CONJUGACIES FOR TILING DYNAMICAL SYSTEMS

by

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Abstract

We consider tiling dynamical systems and topological conjugacies between them. For substitution tiling systems under rather general conditions, including the Penrose and pinwheel systems, we show that conjugacies are generalized sliding block codes, and that the substitution is invertible.

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I. Notation and Main Results

We begin with a definition of tiling dynamical systems, in sufficient generality for this work. Let $\mathcal{A}$ be a nonempty finite collection of compact connected sets in $\mathbb{E}^d$, sets with dense interior and boundary of zero volume. Let $X(\mathcal{A})$ be the set of all tilings of $\mathbb{E}^d$ by congruent copies, which we call tiles, of the elements of the "alphabet" $\mathcal{A}$. We assume $X(\mathcal{A})$ is nonempty, which is automatic for the special class of substitution tiling systems on which we will concentrate below. We label the "types" of tiles by the elements of $\mathcal{A}$. We endow $X(\mathcal{A})$ with the metric

$$m[x,y] \equiv \sup_{n\geq 1} \frac{1}{n} m_H[B_n \cap \partial x, B_n \cap \partial y], \tag{1}$$

where B_n denotes the open ball of radius n centered at the origin $\mathbf{O}$ of $\mathbb{E}^d$, and ∂x the union of the boundaries of all tiles in x. (For a ball centered at $\mathbf{a}$ we use $B_n(\mathbf{a})$.) The Hausdorff metric m_H is defined as follows. Given two compact subsets P and Q of $\mathbb{E}^d$, $m_H[P,Q] = \max\{\tilde{m}(P,Q), \tilde{m}(P,Q)\}$, where

$$\tilde{m}(P,Q) = \sup_{p\in P} \inf_{q\in Q} \|p-q\|, \tag{2}$$

with $\|w\|$ denoting the usual Euclidean norm of w.

Under the metric m two tilings are close if they agree, up to a small Euclidean motion, on a large ball centered at the origin. The converse is also true for tiling systems with finite local complexity (as we require)–closeness implies agreement, up to small Euclidean motion, on a large ball centered at the origin [see RaS1]. Although the metric m depends on the location of the origin, the topology induced by m is Euclidean invariant. A sequence of tilings converges in the metric m if and only if its restriction to every compact subset of $\mathbb{E}^d$ converges in m_H. It is not hard to show [RW] that $X(\mathcal{A})$ is compact and that the natural action of the connected Euclidean group $\mathcal{G}_E$ on $X(\mathcal{A})$, $(g,x) \in \mathcal{G}_E \times X(\mathcal{A}) \mapsto g[x] \in X(\mathcal{A})$, is continuous.

To include some examples it is useful to generalize the above setup, to use what is sometimes called "colored tiles". To make the generalization we assign a "color" from some finite set to each tile, represented on each tile by a "color marking", a line segment in the interior of the tile, of different length for different colors. We then redefine ∂x as the union of the tile boundaries and color markings in the tiling x.

Definition 1. A tiling dynamical system is the action of $\mathcal{G}_E$ on a closed, $\mathcal{G}_E$-invariant subset of $X(\mathcal{A})$.

We emphasize the close connection between such dynamical systems and subshifts. A subshift with $\mathbb{Z}^d$-action is the natural action of $\mathbb{Z}^d$ on a compact, $\mathbb{Z}^d$-invariant subset X of $\mathcal{B}^{\mathbb{Z}^d}$, for some nonempty finite set $\mathcal{B}$. If we associate with each element of $\mathcal{B}$ a "colored" unit cube in $\mathbb{E}^d$, the face-to-face tilings of $\mathbb{E}^d$ by those arrays of such cubes corresponding to the subshift X gives a tiling dynamical system which is basically the suspension of the subshift X (but with rotations of the entire tiling also permitted).

A significant difference between subshifts and tiling dynamical systems is that for (nontrivial) subshifts the group acts on a Cantor set, while the space is typically connected for interesting tiling systems. In fact the spaces for different tiling systems need not be homeomorphic.

A major objective in dynamics is the classification of interesting subclasses up to topological conjugacy. For the class of subshifts a central theorem, due to Curtis, Lyndon and Hedlund, shows that a topological conjugacy can be represented by a sliding block code (see [LM]). For tiling dynamical systems there is a natural analogue of such a representation which we call the "Little Caesar property".

Definition 2. A topological conjugacy $\psi : X_{\mathcal{A}} \mapsto X_{\mathcal{A}'}$ between tiling systems has the "Little Caesar property" if for every $n' > 0$ there is $n > 0$ such that for every $x, y \in X_{\mathcal{A}}$ such that $B_n(\partial x) = B_n(\partial y)$ we have $B_{n'}(\partial \psi x) = B_{n'}(\partial \psi y)$.

Examples have been given recently (see [RS2], [P]) of topological conjugacies of tiling dynamical systems not satisfying the Little Caesar property. One of our main results is:

Theorem 1. Within the subclass of substitution tiling systems with invertible substitution, every topological conjugacy has the Little Caesar property.

Before defining the subclass of "substitution" tiling systems in general we present some relevant examples.

A "Penrose" tiling of the plane, Figure 1, can be made as follows. Consider the 4 (colored) tiles of Figure 2. Divide each tile (also called a "tile of level 0") into 2 or 3 pieces as in Figure 2 and rescale by a linear factor of the golden mean $\tau = (1 + \sqrt{5})/2$ so that each piece is the same size as the original. This yields 4 collections of tiles that we call "tiles of level 1". Subdividing each of these tiles and rescaling gives 4 collections of tiles that we call tiles of level 2. Repeating the process n times gives tiles of level n. A Penrose tiling is a tiling of the plane with the property that every finite subcollection of tiles is congruent to a subset of a tile of some level. A Penrose tiling has only 4 types of tiles, each appearing in 10 different orientations. (For Penrose tilings, and in fact very generally for tilings made by such substitution rules, there are "matching rules" which provide a different method of construction [G].)

A "pinwheel" tiling of the plane, Figure 3, uses two basic tiles: a 1-2-$\sqrt{5}$ right triangle and its mirror image, as shown in Figure 4 with their substitution rule [R]. Notice that at the center of each tile of level 1 there is a tile of level 0 similar to the level 1 tile but rotated by an angle $\alpha = \tan^{-1}(1/2)$. Thus the center tile of a tile of level n is rotated by $n\alpha$ relative to the tile. Since α is an irrational multiple of π, we see, using the fact that within a tile of level 2 there is a tile of level 0 similar to the level 2 tile, that this rotation never ends, and each tiling contains tiles in infinitely many distinct orientations.

Finally, for any integers $m < n$ we consider the "(m,n)-pinwheel" tilings defined (for $m = 2$, $n = 3$) by the substitution of Figure 5, whose tiles are $m{-}n{-}\sqrt{m^2+n^2}$ right triangles. Like the ordinary pinwheel, such variant pinwheel tilings also necessarily have tiles in infinitely many distinct orientations.

It is easy to construct explicit examples of Penrose and pinwheel tilings. Pick a tile to include the origin of the plane. Embed this tile in a tile of level 1 (there are several ways to do this). Embed that tile of level 1 in a tile of level 2, embed that in a tile of level 3, and so on. The union of these tiles of all levels will cover an infinite region, typically—though not necessarily—the entire plane.

In order to generalize from the above examples we need some further notation. A "patch" is a (finite or infinite) subset of a tiling $x \in X(\mathcal{A})$; the set of all finite patches for $\mathcal{A}$ will be denoted $W_{\mathcal{A}}$. A "substitution function" ϕ is a map from $W_{\mathcal{A}}$ to itself defined by a decomposition of each tile type, stretched linearly about the origin by a factor $\lambda_\phi > 1$, into congruent copies of the original tiles. (Recall the Penrose and pinwheel examples .) We assume:

i) For each $k > 0$ there are only a finite number of possible patches, up to Euclidean motion, obtained by taking a ball of radius k around any point inside a tile of level n, $\phi^n(T)$, where T and n are arbitrary. (This is usually called "finite local complexity".)
ii) For each tile $T \in \mathcal{A}$, ϕT contains at least one tile of each type. (This is usually called "primitivity".)
iii) For every tile $T \in \mathcal{A}$ there is $n_T \geq 1$ such that $\phi^{n_T} T$ contains a tile of the same type as T and parallel to it.

Condition i) is significant, the remaining conditions much less so. In interesting cases condition ii) can usually be obtained by replacing the substitution by a power of itself; note that this does not affect the tiling dynamical system at all. We know of no interesting examples of systems not satisfying condition iii).

Definition 3. For a given alphabet $\mathcal{A}$ of (possibly colored) tiles, and a substitution function ϕ, the "substitution tiling space" is the compact subspace $X_\phi \subset X(\mathcal{A})$, invariant under $\mathcal{G}_E$, of those tilings x such that every finite subpatch of x is congruent to a subpatch of $\phi^n(T)$ for some $n > 0$ and $T \in \mathcal{A}$.

The above definition yields spaces of tilings in which the tiles appear in all orientations, although tiles appear in at most countably many different orientations within any *fixed* tiling. Any particular Penrose tiling, for example, has tiles in only 10 distinct orientations, but the space of *all* Penrose tilings contains all rotated versions of that tiling.

For each tiling x and each $r > 0$, consider the set of Euclidean motions g for which x and gx agree exactly on a ball of radius r around the origin. The subgroup of $SO(d)$ generated by the rotational parts of the g's is denoted $\mathcal{G}_{\mathrm{RO}}(r,x)$ and is called the *relative orientation group* of x. In [RS1] it was shown that $\mathcal{G}_{\mathrm{RO}}(r,x)$ is independent of r (and we henceforth write it as $\mathcal{G}_{\mathrm{RO}}(x)$), and that the groups for different tilings $x \in X_\phi$ are related by inner automorphism of $SO(d)$.

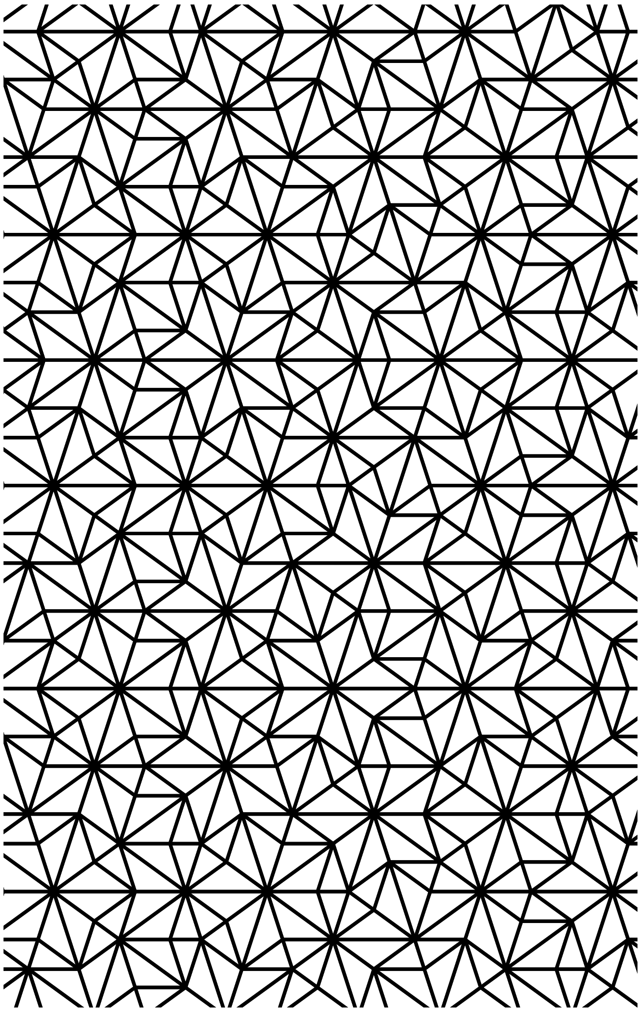

Figure 1. A Penrose tiling

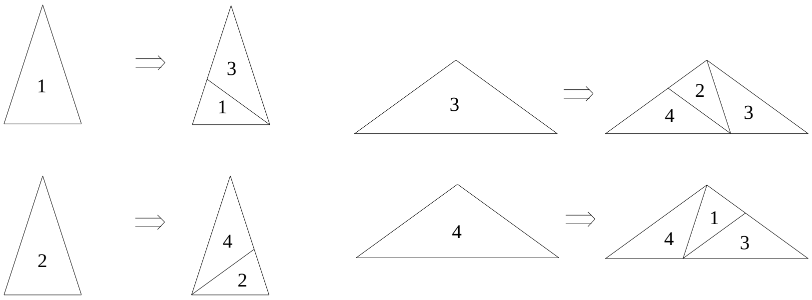

Figure 2. The Penrose substitution

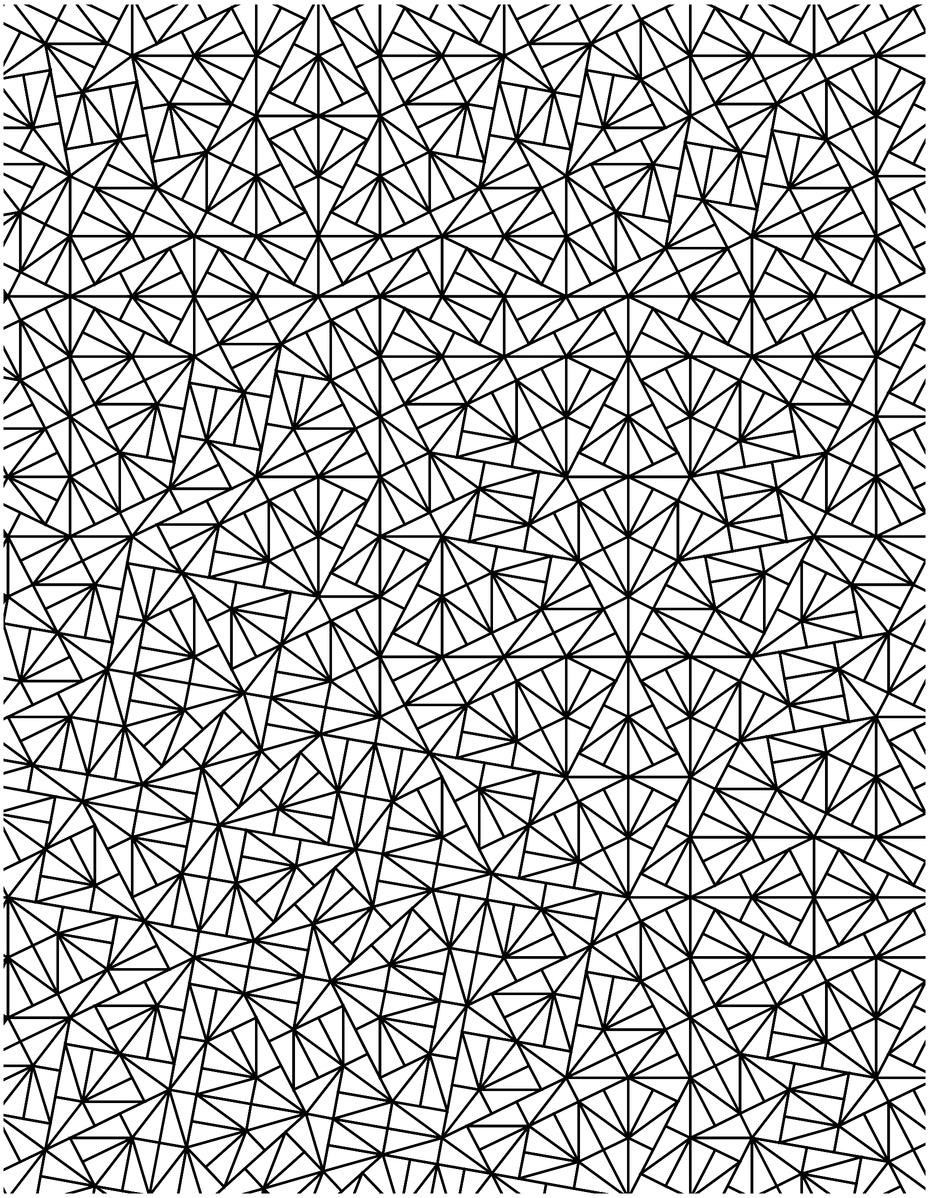

Figure 3. A pinwheel tiling

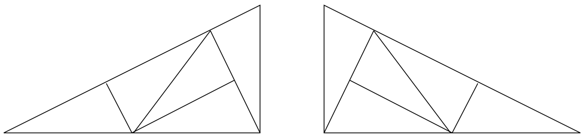

Figure 4. The pinwheel substitution

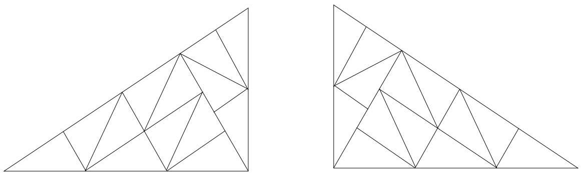

Figure 5. The substitution for (2,3) pinwheel tilings

In 2 dimensions there are no inner automorphisms of $SO(d)$, so the group is exactly the same for all tilings $x \in X_\phi$.

For the Penrose tiling, $\mathcal{G}_{\rm RO} = \mathbb{Z}_{10}$ is the set of rotations by multiples of $2\pi/10$. For the pinwheel tiling, $\mathcal{G}_{\rm RO}(x)$ is generated by rotations of $\pi/2$ and $2\tan^{-1}(1/2)$. For the pinwheel-(m,n) tiling, the group is generated by rotations by $\pi/2$ and $2\tan^{-1}(m/n)$.

From Theorem 1 we deduce that the relative orientation group is a conjugacy invariant:

Corollary 1 If $\psi : X_1 \to X_2$ is a topological conjugacy between substitution tiling systems with invertible substitution, and if $x \in X_1$, then $\mathcal{G}_{\rm RO}(\psi(x)) = \mathcal{G}_{\rm RO}(x)$.

As an application, consider the (1,2)-pinwheel and (3,4)-pinwheel tilings. These have stretching factors $\sqrt{5}$ and 5, and so cannot be distinguished by the homeomorphism invariant of [ORS]. Moreover, their relative orientation groups are each isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}$ as abstract groups. However, these groups are different as subsets of $SO(2)$ (one is an index-2 subgroup of the other), so the two tiling systems cannot be topologically conjugate.

For each tiling $x \in X_\phi$, consider the closure of the orbit of x under translations. Let $\mathcal{G}_{rel}(x)$ be the maximal subgroup of $\mathcal{G}_E$ that maps this orbit closure to itself. It is not hard to see that $\mathcal{G}_{rel}(x)$ is the smallest subgroup of $\mathcal{G}_E$ that contains the closure of $\mathcal{G}_{\rm RO}(x)$ in $SO(d)$ and contains all translations. If the translational orbit of x is dense in X_ϕ, as with the pinwheel and its variations, then $\mathcal{G}_{rel}(x) = \mathcal{G}_E$. We should add that for tiling systems in which tiles only appear in finitely many orientations in any tiling, it can be convenient to consider the dynamical system of translations on the orbit closure of such a tiling. Although our convention is to allow all of the Euclidean group to act on tilings it is easy to obtain corresponding information about these more limited dynamical systems from our results.

Note that the map ϕ extends naturally to a continuous map (again denoted ϕ) from X_ϕ into itself. Our second main result is:

Theorem 2. If for a substitution function ϕ there is some tiling x in X_ϕ not fixed by any translation, for which the orbit under translation is dense in X_ϕ, then the extension of ϕ to X_ϕ is a homeomorphism.

Theorem 2 complements a result of Solomyak [S], which dealt with tiling systems that had finite relative orientation groups. Solomyak's result was itself a generalization of a result of Mossé [M] on one dimensional subshifts. The intermediate case in 3 or more dimensions, where the relative orientation group is infinite but not dense in $SO(d)$, will be treated elsewhere [HRS].

II. Proofs of Theorems 1 and 2, and some further results

For $i = 1, 2$, let ϕ_i be a substitution on alphabet $\mathcal{A}_i$ with linear scaling factor λ_i. Write X_i for X_{ϕ_i} and suppose $\psi : (X_1, \mathcal{G}_E) \mapsto (X_2, \mathcal{G}_E)$ is a topological conjugacy. In Theorem 2 we show that under rather general circumstances $\phi_i : X_i \mapsto X_i$ is a homeomorphism, but for Theorem 1 we simply assume this. This implies that no element of X_i has a nonzero translational period.

Notation. For $r > 0$, $\mathbf{a} \in \mathbb{E}^d$ and a tiling y, $B_r(\mathbf{a})^y$ denotes the patch of y consisting of all tiles in y that intersect the open ball of radius r about $\mathbf{a}$. We abbreviate $B_r(\mathbf{O})^y$ as B_r^y. Pick an "inner radius" m such that every tile in $\mathcal{A}$ contains an open ball of radius m. For patch-valued functions P, Q on X_1 we say P determines Q (or Q is determined by P) if whenever $x, y \in X_1$ and $P(x) = P(y)$ we also have $Q(x) = Q(y)$.

It follows from the fact that $\phi_i : X_i \mapsto X_i$ is a homeomorphism that there is a recognizability radius $D_i > 0$ such that for $x \in X_i$ the patch $B_{D_i}^x$ determines the patch consisting of tiles containing the origin in $\phi_i^{-1}(x)$.

Lemma 1. There is a constant $\rho > 0$ such that if $n \in \mathbb{N}$ and $r > \lambda_2^n \rho$ then for $y \in X_2$ the patch $B_{r/(2\lambda_2^n)}^{\phi_2^{-n}(y)}$ is determined by the patch B_r^y.

Proof. Take $\rho = 2D_2/(\lambda_2 - 1)$. A patch of radius r in y determines a patch of radius $(r - D_2)/\lambda_2$ in $\phi_2^{-1}(y)$, of radius $[(r - D_2)/\lambda_2 - D_2]/\lambda_2$ in $\phi_2^{-2}(y)$, and $r/\lambda_2^n - D_2(\lambda_2^{-1} + \lambda_2^{-2} + \cdots + \lambda_2^{-n})$ in $\phi_2^{-n}(y)$. This last radius is greater than $r\lambda_2^{-n} - D_2/(\lambda_2 - 1)$, which in turn is at least $r/2\lambda_2^n$. ∎

Any element $g \in \mathcal{G}_E$ can be written uniquely as the composition of a rotation and a translation, i.e., there exist unique $\alpha \in SO(d)$ and $\mathbf{s} \in \mathbb{R}^d$ such that

$$g\mathbf{a} = \alpha\mathbf{a} + \mathbf{s} \text{ for all } \mathbf{a} \in \mathbb{E}^d \tag{3}$$

and we set

$$\ell(g) = \|\alpha - I\|_{\text{operator}} + \|\mathbf{s}\|. \tag{4}$$

Notation. For patch-valued functions P, Q on X_1 the phrase "P determines Q up to motion by some $g \in \mathcal{G}_E$ with $\ell(g) \le \eta$" means that if $x, y \in X_1$ are such that $P(x) = P(y)$ then there exists $g \in \mathcal{G}_E$ with $\ell(g) \le \eta$ such that $Q(x) = gQ(y)$.

Lemma 2. There exist a constant $S_0 > 0$ and a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{r\to\infty} \eta(r) = 0$ and if $r > S_0$ then for $y \in X_1$ the patch B_r^y determines the patch $B_{r-S_0}^{\psi(y)}$ up to motion by some $g \in \mathcal{G}_E$ with $\ell(g) \le \eta(r)$.

Proof. By uniform continuity of ψ, there is a radius S_0 such that the patch $B_{S_0}^{y}$ determines the tile at the origin of $\psi(y)$ and its immediate neighbors, up to motion by less than $m/2$. Since ψ is a conjugacy, for any point $\mathbf{a} \in \mathbb{E}^d$, the patch $B_{S_0}(\mathbf{a})^y$ determines the tile at $\mathbf{a}$ in $\psi(y)$ and its nearest neighbors, up to a small motion. Applying this to all points $\mathbf{a} \in B_{r-S_0}$, we have that the patch B_r^y determines $B_{r-S_0}^{\psi(y)}$ up to an overall small motion g. The bound on $\ell(g)$ follows from uniform continuity of ψ. ∎

For $n \in \mathbb{N}$ put

$$k(n) = \left\lfloor n \frac{\log \lambda_1}{\log \lambda_2} \right\rfloor \tag{5}$$

and set

$$\psi_n = \phi_2^{-k(n)} \circ \psi \circ \phi_1^n. \tag{6}$$

Fix $x \in X_1$. Note that $\lambda_1^n / \lambda_2^{k(n)} \in [1, \lambda_2)$, and if $\log \lambda_1 / \log \lambda_2$ is rational then the sequence $(\lambda_1^n \lambda_2^{-k(n)})_{n=1}^{\infty}$ is periodic and 1 is in its range, while if $\log \lambda_1 / \log \lambda_2$ is irrational then the range of this sequence is dense in $[1, \lambda_2]$. Thus there exists a subsequence n_i such that $\psi_{n_i}(x)$ converges, say to $x' \in X_2$, and such that

$$\lambda_2^{-k(n_i)} \lambda_1^{n_i} \to 1 \qquad \text{as } i \to \infty. \tag{7}$$

Theorem 3. The sequence $\{\psi_{n_i}\}_{i=1}^{\infty}$ converges uniformly to a conjugacy $\psi' : (X_1, \mathcal{G}_E) \mapsto (X_2, \mathcal{G}_E)$ such that, for $r > \rho + S_0$ and for $y \in X_1$ the patch B_r^y determines the patch $B_{r/2}^{\psi'(y)}$.

Proof. Step 1. If $g \in \mathcal{G}_E$ then $\psi_{n_i}(gx) \to gx'$ as $i \to \infty$. Indeed, if we write $gx = \alpha x + \mathbf{s}$ as in equation (3) then

$$\begin{aligned}
\psi_{n_i}(gx) &= (\phi_2^{-k(n_i)} \circ \psi)(\phi_1^{n_i}(gx)) \\
&= (\phi_2^{-k(n_i)} \circ \psi)(\alpha \phi_1^{n_i}(x) + \lambda_1^{n_i} \mathbf{s}) \\
&= \phi_2^{-k(n_i)}(\alpha \psi[\phi_1^{n_i}(x)] + \lambda_1^{n_i} \mathbf{s}) \\
&= \alpha \psi_{n_i}(x) + \lambda_2^{-k(n_i)} \lambda_1^{n_i} \mathbf{s}
\end{aligned} \tag{8}$$

which clearly converges to gx' as $i \to \infty$.

Step 2. Suppose $g, g' \in \mathcal{G}_E$ and $r > \rho + S_0$ are such that $B_r^{gx} = B_r^{g'x}$. For each i,

$$B_{\lambda_1^{n_i} r}^{\phi_1^{n_i}(gx)} = B_{\lambda_1^{n_i} r}^{\phi_1^{n_i}(g'x)}. \tag{9}$$

By Lemma 2, there exists $h_i \in \mathcal{G}_E$ with $\ell(h_i) \le \eta(\lambda_1^{n_i} r)$ such that

$$B_{\lambda_1^{n_i} r - S_0}^{h_i \psi[\phi_1^{n_i}(gx)]} = B_{\lambda_1^{n_i} r - S_0}^{\psi[\phi_1^{n_i}(g'x)]}. \tag{10}$$

By Lemma 1

$$B_{(\lambda_1^{n_i} r - S_0)/2\lambda_2^{k(n_i)}}^{\phi_2^{-k(n_i)}(h_i \psi[\phi_1^{n_i}(gx)])} = B_{(\lambda_1^{n_i} r - S_0)/2\lambda_2^{k(n_i)}}^{\phi_2^{-k(n_i)}(\psi[\phi_1^{n_i}(g'x)])}. \tag{11}$$

Let $\alpha, \beta_i \in SO(d)$ and $\mathbf{s}, \mathbf{t}_i \in \mathbb{R}^d$ be such that $g\mathbf{a} = \alpha\mathbf{a} + \mathbf{s}$ and $h_i\mathbf{a} = \beta_i\mathbf{a} + \mathbf{t}_i$ for all $\mathbf{a} \in \mathbb{E}^d$. We have

$$\phi_2^{-k(n_i)}(h_i\psi[\phi_1^{n_i}(gx)]) = \beta_i\alpha\psi_{n_i}(x) + \lambda_2^{-k(n_i)}\lambda_1^{n_i}\beta_i\mathbf{s} + \lambda_2^{-k(n_i)}\mathbf{t}_i, \tag{12}$$

and this converges to gx' as $i \to \infty$. We know from Step 1 that

$$\phi_2^{-k(n_i)}(\psi[\phi_1^{n_i}(g'x)]) \to g'x' \text{ as } i \to \infty. \tag{13}$$

Since

$$\frac{\lambda_1^{n_i} r - S_0}{2\lambda_2^{k(n_i)}} \to \frac{r}{2} \qquad \text{as } i \to \infty, \tag{14}$$

it follows that

$$B_{r/2}^{gx'} = B_{r/2}^{g'x'}. \tag{15}$$

Step 3. Define $\psi' : X_1 \mapsto X_2$ as follows. Given $y \in X_1$ let $\{g_j\}_j$ be a sequence in $\mathcal{G}_E$ such that $g_jx \mapsto y$ as $j \to \infty$. There exist group elements h_j tending to the identity in $\mathcal{G}_E$ and positive real numbers r_j tending to infinity such that $B_{r_j}^{h_jg_jx} = B_{r_j}^{y}$ for each j. By Step 2, $(h_jg_jx')_j$ converges, hence so does $(g_jx')_j$, and we define $\psi'(y) = \lim_{j\to\infty} g_jx$. The existence of this limit ensures that ψ' is well defined and continuous. We see from Step 1 that it is a conjugacy, and the Little Caesar property follows from Step 2.

Step 4. We still need to show that ψ_{n_i} converges to ψ' uniformly on X_1. For $y \in X_1$ and $r > 0$ one can find $g \in \mathcal{G}_E$ such that gx agrees with y on a ball of radius r about the origin. By linear repetitivity (see Lemma 4, below), we can always choose g such that $\ell(g) < Cr$ for some fixed constant C. By the triangle inequality,

$$\begin{aligned} m[\psi_{n_i}(y), \psi'(y)] \le m[\psi_{n_i}(y),&\psi_{n_i}(gx)] + m[\psi_{n_i}(gx), g\psi_{n_i}(x)] \\ &+ m[g\psi_{n_i}(x), \psi'(gx)] + m[\psi'(gx), \psi'(y)]. \end{aligned} \tag{16}$$

Given ϵ, we will show that for i large (with estimates independent of y), and for the correct choice of g, each term on the right hand side is bounded by $\epsilon/4$. The left hand side (which is independent of the choices made) is then bounded by ϵ. Since the estimates on i were independent of y, the left hand side goes to zero as $i \to \infty$ at a rate that is independent of y.

The argument of Step 2 shows that the maps ψ_{n_i} are uniformly continuous with estimates that are independent of i. As a result, the first term can be made small, independent of i (and y), by choosing r large enough. The last term is also small if r is large, since ψ' is uniformly continuous. For fixed r (and hence fixed g), the second term is bounded by $Cr|1 - \lambda_1^{n_i}\lambda_2^{-k(n_i)}|$, which is small if i is large enough. Finally, the third term is small once i is big enough that $\psi_{n_i}(x)$ and $\psi'(x)$ agree up to a small motion on a ball of radius $\gg Cr$ about the origin. ∎

Proposition 1. *For all sufficiently large i we have $\lambda_1^{n_i}\lambda_2^{-k(n_i)} = 1$.*

Proof. Fix a tiling $y \in X_1$ such that for some $\mathbf{t} \in \mathbb{R}^d \setminus \{\mathbf{O}\}$ and for some $r > \rho + S_0$, y and $y + \mathbf{t}$ agree on B_r. Such a y exists by iii). Let $\mu \in [1, \lambda_2]$ be a limit point of $(\lambda_1^n \lambda_2^{-k(n)})_{n=1}^{\infty}$. Then there is an increasing sequence m_i of positive integers such that $\lambda_1^{m_i} \lambda_2^{-k(m_i)} \to \mu$ as $i \to \infty$ and such that $\psi_{m_i}(y)$ converges to a tiling y'. By an argument similar to Step 2 above, y' and $y' + \mu\mathbf{t}$ then agree on $B_{r/2}$. By finite local complexity, however, there are only finitely many values of $\mu \in [1, \lambda_2]$ for which this is possible. Thus the image of the sequence $(\lambda_1^n \lambda_2^{-k(n)})_{n=1}^{\infty}$ is not dense in $[1, \lambda_2]$. From the remarks before Theorem 3 we see that $\log \lambda_1 / \log \lambda_2$ is rational, and the desired result follows from the fact that $(\lambda_1^{n_i} \lambda_2^{-k(n_i)})_{i=1}^{\infty}$ takes only finitely many values and converges to 1. ■

We have just shown that λ_1 and λ_2 are multiplicatively dependent over $\mathbb{Q}$. Note that ψ_{n_i} is a conjugacy whenever $\lambda_1^{n_i} = \lambda_2^{k(n_i)}$. Denote by $\mathcal{O}[y]$ the orbit of a tiling y under translation.

Proposition 2. There exists $I \in \mathbb{N}$ such that

$$\psi_{n_i}(\mathcal{O}[y]) = \psi'(\mathcal{O}[y]) \tag{17}$$

for all $i \geq I$ and for all $y \in X_1$.

Proof. If ε is chosen sufficiently small, then the only way that two tilings $y, y' \in X_2$ can have the property that $d(gy, gy') < \varepsilon$ for all $g \in \mathcal{G}_E$, is if y and y' are small translates of one another, and in particular that $\mathcal{O}[y] = \mathcal{O}[y']$. Choose I large enough so that for all $i \geq I$, $\lambda_1^{n_i} \lambda_2^{-k(n_i)} = 1$ and $d(\psi_{n_i}(y), \psi'(y)) < \varepsilon$ for all $y \in X_1$. For $i \geq I$ and $y \in X_1$ we have

$$d(g\psi_{n_i}(y), g\psi'(y)) = d(\psi_{n_i}(gy), \psi'(gy)) < \varepsilon, \tag{18}$$

thus $\psi_{n_i}(\mathcal{O}[y]) = \mathcal{O}[\psi_{n_i}(y)] = \mathcal{O}[\psi'(y)] = \psi'(\mathcal{O}[y])$. ■

Corollary 2. For each $i \geq I$ and each $y \in X_1$, there exists $g_{i,y}$ in the center of $\mathcal{G}_{rel}(y)$ such that $\psi_{n_i}(y) = g_{i,y}\psi'(y)$. Furthermore, if y' is in the closure of the translational orbit of y, then $g_{i,y'} = g_{i,y}$.

Proof. Fix i and y. By Proposition 2 there exists a translation $g_{i,y}$ such that $\psi_{n_i}(y) = g_{i,y}\psi'(y)$. Since ψ_{n_i} and ψ' are conjugacies and all translations commute, we have $g_{i,y} = g_{i,y'}$ for any y' in the translation orbit of y. By continuity, this last equality holds for all y' in the closure of the translation orbit of y.

To show that $g_{i,y}$ is in the center of $\mathcal{G}_{rel}(y)$, it suffices to show that $\alpha g_{i,y} = g_{i,y}$ for every $\alpha \in SO(d) \cap \mathcal{G}_{rel}(y)$. Fix such α. By definition of $\mathcal{G}_{rel}(y)$ there is a sequence $h_j \in \mathcal{G}_{rel}(y)$ such that $h_j y \to y$ as $j \to \infty$ and such that the rotational part α_j of

h_j converges to α. If $g_{i,y}$ is translation by $\mathbf{s}_{i,y}$ then

$$\begin{aligned} \psi'(y) + \mathbf{s}_{i,y} &= \psi_{n_i}(y) \\ &= \lim_{j\to\infty} \psi_{n_i}(h_j y) \\ &= \lim_{j\to\infty} h_j \psi_{n_i}(y) \\ &= \lim_{j\to\infty} h_j g_{i,y} \psi'(y) \\ &= \lim_{j\to\infty} h_j \psi'(y) + \alpha_j \mathbf{s}_{i,y} \\ &= \lim_{j\to\infty} \psi'(h_j y) + \alpha_j \mathbf{s}_{i,y}, \\ &= \psi'(y) + \lim_{j\to\infty} \alpha_j \mathbf{s}_{i,y}, \end{aligned} \tag{19}$$

and hence $\alpha_j \mathbf{s}_{i,y} \to \mathbf{s}_{i,y}$. It follows that $g_{i,y}$ commutes with α. ■

Corollary 3. The original conjugacy ψ has the Little Caesar property.

Proof. First we show that ψ_{n_i} has the Little Caesar property for $i \geq I$. If two tilings $x, y \in X_1$ agree on a large ball around the origin, they lie in the closure of the same translational orbit. Since ψ_{n_1} and ψ' differ by a (fixed) translation on this orbit closure, and since $\psi'(x)$ and $\psi'(y)$ agree on a (smaller) ball around the origin, $\psi_{n_i}(x)$ and $\psi_{n_i}(y)$ agree on a (still smaller) ball around the origin.

Now note that $\psi = \phi_2^{k(n_i)} \psi_{n_i} \phi_1^{-n_i}$ is a composition of three maps, each of which has the Little Caesar property up to scaling. Each patch in $\psi(x)$ is determined by a (much smaller) patch in $\psi_{n_i} \circ \phi_1^{-n_i}(x)$, which is determined by a patch in $\phi_1^{-n_i}(x)$, which is determined by a (much larger) patch in x. Thus ψ has the Little Caesar property. ■

Corollary 4. If the translational orbit of a tiling is dense in X_1, then ψ intertwines the actions of some powers of ϕ_1 and ϕ_2.

Proof. In this case $\mathcal{G}_{rel}(y)$ of a tiling y has no center, so g_i is the identity and $\psi_{n_i} = \psi'$ for all $i \geq I$. But then

$$\phi_2^{-k(n_i)} \circ \psi \circ \phi_1^{n_i} = \psi_{n_i} = \psi' = \psi_{n_{i+1}} = \phi_2^{-k(n_{i+1})} \circ \psi \circ \phi_1^{n_{i+1}}. \tag{20}$$

Multiplying on the left by $\phi_2^{k(n_{i+1})}$ and on the right by $\phi_1^{-n_i}$ gives

$$\phi_2^{k(n_{i+1})-k(n_i)} \circ \psi = \psi \circ \phi_1^{n_{i+1}-n_i}. \ ■ \tag{21}$$

We now proceed to the proof of Theorem 2. We abbreviate X_ϕ by X and call a patch admissable if there is a tiling $x \in X$ containing it. We assume ϕ has a fixed point in X, not fixed by any translation, whose orbit under translations is dense in X. (The existence of a fixed point for ϕ follows from iii).) Let $H : \mathcal{G}_E \mapsto \mathcal{G}_E$ be defined by $\phi(gx) = H(g)\phi(x)$.

The following four lemmas are proved with standard arguments, as sketched below.

Lemma 3. The extension $\phi : X \mapsto X$ is surjective.

Sketch of proof. Since X is compact, $\phi(X)$ is a closed subset of X. To see that it is dense, note that any admissible patch is a subset of some tile of level n. Thus, for any tiling $x \in X$, and any $r > 0$, $B_r^x = B_r^{\phi(y)}$ for some tiling y. ■

Lemma 4. There is a constant $C > 0$ such that for every $r > 0$ and for every pair of admissable patches P, P' with $\operatorname{supp}(P) \subset B_r$ and $B_{Cr} \subset \operatorname{supp}(P')$ there exists $g \in \mathcal{G}_E$ such that $gP \subset P'$.

Sketch of proof. By minimality, this is true for patches P of size less than 10 times the diameter of the largest tile. By Lemma 3, one can desubstitute (perhaps not uniquely), thereby relating the frequency of large patches to the frequency of small patches. ■

Lemma 5. If $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbb{E}^d$ and $\mathbf{t} = \sum_{i=1}^k a_i \mathbf{b}_i$ with $a_i \in \mathbb{N}$ then there exist $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_\ell$ such that $\mathbf{t}_0 = 0$, $\mathbf{t}_\ell = \mathbf{t}$, and for each $j = 1, \ldots, \ell$,

$$\mathbf{t}_j - \mathbf{t}_{j-1} \in \{\pm \mathbf{b}_i : i = 1, \ldots, k\} \tag{22}$$

and $\mathbf{t}_j$ lies within $(k/2) \max_{1 \le i \le k} \|\mathbf{b}_i\|$ of the straight-line path from $\mathbf{O}$ to $\mathbf{t}$.

Sketch of proof. Each point along the straight-line path from $\mathbf{O}$ to $\mathbf{t}$ is a linear combination of the $\mathbf{b}_i$'s with real coefficients. Round these coefficients to the nearest integer to get the sequence of $\mathbf{t}_j$'s. ■

Lemma 6. If $\mathcal{G}_{rel}(y) = SO(d)$ there is a constant D such that if P, P' are admissable patches in y with $\operatorname{supp}(P) \subset B_r$ and $B_{Dr} \subset \operatorname{supp}(P')$ then there exist $\alpha_1, \ldots, \alpha_n \in SO(d)$ and $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathbb{R}^d$ such that $\alpha_i P + \mathbf{t}_i \subset P'$, $i = 1, 2, \ldots, n$, and such that no proper subspace of $\mathbb{E}^d$ is invariant under all the α_i.

Sketch of proof. This is similar to Lemma 4. We prove it first for individual tiles, and then extend the result to larger patches by desubstitution. ■

Let $M = \max\{\operatorname{diam}(T) : T \in \mathcal{A}\}$. Let us say that a patch P has *period* $g \in \mathcal{G}_E$ if $P \cup gP$ is a patch, i.e., if P and gP agree where their supports overlap (we do not require that they actually overlap). Alternatively, P has period $g \in \mathcal{G}_E$ if and only if whenever $T \in P$ is such that gT° intersects $\operatorname{supp}(P)$ we also have $gT \in P$. Of course any subpatch of a patch of period g has period g.

Lemma 7. If $\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\} \subset \mathbb{E}^d$ is a basis for a lattice $\mathcal{L}$ and P is a patch having all periods in $\mathcal{L}$ and additional translational period $\mathbf{b}_k$ such that $B_r \subset \operatorname{supp}(P)$ where

$$r > (d+1) \max_{1 \le i \le k} \|\mathbf{b}_i\| + 4M \tag{23}$$

then $\langle \mathbf{b}_1, \ldots, \mathbf{b}_k \rangle$ is a lattice of periods for $B_{r/2}^P$.

Proof. Suppose $T \in B^P_{r/2}$ and $T^\circ + \mathbf{t}$ intersects $\operatorname{supp}(B^P_{r/2})$, where $\mathbf{t} = a_1\mathbf{b}_1 + \ldots + a_k\mathbf{b}_k$ with each $a_i \in \mathbb{N}$. Let $\mathbf{b} \in T \cap B_{r/2}$. Then $\mathbf{b} + \mathbf{t} \in B_{r/2+2M}$. Let $\mathbf{t}_0, \ldots, \mathbf{t}_\ell$ be as in Lemma 5. The straight-line path from $\mathbf{b}$ to $\mathbf{b} + \mathbf{t}$ lies in $B_{r/2+2M}$ so each $\mathbf{b} + \mathbf{t}_j$ is in B_r. Thus, for each j we have $(T^\circ + \mathbf{t}_j) \cap P \neq \emptyset$ and by finite induction $T + \mathbf{t}_j \in P$. It follows that $T + \mathbf{t} \in B^P_{r/2}$. ■

Lemma 8. If $k < d$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for a lattice $\mathcal{L}$ and P is an admissable patch having all periods in $\mathcal{L}$ such that $B_r \subset \operatorname{supp}(P)$ where

$$r > D((d+1) \max_{1 \le i \le k} \|\mathbf{b}_i\| + 4M) \tag{24}$$

then there exists $\mathbf{b}_{k+1} \in \mathbb{E}^d \setminus \operatorname{span}(\mathcal{L})$ with $\|\mathbf{b}_{k+1}\| \le \max_{1 \le i \le k} \|\mathbf{b}_i\|$ and such that $\langle \mathbf{b}_1, \ldots, \mathbf{b}_{k+1} \rangle$ is a lattice of periods for $B^P_{r/3D}$.

Proof. Let $P' = B^P_{r/D-M}$. By Lemma 6, there exist $\alpha \in SO(d)$ and $\mathbf{t} \in \mathbb{R}^d$ such that $\alpha P' + \mathbf{t} \subset P$ and $\alpha \operatorname{span}(\mathcal{L}) \neq \operatorname{span}(\mathcal{L})$. It follows that P' has all periods in $\mathcal{L}$ as well as all periods in $\alpha^{-1}\mathcal{L}$. Let $\mathbf{b}_{k+1} \in \alpha^{-1}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} \setminus \operatorname{span}(\mathcal{L})$. By Lemma 7, $\langle \mathbf{b}_1, \ldots, \mathbf{b}_{k+1} \rangle$ is a lattice of periods for $B^P_{r/2D-M/2}$. Since $r/2D - M/2 > r/3D$, the proof is complete. ■

Proposition 3. There is a constant $K > 0$ such that if P is an admissable patch whose support contains a ball of radius r then every non-identity period g of P satisfies $\|g\mathbf{b} - \mathbf{b}\| > Kr$ for some $\mathbf{b} \in \operatorname{supp}(P)$.

Proof. It suffices to prove there is a K which satisfies the conclusion for all sufficiently large r. Recall the notation of m as the inner radius. No tiling in X has a translational period of magnitude less than m. Let $K > 0$ be less than each of the following:

a) $4C\lambda(\lambda+1)(3D)^{d-1}(d + 4M/m)^{-1}$,
b) $\inf\{\|\alpha - I\|_{\text{operator}} : I \neq \alpha \in SO(d) \text{ and } \alpha \text{ fixes some element of } X\}$,
c) $\frac{1}{4C}$.

Suppose there exist $r > 4MC$ and a patch P having a non-identity period $g \in \mathcal{G}_E$ with $\|g\mathbf{b} - \mathbf{b}\| \le Kr$ for all $\mathbf{b} \in \operatorname{supp}(P)$, and such that $B_r \subset \operatorname{supp}(P)$. Let $x \in X$ be a fixed point for ϕ, let $P' = B^x_{r/C-M}$, and let $h \in \mathcal{G}_E$ be such that $hP' \subset P$. Then $h^{-1}gh$ is a period for P' and

$$\|h^{-1}gh\mathbf{b} - \mathbf{b}\| \le Kr, \quad \text{for all } \mathbf{b} \in P'. \tag{25}$$

Since $\phi(P') \supset P'$ and $H(h^{-1}gh)$ is a period for $\phi(P')$, $H(h^{-1}gh)(h^{-1}gh)^{-1}$ is a period for $P' \cap (h^{-1}gh)^{-1}P'$. We have

$$B^x_{r(\frac{1}{C}-K)-M} \subset P' \cap (h^{-1}gh)^{-1}P' \tag{26}$$

and

$$\|H(h^{-1}gh)(h^{-1}gh)^{-1}\mathbf{b} - \mathbf{b}\| \le (\lambda+1)Kr, \text{ for all } \mathbf{b} \in \operatorname{supp}(B^x_{r(\frac{1}{C}-K)-M}). \tag{27}$$

Now $H(h^{-1}gh)(h^{-1}gh)^{-1}$ is a translation, say by $\mathbf{b}_1 \in \mathbb{R}^d$, and if $\mathbf{b}_1 = 0$ then $h^{-1}gh \in SO(d)$ and $h^{-1}ghb = b$. This last is impossible due to our choice of K, hence

$$0 < \|x_1\| \leq (\lambda + 1)Kr. \tag{28}$$

An application of Lemma 7 followed by $d-1$ applications of Lemma 8 (we will see in a moment that r is large enough for this) yields a d-dimensional lattice $\mathcal{L} = \langle \mathbf{b}_1, \ldots, \mathbf{b}_d \rangle$ of periods for $P' = B^x_{r'}$, where

$$\|\mathbf{b}_i\| \leq (\lambda + 1)Kr, \qquad i = 1, \ldots, d, \tag{29}$$

and

$$\begin{aligned} r' &= \frac{1}{2(3D)^{d-1}} r(\frac{1}{C} - K) - M \\ &\geq \frac{r}{4C(3D)^{d-1}} \\ &\geq (d + 4M/m)\lambda(\lambda+1)Kr \\ &\geq (d + 4M/m)\lambda \max_{1 \leq i \leq d} \|x_i\|. \end{aligned} \tag{30}$$

Thus $r' > \lambda(\sum_{i=1}^d \|x_i\| + M)$, so the fundamental domain $F = \{t_1\mathbf{b}_1 + \ldots + t_d\mathbf{b}_d : t_i \in [0,1]\,\}$ for $\mathcal{L}$ is such that $\phi(F^x) \subset B^x_{r'}$. This implies that every tile in $\phi(F^x)$ is a translate by an element of $\mathcal{L}$ of a tile in F^x, and it follows that all tiles in x are translates of tiles in F^x, contradicting that x should have tiles in infinitely many different orientations. ■

For a tile T and $n \geq 0$ let $\mathcal{P}_n(T)$ be the set of admissable patches P for which $\phi^n(T) \subset \phi^n(P)$ and $\phi^n(T) \not\subset \phi^n(P')$ for any proper subpatch P' of P. Then each $\mathcal{P}_n(T)$ is finite and $\{T\} = \mathcal{P}_0(T) \subset \mathcal{P}_1(T) \subset \cdots$.

Lemma 9. For each tile T there is a positive integer N_T such that $\mathcal{P}_{N_T}(T) = \mathcal{P}_{N_T+1}(T) = \cdots$.

Proof. Set $\mathcal{P}(T) = \cup_{n \geq 0} \mathcal{P}_n(T)$ and let $r > 0$, $\mathbf{y} \in \mathbb{E}^d$ such that $B_r(\mathbf{y}) \subset T$. By finite local complexity, $\mathcal{P}(T)$ has only finitely many patches up to rigid motion, since every $P \in \mathcal{P}(T)$ is of the form $(T^\circ)^x$ for some tiling $x \in X$.

If $P, gP \in \mathcal{P}_n(T)$ for some $g \in \mathcal{G}_E$ with $\|g\mathbf{b} - \mathbf{b}\| < Kr$ for all $\mathbf{b} \in P$ then $H^n(g)$ is a period for $\phi^n(P)$ which violates Proposition 3. Thus for each patch $P \in \mathcal{P}(T)$ the set of $g \in \mathcal{G}_E$ for which $gP \in \mathcal{P}(T)$ is discrete and bounded, hence finite. It follows that $\mathcal{P}(T)$ is finite, which is equivalent to the desired result. ■

Lemma 10. Suppose $\mathcal{P}_n(T) = \mathcal{P}_{n+1}(T)$. If $x \in X$ is such that $\phi^{n+1}(T) \subset \phi(x)$ then $\phi^n(T) \subset x$.

Proof. Let $x' \in X$ such that $\phi^n(x') = x$. Then $\phi^{n+1}(T) \subset \phi(x) = \phi^{n+1}(x')$, hence there exists $P \in \mathcal{P}_{n+1}(T)$ such that $P \subset x'$. Since $P \in \mathcal{P}_n(T)$ we have $\phi^n(T) \subset \phi^n(P) \subset \phi^n(x') = x$. ■

Proof of Theorem 2. Set $N = \max\{N_T : T \in \mathcal{A}\}$. Let $x \in X$, and let $x_1, x_2 \in X$ be any tilings such that

$$\phi(x_1) = \phi^{N+1}(x_2) = x. \tag{31}$$

We only need to show that $x_1 = \phi^N(x_2)$.

Let $T \in x_2$ be any tile and let $g \in \mathcal{G}_E$ such that gT is a tile in $\mathcal{A}$. Then

$$\phi^{N+1}(gT) \subset \phi^{N+1}(gx_2) = \phi(H^N(g)x_1). \tag{32}$$

By Lemma 10, $\phi^N(gT) \subset H^N(g)x_1$, and hence $\phi^N(T) \subset x_1$. ■

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