{dist}(d\gamma(t), N) :\, t_k \le t \le t_k + T_k\} \to 0. \tag{3.14}$$

If not, there exist $d > 0$, $T > 0$ and a subsequence t_j of the sequence t_k such that $\mathrm{dist}(d\gamma(t), N) \ge d$ for every j and some $s_j \in [t_j, t_j + T]$. As $\gamma(t)$ is a c-minimal curve, $d\gamma$ lies in a bounded region of $TM \times \mathbb{T}$, the closure of the orbit is compact. Thus, for some subsequence t_i of the sequence t_j, the sequence $d\gamma(t_i)$ and $d\gamma(s_i)$ are convergent to some points $x \in N$ and $y \in TM \times \mathbb{T}$ respectively, where $\mathrm{dist}(y, N) \ge d$. Consequently, $\phi^{t_0}(x) = y$ for some $0 < t_0 \le T$. This contradicts to the invariance of N to the Euler-Lagrange flow.

Let μ_n be the probability measure evenly distributed along $d\gamma[t_k, t_k + T_k]$, μ be an accumulation point of $\{\mu_n\}$. As $d\gamma$ is a c-minimal orbit of the Lagrange system μ is a c-minimal measure, i.e. $\mu = \mu_{\tilde\Gamma}$. From (3.14) we see $\mathrm{dist}(N, \tilde\Gamma) = 0$. As $\tilde\Gamma$ has dense orbit, $N = \tilde\Gamma$, i.e. the ω-limit set of $d\gamma$ has only one minimal invariant subset $\tilde\Gamma$ (in Birkhoff sense). In the same way we can show that the α-limit set of $d\gamma$ has only one minimal invariant subset $\tilde\Gamma$ also.

Let $c \in \mathrm{int}I$ and $d\gamma \in \tilde{\mathcal{N}}(c)$. From the lemma 3.3 we see that $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$ in this case. We claim that for each $\xi \in \pi(\Gamma)$ there exist two sequences $k_{ij} \to \infty$ as $j \to \infty$, $(i = 1, 2)$, such that

$$d\gamma(-k_{1j}), d\gamma(k_{2j}) \to \pi^{-1}(\xi) \in \tilde\Gamma|_{t=0}, \qquad \lim_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \dot\gamma_2(t)dt = 0, \tag{3.15}$$

where $\pi^{-1} : \mathcal{A}(c) \to \tilde{\mathcal{A}}(c)$. In fact, for any $\xi \in \pi(\Gamma)$ there exist two subsequences $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$ such that $d\gamma(-k_{ij}) \to \pi^{-1}(\xi)$ and $d\gamma(k_{2j}) \to \pi^{-1}(\xi)$ as $j \to \infty$. It follows from the fact that γ is c-static that

$$h_c^{k_{1j}}(\gamma(-k_{1j}), \gamma(0)) + h_c^{k_{2j}}(\gamma(0), \gamma(k_{2j})) \to 0.$$

If the equality in (3.15) does not hold, by choosing a subsequence again (we use the same symbol) we would have

$$\left|\lim_{j\to\infty}\int_{-k_1 j}^{k_2 j} \dot\gamma_2(t)dt\right| \ge 2\pi > 0.$$

In this case, let us consider the barrier function $B^*_{c'}$ where $c' = (c_1, c'_2)$. Since $c - c' = (0, c_2 - c'_2)$, we obtain from the proposition 3.2 that $\alpha(c') = \alpha(c)$, so

$$\begin{aligned}
B_{c'}(\gamma(0)) \le & \liminf_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - c_1\dot\gamma_1(t) - c'_2\dot\gamma_2(t) - \alpha(c'))\, dt \\
\le & \liminf_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - c_1\dot\gamma_1(t) - c_2\dot\gamma_2(t) - \alpha(c))\, dt \\
& + (c_2 - c'_2)\lim_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \dot\gamma_2(t)dt \\
\le & -2|c_2 - c'_2|\pi < 0
\end{aligned}$$

as we can choose $c_2' > c_2$ or $c_2' < c_2$ accordingly. But this is absurd since barrier function is non-negative.

Now let us derive from (3.15) that there is no such a c-semi-static orbit that is not contained in $\tilde{\Sigma}$. It is easy to see that there is no other such orbits which stays in the small neighborhood of $\tilde{\Sigma}$ for all $t \in \mathbb{R}$ except for the orbit in $\tilde{\Gamma}$ as there is normally hyperbolic structure nearby. If there is a c-minimal curve γ which is not contained in the support of the minimal measure, guaranteed by the semi-asymptotic property of $d\gamma$, there are still two sequences $t_{1j} \to -\infty$, $t_{2j} \to \infty$ as $j \to \infty$ such that $d\gamma(t_{ij})$ $(i = 1, 2)$ approaches to $\tilde{\Gamma}$ as $j \to \infty$. Let $R = \{(q_2, \dot{q}_2) : |q_2| \le \min\{a_1(\bar{a}_2(a_0^*)), a_2(\bar{a}_1(a_0^*))\}, |\dot{q}_2| \le \delta\}$ (for the definition of a_0^* and δ, see the proof of the lemma 3.1). Forced by (3.15) there is an infinite subsequence t_k such that one lift of $d\gamma$ has the property $d\bar{\gamma}_2(t_k) \in R$. On the other hand, since $d\gamma$ does not stay in $\tilde{\Gamma}$, there exists $\hat{t} \in \mathbb{R}$ such that $d\bar{\gamma}_2(\hat{t}) \notin R$. Therefore, there must be $t_2 > t_1$ such that $d\bar{\gamma}_2$ leaves R at $t = t_1$ and $d\bar{\gamma}_2$ gets into R at $t = t_2$ from the same side of the line $q_2 = 0$ in the following sense:

either $\bar{\gamma}_2(t_1) = a_1(\bar{a}_2(a_0^*))$ and $0 \le \dot{\bar{\gamma}}_2(t_1) < G^+(\bar{\gamma}_2(t_1)) + \delta$ (leaving to right), together with $\bar{\gamma}_2(t_2) = \bar{a}_1(a_0^*)$ and $G^-(\bar{\gamma}_2(t_2)) - \delta < \dot{\bar{\gamma}}_2(t_2) \le 0$ (getting into from right).

or $\bar{\gamma}_2(t_1) = -\bar{a}_2(a_0^*))$ with $G^-(\bar{\gamma}_2(t_1)) - \delta \le \dot{\bar{\gamma}}_2(t_1) < 0$ (leaving to left), together with $\bar{\gamma}_2(t_2) = -a_2(\bar{a}_1(a_0^*))$ with $G^-(\bar{\gamma}_2(t_2)) - \delta > \dot{\bar{\gamma}}_2(t_2) \ge 0$ (getting into from left).

Without losing of generality we study the first alternative, in the same way we can study the second alternative. From (3.9) and (3.10) we find that $d\gamma_2$ shall cross the line $\bar{q}_2 = 2\pi - \bar{a}_2(a_0^*)$ at the time $t = t_1 + T_1$ with

$$T_+(a_1(\bar{a}_2(a_0^*)), \bar{a}_2(a_0^*)) - \tau \le T_1 \le T_+(a_0^*).$$

From (3.9') and (3.10') we find $d\gamma_2$ has crossed the line $\bar{q}_2 = 2\pi - a_2(\bar{a}_1(a_0^*))$ at the time $t = t_2 - T_2$ with

$$T_-(\bar{a}_1(a_0^*)), a_2(\bar{a}_1(a_0^*))) - \tau \le T_2 \le T_-(a_0^*).$$

Let us construct a curve $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ in the way that its lift has the form

$$\bar{\zeta}_2(t) = \begin{cases} \bar{\gamma}_2(t) & t \le t_1 \\ \xi_2(t) & t_1 \le t \le t_1 + T_1 \\ \bar{\gamma}_2(t) - 2\pi & t_1 + T_1 \le t \le t_2 - T_2 \\ \eta_2(t) & t_2 - T_2 \le t \le t_2 \\ \bar{\gamma}_2(t) & t \ge t_2 \end{cases}$$

and $\zeta_1(t) = \gamma_1(t)$, where ξ_2 is the extremal of the Lagrangian ℓ_2 with the boundary condition $\xi_2(t_1) = a_1(\bar{a}_2(a_0^*))$ and $\xi_2(t_1 + T_1) = -\bar{a}_2(a_0^*)$; η_2 is the extremal of the Lagrangian ℓ_2 with the boundary condition $\eta_2(t_2 - T_2) = -a_2(\bar{a}_1(a_0^*))$, $\eta_2(t_2) =$

$\bar{a}_1(a_0^*)$. Thus, $\bar\gamma = \bar\zeta$ at $t = t_1$ as well as at $t = t_2$. The difference between the action of $L - \eta_c$ along γ and ζ is

$$\begin{aligned} A_c(\gamma) - A_c(\zeta) \geq & \int_{t_1}^{t_1+T_1} \Big\{\ell_2(d\gamma_2(t)) - \ell_2(d\zeta_2(t))\Big\}dt \\ & + \int_{t_2-T_2}^{t_2} \Big\{\ell_2(d\gamma_2(t)) - \ell_2(d\zeta_2(t))\Big\}dt - 2(T_1+T_2)\epsilon \\ \geq & \frac{6}{5}C_A^+ - 2(T_+(a_0^*) + T_-(a_0^*) + 2\tau)\epsilon \\ & \qquad\qquad\qquad \text{(by (3.6), (3.8) and (3.12))} \\ > & 0 \end{aligned}$$

if $\epsilon \leq \frac{1}{2}C_A^+(T_+(a_0^*) + T_-(a_0^*) + 2\tau)^{-1}$, but this contradicts the minimality of γ.

Finally, let us consider the case when $c \in \partial I$ and there is an invariant circle containing Γ. In this case, we obtain from the lemma 3.3 that $\mu_{\tilde\Gamma}$ is the only minimal measure still. According to the upper semi-continuity of the set-valued function $c \to \tilde{\mathcal{N}}(c)$ that $\tilde{\mathcal{N}}(c')$ should be in a small neighborhood of $\tilde{\mathcal{N}}(c)$ if c' is close to c. It implies that $\tilde{\mathcal{N}}(c)$ should contain some orbits outside of $\tilde\Sigma$. If this is not true, $\tilde{\mathcal{N}}(c')$ would be in a small neighborhood of $\tilde\Sigma$ for some $c' = (c_1, c_2')$ with $c_2 < a(c_1)$ or $c_2 > b(c_1)$. As we have normally hyperbolic structure in the neighborhood of $\tilde\Sigma$, any invariant set should be on $\tilde\Sigma$, consequently, we would have $\tilde{\mathcal{M}}(c') = \tilde\Gamma$ as the map induced by the Euler-Lagrange flow on this manifold corresponds to a twist area-preserving map on Σ. But this contradicts to the definition of $I(c_1)$.

At the beginning of the proof we have shown that any c-minimal orbits must be semi-asymptotic to the support of the minimal measure if it is uniquely ergodic. What remain to show is such orbit is homoclinic to the invariant circle in this case. As Γ is contained in an invariant circle, denoted by Γ^*, the Aubry set contains a codimension 1 torus $\tilde\Gamma^* = \cup_{t\in[0,1]}(\phi^t(\mathcal{L}(\Gamma^*)), t)$, because $P_\omega(q) = B_c(q)$ for all $q \in \pi(\Gamma^*)$ when $\omega = \partial_1\alpha(c)$ is irrational, and because the necessary and sufficient condition for the existence of invariant circle is the Peierl's barrier function is identically equal to zero. Due to the Lipschitz property of the Aubry set, any c-minimal curve can not cross $\pi(\tilde\Gamma^*)$, so

$$\int_{-k}^{k} \dot\gamma_2(t)dt \leq 2\pi + \mathcal{O}(\|P\|), \qquad \forall\ k \in \mathbb{Z}^+.$$

As $d\gamma$ is semi-asymptotic to $\tilde\Gamma$, $d\gamma$ enters the small neighborhood of $\tilde\Sigma$. If $d\gamma$ does not fall either on the stable manifold or on the unstable manifold, then it will go outside of the neighborhood again. It implies that $d\gamma$ is a multi-bump solution of the Lagrange equation. As we did in the proof of the lemma 3.1, we can construct a curve ζ by cutting off all other bumps and leave only one bump. In this case the c-action of ζ is smaller than that of γ, but this is absurd. Thus, $d\gamma(t) \in W^s_{loc}(\tilde\Gamma^*) \cup W^u_{loc}(\tilde\Gamma^*)\backslash\{\tilde\Gamma^*\}$ when $d\gamma(t)$ is in a small neighborhood of $\tilde\Sigma$. $\square$

We can see from this lemma that only the prime intersection point of the stable and unstable manifolds of $\mathcal{L}(\Gamma)$ can be the candidates for $\tilde{\mathcal{N}}(c)$ when there is an invariant circle containing Γ. In general, the time-1-section $\mathcal{N}(c)|_{t=0}\backslash\pi(\mathcal{L}(\Gamma))$ is homotopically trivial. By definition we mean that there exists an open neighborhood $U = \cup_{i=1}^m U_i$ of $\mathcal{N}(c)|_{t=0}$ such that $U_i \cap U_j = \varnothing$ if $i \neq j$, U_0 is an open neighborhood of $\mathcal{L}(\Gamma)$ and each U_i $(i \neq 0)$ is contractible to one point. In this case we have

$$i_*H_1(U,\mathbb{R}) \subset \operatorname{span}([\partial q_1]),$$

where i is the standard inclusion map. By the Lipschitz property of $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$ in this case, we may choose bounded, mutually disjoint open sets $\tilde{U}_i$ in TM such that $\pi\tilde{U}_i = U_i$ and $\cup\tilde{U}_i \supset \tilde{\mathcal{N}}(c)$. Under this assumption we have

Lemma 3.5. *Assume $c = (c_1, b(c_1))$, $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$ and $\mathcal{N}(c)|_{t=0}\backslash\pi(\tilde{\Gamma})$ is homotopically trivial. Let $c' = (c_1, c_2')$ with $c_2' - b(c_1) > 0$ being sufficiently small. If $\tilde{\mathcal{M}}(c')$ is uniquely ergodic, then there exists a neighborhood $N_{c'}$ of $\mathcal{N}_0(c')$ such that $i_*H(N_{c'},\mathbb{R}) = 0$.*

Proof. By assumption, we can choose $\tilde{U} = \cup_{i=0}^m\tilde{U}_i$, a neighborhood of $\tilde{\mathcal{N}}(c)$ such that $\pi(\tilde{U}_i) \cap \pi(\tilde{U}_j) = \varnothing$ if $i \neq j$, $\tilde{U}_0$ is an open neighborhood of $\mathcal{L}(\Gamma)$ and each U_i $(i \neq 0)$ is contractible to one point. By the upper-semi continuity of $c \to \tilde{\mathcal{N}}(c)$, $\tilde{\mathcal{N}}(c') \subset \tilde{U}$ if $c_2' - b(c_1)$ sufficiently small. We claim that for each $z \in \tilde{U}_0 \cap \tilde{\mathcal{N}}(c')$, $\exists$ an integer $k(z) \in \mathbb{Z}_+$ such that $\phi^{k(z)}(z) \notin \tilde{U}_0$ and there is a uniform upper bound $K \in \mathbb{Z}_+$ for all these $k(z)$. If this is not true, for any $k > 0$, $k \in \mathbb{Z}$ there is $z_k \in \tilde{U}_0$ such that $\phi^l(z_k) \in \tilde{U}_0$, $\forall 0 \leq l \leq k$. Let ν_k be a probability measure distributed evenly on $\phi^t(z)$ $(0 \leq t \leq k)$ and let $k \to \infty$, we find there is an accumulation point ν, $\operatorname{supp}(\nu) \subset \tilde{U}_0$. Obviously, $\nu \in \tilde{\mathcal{M}}(c)$. As there is normally hyperbolic structure on $\tilde{\Sigma}$, the invariant set in $\tilde{U}_0$ must be on $\tilde{\Sigma}$, it follows that $\tilde{\mathcal{M}}(c) \subset \tilde{\Sigma}$, but it contradicts the definition of $I(c_1)$.

By the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ and the assumption on the intersection of the stable and unstable manifolds we see that $\tilde{\mathcal{N}}(c')\backslash\tilde{U}_0$ can be covered by finite mutually disjoint open sets, each of them is homotopic to a point. As each point in $\tilde{U}_0$ shall go outside under the time-1-map ϕ^1, the whole $\tilde{\mathcal{N}}(c')$ can be covered by finite mutually disjoint, homotopically trivial open sets. Because $\tilde{\mathcal{M}}(c')$ is assumed uniquely ergodic we obtain from the lemma 2.5 that $\tilde{\mathcal{N}}(c') = \tilde{\mathcal{A}}(c')$. The Lipschitz property of $\mathcal{A}(c')$ guarantees that $\mathcal{N}(c') = \mathcal{A}(c')$ is also homotopically trivial. $\square$

4, Some Barrier functions

In this section we consider that co-homology class $c = (c_1, b(c_1))$ such that $\tilde{\mathcal{A}}(c)$ contains a 2-torus in $\mathbb{T}^2 \times \mathbb{T}$, i.e. its time-1-sections has an invariant circle on the cylinder, and study the relevant barrier functions introduced in [Ma4]. The study for $c = (c_1, a(c_1))$ is the same. As we have assumed, the rotation number of this circle is irrational. To go further for our proof, let us turn back to the Hamiltonian formalism temporarily to look at something.

Let $\Phi_H=\Phi_H^1$ be the time-1-map of the Hamiltonian flow Φ_H^t, it has invariant cylinder Σ. Restricted to the cylinder Σ this map is clearly twist and area-preserving, thus the invariant circle Γ is at least of Lipschitz. When $P=0$, we have the cylinder $\mathbb{T}\times\mathbb{R}\times\{q_2=p_2=0\}$ as the normally hyperbolic manifold for Φ_{f+g}, each orbit on this manifold makes up an invariant circle and has zero Lyapunov exponent only. Both stable and unstable manifold have two branches, each of them has invariant fibration $\{q_1=p_1=0,p_2=\tilde G^{\pm}(q_2)\}$ if we use $\{q_2,\tilde G^{\pm}(q_2)\}$ to denote the homoclinic loops of Φ_g in the space of (q_2,p_2). Under the small perturbation, the invariant circle on Σ is the graph of small functions, i.e. $\Gamma=\{q_1\in\mathbb{T},p=p_\Gamma(q_1),q_2=q_{2\Gamma}(q_1)\}$. From the theory of normally hyperbolic manifolds we know that the fibration has C^{r-2}-smoothness on the base points. As Γ is an invariant circle, all stable (unstable) fibers with base points on Γ constitute the local stable (unstable) manifold $W_H^{s,u}(\Gamma)$ of Γ. Both the stable and the unstable manifold have two branches corresponding to $(c_1,b(c_1))$ and $(c_1,a(c_1))$ respectively. Let us consider the branch corresponding to $(c_1,b(c_1))$. In the covering space $T(\mathbb{T}\times\mathbb{R})$, one lift of unstable manifold originates from $\{p=p_\Gamma(q_1),q_2=q_{2\Gamma}(q_1)\}$ and extends to right, one lift of stable manifold originates from $\{p=p_\Gamma(q_1),q_2=q_{2\Gamma}(q_1)+2\pi\}$ and extends to left. When $P=0$, these two manifolds totally coincide with each other and keep horizontal when $0\le q_2\le 2\pi$. Thus, for suitably small $a>0$, there exists $\epsilon>0$ such that if $\|P\|\le\epsilon$ the unstable manifold keeps horizontal in the region $\{q_{2\Gamma}(q_1)\le q_2\le 2\pi-a\}$ and the stable manifold keeps horizontal in the region $\{a\le q_2\le q_{2\Gamma}(q_1)+2\pi\}$, i.e. they are the graphs of some functions in the relevant regions,

$$\begin{aligned}W^u(\Gamma)=&\{q,p^u(q):q_1\in\mathbb{T},\ q_{2\Gamma}(q_1)\le q_2\le 2\pi-a\},\\ W^s(\Gamma)=&\{q,p^s(q):q_1\in\mathbb{T},\ a\le q_2\le q_{2\Gamma}(q_1)+2\pi\}.\end{aligned}\tag{4.1}$$

Although each stable (unstable) fiber has C^{r-2}-smoothness, the base points of these fibers fall on a circle for which we can only assume the Lipschitz smoothness, these manifolds are at least of Lipschitz, i.e. $p^{s,u}(q)$ in (4.1) are at least of Lipschitz. We choose suitably small $a>0$ such that the time for any $d\gamma_2$ to cross the strip $\{a\le q_2\le 2\pi-a\}$ is longer than 1. Such assumption is feasible as Φ_H^t is a small perturbation of Φ_{f+g}^t for which this assumption is clearly true.

If there is another invariant circle Γ_1 very close to Γ, by the smoothness of the invariant fibration we see $W_H^{s,u}(\Gamma_1)$ can also keep horizontal in the relevant region. Let $\Gamma(A)$ be the highest circle on Σ where $p_1\le A$, let $\Gamma(B)$ be the lowest circle where $p_1\ge B$. As all invariant circles on Σ make up a closed set, it is reasonable to assert that we have some $\epsilon>0$ such that if $\|P\|\le\epsilon$, the stable and unstable manifolds of all Γ between $\Gamma(A)$ and $\Gamma(B)$ can keep horizontal in the region $\{a\le q_2\le q_{2\Gamma}(q_1)+2\pi\}$ and $\{q_{2\Gamma}(q_1)\le q_2\le 2\pi-a\}$ respectively.

As the Hamiltonian system under study has standard symplectic structure, each horizontal Lagrangian sub-manifold is a graph of some closed 1-form defined on M. We know that the stable (unstable) manifold of some smooth isotropic manifold is a Lagrangian manifold, therefore, if we use $(q,p(q))$ $(p(q)\in C^1)$ to denote such a

smooth manifold, then
$$\frac{\partial p_1}{\partial q_2} = \frac{\partial p_2}{\partial q_1}, \tag{4.2}$$
it follows that there exists a C^2-function $S(q)$ and constant vector $c \in \mathbb{R}^2$ such that
$$\frac{\partial S}{\partial q_1} + c_1 = p_1, \qquad \frac{\partial S}{\partial q_2} + c_2 = p_2. \tag{4.3}$$

If we consider the manifold as the graph of some closed 1-form, $c \in H^1(M, \mathbb{R})$ is the cohomology class of this closed 1-form. Since Lipschitz function is differentiable almost everywhere, we claim that there exists a $C^{1,1}$-function S so that (4.3) holds, here we use $C^{k,\alpha}$ to denote those functions whose k-th derivative is of α-Hölder.

Lemma 4.1. *Let Γ be an invariant circle on the cylinder Σ, let $W^{s,u}(\Gamma)$ be its stable (unstable) manifold, which is horizontal over a connected open set $U \subset M$ with $\pi(\Gamma) \in U$, then there exists $C^{1,1}$ functions $S^{s,u}$: $U \to \mathbb{R}$ and a constant vector $c \in \mathbb{R}^2$ such that $\{W_H^{s,u} : q \in U\} = \{(q, dS^{s,u}(q)) + c : q \in U\}$.*

Proof: Let us consider the case for stable manifold. By the condition there is a Lipschitz function $p = (p_1, p_2)$: $U \to R^2$ such that $W^s(\Gamma) = \{(q, p^s(q)) : q \in U\}$. Let γ be a closed curve which is the boundary of some topological disk σ on W^s. Since γ is on the stable manifold, $\Phi_H^k(\gamma)$ approaches uniformly to Γ, it implies that $\Phi_H^k(\gamma)$ is auch a closed curve going from a point to another point and returning back along almost the same path when k is sufficiently large. As Φ_H is a symplectic diffeomorphism we have
$$\iint_\sigma dp \wedge dq = \oint_\gamma pdq = \oint_{\Phi_H^k(\gamma)} pdq = 0. \tag{4.4}$$

Note p is of Lipschitz, by the theorem of Rademacher ([R]), p is differentiable almost everywhere in U. As γ is arbitrarily chosen, (4.2) holds for almost all $q \in U$. Consequently, there exists a $C^{1,1}$-function S^s and $c \in \mathbb{R}^2$ such that $p^s = dS^s + c$. In the same way, we obtain a $C^{1,1}$-function S^u and $c' \in \mathbb{R}^2$ such that $p^u = dS^u + c'$. As W_H^s intersects W_H^u at the whole Γ, $c' = c$. $\square$

In fact, for almost all initial point $(q, p^s(q))$, p is differentiable at all $\Phi_H^k(q, p^s(q))$ $(\forall k \in \mathbb{Z}^+)$. To see that, let O be an open set in U, for each $k \in \mathbb{Z}^+$, there is a full Lebesgue measure set $O_k \subset \pi(\Phi_H^k\{O, p(O)\})$ where p is differentiable, since Φ is a diffeomorphism, the set
$$O^* = \bigcap_{k=0}^{\infty} \pi\Big(\Phi_H^{-k}\{O_k, p^s(O_k)\}\Big)$$
is a full Lebesgue measure subset of O. For any point $q \in O^*$, p is differentiable at the points $\pi(\Phi_H^k(q, p^s(q)))$ for all $k \in \mathbb{Z}^+$.

Let us consider the Hamiltonian flow. If the locally horizontal stable (unstable) manifold has the form

$$W_H^{s,u} = \{(q, p^{s,u}(q,t), t) : (q,t) \in U \times \mathbb{T}\}$$

and if we call the 2-form $\Omega = \sum dp_i \wedge dq_i - dH \wedge dt$, then $(p^{s,u}, t)^*\Omega = 0$. In the covering space $\mathbb{R}^2 \times \mathbb{R}$ we find that there exists $\bar{S}^{s,u}(q,t)$ such that $d\bar{S}^{s,u} = p^{s,u}(q,t)dq - H(p^{s,u}(q,t), q, t)dt$. By applying the standard argument (see for instance the appendix 2 in [Ma3]), we find that

$$L^{s,u} = L - \langle \partial_q \bar{S}^{s,u}, \dot{q} \rangle - \partial_t \bar{S}^{s,u} \tag{4.5}$$

attains its minimum at $\partial_q \bar{S}^{s,u}$ as the function of $\dot{q}$. Note $L_{\dot{q}}^{s,u} = L_{\dot{q}} - \partial_q \bar{S}^{s,u}$ is of Lipschitz, $dL_{\dot{q}}^{s,u}/dt$ and $L_q^{s,u}$ exist almost everywhere. Since $W^{s,u}$ is a manifold consisting of the trajectories of the Euler-Lagrange flow, it follows from the Euler-Lagrange equation $dL_{\dot{q}}/dt = L_q$ and (4.2) that $L_q^{s,u} = 0$ almost everywhere. The absolute continuity of L implies that $L^{s,u}$ is a function of t alone. Therefore, by adding some function of t to $\bar{S}^{s,u}$, we can make $L^{s,u} = 0$. Note the local stable (unstable) manifold can be thought as the graph of some function defined on $\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : a \le q_2 \le q_{2\Gamma}(q_1,t) + 2\pi\}$ ($\{\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : q_{2\Gamma}(q_1,t) \le q_2 \le 2\pi - a\}$), where $q_{2\Gamma}(q_1,t)$ is such a function that $\pi(\tilde{\Gamma}) = \{(q,t) : q_2 = q_{2\Gamma}(q_1,t)\}$, $q_{2\Gamma}(q_1) = q_{2\Gamma}(q_1, 0)$. The first co-homology group is $\mathbb{R} \times \{0\} \times \mathbb{R}$. Thus, there exists a function $S^u(q,t)$: $\{\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : q_{2\Gamma}(q_1,t) \le q_2 \le 2\pi - a\} \to \mathbb{R}$, $S^s(q,t)$: $\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : a \le q_2 \le q_{2\Gamma}(q_1,t) + 2\pi\} \to \mathbb{R}$ and $(c_1^*, 0, \alpha^*)$ such that $\bar{S}^{s,u}(q,t) = S^{s,u}(q,t) + c_1^* q_1 + \alpha^* t$, where we have used the fact that both the stable and the unstable manifolds coincide with each other at $\tilde{\Gamma}$. In this case we obtain from (4.5) that

$$L^{s,u} = L - \langle (c_1^*, 0), \dot{q} \rangle - \langle \partial_q S^{s,u}, \dot{q} \rangle - \partial_t S^{s,u}$$

attains its minimum at $W^{s,u}$ as the function of $\dot{q}$ with $L^{s,u}|_{W^{s,u}} = \alpha^*$. Thus, for all $d\gamma$ on $\tilde{\Gamma}$ we have

$$\int_{-\infty}^{\infty} \Big(L(d\gamma(t), t) - \langle (c_1^*, 0), \dot{\gamma} \rangle - \langle \partial_q S^{s,u}(\gamma(t), t), \dot{\gamma} \rangle - \partial_t S^{s,u}(\gamma(t), t) - \alpha^* \Big) dt = 0. \tag{4.6}$$

We have mentioned before that the Euler-Lagrange equation for $L - \eta_c$ is the same as that for L if η_c is a closed 1-form. In local coordinates we can write $\eta_c = \langle c(q), \dot{q} \rangle$. If we use $H_{\eta_c}(p,q,t)$ to denote the Legendre transformation

$$H_{\eta_c}(p,q,t) = \max_p \Big\{ \langle p, \dot{q} \rangle - \Big(L - \langle c(q), \dot{q} \rangle \Big) \Big\}$$

then we obtain

$$p + c(q) = \frac{\partial L}{\partial \dot{q}}.$$

It implies that $H_{\eta_c}(p,q,t) = H(p + c(q), q, t)$. As η_c is closed, the coordinate translation $(p,q) \to (p + c(q), q)$ is symplectic. Under such a coordinate translation the horizontal stable (unstable) manifold is the graph of the function $p^{s,u}(q) - c(q)$.

We know that $\tilde{\Gamma}$ is contained in some Aubry set $\mathcal{A}(c) = \{B_c = 0\}$ where $c = (c_1, c_2')$ with $a(c_1) \leq c_2' \leq b(c_1)$. From above arguments and the proposition 3.2 we can see that $c_1 = c_1^*$ and $\alpha^* = \alpha(c)$.

To study the barrier function B_c^*, we consider the covering of $\mathbb{T}^2$ given by $\mathbb{T} \times \mathbb{R}$, let $\tilde{\Gamma}_k$ be the lift of $\tilde{\Gamma}$ which is close to $\mathbb{T} \times \{2k\pi\} \times \{p_1 = \text{const.}, p_2 = 0\} \times \mathbb{T}$. Without lose of generality we single out one lift of the unstable manifold W^u that extends from $\tilde{\Gamma}_0$ and keep horizontal over $\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : q_{2\Gamma}(q_1, t) \leq q_2 \leq 2\pi - a\}$ and single out one lift of the stable manifold W^s that extends from $\tilde{\Gamma}_1$ and keep horizontal over $\{(q,t) \in \mathbb{T}^2 \times \mathbb{T} : a \leq q_2 \leq q_{2\Gamma}(q_1, t) + 2\pi\}$.

Recall $c = (c_1, b(c_1))$. Since $L^{s,u}$ attains its minimum on the local horizontal stable (unstable) manifold, for $q \in \mathbb{T} \times (a, 2\pi - a)$ there exists only one c-minimal orbit $d\gamma_c^s \colon \mathbb{R}^+ \to TM$ as well as only one c-minimal orbit $d\gamma_c^u \colon \mathbb{R}^- \to TM$ such that $\gamma^{s,u}(0) = q$. In fact, such an orbit $d\gamma_c^{s,u}$ lies on the local stable (unstable) manifold. Note $L^{s,u} = \alpha(c)$ on $W^{s,u}$, for arbitrary $T > 0$ we have

$$
\begin{aligned}
\int_{-T}^{0} \Big(L(d\gamma_c^u(t), t) - \langle c, \dot{\gamma}_c^u(t) \rangle - \alpha(c) \Big) dt =& S^u(\gamma_c^u(0), 0) - S^u(\gamma_c^u(-T), -T) \\
& - b(c_1)(\bar{\gamma}_{c2}^u(0) - \bar{\gamma}_{c2}^u(-T)), \\
\int_{0}^{T} \Big(L(d\gamma_c^s(t), t) - \langle c, \dot{\gamma}_c^s(t) \rangle - \alpha(c) \Big) dt =& S^s(\gamma_c^s(T), T) - S^s(\gamma_c^s(0), 0) \\
& - b(c_1)(\bar{\gamma}_{c2}^s(T) - \bar{\gamma}_{c2}^s(0)).
\end{aligned} \tag{4.7}
$$

Since Φ is an area-preserving twist map when it is restricted on the cylinder, from the lemma 2.6 and the corollary 2.7 we see that L_c is regular. Therefore, for any $\varepsilon > 0$, $0 \leq s < 1$, $0 \leq t < 1$ and $q', q^* \in M$, there exists $K_0 \in \mathbb{Z}^+$ such that

$$
|h_c^\infty(q', q^*, s, t) - h_c^K(q', q^*, s, t)| \leq \varepsilon, \qquad \forall K_0 \leq K \in \mathbb{Z}.
$$

Since $\mathcal{M}(c)$ is uniquely ergodic in this case, for any $\delta > 0$, $0 \leq t < 1$, $\gamma^s \colon \mathbb{R}^+ \to M$ with $\gamma^s(0) = q \in \mathbb{T} \times (a, 2\pi - a)$ and $q^* \in \mathcal{M}_t(c)$ there exists a sequence of $\{K_i\}_{i=1}^\infty$ ($K_i \in \mathbb{Z}^+$) such that

$$
d(\gamma^s(t + K_i), q^*) \leq \delta.
$$

It is easy to construct an absolutely continuous curve $\zeta \colon [s, K_i + t] \to M$ such that $\zeta(t) = \gamma^s(t)$ as $s \leq t \leq K_i + t - 2$, $d(d\zeta(t), d\gamma^s(t)) \leq \delta$ as $K_i + t - 2 \leq t \leq K_i + t$ and $\zeta(K_i + t) = q^*$. As $\bar{L}^s$ attains its minimum at W^s for each $(q,t) \in U$, it follows from the convexity of L in $\dot{q}$ and (4.7) that

$$
\begin{aligned}
0 \leq & \int_s^{K_i + t} \Big(L_c(d\zeta(t), t)) - \alpha(c) \Big) dt \\
& - S^s(q^* + (0, 2\pi), t) + S^s(q', s) - b(c_1)(q_2^* - q_2) \\
\leq & o(\delta),
\end{aligned}
$$

where $L_c = L - \langle c, \dot q\rangle$. If $\gamma^s_{K_i}$: $[s, K_i + t] \to M$ is the minimizer of $h^{K_i}_c(q, q^*, s, t)$, then

$$
\begin{aligned}
0 \leq & \int_s^{K_i+t} \Big(L_c(d\gamma^s_{K_i}(t), t)) - \alpha(c)\Big) dt \\
& - S^s(q^* + (0, 2\pi), t) + S^s(q, s) + b(c_1) \int_s^{K_i+t} \dot\gamma^s_{K_i 2}(t) dt \\
\leq & \int_s^{K_i+t} \Big(L_c(d\zeta(t), t)) - \alpha(c)\Big) dt \\
& - S^s(q^* + (0, 2\pi), t) + S^s(q, s) + b(c_1)(q^*_2 - q_2) \\
\leq & o(\delta).
\end{aligned}
$$

It is easy to see that $d\gamma^s_{K_i}(t)$ keeps close to the branch of the stable manifold which corresponds to the cohomology class $c = (c_1, b(c_1))$ if K_i is sufficiently large. Thus, we have

$$
\int_s^{K_i+t} \dot\gamma^s_{K_i 2}(t) dt = q^*_2 + 2\pi - q_2.
$$

Therefore, we assert that for all $q \in \mathbb{T} \times (a, 2\pi - a)$, $q^* \in \pi(\tilde\Gamma_t(c))$ and $s, t \in \mathbb{T}$

$$
\begin{aligned}
& h^\infty_c(q, q^*, s, t) = S^s(q^* + (0, 2\pi), t) - S^s(q, s) - b(c_1)(q^*_2 + 2\pi - q_2), \\
& h^\infty_c(q^*, q, s, t) = S^u(q, s) - S^u(q^*, t) - b(c_1)(q_2 - q^*_2).
\end{aligned} \tag{4.8}
$$

In fact, we have seen that (4.8) holds for $q^* \in \mathcal{M}_t(c)$, $q \in \mathbb{T} \times (a, 2\pi - a)$ or $q \in \pi(\tilde\Gamma|_s)$. As there exists an invariant circle on which the rotation number is irrational, we see that $B_c(q) = P_\omega(q) \equiv 0$ for all $q \in \pi(\Gamma)$, thus $d_c(\hat q, q^*) = 0$ for all $q^* \in \mathcal{M}(c)$ and $\hat q \in \pi(\Gamma)$, where $\omega = \partial_1\alpha(c)$, P_ω is the Peierl's barrier function. Consequently, we have $h^\infty_c(q, \hat q) = h^\infty_c(q, q^*) + h^\infty_c(q^*, \hat q)$. Therefore we obtain (4.8) for any $q \in \mathbb{T} \times (a, 2\pi - a)$ and any $q^* \in \pi(\tilde\Gamma_t)$. As $dS^s|_{\pi(\Gamma)} = dS^u|_{\pi(\Gamma)}$, by adding a constant we can assume that $S^s(q + (0, 2\pi), t) = S^u(q, t)$ if $(q, t) \in \pi(\tilde\Gamma)$. Since the c-minimal measure is uniquely ergodic, by the lemma 3.6, we have the following

Lemma 4.2. *Let $q \in \mathbb{T} \times (a, 2\pi - a)$, then*

$$
B^*_c(q) = S^u(q, 0) - S^s(q, 0) - 2\pi b(c_1). \tag{4.9}
$$

Proof: Since $\tilde{\mathcal{M}}(c)$ is uniquely ergodic, by definition of B^*_c, the property $S^s(q + (0, 2\pi), t) = S^u(q, t)$ if $(q, t) \in \pi(\tilde\Gamma)$ and (4.8) we have

$$
\begin{aligned}
B^*_c(q) = & \min_{\xi,\eta} \Big\{ h^\infty_c(\xi, q) + h^\infty_c(q, \eta) - h^\infty_c(\xi, \eta) : \xi, \eta \in \mathcal{M}(c) \Big\} \\
= & \min_{\xi} \Big\{ h^\infty_c(\xi, q) + h^\infty_c(q, \xi) : \xi \in \mathcal{M}(c) \Big\} \\
= & S^u(q, 0) - S^s(q, 0) - 2\pi b(c_1). \qquad \square
\end{aligned}
$$

Next, we consider the stable (unstable) manifold of all invariant circles. Different invariant circle determines different stable and unstable manifold, so we have a family of these manifolds. We claim that this family of stable (unstable) manifolds can be parameterized by some parameter σ so that both $p_1^{s,u}$ and $p_2^{s,u}$ have $\frac{1}{2}$-Hölder continuity in σ. Indeed we arbitrarily choose one circle Γ_0 and parameterize another circle Γ_σ by the algebraic area between Γ_σ and Γ_0,

$$\sigma = \int_0^1 (\Gamma_\sigma(q_1) - \Gamma_0(q_1))dq_1. \tag{4.10}$$

This integration is in the sense that we pull it back to the standard cylinder by $\psi \circ \psi_1 \in \text{diff}(\Sigma_0, \Sigma)$ (cf. (3.1)). In this way we obtain an one-parameter family curves Γ: $\mathbb{T} \times \mathbb{S} \to \Sigma$ in which $\mathbb{S} \subset [A', B']$ is a closed set. Usually, $\mathbb{S}$ is a Cantor with positive Lebesgue measure, A' and B' correspond to the curves where the action $p_1 \leq A$ and $p_1 \geq B$ respectively. Clearly, for each $\sigma \in \mathbb{S}$, there is only one $c_1 = c_1(\sigma)$ such that $\Gamma_\sigma = \tilde{\mathcal{M}}(c)|_{t=0}$ for all $c \in I(c_1(\sigma))$ as the rotation number is irrational. We can think Γ_σ as a map to function space C^0 equipped with supremum norm Γ: $\mathbb{S} \to C^0(\mathbb{T}, \mathbb{R})$,

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| = \max_{q_1 \in \mathbb{T}} |\Gamma(q_1, \sigma_1) - \Gamma(q_1, \sigma_2)|.$$

Direct calculation shows

$$|\sigma_1 - \sigma_2| \geq \frac{1}{2C_h} \left(\max_{q_1 \in \mathbb{T}} |\Gamma(q_1, \sigma_1) - \Gamma(q_1, \sigma_2)| \right)^2,$$

where C_h is the Lipschitz constant for the twist map, it follows that

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| \leq C_s |\sigma_1 - \sigma_2|^{\frac{1}{2}} \tag{4.11}$$

where $C_s = \sqrt{2C_h}$. Since the stable (unstable) fibers have C^{r-2}-smoothness on their base points on Σ, $p_\sigma^{s,u}$ is also $\frac{1}{2}$-Hölder continuous in σ. Thus, there exist two families of $C^{1,1}$ functions $S_\sigma^u(q,t)$: $\{(q,t) : q_{2\Gamma_\sigma}(q_1,t) \leq q_2 \leq 2\pi - a\} \to M$ and $S_\sigma^s(q,t)$: $\{(q,t) : a \leq q_2 \leq q_{2\Gamma_\sigma}(q_1,t) + 2\pi\} \to M$, which are also $\frac{1}{2}$-Hölder continuous in σ. Remember for each $\sigma \in \mathbb{S}$, $B^*_{c(\sigma)}(q)$ can always take the value zero as its minimum in the region $\{a \leq q_2 \leq 2\pi - a\}$, it follows from the $\frac{1}{2}$-Hölder continuity of $S^{s,u}_{c(\sigma)}$ and the expression of $B^*_{c(\sigma)}$ given by (4.9) that $b(c_1(\sigma))$ also has $\frac{1}{2}$-Hölder continuity in σ. For $z \in \mathbb{T}$, there is unique $z_\sigma(t) \in \pi(\tilde{\Gamma}_{\sigma t})$ such that $z_\sigma(t) = (z, q_{2\Gamma_\sigma}(z,t))$. Let $c(\sigma) = (c_1(\sigma), b(c_1(\sigma)))$, we have:

Lemma 4.3. *For all $q \in \mathbb{T} \times (a, 2\pi - a)$, $z \in \mathbb{T}$ and $s, t \in \mathbb{T}$ the functions $S_\sigma^{s,u}(q)$, $h^\infty_{c(\sigma)}(q, z_\sigma(t), s, t)$, $h^\infty_{c(\sigma)}(z_\sigma(t), q, s, t)$ and $B^*_{c(\sigma)}(q)$ are $\frac{1}{2}$-Hölder continuous in $\sigma \in \mathbb{S}$.*

Different from B^*_c, h^∞_c depends on the choice of the closed 1-form η_c (cf. [Ma4]). To guarantee the Hölder continuity we choose $\eta_c = \langle c(\sigma), \dot{q} \rangle$ in above lemma.

5, Construction of connection orbits

Throughout this section we shall make the following hypotheses, their verification shall be postponed to the section 6.

(H1): For each $\sigma \in \mathbb{S} \subset [A', B']$, the set $\{B^*_{c(\sigma)} = 0\} \cap \{a \le q_2 \le 2\pi - a\}$ consists of discrete points.

Remark: By the choice of a, the set $\{B^*_{c(\sigma)} = 0\} \cap \{a \le q_2 \le 2\pi - a\}$ is not empty since $d\gamma_2$ can not cross the strip $\{a \le q_2 \le 2\pi - a\}$ under one step of the map ϕ, there must be some points on time-1-section of the minimal orbits whose projection fall into the strip. By the definition of $\mathbb{S}$, for each $\sigma \in \mathbb{S}$, $\tilde{\mathcal{A}}(c(\sigma))|_{t=0}$ contains an invariant circle on the cylinder. In this case we have an explicit expression of $B^*_c(q)$ in the strip. The hypothesis (H1) implies the minimal critical point set of $S^s_{c(\sigma)} - S^u_{c(\sigma)}$ consists of discrete points, and there must be some minimal points in the interior of this strip.

(H2): If the rotation number of Γ is rational, then the associated c-minimal measure consists of single periodic orbit. There is no homoclinic loop in Σ to this orbit.

Before making the third hypothesis let us note that all invariant circles on the cylinder form a closed set. These circles do not intersect each other, so the complement set consists of countably many invariant annulus.

(H3): Let Γ be an invariant circle on Σ, associated with co-homology class c. If this circle is on the boundary of a gap, then for sufficiently small $\delta > 0$ there exists $c' = (c_1, c_2')$ with either $0 < c_2' - b(c_1) < \delta$ or $-\delta < c_2' - a(c_1) < 0$ such that $\mathcal{M}(c')$ is uniquely ergodic.

Let $\tilde{\mathcal{N}}_s(c) = \tilde{\mathcal{N}}(c)|_{t=s}$, $\mathcal{N}_s(c) = \mathcal{N}(c)|_{t=s}$. According to the study in the last section we know that $\tilde{\mathcal{N}}_0(c')$ is homotopically trivial, but this does not guarantee that $\mathcal{N}_0(c')$ is also homotopically trivial on M, since the projection from $\tilde{\mathcal{N}}(c') \to \mathcal{N}(c')$ is not necessarily injective. If $\tilde{\mathcal{M}}(c')$ is uniquely ergodic, then $\tilde{\mathcal{N}}(c') = \tilde{\mathcal{A}}(c')$. The Lipschitze property of $\mathcal{A}(c')$ implies that $\mathcal{N}_0(c')$ is homotopically trivial in this case. Given arbitrary small $d > 0$, there are only finite invariant circles which are the boundary of some annulus with width not smaller than d. Actually, we require the third hypothesis only for these tori.

The first task in this section is to build a C-equivalent sequence $\{c^{(i)}\}_{i=1}^m$ where $c_1^{(1)} = c_1(\sigma')$, $a(c_1^{(1)}) \le c_2^{(1)} \le b(c_1^{(1)})$, $c_1^{(m)} = c_1(\sigma^*)$, $a(c_1^{(m)}) \le c_2^{(m)} \le b(c_1^{(m)})$ and $\sigma' < \sigma^*$ correspond to two invariant circles which make up the whole boundary of a gap. Thus a theorem of connecting C-equivalent Mañé sets is used to construct the diffusion orbits crossing this gap. This kind of theorem was discovered by Mather in [Ma4] where the proof was sketched. To make use of this theorem, we shall give a complete proof first. A theorem of connecting different $\mathcal{G}(c)$ was proved by Bernard recently ([Be]).

To any subset A of M we associate a subspace of $H_1(M,\mathbb{R})$

$$V(A)=\bigcap\Big\{i_{U*}H_1(U,\mathbb{R}):U \text{ is an open neighborhood of } A\Big\} \tag{5.1}$$

where i_{U*}: $H_1(U,\mathbb{R})\to H_1(M,\mathbb{R})$ is the map induced by the inclusion. Clearly, there exists an open neighborhood U of A such that $V(A)=i_{U*}H_1(U)$. Let $V^{\perp}(A)$ be the annihilator of $V(A)$. In other words, if $c\in H^1(M,\mathbb{R})$, then $c\in V^{\perp}$ if and only if $\langle c,h\rangle=0$ for all $h\in V(A)$. Given $c\in H^1(M,\mathbb{R})$ we define

$$R(c)=\sum_{t\in\mathbb{T}}(V(\mathcal{N}_t(c)))^{\perp}. \tag{5.2}$$

In [Be] $R(c)$ is defined by using $\mathcal{G}(c)$ instead of using $\mathcal{N}(c)$.

We say a continuous curve Γ: $\mathbb{R}\to H^1(M,\mathbb{R})$ is admissible if for each $t\in\mathbb{R}$ there exists $\delta>0$ such that $\Gamma(t)-\Gamma(t_0)\in R(\Gamma(t_0))$ for all $t\in[t_0-\delta,t_0+\delta]$. We say c, $c'\in H^1(M,\mathbb{R})$ are C-equivalent if there is an admissible curve Γ: $[0,1]\to M$ such that $\Gamma(0)=c$ and $\Gamma(1)=c'$.

Let U be an open subset of $M\times\mathbb{T}$, we can think it as the open subset in $M\times\mathbb{R}$ of points (q,t) such that $(q,t \bmod 1)\in U$. The 1-form μ on $M\times\mathbb{R}$ is called a U-step form if there is a closed form $\bar\mu$ on $M\times\mathbb{T}$, also considered as a periodic 1-form on $M\times\mathbb{R}$, such that the restriction of μ to $t\le 0$ is 0, the restriction of μ to $t\ge 1$ is $\bar\mu$, and such that the restriction of μ to the set $U\cup\{t\le 0\}\cup\{t\ge 1\}$ is closed. In the application in this paper, $\bar\mu$ is chosen as a closed form on M.

If the first de Rham cohomology $d\in R(c)$, then there exist an open neighborhood U of $\mathcal{N}(c)$ and a U-step form μ such that $[\bar\mu]=d$. Such a neighborhood U will be called an adapted neighborhood. Indeed, similar to the arguments in [Be], let us fix a time $t\in[0,1]$ and a cohomology class $d\in V(\mathcal{N}_t(c))^{\perp}$. There exist an open neighborhood Ω of $\mathcal{N}_t(c)$ and a $\delta>0$ such that $V(\Omega)=V(\mathcal{N}_t(c))$ and such that $\mathcal{N}_s(c)\subset\Omega$ for all $s\in[t-\delta,t+\delta]$. As $d\in R(c)$, we can take a closed form $\bar\mu$ on M whose support is disjoint from Ω and such that $[\bar\mu]=d$. Let ρ: $\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\rho=0$ on $(-\infty,t-\delta]$, $\rho=1$ on $[t+\delta,\infty)$ and $0\le\rho\le 1$ for all $t\in\mathbb{R}$ and let $U=M\times((0,t-\delta)\cup(t+\delta,1))\cup\Omega\times[t-\delta,t+\delta]$. Obviously, the form

$$\mu=\rho(t)\bar\mu$$

is an U-step form satisfies the required conditions.

Let Γ: $[0,1]\to H^1(M,\mathbb{R})$ be an admissible curve such that $\Gamma(0)=c$ and $\Gamma(1)=c'$. For each $t\in[0,1]$ and an adapted neighborhood $U(t)$, let $\eta(t)$ be a closed 1-form on M such that $[\eta(t)]=\Gamma(t)$, there exists $\delta(t)>0$ such that $\Gamma(s)-\Gamma(t)\in R(\Gamma(t))$ and a U-step form $\mu(s)$ with $[\bar\mu(s)]=\Gamma(s)-\Gamma(t)$ if $s\in(t-\delta,t+\delta)$. According to the upper semi-continuity $(\eta,\mu)\to\tilde{\mathcal{N}}_{\eta,\mu}$ proved in the lemma 2.4, we can assume that

$$\pi(\tilde{\mathcal{N}}_{\eta(t),\mu(s)})+\epsilon(t)\subset U(t) \tag{5.3}$$

if we take suitably small $\delta(t)$. In this paper we use $U+a$ to denote the set $\{x\in M: \mathrm{dist}(x,U)\le a\}$. Clearly, there is a finite increasing sequence $\{t_i\}_{0\le i\le N}$ such that

$$\bigcup_{i=0}^{N}(t_i-\delta(t_i),t_i+\delta(t_i))\supset[0,1],$$
$$t_{i-1}>t_i-\delta(t_i),\qquad t_{i+1}<t_i+\delta(t_i). \tag{5.4}$$

and (5.3) holds for each t_i, and each $s\in(t_i-\delta(t_i),t_i+\delta(t_i))$. In the following we shall use ϵ_i, δ_i, U_i, η_i and μ_i to denote $\epsilon(t_i)$, $\delta(t_i)$, $U(t_i)$, $\eta(t_i)$ and $\mu(t_i)$ respectively. Thus we have

$$\eta_i=\eta_0+\sum_{j=0}^{i-1}\bar\mu_j. \tag{5.5}$$

Let us fix some $0\le i\le N$ and consider the function $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)$ defined in (2.13). For each small $\epsilon_i^*>0$ and $(m_0,m_1)\in M\times M$ there exists $(\breve T_0^i,\breve T_1^i)=(\breve T_0^i,\breve T_1^i)(\epsilon_i^*,m_0,m_1)\in\mathbb{Z}^+$ such that

$$h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)\ge h^{\infty}_{\eta_i,\mu_i}(m_0,m_1)-\epsilon_i^*,\qquad \forall T_j\ge\breve T_j^i,\ j=0,1. \tag{5.6}$$

Obviously, there are infinite $T_j\ge\breve T_j^i$ $(j=0,1)$ such that

$$|h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)-h^{\infty}_{\eta_i,\mu_i}(m_0,m_1)|\le\epsilon_i^*. \tag{5.7}$$

Let $\gamma_i(t,m_0,m_1,T_0,T_1)$: $[-T_0,T_1]\to M$ be the minimizer of $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)$, it follows from the lemma 2.3 that if $\epsilon_i^*>0$ is sufficiently small, $\breve T_j^i$ $(j=0,1)$ are sufficiently large, and T_0, T_1 are chosen so that (5.7) holds, then

$$d\gamma_i(t,m_0,m_1,T_0,T_1)\in\tilde{\mathcal N}_{\eta_i,\mu_i}(t)+\epsilon_i,\qquad\forall 0\le t\le 1. \tag{5.8}$$

From the Lipschitze property of $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)$ in (m_0,m_1) and the compactness of M, we see that there are $\breve T_j^i=\breve T_j^i(\epsilon_i)$ $(j=0,1)$, independent of (m_0,m_1), so that (5.6) holds for all $T_j\ge\breve T_j^i$. We can see also that there exist $\hat T_j^i(\epsilon_i)>\breve T_j^i(\epsilon_i)$ $(j=0,1)$ so that for any $(m_0,m_1)\in M\times M$, there are T_j with $\breve T_j^i\le T_j\le\hat T_j^i$ $(j=0,1)$ such that (5.7) and consequently (5.8) holds. Note that for different (m_0,m_1), we may need different $T_j\ge\breve T_j^i$.

We are now on the stage to construct a connecting orbit joining $\mathcal N(c_0)$ and $\mathcal N(c_N)$. We consider τ_i as the time translation $(q,t)\to(q,t+\tau_i)$ on $M\times\mathbb R$, and define the modified Lagrangian

$$\tilde L=L-\eta_0-\sum_{i=0}^{N-1}(-\tau_i)^*\mu_i. \tag{5.9}$$

For each $\vec{\tau} = (\tau_0, \tau_1, \cdots, \tau_{N-1})$ the following variational problem

$$\begin{aligned} h_{\tilde{L}}^{T_0,T_N}(m,m',\vec{\tau}) = & \inf_{\substack{\gamma(-T_0)=m \\ \gamma(T_N+\tau_{N-1})=m'}} \int_{-T_0}^{T_N+\tau_{N-1}} \Big(L - \eta_0 - \sum_{i=0}^{N-1}(-\tau_i)^*\mu_i\Big)(d\gamma(t),t)dt \\ & - \sum_{i=1}^{N-1}(\tau_i - \tau_{i-1})\alpha(c_i) - T_0\alpha(c_0) - T_N\alpha(c_N) \end{aligned}$$

has a C^1-minimizer $\gamma(t,m,m',\vec{\tau},T_0,T_N)$ which is clearly the solution of the Euler-Lagrangian equation determined by $\tilde{L}$. We need to show it can be the extremal of L if we suitably choose $\vec{\tau}$, T_0 and T_N. We define

$$\begin{aligned} \Lambda = \Big\{ \vec{\tau} \in \mathbb{Z}^N : & \max\{\breve{T}_0^i, \breve{T}_1^{i-1}+1\} \le \tau_i - \tau_{i-1} \le \max\{\hat{T}_0^i, \hat{T}_1^{i-1}+1\}, \\ & \forall 1 \le i \le N-1, \tau_0 = 0 \Big\}, \end{aligned}$$

and take the minimum of $h_{\tilde{L}}^{T_0,T_N}(m,m',\vec{\tau})$ over Λ

$$F_{\tilde{L}}(m,m',T_0,T_N) = \min_{\vec{\tau}\in\Lambda} h_{\tilde{L}}^{T_0,T_N}(m,m',\vec{\tau}). \tag{5.10}$$

we call $\vec{\tau}^*(T_0,T_N)$ the value where the minimum is reached. If $\gamma(t,m,m',T_0,T_N)$ is the minimizer of $F_{\tilde{L}}(m,m',T_0,T_N)$, we claim that for $t \in [\tau_i, \tau_i+1]$ and $0 < i < N-1$

$$d\gamma(t,m,m',T_0,T_N) \in (-\tau_i)^*\Big(\tilde{\mathcal{N}}_{\eta_{i-1},\mu_i}|_t\Big) + \epsilon_i. \tag{5.11}$$

In fact, let us to choose $m_i = \gamma(\tau_{i-1}+1)$, $m_i' = \gamma(\tau_{i+1})$ for $0 < i < N-1$. Since $\gamma(t,m,m',T_0,T_N)$ is the minimizer of $F_{\tilde{L}}(m,m',T_0,T_N)$, thus

$$\begin{aligned} A_{\eta_{i-1},\mu_i}((-\tau_i)^*\gamma|_{\tau_{i-1}+1}^{\tau_{i+1}}) = & \inf_{\substack{\gamma^*(-T_0)=m_i \\ \gamma^*(T_1)=m_i' \\ \breve{T}_0^i \le T_0 \le \hat{T}_0^i \\ \breve{T}_1^i \le T_1 \le \hat{T}_1^i}} \int_{-T_0}^{T_1} (L - \eta_i - \mu_i)(d\gamma^*(t),t)dt \\ & - T_0\alpha(c_i) - T_1\alpha(c_{i+1}). \end{aligned} \tag{5.12}$$

So, we obtain (5.11) from (5.6∼8), (5.12) and the choice of $\breve{T}_j^i$ as well as $\hat{T}_j^i$ $(j = 0, 1)$. We define the infimum limit of $F_{\tilde{L}}(m,m',T_0,T_N)$

$$h_{\tilde{L}}^{\infty}(m,m') = \liminf_{T_0,T_1\to\infty} F_{\tilde{L}}(m,m',T_0,T_N). \tag{5.13}$$

Let T_j^k $(j = 0, N)$ be the subsequences such that $T_j^k \to \infty$ as $k \to \infty$

$$|F_{\tilde{L}}(m,m',T_0^k,T_N^k) - h_{\tilde{L}}^{\infty}(m,m')| \le \min\{\epsilon_0^*, \epsilon_N^*\}, \qquad \forall\ k,$$

as well as

$$\lim_{k\to\infty} F_{\tilde L}(m,m',T_0^k,T_N^k)=h_{\tilde L}^{\infty}(m,m'),$$

and let $\gamma_k(t,m,m')=\gamma(t,m,m',T_0^k,T_N^k)$ be the minimizer of $F_{\tilde L}(m,m',T_0^k,T_N^k)$. It is easy to see that (5.11) holds also for $i=0,N$. From (5.3), (5.12) and the definition of U_i we obtain that $d\gamma_k(t)$ is the extremal of L with the boundary condition $\gamma_k(-T_0^k)=m$, $\gamma_k(T_N^k+\tau_{N-1}^*)=m'$. Clearly, for any compact interval $[a,b]$ the set $\{\gamma_k\}_{k\ge\bar k}$ is pre-compact in the $C^1([a,b],M)$ topology if $\bar k$ is suitably large. Let γ: $\mathbb{R}\to M$ be the accumulation point of $\{\gamma_k\}$, then $d\gamma$ is the solution of the Euler-Lagrange equation determined by L and

$$\alpha(d\gamma)\subseteq\tilde{\mathcal A}(c_0),\qquad \omega(d\gamma)\subseteq\tilde{\mathcal A}(c_N).$$

Consider a bi-infinite sequence $(\cdots,c_i,\cdots)$ of C-equivalent cohomology classes and a sequence $(\cdots,\varepsilon_i,\cdots)$ of small positive numbers. Let $\{\tau_i\}_{-\infty}^{\infty}$ be a monotone sequence of integers such $\tau_0=0$, $\tau_i\to\pm\infty$ as $i\to\pm\infty$. Let

$$\nu_N=\sum_{i=-N}^{N}(-\tau_i)^*\mu_i.$$

For each $\vec\tau_N=(\tau_{-N},\cdots,\tau_{N-1})$ we consider the following variational problem

$$\begin{aligned} h_{\tilde L}^{T_{-N},T_N}(m,m',\vec\tau_N)=&\inf_{\substack{\gamma(-T_{-N}-\tau_{-N})=m\\ \gamma(T_N+\tau_{N-1})=m'}}\int_{-T_{-N}-\tau_{-N}}^{T_N+\tau_{N-1}}\Big(L-\eta_0-\nu_N\Big)(d\gamma(t),t)dt\\ &-\sum_{i=-N+1}^{N-1}(\tau_i-\tau_{i-1})\alpha(c_i)-T_{-N}\alpha(c_{-N})-T_N\alpha(c_N)\end{aligned}$$

Let Λ_N be the set of $2N$ dimensional integer vectors defined in the same way as for Λ with the subscripts ranging over $(-N,\cdots,N-1)$ instead of $(0,\cdots,N-1)$. Let $\gamma_N(t,m,m',T_{-N},T_N)$ be the minimizer of

$$F_{\tilde L}(m,m',T_{-N},T_N)=\min_{\vec\tau\in\Lambda_N} h_{\tilde L}^{T_{-N},T_N}(m,m',\vec\tau_N).$$

With the same arguments above, we can make $\gamma_N(t,m,m',T_{-N},T_N)$ be the extremal of L by choosing suitably large T_{-N}, and T_N. From (5.3) and (5.11) we can see that $d\gamma_N$ passes within a distance of ε_i of each $\tilde{\mathcal L}(c_i)$ for $-N\le i\le N$ if we set $\hat T_j^i$ suitably large for each $j=0,1$ and each $-N\le i\le N$. Let γ: $\mathbb{R}\to M$ be an accumulation point of the set $\{\Gamma_N\}_{N\ge N_0}^{\infty}$, $d\gamma$ clearly determines a trajectory of the Euler-Lagrange flow of L which passes within a distance of ε_i of each $\tilde{\mathcal A}(c_i)$ for all $i\in\mathbb Z$. Therefore we have proved the theorem

Theorem 5.1. (Mather 1993) *Suppose c_0 and c_N are C-equivalent classes. There there is a trajectory of the Euler-Lagrange flow of L whose α-limit set lies in $\tilde{\mathcal A}(c_0)$ and whose ω-limit set lies in $\tilde{\mathcal A}(c_N)$.*

Consider a bi-infinite sequence $(\cdots, c_i, \cdots)$ of C-equivalent cohomology classes and a sequence $(\cdots, \varepsilon_i, \cdots)$ of small positive numbers. Then there is a trajectory of the Euler-Lagrange flow of L which passes within a distances of ε_i of each $\tilde{\mathcal{A}}(c_i)$ in turn.

The next step is to establish C-equivalence among some Mañé sets of the special L given by (2.2). Let us consider the first de Rham cohomology class $c \in H^1(M, \mathbb{R})$ such that the support of c-minimal measure uniquely sits on $\tilde{\Gamma} \subset \tilde{\Sigma}$. First, we consider the case that Γ is a Denjoy set and there is no invariant circle containing Γ. The rotation number of Γ is irrational. By the well-known knowledge we see that the β-function for the twist map is differentiable at the point of irrational number, it implies that there is only one c_1 such that $\tilde{\Gamma}$ is the support of c-minimal measure if $c \in intI(c_1)$. We see from the lemma 3.4 that $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{M}}(c)$ when $a(c_1) < c_2 < b(c_1)$. By the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ we find that there exists $\delta > 0$, if $c' \in J = ((c_1 - \delta, c_1 + \delta) \times (a(c_1) + \delta, b(c_1) - \delta)$ then $\mathcal{N}(c')$ is in a small neighborhood of $\pi(\tilde{\Gamma})$, thus each of such $\mathcal{N}_0(c)$ is homotopically trivial. Therefore, all $c' \in J$ are C-equivalent.

Next, let us consider the case when Γ consists of single periodic orbit. Since the β-function of the twist map has a corner at the rational rotation number, there is a flat piece of the α-function of the twist map, over the interval $[c_1^-, c_1^+]$. Consequently, there is a rectangle $(c_1^-, c_1^+) \times (a(c_1), b(c_1)) \in H^1(M, \mathbb{R})$ such that all c-minimal measures have their support on $\tilde{\Gamma}$ if c is in this rectangle. When $c_1^- < c_1 < c_1^+$, $a(c_1) < c_2 < b(c_1)$, $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{M}}(c)$. When $a(c_1) < c_2 < b(c_1)$ and $c_1 = c_1^-$ or $c_1 = c_1^+$, $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{M}}(c) \cup \{\text{minimal homoclinic orbit in } \tilde{\Sigma}\}$. Due to the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ and the hypothesis of (H2), we find that there exists $\delta > 0$, if $c' \in J = ((c_1^- - \delta, c_1^+ + \delta) \times (a(c_1) + \delta, b(c_1) - \delta)$ then $\mathcal{N}_0(c)$ is homotopically trivial, thus all $c' \in J$ are C-equivalent.

Finally we consider the case when Γ is contained in an invariant circle on the boundary of a gap. In this case $\tilde{\Gamma}$ is the support of that c-minimal measure with $c \in I(\bar{c}_1) = \{(\bar{c}_1, c_2) : a(\bar{c}_1) \le c_2 \le b(\bar{c}_1)\}$. Because of the hypotheses (H1), (H3) and in virtue of the lemma 3.4∼6, we have $\mathcal{N}_0(c) \subset U = \cup_{i=0}^m U_i$, where $U_i \cap U_j = \varnothing$ if $i \neq j$, U_0 is an open neighborhood of Γ, all other U_i $(i \neq 0)$ are open set contractible to one point. Let $J = (\bar{c}_1 - \delta, \bar{c}_1 + \delta) \times (a(\bar{c}_1) - \delta, b(\bar{c}_1) + \delta)$. Due to the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$, we can see that $\mathcal{N}_0(c) \subset U$ for all $c \in J$ if $\delta > 0$ is sufficiently small. To establish the C-equivalent relationship between any two $c, c' \in J$, let us consider first the special case when $c, c' \in J$ and $c - c' = (0, c_2 - c_2')$. Let $\Gamma(s) = \{(c_1, sc_2 + (1-s))c_2' : 0 \le s \le 1\}$, obviously, $[dq_2]$ is the annihilator of $V_{\Gamma(s)}(t)$ $\forall s \in [0,1]$, $t \in \mathbb{T}$. Thus $\Gamma(s)$ is an admissible curve. Second, let us consider the case when $c = (c_1, c_2) \in J$ but $c_2 > b(\bar{c}_1)$ or $c_2 < a(\bar{c}_1)$. Under the hypotheses (H1) and (H3), for any $\delta > 0$, there exists $c = (\bar{c}_1, c_2)$ with $b(\bar{c}_1) < c_2 < b(\bar{c}_1) + \delta$ or $a(\bar{c}_1) - \delta < c_2 < a(\bar{c}_1)$ such that $\mathcal{N}_t(c)$ is homopotically trivial for any $t \in \mathbb{T}$. Therefore, $\exists$ $\delta' > 0$ such that for all $c' \in B_{\delta'}(c)$, $\mathcal{G}_0(c')$ is homotopically trivial. Replacing δ with δ' in the definition of J, we find that all $c \in J$ are C-equivalent. In fact, given any two c, $c' \in J$, we can construct the admissible curve as follows. Let $\Gamma\colon [0,3] \to H^1(M,\mathbb{R})$,

$$\Gamma(s) = \begin{cases} sc + (1-s)\tilde{c}, & 0 \le s \le 1 \\ (s-1)\tilde{c} + (2-s)\tilde{c}', & 1 \le s \le 2 \\ (s-2)\tilde{c}' + (3-s)c', & 2 \le s \le 3 \end{cases}$$

in which $\tilde{c}$ and $\tilde{c}' \in J$ are defined in the way $\tilde{c}_2 = \tilde{c}'_2 > b(\bar{c}_1)$ or $\tilde{c}_2 = \tilde{c}'_2 < a(\bar{c}_1)$, $\tilde{c}_1 = c_1$ and $\tilde{c}'_1 = c'_1$, both $\mathcal{N}_t(\tilde{c})$ and $\mathcal{N}_t(\tilde{c}')$ are homotopically trivial.

Lemma 5.2. *We assume the hypotheses (H1∼3). Let $\hat{c} = (c_1(\sigma'),0)$ and $\bar{c} = (c_1(\sigma^*),0)$ be two co-homology classes such that $\tilde{\mathcal{N}}(\hat{c})|_{t=0}$ and $\tilde{\mathcal{N}}(\bar{c})|_{t=0}$ make up the whole boundary of some given gap with $\sigma' < \sigma^*$. Then $\hat{c}$ and $\bar{c}$ are C-equivalent.*

Proof: By assumption, there is no other invariant circle between $\mathcal{N}(\hat{c})|_{t=0}$ and $\mathcal{N}(\bar{c})|_{t=0}$. In this case, we have shown that for any $c_1(\sigma') \le c_1 \le c_1(\sigma^*)$ there is an open rectangle $J(c_1) \subset H^1(M,\mathbb{R})$ containing $(c_1,0)$ such that all $c \in J(c_1)$ are C-equivalent. By the compactness of the interval $[c_1(\sigma'), c_1(\sigma^*)]$ there is a sequence $\{c_1^{(i)}\}_{i=0}^m$ such that $\cup_{i=0}^m J(c_1^{(i)}) \supset [c_1(\sigma'), c_1(\sigma^*)] \times \{0\}$. Obviously, the C-equivalence has transitivity. $\square$

This C-equivalence establishes the existence of the diffusion orbits crossing gaps as we have the theorem 5.1.

To go further, we need to know more details of U-step forms. Let η_j be any given closed 1-form such that $[\eta_j] = c^{(j)}$ for $j = 1, k$. A natural question is whether there exists such kind of $\mu(t)$ so that $\mu(t) = \eta_1$ for $t \le 0$ and $\mu(t) = \eta_k$ for $t \ge \tau_k + 1$ even though $c^{(1)}$ is equivalent to $c^{(j)}$? In general, we do not know whether it is true or not, but in our case, the answer is yes.

Lemma 5.3. *Let $c^{(1)} = (c_1^{(1)}, c_2^{(1)})$, $c^{(k)} = (c_1^{(k)}, c_2^{(k)})$ be two cohomology classes connected by an admissible curve Γ, where $a(c_1^{(1)}) \le c_2^{(1)} \le b(c_1^{(1)})$, $a(c_1^{(k)}) \le c_2^{(k)} \le b(c_1^{(k)})$, and $\tilde{\mathcal{M}}(c^{(1)})$, $\tilde{\mathcal{M}}(c^{(2)}) \subset \tilde{\Sigma}$. Let η_1, η_k be two closed one forms such that $[\eta_1] = c^{(1)}$, $[\eta_k] = c^{(k)}$. Then there exists a composition of finite U-step forms $\mu(t)$ such that $\mu(t) = \eta_1$ for $t \le 0$ and $\mu(t) = \eta_k$ for $t \ge \tau_k + 1$.*

Proof. Since Φ is an area-preserving twist map when it is restricted on the cylinder, by the hypothesis (H2), there is some c with $c_1^{(1)} < c_1 < c_1^{(k)}$, $c_2 = 0$ such that its semi-static minimal orbit set consists of single m-periodic orbit with $m > 1$. Thus, for each $s \in \mathbb{T}$, $\mathcal{N}_s$ consists of several points, $\mathcal{N}_s(c) = \cup\{q_i(s)\}$. Consequently, there exist $\delta > 0$, and $0 < s_1 < s_2 < 1$ such that

$$\Big(\cup\{q_i(s_1)\} + 3\delta\Big) \cap \Big(\cup\{q_i(s_2)\} + 3\delta\Big) = \varnothing$$

and there exists $\epsilon > 0$ such that $0 < s_1 - \epsilon < s_1 + \epsilon < s_2 - \epsilon < s_2 + \epsilon < 1$ and $\cup\{q_i(s)\} \subset \cup\{q_i(s_j)\} + \frac{1}{2}\delta$ when $|s - s_j| < \epsilon$ for $j = 1, 2$.

Let η be an any exact 1-form, we claim there exists a U-step form ν such that $\nu(t)=0$ for $t\le 0$ and $\nu(t)=\eta$ for $t\ge 1$, where U is a neighborhood of $\mathcal{N}(c)=\cup_{s\in\mathbb{T}}\cup_i\{q_i(s)\}$. Let F: $M\to\mathbb{R}$ be the function such that $\eta=dF$. Let $\lambda_\delta(q)$: $M\to\mathbb{R}$ be a smooth function $\lambda_\delta=1$ when $\|q\|\le\delta$, $0<\lambda_\delta<1$ when $\delta<\|q\|<2\delta$ and $\lambda_\delta=0$ when $\|q\|\ge 2\delta$. Let

$$F^*=\Big(1-\sum_{i=1}^k\lambda_\delta(q-q_i(s_1))\Big)F,\qquad \tilde F=\Big(\sum_{i=1}^k\lambda_\delta(q-q_i(s_1))\Big)F,$$

obviously, $\mathrm{supp}(dF^*)\cap(\cup\{q_i(s_1)\}+2\delta)=\varnothing$, $\mathrm{supp}(d\tilde F)\cap(\cup\{q_i(s_2)\}+2\delta)=\varnothing$. If we choose

$$\nu=\rho(t-s_1+\epsilon)dF^*+\rho(t-s_2+\epsilon)d\tilde F,$$

where $\rho=0$ for $t\le 0$, $0<\rho<1$ for $0<t<2\epsilon$ and $\rho=1$ for all $t\ge 2\epsilon$, then $\nu(t)=0$ for $t\le 0$ and $\nu(t)=dF$ for $t\ge 1$. Let $U=\cup_{j=1,2}(\cup\{q_i(s_j)\}+2\delta)\times[s_j-\epsilon,s_j+\epsilon]\cup M\times([0,s_1-\epsilon]\cup[s_1+\epsilon,s_2-\epsilon]\cup[s_2+\epsilon,1])$, then $d\nu|_U=0$.

Since both $[\eta_1]$ and $[\eta_2]$ are C-equivalent to c, there are two composition of U-step forms ν_1, ν_2 such that

$$\begin{aligned}
&\eta_1+\nu_1(t)=\langle c,dq\rangle+dF_1, && t\ge\tau_1;\\
&\nu_2(t)=0, && t\le\tau_1+1,\\
&\langle c,dq\rangle+\nu_2(t)=\eta_2+dF_2, && t\ge\tau_2.
\end{aligned}$$

By the demonstration above, there is a U-step form ν such that $\nu(t)=-d(F_1+F_2)$ when $t\ge 1$. Clearly, the 1-form $\mu=(-\tau_1)^*\nu+\nu_1+\nu_2$ is what we are looking for. □

The following work in this section is to join the orbit crossing the gaps smoothly with the orbit constructed via Arnold's mechanism. We shall make use of some ideas developed in [Bs] and in [BCV], it is showed that the diffusion orbits in several examples, constructed by transition chain, are actually the orbits which locally minimize the Lagrange action.

Let us consider the barrier function of those cohomology classes corresponding to an invariant circle Γ_c on the cylinder. In this case, $\mathcal{M}(c)|_{t=0}\subseteq\Gamma_c$ and $d_c(\xi,\xi')=0$ for all $\xi,\xi'\in\pi(\Gamma_c)$. Thus

$$\begin{aligned}
B_c^*(q)=&\min_{\xi,\eta\in\mathcal{M}(c)}\{h_c^\infty(\xi,q)+h_c^\infty(q,\eta)-h_c^\infty(\xi,\eta)\}\\
=&h_c^\infty(\xi,q)+h_c^\infty(q,\xi),\qquad \forall\ \xi\in\pi(\Gamma_c).
\end{aligned}\tag{5.14}$$

Under the hypothesis (H1), the set $\{B^*_{c(\sigma)}=0\}\cap\mathbb{T}\times(a,2\pi-a)$ consists of discrete points for all $\sigma\in\mathbb{S}$. Thus, for any given $\sigma\in\mathbb{S}$ and any $\epsilon>0$, there are finite and mutual disjoint balls $\mathcal{B}_\epsilon(q_i)$ and $\delta=\delta(\sigma,\epsilon)>0$ such that $\cup\mathcal{B}_\epsilon(q_i)\supset\{B^*_{c(\sigma)}=0\}\cap\mathbb{T}\times(a,2\pi-a)$ and

$$\min\{B^*_{c(\sigma)}(q):q\in\partial\mathcal{B}_\epsilon(q_i),\forall i\}\ge 2\delta,\qquad B^*_{c(\sigma)}(q_i)=0.$$

In other words, as a function of q. $B^*_{c(\sigma)}$ reaches its minimum in $\{a \le q_2 \le 2\pi - a\}$ at these discrete points

$$\min_{q\in\partial\mathcal{B}_\epsilon(q_i)} B^*_{c(\sigma)}(q) - \min_{q\in\mathcal{B}_\epsilon(q_i)} B^*_{c(\sigma)}(q) \ge 2\delta. \tag{5.15}$$

Recall for each $z \in \mathbb{T}$, there is unique $z_\sigma \in \pi(\Gamma_\sigma)$ such that $z_\sigma = (z, q_{2\Gamma_\sigma}(z))$. From (5.14), (5.15) and the Hölder continuity guaranteed by Lemma 4.3 we find that for each $z \in \mathbb{T}$

$$\begin{aligned}\min_{q\in\partial\mathcal{B}_\epsilon(q_i)} & h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'}) - \\ & \min_{q\in\mathcal{B}_\epsilon(q_i)} h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'}) \ge \frac{3}{2}\delta.\end{aligned} \tag{5.16}$$

As these functions depend on the choice of closed 1-form η_c, to obtain (5.16) we choose $\eta_c = \langle c(\sigma), \dot q\rangle$. In general, $h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'})$ is also the function of z, but its variation over $z \in \mathbb{T}$ is very small if σ' is sufficiently close to σ, becasue $q_{2\Gamma_\sigma}(z)$ has $\frac{1}{2}$-Hölder continuity in σ. Since $\mathbb{S}$ is compact, there exists $\delta = \delta(\epsilon)$, $\epsilon_1 = \epsilon_1(\epsilon)$ independent of σ, such that (5.15) and (5.16) hold if $|\sigma - \sigma'| \le \epsilon_1$.

We say σ_j is linked with σ_{j+1} by transition torus with some persistency if $\sigma_{j+1} \in \mathbb{S}$ is so close to σ_j such that

$$|c_1(\sigma_j) - c_1(\sigma_{j+1})| \le \frac{1}{4}\delta, \tag{5.17}$$

and (5.16) hold where we replace σ and σ' by σ_j and σ_{j+1} respectively. We say σ_j is linked with σ_k by transition chain with some persistency if there there is a sequence $\sigma_j, \sigma_{j+1}, \cdots, \sigma_{k-1}, \sigma_k$ in $\mathbb{S}$ such that for each $j \le i < k$ σ_i is linked with σ_{i+1} by transition torus with some persistency. To be brief, we shall say in the following that they are linked by transition torus (chain). Note that $\mathbb{S} \subset [A', B']$ is compact, we can find finite $\sigma_k \in \mathbb{S}$ $(0 \le k \le K)$ such that we have one of the following alternatives for each $k < K$: either σ_k is linked with σ_{k+1} by transition chain, or Γ_{σ_k} and $\Gamma_{\sigma_{k+1}}$ make up the boundary of an annulus of Birkhoff instability, i.e. there is no other invariant circle between Γ_{σ_k} and $\Gamma_{\sigma_{k+1}}$. In the following we shall use Γ_i to denote Γ_{σ_i} and use z_i to denote z_{σ_i}.

Let us consider a sequence of invariant circles Γ_i $(i = 0, 1, \cdots, \ell, \ell+1)$ on the cylinder Σ such that Γ_1 is linked with Γ_ℓ through the transition chain $\Gamma_2, \cdots, \Gamma_{\ell-1}$, and there are two annulus of Birkhoff instability, one has Γ_0 and Γ_1 as its boundary, another one has Γ_ℓ and $\Gamma_{\ell+1}$ as its boundary. By the construction of this transition chain we know that for each $1 \le i < \ell$ there is $x_i \in \{B^*_{c(\sigma_i)} = 0\} \cap (a, 2\pi - a)$ such that for any $z \in \mathbb{T}$

$$\begin{aligned}\min_{q\in\partial\mathcal{B}_\epsilon(x_i)} & h^\infty_{c(\sigma_i)}(z_i, q) + h^\infty_{c(\sigma_{i+1})}(q, z_{i+1}) - \\ & \min_{q\in\mathcal{B}_\epsilon(x_i)} h^\infty_{c(\sigma_i)}(z_i, q) + h^\infty_{c(\sigma_{i+1})}(q, z_{i+1}) \ge \frac{3}{2}\delta.\end{aligned} \tag{5.16i}$$

As in [BCV], let us consider the covering of $\mathbb{T}^2$ given by $\bar{M} = \mathbb{T} \times \mathbb{R}$. For each x_i we identify it with its lift in the region $\mathbb{T} \times (0, 2\pi)$ and single out a point on its lift, $\bar{x}_i = x_i + (0, 2i\pi)$, we also identify each z_i with its lift $z_i + (0, 2i\pi)$. For $i \in (1, 2, \ldots, \ell - 1)$ we introduce a smooth function Ψ_i: $\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ which vanishes outside $\{q : |q - \bar{x}_i| \le 2\epsilon\}$ and such that

$$\nabla \Psi_i(q) = c(\sigma_{i+1}) - c(\sigma_i) \qquad \forall q : \ |q - \bar{x}_i| \le \epsilon. \tag{5.18}$$

If we set

$$\bar{c}_i(q) = c(\sigma_i) + \nabla \Psi_i(q), \tag{5.19}$$

then

$$h^\infty_{\bar{c}_i}(z, q) + h^\infty_{\bar{c}_{i+1}}(q, z) = h^\infty_{c(\sigma_i)}(z, q) + h^\infty_{c(\sigma_{i+1})}(q, z) + \Psi_{i+1}(q) - \Psi_i(q).$$

Note $\Psi_{i+1}(q) = 0$ as $q \in \mathcal{B}_\epsilon(q_i)$. If we require further that σ_{i+1} is so close to σ_i that (5.17) holds, we obtain from (5.16i) and (5.18) that

$$\begin{aligned} \min_{q \in \partial \mathcal{B}_\epsilon(x_i)} & h^\infty_{\bar{c}_i}(z_i, q) + h^\infty_{\bar{c}_{i+1}}(q, z_{i+1}) - \\ \min_{q \in \mathcal{B}_\epsilon(x_i)} & h^\infty_{\bar{c}_i}(z_i, q) + h^\infty_{\bar{c}_{i+1}}(q, z_{i+1}) \ge \delta. \end{aligned} \tag{5.20}$$

Let $\mathbb{B} = \mathcal{B}_\epsilon(x_1) \times \mathcal{B}_\epsilon(x_2) \times \cdots \times \mathcal{B}_\epsilon(x_{\ell-1})$, $Q = (q_1, \ldots, q_{\ell-1})$, $\vec{n} = (n_0, n_1, \cdots, n_\ell) \in \mathbb{Z}^{\ell+1}$ and define

$$\begin{aligned} \mathfrak{h}(Q, z_1, z_\ell, \vec{n}) = & \sum_{i=1}^{\ell-2} h^{n_{i+1} - n_i}_{\bar{c}_{i+1}}(q_i, q_{i+1}) \\ & + h^{n_1 - n_0}_{\bar{c}_1}(z_1, q_1) + h^{n_\ell - n_{\ell-1}}_{\bar{c}_\ell}(q_{\ell-1}, z_\ell), \end{aligned} \tag{5.21}$$

We see that $\mathfrak{h}$, as the function of Q, takes its local minimum in the interior of $\mathbb{B}$ if $n_{i+1} - n_i$ is sufficiently large for all $1 \le i \le \ell - 1$. In fact, let x^*_i be the point where the function of q $h^\infty_{\bar{c}_i}(z_i, q) + h^\infty_{\bar{c}_{i+1}}(q, z_{i+1})$ attains its local minimum in $\mathcal{B}_\epsilon(x_i)$, we find that the function of Q

$$\sum_{i=1}^{\ell-1} h^\infty_{\bar{c}_i}(z_i, q_i) + h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1})$$

takes its local minimum at the point $(x^*_1, x^*_2, \ldots, x^*_{\ell-1})$ which is obviously in the interior of $\mathbb{B}$. Thus, the local minimum of $\mathfrak{h}$ is in the interior of $\mathbb{B}$ if the following holds

$$\lim_{n_{i+1} - n_i \to \infty} h^{n_{i+1} - n_i}_{\bar{c}_{i+1}}(q_i, q_{i+1}) = h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1}) + h^\infty_{\bar{c}_{i+1}}(z_{i+1}, q_{i+1}). \tag{5.22}$$

To show this let us state a lemma:

Lemma 5.4. *Assume $\tilde{\mathcal{M}}(c)$ has a dense orbit. For any $m_0, m_1 \in M$, let γ: $[0,K] \to M$ be c-minimal curve connecting m_0 and m_1, $\gamma(0) = m_0$, $\gamma(K) = m_1$. For any $\delta > 0$, any $K_1 \in \mathbb{Z}^+$ and any $z \in \mathcal{M}(c)|_{t=0}$, $\exists$ $K_0 \in \mathbb{Z}^+$, if $K \geq K_0$ then there exists $T \in \mathbb{Z}^+$ such that $\gamma(T) \in B_\delta(z)$, $T \geq K_1$ and $K_0 - T \geq K_1$.*

Proof: For any $\delta^* > 0$ there is $K_0 \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^+$ such that $d\gamma(k) \in \tilde{\mathcal{M}}(c)|_{t=0} + \delta^*$ if $K \geq K_0$, otherwise there would be another c-minimal measure. For any $z \in \mathcal{M}(c)|_{t=0}$, by choosing sufficiently small δ^* and sufficiently large K_0 there is some $T \in \mathbb{Z}^+$ so that $\gamma(T)$ is in δ-neighborhood of z. Clearly, for any $K_1 \in \mathbb{Z}^+$ there exists such T so that $T \geq K_1$ and $K_0 - T \geq K_1$, otherewise there would be another c-minimal measure also. $\square$

Applying the lemma to this problem we find that for each $m \in \mathcal{M}(c_i)$, each small $\delta > 0$ and each large $K > 0$ there exist $n^* \in \mathbb{Z}^+$ such that if $n \geq n^*$ then there exists $z_n \in \mathcal{B}_\delta(m)$ such that

$$h^n_{c_i}(q, q') = h^{n_1}_{c_i}(q, z_n) + h^{n_2}_{c_i}(z_n, q')$$

where $n = n_1 + n_2$ with $n_1, n_2 \geq K$. Using the Lipschitz property of $h^n_{c_i}(m, m')$ in (m, m') we find that for each small $\epsilon > 0$ the following holds

$$\left| h^n_{c_i}(q, q') - h^{n_1}_{c_i}(q, m) + h^{n_2}_{c_i}(m, q') \right| < \epsilon$$

if δ is sufficiently small and n^* is sufficiently large. Since we consider those $\mathcal{M}(c_i)$ which is on the cylinder with irrational number, thanks to the corollary 2.7, we know that $L - \langle \bar{c}_i(q), \dot{q} \rangle$ is regular for each $0 \leq i \leq \ell + 1$, (5.22) follows from the property that $d_{c_i}(m, m') = 0$ for all $m, m' \in \pi(\Gamma_i)$. Denote the corresponding minimizer by γ: $[n_0, n_\ell] \to M$, we use $\gamma_i(t)$ to denote its restriction on the time interval $[n_i, n_{i+1}]$. Once $\gamma(t)$ reaches its local minimum in the interior of $\mathbb{B}$, standard argument shows that

$$\frac{\partial L_{\bar{c}_i}}{\partial \dot{q}}(d\gamma_i(t), t) = \frac{\partial L_{\bar{c}_{i+1}}}{\partial \dot{q}}(d\gamma_{i+1}(t), t)$$

holds at $t = n_{i+1}$. Note that $L_{\bar{c}_i} = L_{\bar{c}_{i+1}}$ in the neighborhood of $\mathcal{B}_\epsilon(\bar{x}_i)$ by the definition of $\Psi_i(q)$, we get

$$\dot{\gamma}_i(n_{i+1}) = \dot{\gamma}_{i+1}(n_{i+1}),$$

thus $\gamma(t)$ is a solution of the Euler-Lagrange equation over the time interval $[n_0, n_\ell]$.

In fact, we can remove the restriction on z_1 and z_ℓ that there is $z \in \mathbb{T}$ so that $z_j = (z, q_{2\Gamma_j}(z))$ for $j = 1, \ell$, We can replace z_j by any point $z^*_j \in \mathcal{M}(c_j)$ simply because $d_{c_j}(m, m') = 0$ for all $m, m' \in \pi(\Gamma_j)$ thus the function of $Q = (q_1, \ldots, q_{\ell-1})$

$$\begin{aligned}
&\sum_{i=2}^{\ell-2} h^\infty_{\bar{c}_i}(z_i, q_i) + h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1}) + h^\infty_{\bar{c}_1}(z^*_1, q_1) + h^\infty_{\bar{c}_\ell}(q_\ell, z^*_\ell) \\
=&\sum_{i=1}^{\ell-1} h^\infty_{\bar{c}_i}(z_i, q_i) + h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1}) + h^\infty_{\bar{c}_1}(z^*_1, z_1) + h^\infty_{\bar{c}_\ell}(z_\ell, z^*_\ell)
\end{aligned}$$

also reaches its local minimum at the point $(x_1^*, x_2^*, \cdots, x_{\ell-1}^*)$. Thus, the Lipschitz property of h_c^n enable us to assert that there exist large $(\Delta n_1, \Delta n_2, \cdots, \Delta n_\ell)$ and small $\delta^* > 0$, if $n_i - n_{i-1} \geq \Delta n_i$, $z_j \in \mathcal{B}_{\delta^*}(z_j^*)$ for $j = 1, \ell$, then as the function of Q, $\mathfrak{h}(Q, z_1, z_\ell, \vec{n})$ reaches its local minimum in the interior of $\mathbb{B}$.

Now we are ready to construct some diffusion orbit γ: $\mathbb{R} \to M$ such that $\alpha(d\gamma) \supset \Gamma_0$ and $\omega(d\gamma) \supset \Gamma_{\ell+1}$.

By the condition, Γ_i and Γ_{i+1} make up the boundary of the resonant zone Z_i for $i = 0, \ell$. For $i = 0, \ell+1$ we take $c^{(i)}$ be such a co-homology class that $\{B^*_{c^{(i)}} = 0\} = \Gamma_i$. For $i = 1, \ell$ we take $c^{(i)}$ be such a co-homology class that $c^{(i)} = (c_1^{(i)}, b(c_1^{(i)}))$, in this case, $\{B^*_{c^{(i)}} = 0\} = \Gamma_i \cup \{$its $c^{(i)}$-minimal homoclinic orbits$\}$. Since the C-equivalence between $c^{(i)}$ and $c^{(i+1)}$has been established for $i = 0, \ell$, analogous to the proof of Theorem 5.1 we can find the composition of finite U-step forms ν_j

$$\nu_j = \sum_{i=0}^{N_j} (-\tau_i^j)^* \mu_i^j, \qquad (j = 1, 2)$$

such that their cohomology classes are $[\nu_1(t)|_{t \leq 0}] = 0$, $[\nu_1(t)|_{t \geq \tau^1_{N_1}+1}] = c^{(1)} - c^{(0)}$, $[\nu_2(t)|_{t \leq 0}] = 0$ and $[\nu_2(t)|_{t \geq \tau^2_{N_2}+1}] = c^{(\ell+1)} - c^{(\ell)}$, where τ_i^j is the time translation $(q, t) \to (q, t + \tau_i^j)$. Moreover, by the lemma 5.3, we can choose those ν_j such that $\nu_1(t)|_{t \leq 0} = 0$, $\nu_1(t)|_{t \geq \tau^1_{N_1}+1} = \langle \bar{c}_1(q) - \bar{c}_0(q), dq \rangle$, $\nu_2(t)|_{t \leq 0} = 0$ and $\nu_2(t)|_{t \geq \tau^2_{N_2}+1} = \langle \bar{c}_{\ell+1}(q) - \bar{c}_\ell(q), dq \rangle$, (see (5.18) for the definition of $\bar{c}_i(q)$). Let $\eta_0^1 = \langle \bar{c}_0(q), dq \rangle$, $\eta_0^2 = \langle \bar{c}_\ell(q), dq \rangle$, $\eta_i^j = \eta_0^j + \sum_{k=0}^{i-1} \bar{\mu}_k^j$ and $c_i^j = [\eta_i^j]$, then $\eta^1_{N_1+1} = \langle \bar{c}_1(q), dq \rangle$, $\eta^2_{N_2+1} = \langle \bar{c}_{\ell+1}(q), dq \rangle$. Based on the proof of Theorem 5.1 we can choose each μ_i^j, the adapted neighborhood U_i^j and $\epsilon_i^j > 0$ so that

$$\pi(\tilde{\mathcal{N}}_{\eta_i^j, \mu_i^j}) + \epsilon_i^j \subset U_i^j, \qquad \forall\ j = 1, 2,\ 0 \leq i \leq N_j. \tag{5.6ij}$$

For each $\epsilon_i^{j*} > 0$, there exist $\hat{T}^j_{ki}, \check{T}^j_{ki} \in \mathbb{Z}_+$ with $\hat{T}^j_{ki} > \check{T}^j_{ki}$, $(k = 0, 1)$ such that

$$h^{T_0, T_1}_{\eta_i^j, \mu_i^j}(m_0, m_1) \geq h^{\infty}_{\eta_i^j, \mu_i^j}(m_0, m_1) - \epsilon_i^{j*}, \qquad \forall\ T_k \geq \check{T}^j_{ki}, (k = 0, 1),$$
$$\forall\ (m_0, m_1) \in M \times M;$$

for any given $(m_0, m_1) \in M \times M$ there exists $T_k = T_k(m_0, m_1)$ with $\check{T}^j_{ki} \leq T_k \leq \hat{T}^j_{ki}$ such that

$$\left| h^{T_0, T_1}_{\eta_i^j, \mu_i^j}(m_0, m_1) - h^{\infty}_{\eta_i^j, \mu_i^j}(m_0, m_1) \right| \leq \epsilon_i^{j*}. \tag{5.7ij}$$

Let $\gamma_i^j(t, m_0, m_1, T_0, T_1)$: $[-T_0, T_1] \to M$ be the minimizer of $h^{T_0, T_1}_{\eta_i^j, \mu_i^j}(m_0, m_1)$. Let $\check{T}^j_{ki}$ be set so large and $\epsilon_i^{j*} > 0$ be set so small such that if (5.7ij) holds, then

$$d\gamma_i^j(t, m_0, m_1, T_0, T_1) \in \tilde{\mathcal{N}}_{\eta_i^j, \mu_i^j} + \epsilon_i^j, \qquad \forall\ 0 \leq t \leq 1. \tag{5.8ij}$$

We define the index set for $\vec{\tau}^j=(\tau_0^j,\tau_1^j,\cdots,\tau_{N_j}^j)$

$$\Lambda^j=\Big\{\vec{\tau}^j\in\mathbb{Z}^{N_j}:\max\{\breve{T}_{0i}^j,\breve{T}_{1i}^j+1\}\le\tau_i^j-\tau_{i-1}^j\le\max\{\hat{T}_{0i}^j,\hat{T}_{1i}^j+1\},\\ \forall 1\le i\le N_j,\tau_0^j=0\Big\},$$

and introduce a modified Lagrangian depending on the parameters $\vec{\tau}^j$ $(j=1,2)$ and $\vec{n}$

$$\tilde{L}=\begin{cases} L-\langle\bar{c}_0(q),\dot{q}\rangle-(\tau_{N_1}^1+1)^*\nu_1, & t\le n_1,\\ L-\langle c(\sigma_j)+\varrho_j(t)\nabla\Psi_j(q),\dot{q}\rangle, & n_j\le t\le n_{j+1},\ 1\le j\le \ell-2,\\ L-\langle\bar{c}^{\ell-1}(q),\dot{q}\rangle+(n_\ell+\tau_{N_2}^2+1)^*\nu_2, & t\ge n_{\ell-1},\end{cases}$$

where ϱ_j is such a smooth function that $\varrho_j(t)=0$ for $t\le\frac{1}{2}(n_{j+1}-n_j)$, $0<\varrho_j<1$ when $\frac{1}{2}(n_{j+1}-n_j)<t<\frac{1}{2}(n_{j+1}-n_j)+1$ and $\varrho_j=1$ when $t\ge\frac{1}{2}(n_{j+1}-n_j)+1$, this function is well defined if $n_j-n_{j-1}\ge 4$. Clearly, $\tilde{L}$ is smooth in

$$(\dot{q},q,t)\in TM\times\Big\{\mathbb{R}\backslash\bigcup_{i=1}^{\ell-1}\{n_i\}\Big\}\cup\bigcup_{i=1}^{\ell-1}TB_\epsilon(x_i)\times\Big(n_i-1,n_i+1\Big)$$

For each $(m,m')\in M\times M$, $Q=(q_1,\cdots,q_{\ell-1})\in\mathbb{B}$ let

$$\begin{aligned} h_{\tilde{L}}^{T_0,T_{\ell+1}}(m,m',Q,\vec{\tau}^1,\vec{\tau}^2,\vec{n})=&\inf_{\substack{\gamma(-T_0^*)=m\\ \gamma(T_{\ell+1}^*)=m'\\ \gamma(n_j)=q_j\\ j=1,\cdots,\ell-1}}\int_{-T_0^*}^{T_{\ell+1}^*}\tilde{L}(d\gamma(t),t)dt\\ &-\sum_{\substack{1\le i\le N_j\\ j=1,2}}(\tau_i^j-\tau_{i-1}^j)\alpha(c_i^j)-n_0\alpha(c_1)\\ &-(n_\ell-n_{\ell-1})\alpha(c_\ell)-T_0\alpha(c_0)-T_{\ell+1}\alpha(c_{\ell+1})\end{aligned}$$

where $T_0^*=T_0+\tau_{N_1}^1+1$, $T_{\ell+1}^*=T_{\ell+1}+n_\ell+\tau_{N_2}^2+1$ and $T_0,T_{\ell+1}>0$. In virtue of the lemma 5.4, we can take sufficiently large n_1' so that any $c(\sigma_1)$-minimal curve γ_1: $[0,n_1]\to M$ with $n_1\ge n_1'$ has a point $\gamma_1(n_0)\in B_{\delta^*}(z_1^*)$ with $n_1-n_0\ge\Delta n_1$. Similarly, we can take sufficiently large $\Delta n_\ell'$ so that any $c(\sigma_\ell)$-minimal curve γ_ℓ: $[n_{\ell-1},n_\ell']\to M$ with $n_\ell'-n_{\ell-1}\ge\Delta n_\ell'$ has a point $\gamma_\ell(n_\ell)\in B_{\delta^*}(z_{\ell+1})$ with $n_\ell'-n_{\ell-1}>n_\ell-n_{\ell-1}\ge\Delta n_\ell$. We can also take suitable large n_i $(i=2,3,\cdots,\ell-1)$ so that $n_{i+1}-n_i\ge\Delta n_i$ for each $1\le i\le\ell-1$. Under these conditions we take the minimum of $h_{\tilde{L}}^{T_0,T_{\ell+1}}(m,m',Q,\vec{\tau}^1,\vec{\tau}^2,\vec{n})$ over $\mathbb{B}$

$$h_{\tilde{L}}^{T_0,T_{\ell+1}}(m,m',\vec{\tau}^1,\vec{\tau}^2,\vec{n})=\min_{Q\in\mathbb{B}}h_{\tilde{L}}^{T_0,T_{\ell+1}}(m,m',Q,\vec{\tau}^1,\vec{\tau}^2,\vec{n}).$$

Let $\gamma(t)=\gamma(t,m,m',\vec\tau^1,\vec\tau^2,\vec n)$ be the minimizer of $h_{\tilde L}^{T_0,T_{\ell+1}}(m,m',\vec\tau^1,\vec\tau^2,\vec n)$. Recall the support of Ψ_i is a small ball. For each cohomology class under our consideration here, the support of the minimal measure is on the cylinder, the hyperbolicity of the cylinder let us see that $\gamma(t)$ is outside of the support of $\nabla\Psi_i$ if both $t-n_i$ and $n_{i+1}-t$ are suitably large, in other words, for $t\in[\frac12(n_{i+1}-n_i),\frac12(n_{i+1}-n_i)+1]$, $\gamma(t)$ falls into the area where $\langle\varrho_i(t)\nabla\Psi_i(q),dq\rangle$ is exact. Thus, $d\gamma$ solves the Euler-Lagrange equation of L for $t\in[n_0,n_\ell]$ if we repeat the argument for the function $\mathfrak{h}(Q,z_1,z_\ell,\vec n)$.

Next, by choosing sufficiently larger value for $\breve T^1_{1N_1}$, $\hat T^1_{1N_1}$, $\breve T^2_{00}$ and $\hat T^2_{00}$ we can assume $\breve T^1_{1N_1}\ge n'_1$ and $\breve T^2_{00}\ge n_{\ell-1}+\Delta n'_\ell$. In this case, let us consider the minimum of $h_{\tilde L}^{T_0,T_{\ell+1}}(m,m',\vec\tau^1,\vec\tau^2,\vec n)$ over $\Lambda^1\times\Lambda^2\times\{\breve T^1_{1N_1}\le n_1\le\hat T^1_{1N_1}\}\times\{\breve T^2_{00}+n_{\ell-1}\le n_\ell\le \hat T^2_{00}+n_{\ell-1}\}$

$$h_{\tilde L}^{T_0,T_{\ell+1}}(m,m')=\min_{\substack{\vec\tau^1\in\Lambda^1,\vec\tau^2\in\Lambda^2\\ \breve T^1_{1N_1}\le n_1\le\hat T^1_{1N_1}\\ \breve T^2_{00}+n_{\ell-1}\le n_\ell\le\hat T^2_{00}+n_{\ell-1}}} h_{\tilde L}^{T_0,T_{\ell+1}}(m,m',\vec\tau^1,\vec\tau^2,\vec n).$$

Denote by $\vec\tau^{j*}$, n_1^* and n_ℓ^* where the the minimum is reached. Let $\gamma(t)=\gamma(t,m,m',T_0,T_{\ell+1})$ be the minimizer of $h_{\tilde L}^{T_0,T_{\ell+1}}(m,m')$. Let $\tau_{1j}=\tau^1_j-\tau^1_{N_1}-1$, $\tau_{2j}=\tau^2_j+\tau^2_{N_2}+1+n_\ell$. From the proof of Theorem 5.1 we can see that (5.6ij), (5.7ij) and (5.8ij) hold for $(-\tau_{ij})^*\gamma$ at $j=1,2,\ 0\le i\le N_j$ except for $(i,j)=(0,1),(N_2,2)$.

As the third step we consider the limit infimum

$$h^\infty_{\tilde L}(m,m')=\liminf_{\substack{T_0\to\infty\\ T_{\ell+1}\to\infty}} h_{\tilde L}^{T_0,T_{\ell+1}}(m,m').$$

Let T_0^k, $T^k_{\ell+1}$ be the subsequence so that $T_0^k\to\infty$, $T^k_{\ell+1}\to\infty$ as $k\to\infty$,

$$|h_{\tilde L}^{T_0^k,T^k_{\ell+1}}(m,m')-h^\infty_{\tilde L}(m,m')|\le\min\{\epsilon_0^{1*},\epsilon_{N_2}^{2*}\},\qquad\forall\ k,$$

and

$$\lim_{k\to\infty}h_{\tilde L}^{T_0^k,T^k_{\ell+1}}(m,m')=h^\infty_{\tilde L}(m,m'),$$

and let γ_k: $[-T_0^{k*},T^{k*}_{\ell+1}]\to M$ be the minimizer of $h_{\tilde L}^{T_0^k,T^k_{\ell+1}}(m,m')$, where $T_0^{k*}=T_0^k+\tau^{1*}_{N1}+1$, $T^{k*}_{\ell+1}=T^k_{\ell+1}+n^*_\ell+\tau^{2*}_{N_2}+1$. By the similar argument to prove Theorem 5.1 we can see (5.6ij), (5.7ij) and (5.8ij) hold for $(-\tau_{ij})^*\gamma_k$ at $(i,j)=(0,1),(N_2,2)$ also. In this case, $d\gamma_k$ is a solution of the Euler-Lagrange equation induced by L. For each small δ, $d\gamma_k$ connects $\tilde\Gamma_0+\delta$ with $\tilde\Gamma_{\ell+1}+\delta$ if k is sufficiently large. Let γ: $\mathbb R\to M$ be the accumulation point of $\{\gamma_k\}_{k\in\mathbb Z^+}$, then $\alpha(d\gamma)=\Gamma_0$ and $\omega(d\gamma)=\Gamma_{\ell+1}$ since $\tilde{\mathcal A}(c_i)=\Gamma_i$ for $i=0,\ell+1$.

In the case that there are finite resonant gaps, the construction of diffusion orbits can be done in the same way.

6, Generic property

Up to now we have shown the existence of diffusion orbits under the hypotheses (H1), (H2) and (H3). The task of this section is to show these hypotheses are actually dense properties in C^r-topology for $r \geq 3$. Since we are interested in the diffusion from $p_1 < A$ to $p_1 > B$, a compact domain for $\{\|p\| \leq K\} \times \mathbb{T}^2$ satisfies such an requirement if $K > 0$ is sufficiently large. The C^r-topology is endowed in the usual sense for functions $\{\|p\| \leq K\} \times \mathbb{T}^2 \to \mathbb{R}$.

The hypothesis (H1) is made only for those co-homology classes $c = (c_1, b(c_1))$, the time-1-section of the minimal measure is contained in an invariant circle on the cylinder, and its semi-static orbit set $\tilde{\mathcal{N}}(c)$ consists of the invariant circle and its minimal homoclinic orbits. It is precisely the set $\{B_c^* = 0\}$. Let us look at this issue from the Hamiltonian dynamics point of view.

Since the system is positive definite in the action variable p, it has a generating function $G(q, q')$

$$G(q, q') = \inf_{\substack{\gamma \in C^1([0,1], \bar{M}) \\ \gamma(0)=q, \gamma(1)=q'}} \int_0^1 L(\gamma(s), \dot{\gamma}(s), s) ds, \tag{6.1}$$

where (q, q') is in the covering space $\bar{M} = \mathbb{R}^2 \times \mathbb{R}^2$. Clearly, $G(q + 2m\pi, q' + 2m\pi) = G(q, q')$ for all $m \in \mathbb{Z}^2$. The map Φ_H: $(p, q) \to (p', q')$ is given by

$$p' = \partial_{q'} G(q, q'), \qquad p = -\partial_q G(q, q'). \tag{6.2}$$

Let π_1 be the standard projection from $\mathbb{R}^2$ to $\mathbb{T}^2$, let $c \in \mathbb{R}^2$ and

$$G_c(q, q') = G(q, q') - \langle c, q' - q \rangle$$

then

$$h_c(x, x') = \min_{\substack{\pi_1(q)=x \\ \pi_1(q')=x'}} G_c(q, q') - \alpha(c). \qquad \text{(see (2.6))} \tag{6.3}$$

As the system is nearly integrable, the matrix $\partial^2_{q'q} G$ is non-degenerate everywhere. Thus we can solve the second equation in (6.2) and obtain somehow more explicit form of the map (6.2)

$$p' = \frac{\partial G}{\partial q'}(q, q'(p, q)), \qquad q' = q'(p, q). \tag{6.4}$$

Let us consider a small perturbation of the generating function $G(q, q') \to G(q, q') + \kappa(q - q') G_1(q')$ in which $0 \leq \kappa(q - q') \leq 1$, $\kappa(q - q') = 1$ if $|q - q'| \leq K$ and $\kappa(q - q') = 0$ if $|q - q'| \geq K + 1$. We choose sufficiently large K so that $\{\|p\| \leq \max(|A|, |B|) + 1\}$ is contained in the set where $|q - q'| \leq K$. In this set the map will have the form

$$p' = \frac{\partial G}{\partial q'}(q, q'(p, q)) + \frac{\partial G_1}{\partial q'}(q'(p, q)), \qquad q' = q'(p, q). \tag{6.5}$$

Note that both stable and unstable manifolds of Γ keep horizontal over the strip $U=\{a\le q_2\le 2\pi-a\}$, restricting Φ to W^s and to W^u where they keep horizontal, and projecting it to the underline manifold M along the fibers we obtain two maps f^s and f^u on M such that $\pi\circ\Phi=f^{s,u}\circ\pi$. We choose $G_1\in C^r$ satisfying its support $supp(G_1)=\mathcal{B}_b(q^*)$ mod $2\pi m\subset U$ mod $2\pi m$ where $m\in\mathbb{Z}^2$. We see that $(f^u)^{-1}(\mathcal{B}_b(q^*))\cap\mathcal{B}_b(q^*)=\varnothing$ and $f^s(\mathcal{B}_b(q^*))\cap\mathcal{B}_b(q^*)=\varnothing$ if $b>0$ is chosen suitably small. Let us consider the problem in the covering space $\mathbb{T}\times\mathbb{R}$ and assume one lift of the unstable manifold starting from $q_2=0$ to the right, one lift of the stable manifold starting from $q_2=2\pi$ to the left. From (6.5) we can see that the local stable manifold is not deformed $W^s|[q_2^*-b\le q_2\le 2\pi+q_{2\Gamma}(q_1)]=\{q,dS^s+c(\sigma):q_2^*-b\le q_2\le 2\pi+q_{2\Gamma}(q_1)\}$, but the unstable manifold undergoes slight deformation, $W^u|[q_{2\Gamma}(q_1)\le q_2\le q_2^*+b]=\{q,dS^u+dG_1+c(\sigma):\, q_{2\Gamma}(q_1)\le q_2\le q_2^*+b\}$. It is easy to see that the barrier function has the form:

$$B^*_{c(\sigma)}(q)=S^u_{c(\sigma)}(q)-S^s_{c(\sigma)}(q)-G_1(q)+2\pi b(c_1(\sigma)),\qquad \textit{if}\ q\in B_b(q^*)\tag{6.6}$$

We should note the total action of the minimal orbit may be changed because of the perturbation, in other words, the associated cohomology class may be subjected to a small perturbation $(c_1,b(c_1))\to(c_1,b(c_1)\pm\varepsilon)$.

Let $R_d=\{q\in M:|q_1-q_1^*|\le d,\,|q_2-q_2^*|\le d\}\subset\mathcal{B}_b(q^*)$, let $S_\sigma=S^u_{c(\sigma)}-S^s_{c(\sigma)}-G_1$ we define

$$Z(\sigma)=\{q\in R_d:\ S_\sigma(q)=\min_{q\in R_d}S_\sigma\}.$$

We say a connected set V is non-trivial for R_d if either $\Pi_1(V\cap R_d)=\{q_1^*-d\le q_1\le q_1^*+d\}$ or $\Pi_2(V\cap R_d)=\{q_2^*-d\le q_2\le q_2^*+d\}$, where Π_i is the standard projection from $\mathbb{T}^2$ to its i-th component (i=1,2). Let $M_{d,q^*}(S)=\{q: S(q)=\min_{q\in R_d(q^*)}S\}$, we define a set in the function space $\mathfrak{F}(d,q^*)=C^0(R_d(q^*),\mathbb{R})$,

$$\mathfrak{Z}(d,q^*)=\Big\{S\in\mathfrak{F}(d,q^*):\ M_{d,q^*}(S)\,\text{contains a set non-trivial for}\ R_d(q^*)\Big\}.$$

Let

$$\mathfrak{Z}_1=\Big\{S\in\mathfrak{Z}(d,q^*):\ \Pi_1(M_{d,q^*}(S))=\{q_1^*-d\le q_1\le q_1^*+d\}\Big\},$$
$$\mathfrak{Z}_2=\Big\{S\in\mathfrak{Z}(d,q^*):\ \Pi_2(M_{d,q^*}(S))=\{q_2^*-d\le q_2\le q_2^*+d\}\Big\},$$

then

$$\mathfrak{Z}(d,q^*)=\mathfrak{Z}_1\cup\mathfrak{Z}_2.$$

Our first task is to show for each generating function $G\in C^r(M\times M,\mathbb{R})$ and each $\epsilon>0$, there is an open and dense set $\mathfrak{H}(d,q^*)$ of $\mathcal{B}_\epsilon(0)\subset C^r(R_d(q^*),\mathbb{R})$, for each $G_1\in\mathfrak{H}(d,q^*)$, the image of S_σ from $[A',B']$ to $\mathfrak{F}$ has no intersection with the set $\mathfrak{Z}_i$.

Obviously, the set $\mathfrak{Z}_1$ is a closed set and has infinite co-dimensions in the following sense, there exists $\mathfrak{N}$, an infinite dimension subspace of $\mathfrak{F}$, such that $(S+F)\notin\mathfrak{Z}$ for all $S\in\mathfrak{Z}_1$ and $F\in\mathfrak{N}\backslash\{0\}$. In fact, for each non constant function $F(q_1)\in C^0([q_1^*-d,q_1^*+d],\mathbb{R})$ with $F(q_1^*)=0$ and each $S\in\mathfrak{Z}_1$, we have $S+F\notin\mathfrak{Z}_1$. Thus, we can choose $\mathfrak{N}=C^0([q_1^*-d,q_1^*+d],\mathbb{R})/\mathbb{R}$, which we think as the subspace of $C^0(R_d(q^*),\mathbb{R})$ consisting of those continuous functions independent of q_2.

On the other hand, as S_σ: $[A',B']\to\mathfrak{F}$ has $\frac{1}{2}$-Hölder continuity, the image is compact and its box dimension is not bigger than 2

$$D_B\left(\mathfrak{F}_\sigma\right)\le 2. \tag{6.7}$$

where $\mathfrak{F}_\sigma=\{S_\sigma:\sigma\in[A',B']\}$. Clearly, this set is determined by the generating function G.

Lemma 6.1. *There is an open and dense set $\mathfrak{N}^*\subset\mathfrak{N}$ such that for all $F\in\mathfrak{N}^*$*

$$(\mathfrak{F}_\sigma+F)\cap\mathfrak{Z}=\varnothing. \tag{6.8}$$

Proof: The open property is obvious. If there is no density property, there would be n-dimensional ε-ball $\mathcal{B}_\varepsilon\subset\mathfrak{N}$ for some $\varepsilon>0$, for each $F\in\mathcal{B}_\varepsilon$, there exists $S\in\mathfrak{F}_\sigma$ such that $F+S\in\mathfrak{Z}_1$. For each $S\in\mathfrak{F}_\sigma$ there is only one $F\in\mathcal{B}_\varepsilon$ so that $S+F\in\mathfrak{Z}_1$, for, otherwise, there would be $F'\neq F$ such that $F'+S\in\mathfrak{Z}_1$, but we can write $F'+S=F'-F+F+S$ where $F+S\in\mathfrak{Z}_1$ and $F'-F\in\mathfrak{N}\backslash\{0\}$, it contradicts to the definition of $\mathfrak{N}$. Given $F\in\mathcal{B}_\varepsilon$, there might be more than one element in $\mathfrak{S}_F=\mathfrak{S}_F=\{S\in\mathfrak{F}_\sigma:S+F\in\mathfrak{Z}_1\}$. Given any two $F_1,F_2\in\mathcal{B}_\varepsilon$, for any $S_1\in\mathfrak{S}_{F_1}$ and any $S_2\in\mathfrak{S}_{F_2}$, we have

$$\begin{aligned}
d(S_1,S_2)=&\max_{q\in R_d(q^*)}|S_1(q)-S_2(q)|\\
\ge&\max_{|q_1-q_1^*|\le d}|\min_{|q_2-q_2^*|\le d}S_1(q_1,q_2)-\min_{|q_2-q_2^*|\le d}S_2(q_1,q_2)|\\
=&\max_{|q_1-q_1^*|\le d}|F_1(q_1)-F_2(q_1)|\\
=&d(F_1,F_2)
\end{aligned} \tag{6.9}$$

where $d(\cdot,\cdot)$ is the C^0-metric. It follows from (6.9) and the definition of box dimension that

$$D_B(\mathfrak{F}_\sigma)\ge D_B(\mathcal{B}_\varepsilon)=n,$$

but this is absurd if we choose $n>2$. The same argument can be applied to the set $\mathfrak{Z}_2$. $\square$

As C^r is dense in C^0, an open and dense set $\mathfrak{H}(d,q^*)$ of $\mathcal{B}_\epsilon\subset C^r(R_d(q^*),\mathbb{R})$ clearly exists such that for each perturbation of generating function $G_1\in\mathfrak{H}(d,q^*)$, we have

$$\mathfrak{F}_\sigma\cap\mathfrak{Z}(d,q^*)=\varnothing,\qquad\forall\ \sigma\in\mathbb{S}.$$

where by abuse of terminology we continue to denote S_σ and its restriction $R_d(q^*)$ by the same symbol.

Recall we have defined the set $U = \mathbb{T} \times [a, 2\pi - a]$ before. Let $M_U(S) = \{q : S(q) = \min_{q\in U} S\}$ and

$$\mathfrak{Z} = \Big\{S \in C^0(U, \mathbb{R}) : M_U(S) \,\text{consists of discrete points}\Big\}.$$

Given $d_i > 0$, there are finite q_{ij} such that $\cup_j R_{d_i}(q_{ij}) \supset U$. Thus, there exists a sequence $d_i \to 0$ and a countable set $\{q_{ij}\}$ such that

$$\left(\bigcap_{i=1,j=1}^{\infty} \mathfrak{H}(d_i, q_{ij})\right) \bigcap \mathfrak{Z} = \varnothing.$$

Therefore there is generic set in $\mathcal{B}_\epsilon \subset C^r(U, \mathbb{R})$, the hypothesis (H1) holds for each G_1 in this generic set. Note U is an annulus, we can write $G_1 = G_1' + G_1^*$ so that both G_1' and G_1^* have simply connected support.

The perturbation to the generating function G can be achieved by perturbing the Hamiltonian function $H \to H' = H + \delta H$. Let Φ' be the map determined by the generating function $G + \kappa G_1'$, the symplectic diffeomorphism $\Psi = \Phi' \circ \Phi^{-1}$ is close to identity. We choose a smooth function $\rho(s)$ with $\rho(0) = 0$ and $\rho(1) = 1$, let Φ_s' be the symplectic map determined by $G + \rho(s)\kappa G_1'$ and let $\Psi_s = \Phi_s' \circ \Phi^{-1}$. Clearly, Ψ_s defines a symplectic isotopy between identity map and Ψ. Thus there is a unique family of symplectic vector field X_s: $T^*M \to TT^*M$ such that

$$\frac{d}{ds}\Psi_s = X_s \circ \Psi_s.$$

By the choice of perturbation, there is a simply connected and compact domain D_K such that $\Psi_s|_{T^*M\setminus D_K} = id$. It follows that there is a Hamiltonian $H_1(p, q, s)$ such that $dH_1(Y) = dp \wedge dq(X_s, Y)$ holds for any vector field Y. Re-parametrizing s by t we can make H_1 smoothly and periodically depend on t. To see that dH_1 is also small, let us make use of a theorem of Weistain [W]. A neighborhood of the identity in the symplectic diffeomorphism group of a compact symplectic manifold $\mathbf{M}$ can be identified with a neighborhood of the zero in the vector space of closed 1-forms on $\mathbf{M}$. Since Hamiltomorphism is a subgroup of symplectic diffeomorphism, there is a function H', sufficiently close to H, such that $\Phi_{H_1} \circ \Phi_H = \Phi_{H'}^t|_{t=1}$.

Thus the density of (H1) is proved.

For the hypothesis (H2) let us consider the twist map on the cylinder. In this case, each co-homology class corresponds to a unique rotation number. Given any rational number $p/q \in \mathbb{Q}$, it is obvious that there is a open dense set in the space of area-preserving twist map such that there is only one minimal (p, q)-periodic orbit without homoclinic loop. Take the intersection of countably open dense set we obtain that (H2) is a generic property.

To verify the (H3), let us consider an invariant circle Γ_σ on Σ. There is an interval $I(c_1) = \{c = (c_1, c_2) \in \mathbb{R}^2 : a(c_1) \le c_2 \le b(c_1)\}$ such that $\text{supp}(\mathcal{M}(c)|_{t=0}) \subseteq \Gamma_\sigma$ iff $c \in I(c_1)$. Let U be a small neighborhood of $\pi(\Gamma_\sigma)$. Under the hypothesis (H1), the set $\{B^*_c = 0\}\backslash U$ is homotopically trivial for $c = (c_1, a_1(c_1))$ and for $c = (c_1, b_1(c_1))$. By the upper semi-continuity of Mañé sets $c \to \tilde{\mathcal{N}}(c)$, the set $\mathcal{N}(c')|_{t=0}$ is in a small neighborhood of $\{B^*_c = 0\}$ if $c' = (c_1, b(c_1) + \delta)$ with $\delta > 0$ sufficiently small. Let us consider such a minimal measure $\tilde{\mathcal{M}}(c')$. Let μ be an ergodic component of $\tilde{\mathcal{M}}(c')$, there exists $\epsilon^* > 0$ such that $\text{dist}(\text{supp}\mu, \tilde{\Gamma}_\sigma) \ge 3\epsilon^*$ for all $\sigma \in \mathbb{S}$. For any $\epsilon > 0$ with $\epsilon \le \epsilon^*$ we can define a C^r-smooth function $L^\sigma_{k,\epsilon}$: $TM \times \mathbb{T} \to \mathbb{R}$ so that $L^\sigma_{k,\epsilon}(z,t) = 0$ if $(z,t) \in \text{supp}\mu + 2^{-k-1}\epsilon$, $L^\sigma_{k,\epsilon} = 2^{-k}\epsilon^{r+1}$ if $(z,t) \notin \text{supp}\mu + 2^{-k}\epsilon$ and $L^\sigma_{k,\epsilon}$ takes the value between $2^{-k}\epsilon^{r+1}$ and 0 elsewhere. Obviously, μ is the unique ergodic component of c'-minimal measure of the Lagrangian

$$L^\sigma_\epsilon = L + \frac{1}{r!}\sum_{k=1}^{\infty} L^\sigma_{k,\epsilon}$$

and $\|L^\sigma_\epsilon - L\|_{C^r} \le \epsilon$. Since (H3) is required only for countable $\sigma \in \mathbb{S}$, we can choose even smaller ϵ_σ so that the supports of these $L^\sigma_{\epsilon_\sigma} - L$ have no intersection.

Note the perturbation we introduced for (H1) has compact support which has no intersection with the cylinder, the perturbation we introduced for (H3) does not touch the set $\{B^*_{c(\sigma)} = 0\}$ for all $\sigma \in \mathbb{S}$, there is a dense set for P such that (H1), (H2) and (H3) hold. Thus we obtain the density of the perturbation. Since the time for each orbit drifts from $p_1 < A$ to $p_1 > B$ is finite, the smooth dependence of solutions of ODE's on parameter guarantees the openness.

Therefore, the proof of the theorem 1.1 is completed.

Appendix

In this appendix we present the proof of the lemma 2.6, given by Bernard in [Be], for the completeness sake.

Lemma 2.6. *If $\tilde{\mathcal{M}}(c)$ is minimal in the sense of topological dynamics and if there exists a sequence γ_n of n-periodic curves such $A_c(\gamma_n) \to 0$ as $n \to \infty$, then L_c is regular, hence $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c)$.*

Proof: As the first step we show that the following limit exists for all $(x,t) \in \mathcal{M} \times \mathbb{T}$:

$$\lim_{n\to\infty} F_c(x, x, t, t+n) = 0. \tag{A.1}$$

By the condition, we can suppose these n-periodic curves γ_n are minimizers, their n-periodic orbits $X_n(t) = (d\gamma_n(t), t)$ is a compact subset of $TM \times \mathbb{T}$. Each subsequence of X_n has a convergent subsequence in the sense of Hausdorff topology. The limit set of such a sequence is obviously an invariant subset of $\tilde{\mathcal{M}}(c)$. Since $\tilde{\mathcal{M}}(c)$ is minimal, this limit set has to be $\tilde{\mathcal{M}}(c)$ itself. Therefore, the sequence of subsets X_n converges to $\tilde{\mathcal{M}}(c)$ in the Hausdorff topology. It follows that each point $(x, s) \in \tilde{\mathcal{M}}(c)$ is the limit of a sequence $(\gamma(t_n), s)$ with $t_n = s \bmod 1$ for each n. As F_c is of Lipschitz we have

$$\begin{aligned}\limsup_{n\to\infty} F_c(x, x, t, t+n) =& \limsup_{n\to\infty} F_c(\gamma_n(t_n), \gamma_n(t_n), t, t+n)\\ =& \limsup_{n\to\infty} A_c(\gamma_n)\\ =&0,\end{aligned}$$

which implies (A.1).

Next, we claim that (A.1) implies that $L-\eta_c$ is regular, i.e. for any (x, s), $(x', s') \in M \times \mathbb{T}$, $\epsilon > 0$, there exists T such that

$$F_c(x, x', t, t') \leq h_c(x, x', t, t') + \epsilon$$

if t and t' satisfy $t = s \bmod 1$, $t' = s' \bmod 1$ and $t' \geq t + T$. Indeed, let K be the common Lipschitz constant of all functions $F_c(\cdot, \cdot, t, t')$ with $t' \geq t+1$, let $t_0 = s \bmod 1$, $t_0' = s' \bmod 1$, let γ: $[t_0, t_0'] \to M$ be a minimizer with $\gamma(t_0) = x$ and $\gamma(t_0') = x'$, i.e. $A_c(\gamma) = F_c(x, x', t_0, t_0')$. We can make $t_0' - t_0$ is sufficiently large so that $\exists\ t_1 \in [t_0, t_0']$ such that $\mathrm{dist}(\gamma(t_1), y) \leq \epsilon/4K$ for some $y \in \mathcal{M}(c)|_{t=t_1}$, in virtue of standard argument of topological dynamics. Since $h_c(x, x', s, s') = \liminf F_c(x, x', t, t')$, we can suppose in addition that

$$F_c(x, x', t_0, t_0') \leq h_c(x, x', s, s') + \frac{\epsilon}{2}.$$

Let $x_1 = \gamma(t_1)$ we have

$$F_c(x, x', t_0, t_0') = F_c(x, x_1, t_0, t_1) + F_c(x_1, x', t_1, t_0'),$$

It follows that

$$|F_c(x, x', t_0, t_0') - F_c(x, y, t_0, t_1) - F_c(y, x', t_1, t_0')| \leq \frac{\epsilon}{2},$$

thus

$$F_c(x, y, t_0, t_1) + F_c(y, x', t_1, t_0') \leq h_c(x, x', s, s') + \epsilon.$$

By the choice of t and t' we know that $\exists\ n \in \mathbb{N}$ such that $t' - t = t_0' - t_0 + n$, so we have

$$\begin{aligned}F_c(x, x', t, t') =&F_c(x, x', t_0, t_0 + n)\\ \leq&F_c(x, y, t_0, t_1) + F_c(y, y, t_1, t_1 + n)\\ &+ F_c(y, x', t_1 + n, t_0' + n).\end{aligned}$$

Let $n \to \infty$, thanks to (A.1), we obtain

$$\limsup F_c(x, x', t, t') \leq h_c(x, x', s, s') + \epsilon.$$

As this holds for arbitrary $\epsilon > 0$, we see that L is regular.

As the third step, we claim that L is regular implies that $\tilde{\mathcal{G}} = \tilde{\mathcal{N}}$. Let $\gamma \in C^1(\mathbb{R}, M)$ be a minimizing curve, let $t_k \to -\infty$ be a sequence such that $s = t_k$ mod 1 for all $k \in \mathbb{Z}$ and such that $\alpha = \lim \gamma(t_k)$, let $t'_k \to \infty$ be a sequence such that $s' = t'_k$ mod 1 and such that $\omega = \lim \gamma(t'_k)$. In this case

$$A(\gamma|_{[t_k,t'_k]}) = F(\gamma(t_k), \gamma(t'_k), t_k, t'_k) \to h(\alpha, \omega, s, s').$$

Let us consider a compact interval of times $[a, b]$, where $s' = a$ mod 1 and $s = b$ mod 1. For k sufficiently large we have

$$A_c(\gamma|_{[a,b]}) = A_c(\gamma|_{[t_k,t'_k]}) - A_c(\gamma|_{[t_k,a]}) - A_c(\gamma|_{[b,t'_k]}),$$

taking the limit we obtain

$$A_c(\gamma|_{[a,b]}) = h_c(\alpha, \omega, s, s') - h_c(\alpha, \gamma(a), s, s) - h_c(\gamma(b), \omega, s', s').$$

On the other hand, we observe that if L is regular then

$$h_c(\alpha, \omega, s, s') \le h_c(\alpha, \gamma(a), s, s) + \Phi_c(\gamma(a), \gamma(b), s, s') + h_c(\gamma(b), \omega, s', s')$$

it follows that

$$A(\gamma|_{[a,b]}) \le \Phi_c(\gamma(a), \gamma(b), s, s'),$$

hence γ is semi-static. It has been shown in [Ma4] that $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$. $\square$

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