On the Exponential Decay of Magnetic Stark Resonances

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Abstract

We study the time decay of magnetic Stark resonant states. As our main result we prove that for sufficiently large time these states decay exponentially with the rate given by the imaginary parts of eigenvalues of certain non-selfadjoint operator. The proof is based on the method of complex translations.

1 Introduction

The purpose of this paper is to study the decay properties of resonances in two dimensions in the presence of crossed magnetic and electric fields and a potential type perturbation. We assume that the magnetic field acts in the direction perpendicular to the electron plane with a constant intensity B and that the electric field of constant intensity F points in the $x-$direction. The perturbation $V(x,y)$ is supposed to satisfy certain localisation conditions. The corresponding quantum Hamiltonian reads as follows

$$H(F) = H(0) - Fx = H_L + V - Fx,$$

where H_L is the Landau Hamiltonian of an electron in a homogeneous magnetic field of intensity B.

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We begin with the definition of a resonance in terms of an exponential time decay of the corresponding resonant states. In Section 3 we show the connection between these time decaying states and the usual spectral deformation notion of resonance. The basic mathematical tool we use is the method of complex translations for Stark Hamiltonians, which was introduced in [AH] as a modification of the original theory of complex scaling [AC], [BC]. Following [AH] we consider the transformation $U(\theta)$, which acts as a translation in $x-$direction; $(U(\theta)\psi)(x) = \psi(x+\theta)$. For non real θ the translated operator $H(F,\theta) = U(\theta)H(F)U^{-1}(\theta)$ is non-selfadjoint and therefore can have some complex eigenvalues. The main result of Section 3, Theorem 3.1, tells us that if ϕ is an eigenfunction of $H(0)$, then $(\phi, e^{-itH(F)}\,\phi)$ decays exponentially at the rate given by the imaginary parts of the eigenvalues of $H(F,\theta)$. Theorem 3.1 thus can be regarded as a generalisation of the result obtained in [He], where the exponential decay was proved for the Stark Hamiltonians without magnetic field.

Of course on would like to know how the resonance widths behave as functions of F. This question is discussed in the forthcoming paper, in which we prove that for $F \to 0$ the resonance widths decay as $\exp[-\frac{B}{F^2}]$ in contrast with the usual Stark resonances, where the behaviour is exponential. However, the technique used in our next paper requires some specific properties of the Green function $G_1(\mathbf{x},\mathbf{x}';z)$ of the operator

$$H_1(F) = H_L - Fx,$$

in the limit $F \to 0$. In particular, on need to know that $G_1(\mathbf{x},\mathbf{x}';z)$ is exponentially decaying with respect to $(x'-x)^2$ and $|y'-y|$. While similar behaviour is well known in case of purely magnetic Hamiltonian, where the Green function is given explicitly, to the best of our knowledge there is no explicit formula for the Green function of the crossed fields Hamiltonian $H_1(F)$. The direct application of these results on the crossed fields Green function motivates us to include them as a second part of the present paper. However, the estimations of $G_1(\mathbf{x},\mathbf{x}';z)$ could be of general interest for other problems dealing with simultaneous electric and magnetic fields.

2 The Model

We work in the system of units, where $m = 1/2$, $e = 1$, $\hbar = 1$. The crossed fields Hamiltonian is then given by

$$H_1(F) = H_L - Fx = (-i\partial_x + By)^2 - \partial_y^2 - Fx, \quad \text{on} \quad L^2(\mathbb{R}^2). \tag{2.1}$$

Here we use the Landau gauge with $\mathbf{A}(x,y) = (-By, 0)$. A straightforward application of [RS, Thm. X.37] shows that $H_1(F)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, see also [RS, Prob. X.38]. Moreover, one can easily check that

$$\sigma(H_1(F)) = \sigma_{ac}(H_1(F)) = \mathbb{R} \tag{2.2}$$

As mentioned in the Introduction we employ the translational analytic method developed in [AH]. We introduce the translated operator $H_1(F,\theta)$ as follows:

$$H_1(F,\theta) = U(\theta) H_1(F) U^{-1}(\theta) \tag{2.3}$$

where

$$\left(U(\theta)f\right)(x,y) := \left(e^{ip_x\theta} f\right)(x,y) = f(x+\theta, y) \tag{2.4}$$

An elementary calculation shows that

$$H_1(F,\theta) = H_1(F) - F\theta \tag{2.5}$$

Operator $H_1(F,\theta)$ is clearly analytic in θ. Following [AH] we define the class of $H_1(F)-$translation analytic potentials.

Definition 2.1. *Suppose that $V(z,y)$ is analytic in the strip $|\Im z| < \beta$, $\beta > 0$ independent of y. We then say that V is $H_1(F)-$translation analytic if $V(x+z,y)(H_1(F)+i)^{-1}$ is a compact analytic operator valued function of z in the given strip.*

We can thus formulate the conditions to be imposed on V:

(a) $V(x,y)$ is $H_1(F)-$translation analytic in the strip $|\Im z| < \beta$.

(b) There exists $\beta_0 \leq \beta$ such that for $|\Im z| \leq \beta_0$ the function $V(x+z,y)$ is uniformly bounded and

$$\lim_{x,y\to\pm\infty} |V(x+z,y)| = 0$$

(c) The operator $H(F) = H_1(F) + V$ has purely absolutely continuous spectrum.

In order to characterise the potential class for which the above conditions are fulfilled let us assume for the moment, that the integral kernel of $(H_1(F)+i)^{-1}$ has a local logarithmic singularity at the origin. This is a very plausible hypothesis, see Lemma 4.3, it then follows that any $L^2(\mathbb{R}^2)$ function which tends to zero at infinity and can be analytically continued in a given strip $|\Im z| < \beta$ satisfies the conditions (a) and (b).

We can take a Gaussian as an elementary example. The condition (c) is more delicate. Although the quantum tunnelling phenomenon leads us to believe that all the impurity states becomes unstable once the electric field is added, there is no rigorous results on the potential class that satisfies (c).

From the well known perturbation argument, [Ka], we see that under assumption (b)

$$H(F,\theta) = U(\theta)H(F)\,U^{-1}(\theta) = H_1(F,\theta) + V\,(x+\theta, y) \tag{2.6}$$

forms an analytic family of type A.
Furthermore, since $V(x+\theta,y)(H_1(F)+i)^{-1}$ is compact by (a), we have

$$\sigma_{ess}(H(F,\theta)) = \sigma_{ess}(H_1(F,\theta)) = \mathbb{R} - ibF \tag{2.7}$$

where $\theta = ib$, $b \in \mathbb{R}$. By standard arguments [RS, Prob. XIII.76], all eigenvalues of $H(F,ib)$ lie in the strip $-bF < \Im z \leq 0$ and are independent of b as long as they are not covered by the essential spectrum.

3 Exponential decay

The resonant states for our model are defined in the following way:

Definition 3.1. *We say that φ is a resonant state of $H(F)$ with width Γ, if there exists some $\epsilon > 0$, such that*

$$|(\varphi, e^{-itH(F)}\varphi)|^2 = e^{-t\Gamma}(1+R(t)),$$

where

$$|R(t)| = \mathcal{O}(e^{-t\,\epsilon}), \quad as \quad t \to \infty.$$

We remark that for a bounded below Hamiltonian the decay law can be exponential only for times neither too small nor too large, [Ex]. However, in our case, due to the fact that $H(F)$ is unbounded from below, the above definition makes sense. For a detailed discussion of the problem of definition of resonance see also [Si]. The goal of this section is to prove that the resonance width Γ is given by an imaginary part of the associated complex eigenvalue of $H(F,\theta)$. We will borrow the ideas from [He] where a similar problem in three dimensions was treated in the absence of magnetic field. The main ingredient of our analysis is the proof of the fact that $H(F,\theta)$ can have only a finite number of eigenvalues in a given strip. We will need the following claim.

Proposition 3.1. *Let f, g be bounded functions with compact support in $\mathbb{R}^2$. Then*

$$\lim_{\lambda\to\pm\infty} \|f(H_1(F)-\lambda-i\gamma)^{-1}g\| = 0$$

for $F \geq 0$ and uniformly for γ in the compacts of $\mathbb{R}\setminus\{0\}$.

Proof. We take $\gamma < 0$ and write[1]

$$\begin{aligned} f(H_1(F)-\lambda-i\gamma)^{-1}g &= -i\int_0^\infty (fe^{itH_1(F)}g)e^{\gamma t}e^{-i\lambda t}\,\mathrm{d}t := \int_0^\infty G(t)e^{-i\lambda t}\,\mathrm{d}t \\ &= \int_0^\epsilon G(t)e^{-i\lambda t}\,\mathrm{d}t + \sum_{n\in\mathbb{N}}\int_{n\pi/B-\epsilon}^{n\pi/B+\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t \\ &+ \sum_{n\in\mathbb{N}_0}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t \end{aligned} \tag{3.1}$$

The first term on the right hand side is bounded from above by $\|f\|_\infty\,\|g\|_\infty\,\epsilon$. For the second we have

$$\left\|\sum_{n\in\mathbb{N}}\int_{n\pi/B-\epsilon}^{n\pi/B+\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t\right\| \leq 2\epsilon\,\|f\|_\infty\,\|g\|_\infty \sum_{n\in\mathbb{N}} e^{\gamma(n\pi/B-\epsilon)}$$

which implies

$$\begin{aligned} \|f(H_1(F)-\lambda-i\gamma)^{-1}g\| &\leq \epsilon\,\|f\|_\infty\,\|g\|_\infty\left(\frac{2e^{-\gamma\epsilon}}{1-e^{\gamma\pi/B}}+1\right) \\ &+ \left\|\sum_{n\in\mathbb{N}_0}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t\right\| \end{aligned} \tag{3.2}$$

All terms in the sum on the r.h.s. of (3.2) can be integrated by parts to give

$$\begin{aligned} \int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t &= \frac{1}{i\lambda}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G'(t)e^{-i\lambda t}\,\mathrm{d}t \\ &- \left[\frac{1}{i\lambda}\,G(t)e^{-i\lambda t}\right]_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} \end{aligned} \tag{3.3}$$

where the second term on the r.h.s. is bounded above by $2\|f\|_\infty\,\|g\|_\infty|\lambda|^{-1}$. In order to estimate the first term we use the integral kernel of the evolution operator $e^{-itH_1(F)}$ in the gauge where $H_L = p_x^2 + (p_y - Bx)^2$ (keeping in mind that the norm is gauge-invariant). From the formula (A.11) given in Appendix A we then deduce the integral kernel of $G'(t)$

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[1] here $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$

$$
\begin{aligned}
(x,y|G'(t)|x_0,y_0) = {} & \frac{1}{2\pi i}\sqrt{\frac{B}{2}}e^{\gamma t}f(x,y)g(x_0,y_0)e^{iS_{-t}[w_{cl}(\cdot)]}\frac{1}{\sin(Bt)} \times \\
\times & \left\{\gamma + B\cot(Bt) + \frac{i}{4}\Big(u^2 - 2F(x+x_0) - \frac{B^2}{\sin^2(Bt)}[(x-x_0)^2 + (y-y_0+ut)^2]\right. \\
+ & \left. 2F\cot(Bt)(y-y_0+ut)\Big)\right\}
\end{aligned} \tag{3.4}
$$

with $u = \frac{F}{B}$. After some manipulations we find an upper bound on the Hilbert-Schmidt norm of $G'(t)$

$$
\|G'(t)\|_{HS} \leq \frac{C\,e^{\delta\, t}}{|\sin^3(Bt)|}
$$

where $\gamma < \delta < 0$ and the constant C is uniform in t and depends on f, g, F, B. The last inequality yields the following estimate

$$
\left\|\sum_{n\in\mathbb{N}_0}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t\right\| \leq |\lambda|^{-1}\left[2\,\|f\|_\infty\,\|g\|_\infty + C(\delta)\int_\epsilon^{\pi/B-\epsilon}\frac{1}{|\sin^3(Bt)|}\,\mathrm{d}t\right]
$$

here we have put

$$
C(\delta) = \frac{C\,e^{\delta\epsilon/2}}{1-e^{\delta\pi/2B}}\,, \qquad (\delta < 0)
$$

Finally, we can sum up all the contributions on the r.h.s. of (3.1) to write

$$
\begin{aligned}
\|f(H_1(F)-\lambda - i\,\gamma)^{-1}g\| \;\leq\; & \|f\|_\infty\,\|g\|_\infty\left\{\left(1+\frac{2e^{-\gamma\epsilon}}{1-e^{\gamma\pi/B}}\right)\,\epsilon + 2|\lambda|^{-1}\right\} \\
+ \;\; & C(\delta)|\lambda|^{-1}\int_\epsilon^{\pi/B-\epsilon}\frac{1}{|\sin^3(Bt)|}\,\mathrm{d}t
\end{aligned} \tag{3.5}
$$

Sending ϵ to zero in a suitable way, for example as $|\lambda|^{-\alpha}$ with $\alpha > 0$ and sufficiently small, we can make sure that the last term in (3.5) tends to zero as $\lambda \to \pm\infty$ and the claim of the Proposition then follows. The case $\gamma > 0$ can be proved in a similar way. $\square$

Armed with Proposition 3.1 we can prove the promised result about the finite number of eigenvalues in the vicinity of real axis.

Proposition 3.2. *Suppose that assumptions (b) and (c) hold true. Then for any* $aF < bF < \beta_0$ *there exists some* $M(a)$ *such that* $H(F, ib)$ *has no eigenvalues in the strip* $S_a := \{0 \geq \Im z \geq -aF,\ |\Re z| \geq M(a)\}$.

Proof. We write $V_1 := |V(x+ib,y)|^{1/2}$, $\quad V_2 := |V(x+ib,y)|^{1/2}\,\mathrm{phase}V(x+ib,y)$ and, for $z \in S_a$, $R_1(z) = (z - H_1(F,ib))^{-1}$, $\quad R(z) = (z - H(F,ib))^{-1}$. Then, by an approximation argument and Proposition 3.1

$$\lim_{\lambda\to\pm\infty} \|V_1(H_1(F,ib) - \lambda - i\gamma)^{-1}V_2\| = 0, \quad \gamma > F(b-a) > 0 \tag{3.6}$$

so that the Neumann series

$$R(z) = \sum_{n=0}^{\infty} R_1(z)(VR_1(z))^n = R_1(z) + R_1(z)V_1\left(\sum_{n=0}^{\infty}(V_2R_1(z)V_1)^n\right)V_2R_1(z)$$

converges for $z \in S_a$. Moreover, since $\|R_1(z)\| \le ((b-a)F)^{-1}$, we can conclude that

$$\sup_{z\in S_a} \|(z - H(F,ib))^{-1}\| < \infty$$

□

The following definition is a "translational version" of the notion of analytic vectors for dilatation group introduced in [AC].

Definition 3.2. *Let A be any open complex domain having non-empty intersection with $\mathbb{R}$. Then we denote by $\mathcal{D}(A)$ a set of those vectors f, for which $f_\theta = U(\theta)f$, $\theta \in \mathbb{R}$ can be analytically continued to A.*

We are now able to state the main theorem of this section. Since a similar analysis was made in [He] for a non magnetic case, we skip some details of the proof referring to the latter.

Theorem 3.1. *Take $\alpha := \alpha_0 F > 0$ sufficiently small such that the conditions (a), (b) and (c) are satisfied for $\min(\beta,\beta_0) > \alpha$. Assume moreover that $bF > \alpha$ and let ψ, ϕ, $H_1(F)\psi$, $H_1(F)\phi \in \mathcal{D}(\{z \in \mathbb{C} : |\Im z| \le bF\})$. Then for any $t \ge 0$*

$$(\psi, e^{-itH(F)}\phi) = \sum_{-\Im E_j \le \alpha} (\psi_{-ib}, P_j(ib)\phi_{ib})\, e^{-itE_j} + R(t)$$

where

$$R(t) \le \mathcal{C}\, e^{-t(\alpha+\epsilon)}$$

for some $\epsilon > 0$. Here $P_j(ib)$ is the spectral projector of $H(F,ib)$ associated with the eigenvalue E_j.

Proof. Following [He] we put $K_1(z) = (\psi, (z - H(F))^{-1}\phi)$ for $\Im z > 0$ and note that $K_1(z)$ has a meromorphic continuation to $\mathbb{C}$, which is for $\Im z > -bF$ given by $K_1(z) = (\psi_{-ib}, (z - H(F, ib))^{-1}\phi_{ib})$. Similarly $K_2(z) = (\psi, (z - H(F))^{-1}\phi)$, $\Im z < 0$ has for $\Im z < bF$ a meromorphic continuation given by $K_2(z) = (\psi_{ib}, (z - H(F, -ib))^{-1}\phi_{-ib})$.

From the spectral theorem it follows that

$$(\psi, e^{-itH(F)}\phi) = \int_{-\infty}^{\infty} Q(\lambda)\, e^{-it\lambda}\, \mathrm{d}\lambda \tag{3.7}$$

where $Q(\lambda)$ is the spectral density. We have

$$\begin{aligned} Q(\lambda) &= \lim_{\delta\to 0} \frac{i}{2\pi}(\psi, [\lambda + i\delta - H(F))^{-1} - (\lambda - i\delta - H(F))^{-1}]\phi) \\ &= -(2\pi i)^{-1}(K_1(\lambda) - K_2(\lambda)), \quad \lambda \in \mathbb{R} \end{aligned} \tag{3.8}$$

Let us now take a such that $\alpha < aF < bF$. By Proposition 3.2 and assumption (c), the meromorphic continuation of $Q(\lambda)$ to $\mathbb{C}$, which is given by

$$Q(z) = -(2\pi i)^{-1}(K_1(z) - K_2(z))$$

is then analytic in the strip S_a and on the real axis. In addition, the argument of [He] shows that for $0 < \gamma < aF$ and $|E|$ large enough

$$Q(E - i\gamma) = \mathcal{O}(|E|^{-2}) \tag{3.9}$$

This allows us to shift the integration in (3.7) from the real axis downwards to the lower complex half-plane by

$$\lambda \to \lambda - i\,(\alpha + \epsilon) \qquad \alpha + \epsilon < aF$$

so that

$$\begin{aligned} (\psi, e^{-itH(F)}\phi) &= 2\pi i \sum_{-\Im E_j \le \alpha} \operatorname{Res} K_1(z)|_{z=E_j}\, e^{-itE_j} \\ &+ e^{-t(\alpha+\epsilon)} \int_{-\infty}^{\infty} Q(\lambda - i(\alpha+\epsilon))\, e^{-it\lambda}\, \mathrm{d}\lambda \end{aligned} \tag{3.10}$$

For the residues of $K_1(z)$ we have

$$\operatorname{Res} K_1(z)|_{z=E_j} = \frac{1}{2\pi i}\int_{|z-E_j|=\varepsilon} \mathrm{d}z(\psi_{-ib}, (z - H(F, ib))^{-1}\phi_{ib}) = (\psi_{-ib}, P_j(ib)\phi_{ib})$$

However, $f_j(z) = (\psi_{\bar z}, P_j(z)\phi_z)$ is by assumption an analytic function of z for $-F\Im z < \Im E_j$. Since $f_j(z)$ is constant for z real, we can conclude that $f_j(z)$ is independent of z as long as $-F\Im z < \Im E_j$. □

4 Green function of $H_1(F, ib)$

As already announced, we now proceed to the estimations of the Green function of the crossed fields Hamiltonian $H_1(F, ib)$. Results of this Section have a technical character and will be used in the announced forthcoming paper, in which we prove an upper bound on the resonance widths.

4.1 General solution

We want to find an upper bound on the Green function (and its first derivatives) of

$$H_1(ib) := H_1(F, ib) = -\partial_x^2 + (-i\partial_y - Bx)^2 - Fx - Fib \tag{4.1}$$

Since $H_1(ib)$ is translationally invariant in $y-$direction, it can be written as

$$H_1(ib) \simeq \int_{\mathbb{R}}^{\oplus} H_1(ib, k)\, \mathrm{d}k \tag{4.2}$$

where

$$H_1(ib, k) = -\partial_x^2 + (k - Bx)^2 - Fx - Fib \tag{4.3}$$

is the corresponding fiber Hamiltonian on $L^2(\mathbb{R}, \mathrm{d}x)$. Its spectral equation

$$H_1(ib, k)\psi(x, k) = z\psi(x, k) \tag{4.4}$$

can be solved explicitly to give two linearly independent solutions. Namely, with the notation

$$x(k) := x - \frac{k}{B} - \frac{F}{2B^2}, \quad z(k) := z + ibF + \frac{F}{B}k + \frac{F^2}{4B^2} \tag{4.5}$$

we get for $x(k) > 0$:

$$\begin{aligned}
\psi_1(x, k) &= e^{-Bx^2(k)/2}\, U\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) && (4.6)\\
\psi_2(x, k) &= e^{-Bx^2(k)/2}\, V\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) && (4.7)\\
&= e^{-Bx^2(k)/2}\sqrt{\pi}\left[\frac{M\left(\frac{B-z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right)}{\Gamma\left(\frac{3B-z(k)}{4B}\right)} + 2\sqrt{B}\,x(k)\,\frac{M\left(\frac{3B-z(k)}{4B}, \frac{3}{2}, B\,x^2(k)\right)}{\Gamma\left(\frac{B-z(k)}{4B}\right)}\right]
\end{aligned}$$

and for $x(k) \leq 0$:

$$\begin{aligned}
\psi_1(x, k) &= e^{-Bx^2(k)/2}\, V\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) && (4.8)\\
\psi_2(x, k) &= e^{-Bx^2(k)/2}\, U\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) && (4.9)
\end{aligned}$$

where U and M are solutions to Kummer's equation, see [AS, chap. 13]. Here we have followed the analysis made in [EJK] for purely magnetic Hamiltonian. Clearly, $V\left((B-z(k))/4B, 1/2, B\,x^2(k)\right)$ is analytical continuation of $U\left((B-z(k))/4B, 1/2, B\,x^2(k)\right)$ for $x(k)<0$. We note that $\psi_1(x,k) \in L^2([0,\infty))$ and $\psi_2(x,k) \in L^2((-\infty,0])$. The Green function of $H_1(ib,k)$ is thus given by

$$G(x,x';z,k) = \frac{\psi_1(x_>,k)\,\psi_2(x_<,k)}{W(\psi_1,\psi_2)} \tag{4.10}$$

with

$$x_> = \max(x,x'), \quad x_< = \min(x,x') \tag{4.11}$$

With the help of [AS, p. 505] one can calculate the Wronskian

$$W(\psi_1,\psi_2) = \sqrt{\pi B}\, 2^{\frac{3}{2}-\frac{z(k)}{2B}}\, \Gamma^{-1}\left(\frac{B-z(k)}{2B}\right) \tag{4.12}$$

Green's function of $H_1(ib)$ then reads

$$G_1(\mathbf{x},\mathbf{x}';z) = (\pi B)^{-1/2} \int_{\mathbb{R}} 2^{-\frac{3}{2}+\frac{z(k)}{2B}} \psi_1(x_>,k)\,\psi_2(x_<,k)\,\Gamma\left(\frac{B-z(k)}{2B}\right) e^{ik(y-y')}\,\mathrm{d}k \tag{4.13}$$

To discuss the convergence of the integral in the definition of $G_1(\mathbf{x},\mathbf{x}';z)$ we recall the behaviour of the hypergeometric functions U and M, see [AS, p. 504]. The latter gives the asymptotic of the integrand in (4.13) in the form:

$$e^{-k[|x'-x|-i(y'-y)]} \left(\frac{x-kB^{-1}}{x'-kB^{-1}}\right)^{\frac{z(k)}{2B}} \frac{1}{\sqrt{(x-kB^{-1})(x'-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})]$$

as $k \to \infty$, and

$$e^{k[|x'-x|-i(y'-y)]} \left(\frac{x'-kB^{-1}}{x-kB^{-1}}\right)^{\frac{z(k)}{2B}} \frac{1}{\sqrt{(x-kB^{-1})(x'-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})]$$

as $k \to -\infty$. Thus, for $x' \neq x$ the integral converges independently on the value of y', y, for in that case the asymptotic is given by

$$e^{-|k||x'-x|}\,\alpha(k)^k\,k^{-1}, \quad |k| \to \infty \tag{4.14}$$

with $\lim_{|k|\to+\infty} \alpha(k) = 1$. Similarly, when $y' \neq y$ the integral converges even for $x' = x$, since the asymptotic then reads

$$e^{-ik(y'-y)}\,\frac{1}{\sqrt{(x-kB^{-1})(x-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})], \quad |k| \to \infty, \tag{4.15}$$

and simple integration by parts shows that $G_1(\mathbf{x}, \mathbf{x}'; z)$ converges pointwise for any $y' \neq y$.

From the definition of hypergeometric functions and the construction of ψ_1 and ψ_2 it follows, that the product $\psi_1(x,k)\,\psi_2(x,k)$ is analytic w.r.t. k. The integrand of (4.13) is thus a meromorphic function with poles at

$$k_2 = -BF^{-1}(z_2 + bF), \quad k_1(n) = BF^{-1}\left[(2n+1)B - z_1 - F^2/(4B)\right], n \geq 0 \tag{4.16}$$

where we write $k = k_1 + ik_2$ and $z = z_1 + iz_2$. Moreover the integrand vanishes in the limit $|k_1| \to \infty$, see (4.14), (4.15). Therefore we can shift the integration to the lower complex half-plane by substituting

$$p := -\frac{k}{B} - \frac{F}{2B^2} - i\,\frac{z_2 + bF}{2F}\,\delta \quad , \quad \delta = \frac{y - y'}{|y - y'|}, \tag{4.17}$$

so that

$$x(p) = x + p + i\,\Delta, \quad x'(p) = x' + p + i\,\Delta, \quad \Delta = \frac{z_2 + bF}{2F}\,\delta \tag{4.18}$$

Since $U(a,b,t)$ is a many-valued function with a principal branch $-\pi < \arg t \leq \pi$, we have to consider its analytical continuation, see [AS, p. 504]. The fundamental solutions $\psi_1(x_>, p)$ and $\psi_2(x_<, p)$ will be given by different combinations of hypergeometric functions corresponding to different values of quasimomentum p;

1. For $p < -x' < -x$:

$$\psi_1(x',p) = e^{-Bx'^2(p)/2}\,V\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x'^2(p)\right) \tag{4.19}$$

$$\psi_2(x,p) = e^{-Bx^2(p)/2}\,U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x^2(p)\right) \tag{4.20}$$

2. For $-x' < p < -x$:

$$\psi_1(x',p) = e^{-Bx'^2(p)/2}\,U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x'^2(p)\right) \tag{4.21}$$

$$\psi_2(x,p) = e^{-Bx^2(p)/2}\,U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x^2(p)\right) \tag{4.22}$$

3. For $-x' < -x < p$:

$$\psi_1(x',p) = e^{-Bx'^2(p)/2}\,U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x'^2(p)\right) \tag{4.23}$$

$$\psi_2(x,p) = e^{-Bx^2(p)/2}\,V\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\,x^2(p)\right) \tag{4.24}$$

The Cauchy theorem now yields

$$G_1(\mathbf{x},\mathbf{x}';z) = (\pi B)^{-1/2} e^{-\frac{z_2+bF}{2F}\,|y-y'|}\, e^{-iF(y-y')/2+B(z_2+bF)^2/(4F)} \tag{4.25}$$
$$\times \int_{\mathbb{R}} 2^{-\frac{3}{2}+\frac{z(k(p))}{2B}}\, \psi_1(x',k(p))\, \psi_2(x,k(p))\, \Gamma\left(\frac{B-z(k(p))}{2B}\right)\, e^{ipB(y'-y)}\, \mathrm{d}p$$

with $k(p)$ defined through (4.17).

4.2 Long distances: $G_1(\mathbf{x},\mathbf{x}';z)$

Let us suppose, for definiteness, that $x' > x$ and examine the case where $|x'-x| > 1$. For x and x' we have to consider the following three cases: $x' > x > 0$, $x' > 0 > x$ and $0 > x' > x$. In each case we perform the integral (4.25) by dividing it in several pieces depending on the value of p. Before doing so we give some general estimates on the hypergeometric functions which will be used throughout the text.

Remark 4.1. *The symbol C below denotes a positive real number, which depends on the energy z, but not on the size of the electric field F.*

For the product $U(a,b,t)\,M(a,b,t)$ we use the asymptotic expressions, [AS, p. 504], and the corresponding estimate of the error term to get

$$\left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, V\left(\frac{B-z(p)}{4B},\frac{1}{2},B\,x'^2(p)\right)\, U\left(\frac{B-z(p)}{4B},\frac{1}{2},B\,x^2(p)\right)\Gamma\left(\frac{B-z(p)}{2B}\right)\right| \leq$$
$$C\,e^{Bx'^2(p)} \left|\frac{p+x+i\Delta}{p+x'+i\Delta}\right|^{z(p)/2B} B^{-1/2}|(x+p+i\Delta)(x'+p+i\Delta)|^{-1/2}\,[1+C\Delta^{-2}] \tag{4.26}$$

where we have used the doubling formula for the gamma function, [AS, p. 256]

$$\Gamma(2z) = \pi^{-\frac{1}{2}}\,2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) \tag{4.27}$$

Henceforth we will work only with the leading term and drop the factor $[1+C\Delta^{-2}]$. Moreover, as the asymptotic behaviour of both summands in the definition of V is identical, we will consider only the first one.

The following bound can be easily found

$$|(x+p+i\Delta)(x'+p+i\Delta)|^{-1/2} \leq \Delta^{-1}\,. \tag{4.28}$$

We have

$$\left|\frac{p+x+i\Delta}{p+x'+i\Delta}\right|^{z(p)/2B} \quad = \quad \left(1+\frac{(x-x')^2}{(p+x')^2+\Delta^2}+\frac{2(x'-x)(p+x')}{(p+x')^2+\Delta^2}\right)^{\frac{\tilde{z}_1-Fp}{4B}} \tag{4.29}$$

with $\tilde{z}_1 = z_1 - F^2/4B^2$. Remark that $|\cdots| > 1$, thus for $\tilde{z}_1 \leq 0$ and $p \geq 0$ this term can be neglected. For $\tilde{z}_1 > 0$ we can apply the following inequality

$$1+\frac{(x-x')^2}{(p+x')^2+\Delta^2}+\frac{2(x'-x)(p+x')}{(p+x')^2+\Delta^2} \leq 1+\frac{2(x-x')^2}{\Delta^2}\,. \tag{4.30}$$

For $p<0$ we write $|\cdots|^{-\frac{Fp}{2B}} = e^{-\frac{Fp}{2B}\ln|\cdots|}$. Finally, note that the same result holds true if we interchange x and x', which correspond to interchange the functions U and V.

Let $x' > x > 0$

We divide the interval of integration in five parts as follows

$$\mathbb{R} = (-\infty,-2x'] \cup (-2x',-x'] \cup (-x',-x] \cup (-x,-x/2] \cup (-x/2,\infty)$$

For $p \in (-\infty,-2x']$:
Keeping in mind that $F \to 0$ one gets from (4.26)

$$\begin{aligned}
&\int_{-\infty}^{-2x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right)\right| \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}}\,\Delta^{-1}\left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2-x^2)} \int_{-\infty}^{-2x'} e^{p\,B(x'-x)}\, e^{\frac{-Fp}{4B}\,\ln|\cdots|}\,\mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}}\,\Delta^{-1}\left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2-x^2)} \int_{-\infty}^{-2x'} e^{pB(x'-x)/2}\,\mathrm{d}p \\
\leq\ & \frac{C}{B^{3/2}}\Delta^{-1}\left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2}
\end{aligned} \tag{4.31}$$

For $p \in (-x/2,\infty)$:

(4.26) (with x and x' interchanged) and the bounds given before lead to

$$\begin{aligned}
&\int_{-x/2}^{\infty} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, \psi_1(x',x,p)\, \psi_2(x',x,p) \Gamma\left(\frac{B-z(p)}{2B}\right) \right| \,\mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}(x^2-x'^2)} \times \\
\times\ & \left\{ \int_{-x/2}^{0} e^{-Bp(x'-x)} e^{-\frac{Fp}{2B}\ln|\ldots|} \,\mathrm{d}p + \int_0^{\infty} e^{-Bp(x'-x)} \,\mathrm{d}p \right\} \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}(x^2-x'^2)} \left\{ \int_{-x/2}^{0} e^{-2Bp(x'-x)} \,\mathrm{d}p + \int_0^{\infty} e^{-Bp(x'-x)} \,\mathrm{d}p \right\} \\
\leq\ & \frac{C}{B^{3/2}} \Delta^{-1} \left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} 2e^{-\frac{B}{2}(x'-x)^2} \qquad (4.32)
\end{aligned}$$

For $p \in (-2x', -x']$*:*

Here the estimate (4.26) does not give us the sought result. Instead we will rewrite the corresponding part of the integration in (4.25) in the following way,

$$\begin{aligned}
&\int_{-2x'}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right) \right| \,\mathrm{d}p \\
&\equiv \Delta^{-1}\, x'^{-1}\, (x'-x)^{\frac{z_1}{2B}} e^{-\frac{B}{4}(x'-x)^2} \int_{-2x'}^{-x'} \Phi(x',x,p)\,\mathrm{d}p \qquad (4.33)
\end{aligned}$$

and look at the maximum of the function $\Phi(x',x,p)$ in the interval $[-2x',-x']$. We denote the maximum value by $\Phi_0(x',x)$. In particular we want to show that Φ_0 is bounded above by certain function of F, which does not grow faster than a power function of F^{-1} as $F \to 0$. To be more precise, we want to show, that there exist some positive constants Θ_0, θ_1, such that

$$|\Phi(x',x,p)| \leq \Theta_0\, F^{-\theta_1}$$

holds uniformly for $p \in (-2x', -x']$ and F small enough. This procedure will used below also for other values of p.

We recall the asymptotic properties of the gamma function, see [AS, p. 257]

$$\Gamma(az+b) \sim \sqrt{2\pi}\, e^{-az}\, (az)^{az+b-\frac{1}{2}}, \quad |z| \to \infty,\ |\arg z| < \pi,\ a > 0 \qquad (4.34)$$

It is then easy to see, that $\Phi(x',x,p)$ is bounded at the endpoints of the interval $[-2x',-x']$. We can thus confine ourselves to the case when Φ acquires its maximum inside the considered interval. Let us denote the corresponding extremal point by

$$p_0(x') = -x' - j(x')$$

First of all we note that if $j(x')$ is bounded, one can show the boundedness of $\Phi(x',x,p_0(x'))$ in the same way as that of $\Phi(x',x,-x')$. Without loss we may thus assume that $j(x')$ is unbounded. We shall distinguish two different situations according to different behaviour of the function $j(x')$.

1. $j^2(x')/x'$ bounded as $x' \to \infty$. In this case the first parameter of
$$M\left(\frac{B - z(p_0(x'))}{4B}, \frac{1}{2}, B\, x'^2(p_0(x'))\right) \tag{4.35}$$
does not grow more slowly than its argument, for
$$\begin{aligned} z(p_0(x')) &= z_1 + F(x' + j(x')) - \frac{F^2}{4B^2} + \frac{i}{2}(z_2 + bF)(2-\delta) && (4.36)\\ B\, x'^2(p_0(x')) &= B\,(j(x') + i\Delta)^2. && (4.37)\end{aligned}$$
We observe that in our case real parts of $z(p_0(x'))$ and $x'^2(p_0(x'))$ increase faster than their imaginary parts in the limit $x' \to \infty$. It then follows from the definition of function M, [AS, p. 504], that the behaviour of (4.35) at infinity will be governed by
$$M\left(\frac{B - \Re z(p_0(x'))}{4B}, \frac{1}{2}, \Re B\, x'^2(p_0(x'))\right) \tag{4.38}$$
The application of a suitable asymptotic expansion, [Bu, p. 105], also [AS, p. 509, 13.5.21], thus gives us the following inequality for $x' \to \infty$
$$\left| M\left(\frac{B - \Re z(p_0(x'))}{4B}, \frac{1}{2}, \Re B\, x'^2(p_0(x'))\right)\right| \le C\, F^{-1}\, e^{\frac{j^2(x')}{2}} \tag{4.39}$$
Recalling (4.34) we can conclude that
$$\begin{aligned}&\Phi(x',x,p_0(x')) \le C\,\Delta\, x' \,\exp\left[-\frac{B}{4}\,\left((x'-x)^2 + 2j^2(x') + 4j(x')(x'-x)\right)\right]\\ &|\,B(x' - x + j(x'))|^{\frac{F(x'+j(x'))}{2B}}\,\left|\Gamma\left(\frac{B - z(p_0(x'))}{4B}\right)\right| \qquad (4.40)\end{aligned}$$
is bounded above by a constant times $\Delta\, F^{-1}$.

2. $j^2(x')/x'$ unbounded. Here we can use again (4.26) and the boundedness of $\Phi(x',x,p_0(x'))$ then follows after some elementary manipulations.

To sum up we have
$$\begin{aligned}&\int_{-2x'}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, \Gamma\left(\frac{B - z(p)}{2B}\right)\right| \mathrm{d}p\\ &\le C\,(F^{-1} + \Delta^{-1})\,(x'-x)^{\frac{z_1}{2B}}\, e^{-\frac{B}{4}\,(x'-x)^2} \qquad (4.41)\end{aligned}$$

For $p \in (-x, -x/2]$:
Same estimations as for $p \in (-2x', -x']$.

For $p \in (-x', -x]$:
We show that the function to be integrated is bounded by some constant uniform in x, x' times $e^{-\frac{B}{4}(x-x')^2}$. At the boundary it has been shown above that the function is bounded, we suppose that there is an extremal point $p_0 = p_0(x, x') \in (-x', x]$. Denote

$$d(x, x') = |p_0 + x| \qquad \text{and} \qquad d'(x, x') = |p_0 + x'|$$

the distances between the end points and the extremum p_0.
We have to consider the following cases, which correspond to the different behaviours of the argument of U: $d(x, x')$ unbounded, $d(x, x') < C$ and the same for $d'(x, x')$.

1) $d(x, x')$, $d'(x, x')$ unbounded: we have for $p = p_0$

$$\begin{aligned} \mathcal{A}_1(x, x') &:= e^{\frac{B}{4}(x+p_0+i\Delta)^2} \sqrt{\left|W^{-1}\left(\psi_1, \psi_2\right)\right|} \, |\psi_1(x, p)| \\ &= \left| 2^{\frac{z(p_0)}{4B}} e^{-\frac{B}{4}(x+p_0+i\Delta)^2} B(x + p_0 + i\Delta)^{\frac{z(p)-B}{2B}} \right| \left| \Gamma\left(\frac{B - z(p_0)}{2B}\right) \right|^{1/2} \\ &\leq 2^{\frac{\tilde{z}_1 - Fp_0}{4B}} e^{-\frac{B}{4}(x+p_0)^2} (B\,|x + p_0 + i\Delta|)^{\frac{\tilde{z}_1 - B}{2B}} \left| \Gamma\left(\frac{B - \tilde{z}_1 - Fp_0}{2B} + i\eta\right) \right|^{1/2} \end{aligned} \tag{4.42}$$

where η denote the imaginary part of the argument in the gamma function. $\mathcal{A}_2(x, x')$ is defined in the same way where ψ_1 is replaced with ψ_2 and x, x' are interchanged. In the limit $x', x \to \infty$ we consider the following cases.
a)

$$B(d^2(x, x') + \Delta^2), \; B(d'^2(x, x') + \Delta^2) > \nu_0 \frac{\tilde{z}_1 - Fp_0}{4B} : \tag{4.43}$$

where $\nu_0 = 4(1 + \ln 2) f_0^{-1} > 1$ and $f_0 > 0$ is the global minimum of $(1 - t \ln(2/t))$ for $t \geq 0$. Using the asymptotic properties of the gamma function we get for the leading term of (4.42):

$$\exp\left\{ -\frac{B}{4}(x + p_0)^2 \left[1 + f(x, x') \ln\left(-2f^{-1}(x, x')\right)\right] + (1 + \ln 2)\, \frac{\tilde{z}_1 - Fp_0}{4B} \right\} \tag{4.44}$$

where

$$f(x, x') = \frac{F\, p_0(x, x')}{B^2\,(x + p_0(x, x'))^2} < 0 \tag{4.45}$$

The boundedness of $\mathcal{A}_1(x, x')$ follows from (4.43). Same analysis for $\mathcal{A}_2(x, x')$ then gives

$$\begin{aligned} &\left|\, \psi_1(x', x, p)\, \psi_2(x', x, p)\, W^{-1}(\psi_1, \psi_2) \right| \leq e^{-\frac{B}{4}(x+p_0)^2} e^{-\frac{B}{4}(x'+p_0)^2} \mathcal{A}_1 \mathcal{A}_2 \\ &\leq C\, e^{-\frac{B}{8}\,(x'-x)^2} \end{aligned} \tag{4.46}$$

To continue we recall again the asymptotic behaviour of $U(a,b,z)$, see [AS, p. 504], to assure that

$$\begin{aligned}&\left|U\left(\frac{B-z(p_0(x',x))}{4B},\frac{1}{2},B\,(x'+p+i\Delta)^2)\right)\right| \leq \\ &C\left|U\left(\frac{B-\Re z(p_0(x',x))}{4B},\frac{1}{2},B\,((x'+p)^2+\Delta^2)\right)\right|\,[1+C\Delta^{-2}] \qquad (4.47)\end{aligned}$$

Let us now consider
b)

$$B(d^2(x,x')+\Delta^2) > \nu_0\,\frac{\tilde z_1 - Fp_0}{4B}, \quad B(d'^2(x,x')+\Delta^2) = \nu\,\frac{\tilde z_1 - Fp_0}{4B},\ \nu\in[1,\nu_0],$$

in which case the part corresponding to $\mathcal{A}_1(x,x')$ can be treated as above and for the rest of the integrand we use [AS, p. 509, 13.5.20] to get

$$\left|e^{-\frac{B}{2}(x'+p_0)^2}U\left(\frac{B-\Re z(p_0(x'))}{4B},\frac{1}{2},B\,((x'+p)^2+\Delta^2)\right)\right| \leq C\,e^{-\frac{B}{4\nu}\,(x'+p_0)^2} \qquad (4.48)$$

and consequently

$$\begin{aligned}&\left|\,\psi_1(x',x,p)\,\psi_2(x',x,p)\,W^{-1}(\psi_1,\psi_2)\right| \leq e^{-\frac{B}{4}(x+p_0)^2}e^{-\frac{B}{4\nu}(x'+p_0)^2}\mathcal{A}_1\\ &\leq C\,e^{-\frac{B}{8\nu}\,(x'-x)^2} \qquad (4.49)\end{aligned}$$

c)

$$B(d^2(x,x')+\Delta^2) \geq \frac{\tilde z_1 - Fp_0}{4B}, \quad B(d'^2(x,x')+\Delta^2) < \frac{\tilde z_1 - Fp_0}{4B}, \qquad (4.50)$$

The part which includes $\psi_1(x,p)$ can be controlled by one of the estimates given above. For the second part we observe that, [AS, p. 509, 13.5.22], $|\psi_2(x',p)|$ is uniformly bounded for p in $(-x',-x]$. The properties of gamma function then lead to the following inequality for the Wronskian

$$\begin{aligned}|W^{-1/2}\,(\psi_1,\psi_2)| \quad &\leq C\,\exp\left[\frac{Fp_0}{4B}(\ln(\sqrt{-Fp_0/2B})-1-\ln 2)\right]\,e^{\frac{Fp_0}{4B}\,\ln(\sqrt{-Fp_0/2B})}\,\left|\frac{Fp_0}{2B}\right|^{-\frac{\tilde z_1}{4B}}\\ &\leq C\,\exp[-B((x'+p_0)^2+\Delta^2)(\ln(\sqrt{(x'+p_0)^2+\Delta^2})-1-\ln 2)]\\ &\leq C\,e^{-B(x'+p_0)^2}, \qquad (4.51)\end{aligned}$$

so that

$$\begin{aligned}&\left|\,\psi_1(x',x,p)\,\psi_2(x',x,p)\,W^{-1}(\psi_1,\psi_2)\right| \leq e^{-\frac{B}{4\nu}(x+p_0)^2}\,|W^{-1/2}(\psi_1,\psi_2)|\\ &\leq C\,e^{-\frac{B}{8\nu}\,(x'-x)^2} \qquad (4.52)\end{aligned}$$

d)

$$B(d^2(x,x') + \Delta^2) < \frac{\tilde{z}_1 - Fp_0}{4B}, \quad B(d'^2(x,x') + \Delta^2) < \frac{\tilde{z}_1 - Fp_0}{4B}$$

Here both the functions $|\psi_2(x',p)|$ and $|\psi_1(x,p)|$ are uniformly bounded and the exponential decay then comes from the Wronskian in the same way as in the case *c*).

2) One of $d(x,x')$, $d'(x,x')$ bounded.

Let us suppose for definiteness, that $d(x,x')$ is bounded. At the point $p = p_0(x,x')$ we apply again (4.47) and [AS, p. 508, 13.5.16] to find that

$$|\psi_1(x,p)| \leq C \left| \Gamma\left(\frac{1}{2} - \frac{B - z(p_0(x',x))}{4B}\right) \right| \tag{4.53}$$

For the function $\psi_2(x',p)$ and for the Wronskian we use the suitable estimate given above in one of the cases *a*), *b*), *c*), *d*), which gives the desired result.

In all these cases the same analysis can be made when $d(x,x')$ and $d'(x,x')$ interchange their roles.

3) Both $d(x,x')$ and $d'(x,x')$ bounded.

Since this can only happen when $|x'-x| \leq C$, it suffices to show that the integrand is bounded. The latter however follows immediately from (4.53) and

$$\left| \Gamma^2\left(\frac{1}{2} - \frac{B - z(p_0(x',x))}{4B}\right) W^{-1}(\psi_1,\psi_2) \right| \leq C, \quad \forall\, p \in (-x',-x]$$

Finally we conclude that there exists certain constant $\omega > 0$, which depends on B but not on F, such that

$$\int_{-x'}^{-x} |\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2)|\, \mathrm{d}p \leq \int_{-x'}^{-x} e^{-\frac{B}{4}(x+p_0)^2} e^{-\frac{B}{4}(x'+p_0)^2} \mathcal{A}_1 \mathcal{A}_2 \, \mathrm{d}p$$
$$\leq C\, \Delta^{-1}\, (x'-x)\, e^{-\omega\, (x'-x)^2} \tag{4.54}$$

Remark 4.2. *We do not present the analysis of all the possible combinations, because the in the remaining cases one can proceed in a completely analogous way as above.*

Let $x' > 0 > x$

In this case we divide the interval of integration in four parts as

$$\mathbb{R} = (-\infty,-2x'] \cup (-2x',-x'] \cup (-x',-x] \cup (-x,\infty)$$

The intervals $(-\infty, -2x']$, $(-2x', -x']$ can be treated exactly as in the previous case. For $p \in (-x, \infty)$ we proceed in the same way as for $p \in (-x/2, \infty)$ in the previous case, keeping in mind that since $x < 0$ one has $p > 0$.

For $p \in (-x', -x]$ we separate the analysis of the integrand in two pieces.
(1) $p \in (-x', 0]$: Same argument as for the interval $(-x', -x]$ when x', x are both positive.
(2) $p \in (0, -x]$: We divide the interval in $(0, p_c + 1] \cup (p_c + 1, -x]$, where $p_c = \frac{\tilde{z}_1 - B}{F}$. For $p > p_c$ we have $\Re a(p) > 0$ with $a(p)$ the first parameter of the function U. In this case we can use the integral representation of U to get [DMP]

$$|U(a(p), \tfrac{1}{2}, \rho(p))| \leq \frac{C}{\Re a(p)} |\Gamma(a(p))|^{-1} \qquad \text{for } \Re\rho(p) > 0, \;\; \Re a(p) > 0 \tag{4.55}$$

In $(0, p_c + 1]$ the analysis of the maximum of

$$|x' + p + i\Delta|^2 |x + p + i\Delta|^2$$

shows that it is a power function in $(x' - x)$. Thus, since the Γ function remains in this interval bounded, we get the bound $e^{-\frac{B}{2}(x'-x)^2}$ times a polynomial in $(x' - x)$.
In $(p_c + 1, -x - |\Delta|]$ we use the bounds (4.55) and the asymptotic behaviour of the gamma function to get a uniform upper bound. In $(-x - |\Delta|, -x]$ we use (4.55) for the function U depending on x' while for the other U we use its expression in term of a sum of function M. In this case we get a uniform estimate since the argument of M is bounded.

Let $0 > x' > x$

We divide the interval of integration in four parts as follows

$$\mathbb{R} = (-\infty, 0] \cup (0, -x'] \cup (-x', -x] \cup (-x, \infty)$$

For the interval $(-x, \infty)$ the remarks above hold. When $p \in (-x', -x]$ a slight modification of the analysis done in $(0, -x]$ above leads to the desired bound.

For $p \in (-\infty, 0]$:

$$\begin{aligned}
&\int_{-\infty}^{0} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x',x,p)\, \psi_2(x',x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right)\right| \, \mathrm{d}p \\
\leq \;& \frac{C}{\sqrt{B}}\, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2-x^2)} \int_{-\infty}^{0} e^{p\,B(x'-x)}\, e^{\frac{-Fp}{4B}\,\ln|\cdots|}\, \mathrm{d}p \\
\leq \;& \frac{C}{\sqrt{B}}\, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2-x^2)} \int_{-\infty}^{0} e^{pB(x'-x)/2}\, \mathrm{d}p \\
\leq \;& \frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2} \qquad (4.56)
\end{aligned}$$

For $p \in (0, -x']$:

$$\begin{aligned}
&\int_{0}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x',x,p)\, \psi_2(x',x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right)\right| \, \mathrm{d}p \\
\leq \;& \frac{C}{\sqrt{B}}\, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2-x^2)} \int_{0}^{-x'} e^{p\,B(x'-x)}\, \mathrm{d}p \\
\leq \;& 2\frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2} \qquad (4.57)
\end{aligned}$$

Let us finally formulate the results in

Lemma 4.1. *For* F *small enough and* $|x'-x| \geq 1$ *there exist some strictly positive constants* C_1, C_2, $\tilde{\omega}$, *which depends on* B *and* z, *such that the following inequality holds true*

$$|G_1(\mathbf{x},\mathbf{x};z)| \leq C_1\, \Delta^{-1} e^{-\Delta\,|y-y'|}\, e^{-\tilde{\omega}\,(x'-x)^2} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} \left[1 + C_2 \Delta^{-2}\right] \qquad (4.58)$$

with $\Delta = \frac{z_2+bF}{2F}$.

4.3 Long distances: $\partial_{x,y}\, G_1(\mathbf{x},\mathbf{x}';z)$

In this section we want to prove similar result to that one described in Lemma 4.1 also for the derivatives of the Green function w.r.t. x and y. We suppose again that $x' > x$ and $|x'-x| > 1$. As we have already seen the most general and complicated case is the one where $x', x > 0$ and the all the others can be regarded as its simplification. Therefore here we confine ourselves to the situation when both x', x are positive.

We start with the derivative w.r.t. x. For $|x'-x| > 1$ the integral

$$\int_{\mathbb{R}} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \psi_1(x',p)\, \partial_x \psi_2(x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right)\right| \, \mathrm{d}p$$

converges uniformly with respect to x, see (4.14). We can thus interchange the differentiation and integration in (4.25) to get the following inequality for the derivative of $G_1(\mathbf{x},\mathbf{x};z)$:

$$\begin{aligned} &|\partial_x\, G_1(\mathbf{x},\mathbf{x};z)| \le \\ &C\, e^{-\Delta|y'-y|} \int_{\mathbb{R}} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \psi_1(x',p)\, \partial_x \psi_2(x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right)\right| \,\mathrm{d}p \end{aligned} \tag{4.59}$$

We split again the integration in (4.25) into five intervals:

$$\mathbb{R} = (-\infty,-2x'] \cup (-2x',-x'] \cup (-x',-x] \cup (-x,-x/2] \cup (-x/2,\infty)$$

and use [AS, p. 507, 13.4.8/21] to calculate the derivatives of hypergeometric functions. When $p \in (-x/2,\infty)$ we get for the corresponding integrand in (4.59)

$$\begin{aligned} &-B(x+p+i\Delta)\,\psi_1(x',p)\,\psi_2(x,p)\,W^{-1}(\psi_1,\psi_2) + 2B(x+p+i\Delta)e^{-B(x+p+i\Delta)^2/2}\,a(p)\sqrt{\pi} \\ &\times\left[\frac{M(a(p)+1,\frac{3}{2},B(x+p+i\Delta)^2)}{\frac{1}{2}\,\Gamma(a(p)+1/2)} + 2\sqrt{B}(x+p+i\Delta)\,\frac{M(a(p)+\frac{3}{2},\frac{5}{2},B(x+p+i\Delta)^2)}{\frac{3}{2}\,\Gamma(a(p))}\right. \\ &\left. +2\sqrt{B}\,\frac{M(a(p)+1,\frac{3}{2},B(x+p+i\Delta)^2)}{a(p)\Gamma(a(p))}\right]\psi_1(x',p)\,W^{-1}(\psi_1,\psi_2) \end{aligned} \tag{4.60}$$

where

$$a(p) = \frac{B-z(p)}{4B}\,. \tag{4.61}$$

The first term can be controlled in the same way as the Green function itself due to (4.26) and the fact that

$$\left|\frac{x+p+i\Delta}{x'+p+i\Delta}\right|^2 \le 1 + \frac{2(x-x')^2}{\Delta^2} \tag{4.62}$$

As for the term which includes the derivative of the function M, using [AS, p. 504] and $\Gamma(a+1) = a\Gamma(a)$, we note that the asymptotic behaviour of

$$\frac{a(p)\,M(a(p)+1,\frac{3}{2},B(x+p+i\Delta)^2)}{\Gamma(a(p)+1/2)}\,W^{-1}(\psi_1,\psi_2) \tag{4.63}$$

is the same as that of

$$\frac{M(a(p),\frac{1}{2},B(x+p+i\Delta)^2)}{\Gamma(a(p)+1/2)}\,W^{-1}(\psi_1,\psi_2) \tag{4.64}$$

The rest of the analysis is then identical with the case of $G_1(\mathbf{x},\mathbf{x}';z)$ itself.

For $p < -x'$ are x, x' interchanged and we have to differentiate the function U:

$$\partial_x U\left(a(p), \frac{1}{2}, B(x+p+i\Delta)^2\right) = -2B(x+p+i\Delta)\, a(p)\, U\left(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2\right) \tag{4.65}$$

The pre-factor $(x+p+i\Delta)$ is again well controlled due to (4.62). In addition we observe that for the product

$$a(p)\, U\left(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2\right)\; V\left(a(p), \frac{1}{2}, B(x+p+i\Delta)^2\right) \tag{4.66}$$

we get the upper bound (4.26) multiplied by

$$\left|\frac{a(p)}{(x+p+i\Delta)^2}\right| \tag{4.67}$$

and that for $p < -2x'$ is the latter uniformly bounded w.r.t. to x, x'. Thus, for $x \in (-\infty, -2x']$ we can use the same estimations as for $G_1(\mathbf{x}, \mathbf{x}'; z)$.

For $p \in (-2x', -x'] \cup (-x, -x/2]$ we multiply the function $\Phi(x', x, p)$ introduced in (4.33) by $a(p)$, which leads to an additional factor F^{-1} in the estimate (4.41).

Similarly is for $p \in (-x', -x]$ the factor (4.67), coming from the derivative of U, controlled by the decay of the upper bounds that we have found above. More exactly, for the case 1*a*) we see from the inequality (4.43) that (4.67) is uniformly bounded in the interval $(-x', -x]$. The case 1*b*) is treated in an analogous way. As for 1*c*), we note that

$$a(p_0)\, e^{-\frac{B}{8}(x+p_0)^2}$$

is bounded due to (4.50). The result then follows from (4.46). When the inequalities of the case 1*d*) hold, then following (4.51) we get

$$|W^{-1}(\psi_1, \psi_2)\, a(p_0)| \le C e^{-\frac{B}{2}(x'+p_0)^2}\, e^{-\frac{B}{2}(x+p_0)^2},$$

which gives again the exponential decay of the integrand. In the cases 2) and 3) we proceed in the same way as for the Green function itself noting that both

$$\left|a(p_0)\Gamma(1/2 - a(p_0)) W^{-1/2}(\psi_1, \psi_2)\right|, \quad \left|a(p_0)\Gamma^2(1/2 - a(p_0)) W^{-1}(\psi_1, \psi_2)\right|$$

are uniformly bounded. We thus conclude that

$$\left|\partial_x\, \psi_1(x', x, p)\, \psi_2(x', x, p)\, W^{-1}(\psi_1, \psi_2)\right| \le C\, e^{-\frac{B}{16\nu}(x'-x)^2} \tag{4.68}$$

for $p \in (-x', -x]$.

Same arguments can be then used for $\partial_y\, G_1(\mathbf{x},\mathbf{x}';z)$. Since the substitution $k\to p$ is not analytic in y, the differentiation w.r.t. y has to be done before this substitution is made. In other words, we have to differentiate the formula (4.13) and then substitute p for k through (4.17). This leads to a multiplication of the integrand in (4.59) by the factor Bp, which is well controlled by the previously given arguments, noting that

$$\left|\frac{p}{\sqrt{(x+p+i\Delta)(x'+p+i\Delta)}}\right|$$

is uniformly bounded on $(-\infty,-2x']\cup(-x/2,\infty)$.

Finally we get

Lemma 4.2. *For F small enough and $|x'-x|\geq 1$ there exist some strictly positive constants C_3, C_4, $\tilde{\omega}$, which depends on B and z, such that the following inequality holds true*

$$|\partial_{x,y}\, G_1(\mathbf{x},\mathbf{x}';z)| \leq C_3\, F^{-2}\, \Delta^{-1} e^{-\Delta\,|y-y'|}\, e^{-\tilde{\omega}\,(x'-x)^2}\left[1+\frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}+\frac{1}{4}}[1+C_4\Delta^{-2}] \tag{4.69}$$

with $\Delta=\frac{z_2+bF}{2F}$.

4.4 Short distances

Up to now we have considered that $|x'-x|\geq 1$ and $|y'-y|$ was arbitrary. Here we want to investigate the case where $|x'-x|<1$ for any value of $|y'-y|$. Since our system is two-dimensional, we expect the Green function $G_1(\mathbf{x},\mathbf{x}';z)$ to have a logarithmic singularity as $x\to x'$ and $y\to y'$ of the following type:

$$G_1(\mathbf{x},\mathbf{x}';z)\sim\ln(|\mathbf{x}'-\mathbf{x}|)$$

Our goal in this section is to show that

$$\int_{\mathbb{R}}\int_{|x'-x|\leq 1}|\partial^n_{x,y}G_1(\mathbf{x},\mathbf{x}';z)|e^{\frac{\Delta}{2}|y-y'|}\,\mathrm{d}x'\,\mathrm{d}y' \quad n=0,1 \tag{4.70}$$

is bounded as a function of x and y. We will work only with the derivatives of $G_1(\mathbf{x},\mathbf{x}';z)$, noting that same arguments then apply also to $G_1(\mathbf{x},\mathbf{x}';z)$ itself.

We divide the real axis as above and present again only the case $x',x>0$.

$\underline{\partial_x G_1(\mathbf{x},\mathbf{x}';z)}$

From the asymptotic expansion for the integrand of $G_1(\mathbf{x}, \mathbf{x}; z)$, see (4.14), (4.15), it follows that

$$\int_{\mathbb{R}} |\partial_x \, \psi_1(x', x, p) \, \psi_2(x', x, p) \, W^{-1}(\psi_1, \psi_2)| \, \mathrm{d}p$$

converges only if $x' \neq x$. This reflects the usual behaviour of the Green function, i.e. the discontinuity of the derivative for $x' = x$. We will thus investigate $\partial_x G_1(\mathbf{x}, \mathbf{x}'; z)$ separately for $(x' - x)$ in the compacts of $(0, 1)$ and $(-1, 0)$.

Assume first that $(x' - x) \in (0, 1)$. For the derivative w.r.t. x we write

$$|\partial_x \, G_1(\mathbf{x}, \mathbf{x}'; z)| = C \, e^{-\Delta |y' - y|} \left| \int_{\mathbb{R}} g(x', x, p) \, e^{ipB(y' - y)} \, \mathrm{d}p \right| \tag{4.71}$$

where for $p > -x$

$$g(x', x, p) = \psi_1(x', p) \, \partial_x \psi_2(x, p) \, W^{-1}(\psi_1, \psi_2) \tag{4.72}$$

Let us perform first the integration in the interval $p \in (-x/2, \infty)$. We have

$$\begin{aligned}
&\partial_x \psi_2(x, p) = -B(x + p + i\Delta) \, \psi_2(x, p) + e^{-\frac{B}{2}(x+p+i\Delta)^2} \, \partial_x V\left(a(p), \frac{1}{2}, B(x + p + i\Delta)^2\right) \\
&=: \phi_1(x, p) + \phi_2(x, p)
\end{aligned} \tag{4.73}$$

Using the asymptotic expansions for M and U and integrating by parts we find

$$\begin{aligned}
&\left| \int_{-x/2}^{\infty} \psi_1(x', p) \, \phi_1(x, p) \, W^{-1}(\psi_1, \psi_2) \, e^{ipB(y' - y)} \, \mathrm{d}p \right| = C \, e^{-B(x'^2 - x^2)/2} \\
&\times \left| \int_{-x/2}^{\infty} e^{-pB[(x' - x) - i(y' - y)]} \left(\frac{p + x' + i\Delta}{p + x + i\Delta} \right)^{\frac{z(p)}{2B}} \frac{p + x + i\Delta}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right. \\
&\left. [1 + \mathcal{O}(|p + x + i\Delta|^{-2})][1 + \mathcal{O}(|p + x' + i\Delta|^{-2})]] \, \mathrm{d}p \right| \\
&\leq \frac{C}{B|(x' - x) - i(y' - y)|} \left[\Delta^{-1} + e^{-B(x'^2 - x^2)/2} \int_{-x/2}^{\infty} e^{-pB(x' - x)} \, w(x', x, p) \, \mathrm{d}p \right] [1 + C \, \Delta^{-2}]
\end{aligned} \tag{4.74}$$

where

$$w(x', x, p) = \partial_p \left\{ \left(\frac{p + x' + i\Delta}{p + x + i\Delta} \right)^{\frac{z(p)}{2B}} \frac{p + x + i\Delta}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right\} \tag{4.75}$$

Here we have used the fact that the integrand of (4.74) is an analytic function of p and therefore we can differentiate the term

$$[1 + \mathcal{O}(|p + x + i\Delta|^{-2})][1 + \mathcal{O}(|p + x' + i\Delta|^{-2})]]$$

w.r.t. p. It then follows from the Cauchy formula, that the derivative is an $L^1[(-x/2,\infty)]$ function with the corresponding norm smaller than a constant times Δ^{-1}. The first term on the last line of (4.74) gives the expected result. The point is now that, as one can easily verify, the function $w(x',x,p)$ is proportional to $(x'-x)$ in the sense that

$$\frac{w(x',x,p)}{x'-x}$$

is uniformly bounded. In other words

$$\left| e^{-B(x'^2-x^2)/2} \int_{-x/2}^{\infty} e^{-pB(x'-x)}\, w(x',x,p)\,\mathrm{d}p \right| \le C \tag{4.76}$$

and

$$\left| \int_{-x/2}^{\infty} \psi_1(x',p)\,\phi_1(x,p)\,W^{-1}(\psi_1,\psi_2)\,e^{ipB(y'-y)}\,\mathrm{d}p \right| \le \frac{C\,\Delta^{-1}}{|(x'-x)-i(y'-y)|}\,[1+C\,\Delta^{-2}] \tag{4.77}$$

All constants in the latter inequality are uniform for $(x'-x)$ in the compacts of $(0,1)$. Same analysis can be made also for the term $\phi_2(x,p)$, which includes the derivative of the function M, see the remarks below (4.62).

For p in the interval $(-\infty,-2x']$ are x' and x interchanged and we have

$$g(x',x,p) = \psi_2(x',p)\,\partial_x\psi_1(x,p)\,W^{-1}(\psi_1,\psi_2) \tag{4.78}$$

so that $\phi_1(x,p)$ is unchanged and instead of $\phi_2(x,p)$ we get

$$\tilde{\phi}_2(x,p) = e^{\frac{B}{2}(x+p+i\Delta)^2}\,\partial_x U\left(a(p),\frac{1}{2},B(x+p+i\Delta)^2\right) \tag{4.79}$$

Using (4.65) and (4.67) we can proceed as above replacing $w(x',x,p)$ with

$$\begin{aligned}\tilde{w}(x',x,p) &= w(x',x,p)\,\frac{a(p)}{(x+p+i\Delta)^2} \\ &+ \left(\partial_p\,\frac{a(p)}{(x+p+i\Delta)^2}\right)\left(\frac{p+x+i\Delta}{p+x'+i\Delta}\right)^{\frac{z(p)}{2B}}\,\frac{p+x+i\Delta}{\sqrt{(p+x+i\Delta)(p+x'+i\Delta)}}\end{aligned} \tag{4.80}$$

It is now sufficient to realize that

$$\partial_p\left(\frac{a(p)}{(x+p+i\Delta)^2}\right) \in L^1((-\infty,-2x']) \tag{4.81}$$

with the corresponding L^1 norm being uniformly bounded from above by a constant times Δ^{-1}, and that

$$e^{\frac{pB}{2}(x'-x)}\,\left(\frac{p+x+i\Delta}{p+x'+i\Delta}\right)^{\frac{z(p)}{2B}}\,\frac{p+x+i\Delta}{\sqrt{(p+x+i\Delta)(p+x'+i\Delta)}} \tag{4.82}$$

is uniformly bounded for $p \in (-\infty, -2x']$ provided F is small enough. This follows from

$$\ln \left| \frac{p + x + i\Delta}{p + x' + i\Delta} \right| \leq C, \quad \forall\, p \in (-\infty, -2x'] \tag{4.83}$$

Then

$$\begin{aligned} &\left| \int_{-\infty}^{-2x'} \psi_1(x,p)\, \tilde{\phi}_2(x',p)\, W^{-1}(\psi_1,\psi_2)\, e^{ipB(y'-y)}\, \mathrm{d}p \right| \qquad (4.84) \\ &\leq \frac{C}{B|(x'-x) - i(y'-y)|} \left[\Delta^{-1} + e^{B(x'^2-x^2)/2} \int_{-\infty}^{-2x'} e^{pB(x'-x)}\, \tilde{w}(x',x,p)\, \mathrm{d}p \right] [1 + C\,\Delta^{-2}] \\ &\leq \frac{C\,\Delta^{-1}}{|(x'-x) - i(y'-y)|} [1 + C\,\Delta^{-2}] \end{aligned}$$

uniformly for $(x'-x)$ in the compacts of $(0,1)$, since both

$$e^{B(x'^2-x^2)/2} \int_{-\infty}^{-2x'} e^{pB(x'-x)} |w(x',x,p)|\, \mathrm{d}p, \quad e^{B(x'^2-x^2)/2} e^{-Bx'(x'-x)} \tag{4.85}$$

are bounded. Same bounds on $|\partial_x G_1(\mathbf{x},\mathbf{x};z)|$ can be found for $(x'-x) \in (-1,0)$.

$\underline{\partial_y G_1(\mathbf{x},\mathbf{x}';z)}$

As it was already noticed, differentiation w.r.t. y leads to a multiplication of the corresponding integrand by the factor iBp:

$$|\partial_y\, G_1(\mathbf{x},\mathbf{x}';z)| = C\, e^{-\Delta|y'-y|} \left| \int_{\mathbb{R}} h(x',x,p)\, e^{ipB(y'-y)}\, \mathrm{d}p \right| \tag{4.86}$$

where for $p > -x$

$$h(x',x,p) = iBp\, \psi_1(x',p)\, \psi_2(x,p)\, W^{-1}(\psi_1,\psi_2) \tag{4.87}$$

and for $p < -x'$

$$h(x',x,p) = iBp\, \psi_1(x,p)\, \psi_2(x',p)\, W^{-1}(\psi_1,\psi_2). \tag{4.88}$$

We can thus proceed in the same way as for $\partial_x\, G_1(\mathbf{x},\mathbf{x}';z)$. The only new ingredient which we need is the fact that that

$$\left(\partial_p\, \frac{p}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right) \in L^1\left((-\infty, -2x'] \cup (-x/2, \infty) \right), \tag{4.89}$$

where the L^1 norm is again bounded by a constant times Δ^{-1}.

For $p \in (-2x', -x/2]$ we apply to both $\partial_x G_1(\mathbf{x}, \mathbf{x}'; z)$ and $\partial_y G_1(\mathbf{x}, \mathbf{x}'; z)$ the same arguments as for $|x' - x| \geq 1$ noting that these are independent on the value of $(x' - x)$. We have thus proved

Lemma 4.3. *For F small enough there exists some strictly positive constant G_0' such that the following inequality holds true*

$$\int_{\mathbb{R}} \int_{|x'-x|<1} |\partial_{x,y}^m G_1(\mathbf{x}, \mathbf{x}'; z)| e^{\frac{\Delta}{2}|y-y'|} \, \mathrm{d}x' \mathrm{d}y' \leq G_0' \, \Delta^{-3}, \tag{4.90}$$

where $m = 0, 1$.

A Integral kernel of e^{-itH_1}

Here we sketch the calculation of the integral kernel of evolution operator e^{-itH_1} in the gauge $H_L = p_x^2 + (p_y - Bx)^2$. We employ the functional integration to write

$$(x, y|e^{-itH_1}|x_0, y_0) = \int_{x_0,y_0;0}^{x,y;t} \mathrm{d}[w(\cdot)] \exp\left\{ i \int_0^t \mathrm{d}\, s \, L[w(s), \dot{w}(s)] \right\} \tag{A.1}$$

where

$$L[w(s), \dot{w}(s)] = \frac{1}{4} |\dot{w}(s)|^2 + F w_x(s) - \dot{w}_y(s) B w_x(s)$$

is the Lagrangian and

$$S_t[w(\cdot)] = \int_0^t \mathrm{d}\, s \, L[w(s), \dot{w}(s)] \tag{A.2}$$

the corresponding action. The integral in (A.1) is then taken over all trajectories $w(s)$ which satisfy the boundary conditions

$$w(0) = (x_0, y_0), \quad w(t) = (x, y) \tag{A.3}$$

We will write w as a sum of a classical trajectory plus certain fluctuation:

$$w(s) = w_{cl}(s) + \xi(s)$$

and evaluate $S_t[w(\cdot)]$ in the vicinity of the classical action $S_t[w_{cl}(\cdot)]$. As $L[w(s), \dot{w}(s)]$ is a quadratic function of canonical variables, all higher variations of $S_t[w_{cl}(\cdot)]$ are identically zero and

$$S_t[w(\cdot)] = S_t[w_{cl}(\cdot)] + \delta^{(1)} S_t[w_{cl}(\cdot)] + \delta^{(2)} S_t[w_{cl}(\cdot)] \tag{A.4}$$

Moreover, since $w_{cl}(s)$ minimises the classical action, the second term on the r.h.s. of (A.4) vanishes and for the last term we have

$$\delta^{(2)} S_t[w_{cl}(\cdot)] = \int_0^t \mathrm{d}\, s \left\{ \frac{1}{4}\, |\dot{\xi}(s)|^2 - \dot{\xi}_y(s) B \xi_x(s) \right\}$$

From the Van Vleck formula it then follows that the kernel (A.1) can be expressed in terms of the classical action only:

$$(x, y|e^{-i\,tH_1}|x_0, y_0) = \frac{1}{2\pi i}\, e^{i\, S_t[w_{cl}(\cdot)]} \left[\det \left\{ -\frac{\partial^2 S_t[w_{cl}(\cdot)]}{\partial\alpha\partial\beta_0} \right\}_{\alpha,\beta} \right]^{1/2} \tag{A.5}$$

with $\alpha, \beta \in \{x, y\}$.

To compute $S_t[w_{cl}(\cdot)]$ we have to find the solution of the classical equations of motion

$$\begin{aligned} \frac{1}{2}\, \ddot{w}_x^{cl} &= -B \dot{w}_y^{cl} + F \\ \frac{1}{2}\, \ddot{w}_y^{cl} &= B \dot{w}_x^{cl} \end{aligned} \tag{A.6}$$

It is not difficult to verify that the general solution of (A.6) reads

$$\begin{aligned} w_x^{cl}(s) &= C_1(t) \cos(2Bs) + C_2(t) \sin(2Bs) + C_3(t) \\ w_y^{cl}(s) &= -C_2(t) \cos(2Bs) + C_1(t) \sin(2Bs) + u\, s + B^{-1}\, C_4(t) \end{aligned} \tag{A.7}$$

where $u = \frac{F}{B}$ is the drift velocity in $y-$direction and the "constants" $\{C_i(t), i = 1, 2, 3, 4\}$ depend on t through the boundary conditions (A.3). A straightforward calculation gives

$$\begin{aligned} w_x^{cl}(s) &= \tfrac{1}{2}\, [(y - y_0 - ut) + (x - x_0) \cot(Bt)]\, \sin(2Bs) \\ &- \tfrac{1}{2}\, [(x - x_0) - (y - y_0 - ut) \cot(Bt)]\, \cos(2Bs) \\ &+ \tfrac{1}{2}\, [(x + x_0) - (y - y_0 - ut) \cot(Bt)] \end{aligned} \tag{A.8}$$

and similarly

$$\begin{aligned} w_y^{cl}(s) &= -\tfrac{1}{2}\, [(x - x_0) - (y - y_0 - ut) \cot(Bt)]\, \sin(2Bs) \\ &- \tfrac{1}{2}\, [(y - y_0 - ut) + (x - x_0) \cot(Bt)]\, \cos(2Bs) \\ &+ \tfrac{1}{2}\, [(y + y_0 - ut) + (x - x_0) \cot(Bt)] + u\, s \end{aligned} \tag{A.9}$$

The action then takes the form

$$\begin{aligned} S_t[w_{cl}(\cdot)] &= \tfrac{1}{4} \tfrac{F^2}{B^2}\, t + \tfrac{1}{2} \tfrac{F}{B} \left(y - y_0 - \tfrac{F}{B} t \right) - \tfrac{1}{2}\, B(x + x_0) \left(y - y_0 - \tfrac{F}{B} t \right) \\ &+ \tfrac{1}{4}\, B \cot(Bt) \left[\left(y - y_0 - \tfrac{F}{B} t \right)^2 + (x - x_0)^2 \right] \end{aligned} \tag{A.10}$$

and Van Vleck's determinant is thus easily calculated to give the integral kernel of $e^{-i\,tH_1}$

$$(x, y|e^{-itH_1}|x_0, y_0) = \frac{1}{2\pi i} \sqrt{\tfrac{B}{2}}\, e^{i\, S_t[w_{cl}(\cdot)]}\, \frac{1}{\sin(Bt)} \tag{A.11}$$

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