THE OBSTRUCTION CRITERION FOR NON EXISTENCE OF INVARIANT CIRCLES AND RENORMALIZATION

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Abstract. We formulate a conjecture which supplements the standard renormalization scenario for the breakdown of golden circle in twist maps.

We show rigorously that if the conjecture was true then:

(1) The stable manifold of the non-trivial fixed point would indeed be a boundary between the existence of smooth invariant tori and hyperbolic orbits with golden mean rotation number.
In particular, the boundary of the set of twist maps with a torus with a golden mean rotation number would include a smooth submanifold in the space of analytic mappings.
(2) The obstruction criterion of [Olvera-Simo] would be sharp in the universality class of the renormalization group.
(3) The criterion of [Greene-79] for existence of invariant circles if and only if there the residues of approximating orbits are finite would be valid for maps in the universality class.
(4) If there is no invariant circle, there are hyperbolic sets with golden mean rotation number.

We also provide numerical evidence which suggests that the conjecture is true and discuss briefly the possibilities of providing a computer-assisted proof. We also discuss tentatively phenomena for other rotation numbers.

1. Introduction

The main goal of this paper is to formulate precisely a conjecture (see Conjecture 3.3) that supplements the standard scenario of renormalization description of the breakdown of invariant circles with golden mean rotation number in twist mappings. We will also present some rigorous implication of Conjecture 3.3 as well as some numerical evidence supporting it.

Even if Conjecture 3.3 requires some notation to be formulated, the consequences can be formulated with a minimum of specialized notation. Hence, in this section, we will give a quick preview of the consequences. We refer to Section 2 for a review of rather standard results and setting of the notation.

We refer to [McK82] and references therein for the original formulation of the renormalization group description of the break down of invariant

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circles, or [Koc99] for a different formulation which is better suited for rigorous proofs. A recent paper [Koc02b] includes a proof of the existence of a non-trivial fixed point for the renormalization group formulated in [Koc99]. This non-trivial fixed point controls the destruction of invariant curves. In [Koc02a] it is shown that the maps in the stable manifold of the non-trivial fixed point have an invariant circle, which has a finite degree of smoothness. Similar results were obtained in [Sti93], which also showed that qualitative assumptions on the behavior of the fixed point imply that the invariant circles for maps in the stable manifold of the non-critical point are not very regular.

The paper [CJ02] is a review on the renormalization approach for invariant circles. It contains an extensive bibliography emphasizing numerical results and approximations.

Even if there are renormalization description for many rotation numbers, we will describe mainly the description for numbers with a periodic continued fraction expansion. In such case, one can reduce the normalization group predictions to the study of fixed points and their stable manifolds. We will pay special attention to the circles with a golden mean rotation.

Roughly, we show that renormalization theory can team up very well with the *obstruction criterion* of [OS87a],[OS87b], [OS93]. This criterion – which will be explained below (Theorem 2.1) – gives a sufficient condition for the non-existence of invariant circles with a given rotation number.

Our main result is that, by combining the obstruction criterion with renormalization, it is possible to show that, provided Conjecture 3.3 on the behavior of the renormalization group operator is true, the stable manifold of renormalization is a boundary for the existence of invariant circles.

More explicitly: We will formulate Conjecture 3.3 and show rigorously that this conjecture implies that if a family of maps in the universality domain [1] crosses the stable manifold of the non-trivial fixed point, the obstruction criterion applies on one side and, hence, the maps of the family on one side of the stable manifold do not have a invariant circle. (Variants of Conjecture 3.3 which we also discuss are Conjectures 3.5, 3.6.)

We do not prove rigorously Conjecture 3.3 (but we think that a computer-assisted proof may be within reach, see Remark 4.6). Nevertheless, in Section 5 we provide numerical evidence in favor of the validity of Conjecture 3.3 by verifying to very high accuracy several quantitative predictions of the Conjecture.

The fact that the stable manifold of the fixed point separates the existence of circles from the non-existence was known empirically in [McK82]. Providing a justification for this observation was formulated as open question (41) in [McK82]. In particular, Conjectures 3.3, 3.5, 3.6 imply that

[1] We recall that a property is universal or that it holds for a universal family when there is an open set of families including the fixed point of renormalization such that the property holds for all the families in the neighborhood. This is standard notation for all the formulations of renormalization group.

the boundary (in the analytic topology) of the set of twist mappings which posses a smooth circle with rotation number golden mean contains a smooth submanifold of analytic mappings. The question of whether the boundary of the set of maps contains a smooth submanifold is a problem proposed in [Her]. In the formulation of [Her], the boundaries are to be understood in a Sobolev topology. See Remark 4.5 for a discussion of the results of the boundary in other topologies.

Another consequence of Conjecture 3.5 is that, for families in the universality class, once the invariant circle is destroyed, then there are uniformly hyperbolic Aubry–Mather sets with rotation number golden mean. (See [Gor85], [Fat89], [Sta88] for a discussion of the appearance of hyperbolic Aubry-Mather sets with golden mean rotation number). Using the results of [FdlL92b], we will see that the Conjecture 3.5 would imply the full statement of Greene's criterion for families in the universality class.

2. Some preliminary background and notation

In this section, we collect some rather well known results. All of them will be undoubtedly well known to experts. Besides serving as a quick reference, this will serve us to establish our notation. A general reference for this background material is [dlL01].

2.1. **Twist maps.** An area–preserving twist map of the cylinder is a C^1 diffeomorphism $T : \mathbb{T}^1 \times \mathbb{R}$ to itself, which preserves area, orientation and the topological ends of $\mathbb{T}^1 \times \mathbb{R}$ and which satisfies the twist condition:

$$\frac{\partial \bar{x}}{\partial y} > 0 \quad \text{where} \quad (\bar{x}, \bar{y}) = T(x, y) \ . \tag{1}$$

We also assume that the net flux of T (the oriented area between a non-contractible circle and its image under T, by the area preservation of T, this oriented area is independent of the circle chosen) is zero. Otherwise, it is clear that there cannot be any non-trivial invariant circles.

One example that we will use very often is the generalized standard map, given by:

$$\begin{aligned} \bar{y} &= y + f(x) \\ \bar{x} &= x + \bar{y} \end{aligned} \tag{2}$$

where $f(x+1) = f(x)$. The standard map corresponds to taking: $f(x) = \frac{\lambda}{2\pi} \sin(2\pi x)$.

In (2), the net flux is $\int_0^{2\pi} f(x)dx$, so that the condition of zero net flux corresponds to the integral vanishing.

Given such map it is often convenient to consider a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ to the universal cover. Thus, we define a lift of T: Let $p : \mathbb{R} \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be the covering map $p(x, y) = ((x(\mathrm{mod} 1)), y)$. Then a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of T

is a continuous map such that $T \circ p = p \circ F$. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the unit translation $R(x, y) = (x - 1, y)$, then F commutes with R.

2.2. **Reversible mappings and critical lines.** We also recall [Gre79] and [DV58] that many maps F of interest – e.g. (2) – have the property that they can be written as $F = T \circ S$ with $T^2 = S^2 = \mathrm{Id}$ (equivalently $F^{-1} = T \circ F \circ T^{-1}$)

For such maps, the lines $\mathcal{L} = \{x | T(x) = x\}$ are called the critical lines.

The importance of the critical lines is that it is easy to show that if $x \in \mathcal{L}$ is such that $F^N(x) \in \mathcal{L}$, it is easy to show that $F^{2N}(x) = x$. Hence, one can search for periodic orbits by one-dimensional searches.

It is moreover found empirically that these orbits lying in critical lines are much more orderly and have better properties. A systematic study of critical lines is in [PnJL87].

2.3. **Bifurcation surfaces.** One way to describe bifurcations that is very convenient for renormalization is the discussion in terms of bifurcation surfaces. (See e.g. [CH82] p. 42, [AR67].)

These bifurcation surfaces are codimension one surfaces in the space of mappings. When a map belongs to the bifurcation surfaces, it possesses a periodic orbit which experiences a bifurcation.

If we specify the orbit and the type of bifurcation taking place, bifurcation surfaces are smooth manifolds in the space of mappings considered.

For example, the set of maps experiencing a saddle-node in a certain neighborhood can be described as the set of maps for which the graph is tangent to the diagonal. By the transversality theorem (See Theorem 18.2 of [AR67], this shows that the set of mappings experiencing a bifurcation is locally a smooth manifold.

In our case, we will consider the bifurcation manifolds Γ_n, which are the sets of area preserving maps for which a specified periodic orbit – selected by the index n – has one eigenvalue 1 (by the area preservation, then the two eigenvalues should be 1).

These bifurcating maps can be described as the maps in which the graphs of an iterate has a non-transversal intersection with the diagonal in a specified neighborhood.

For numerical applications, it will also be convenient to consider the sets $\Gamma_n(\rho_n)$, in which the trace of the derivative at an specified periodic orbit is specified to be a value different from 2 – equivalently, the eigenvalues are specified to be a number different from 1 –. In such a case, the fact that the set of mappings is a submanifold of codimension 1 is an easy consequence of the implicit function theorem. The periodic orbit persists and the eigenvalues at the periodic orbit can be used as a functional. Setting this functional at an specified level defines a smooth manifold in the space of maps since, clearly the gradient of the functional is not zero.

Other manifolds that we will consider are the manifolds in the space of maps associated to intersections of the stable and unstable manifolds of specified periodic orbits in a specified segment. For example, we will consider Σ_n which is the set of maps in which two segments of stable and unstable manifolds of two orbits – selected by the index n – have a transverse intersection and also the sets $\Lambda_n(\rho_n)$ in which the stable and unstable manifolds intersect enclosing a lobe of area ρ.

The fact that the sets $\Sigma_n, \Lambda_n(\rho_n)$ are smooth manifolds of codimension 1 in the space of smooth area preserving mappings follows because of the results mentioned in Section 2.5 show that the dependence of the observables defining the manifold with respect to the map has a non-zero gradient.

2.4. **The obstruction criterion.** We recall that the obstruction criterion reads (See [OS87a], [OS87b].)

Theorem 2.1. *Let T be a C^1 area preserving twist map of $\mathbb{T}^1 \times \mathbb{R}$ to itself.*

Assume that T has hyperbolic periodic orbits p_1, p_2 of type m_1/n_1, m_2/n_2 respectively ($m_1, m_2, n_1, n_2 \in \mathbb{N}$) with $m_1/n_1 > m_2/n_2$. Assume, moreover that the stable and unstable manifolds intersect transversally. Then, there is no homotopically non-trivial invariant circle with a rotation number $\rho \in [m_1/n_1, m_2/n_2]$.

The proof of Theorem 2.1 developed in [OS87b] is based on the observation that, for a twist mapping, the existence of an homotopically non-trivial invariant circle with rotation number ρ is incompatible with the existence of an orbit which is asymptotic in the future to p_1 and in the past to p_2.

Remark 2.2. We also note that, if the intersection of these stable and unstable manifolds were transversal, then there is a horseshoe between the orbits. Using the symbolic dynamics of the horseshoe, one can produce hyperbolic orbits with rotation number ρ for any $\rho \in [m_1/n_1, m_2/n_2]$. Clearly, for twist mappings the existence of a hyperbolic orbit of rotation number ρ is incompatible with the existence of an invariant circle of rotation number ρ.

Remark 2.3. Another implementation of the obstruction criterion is in [OS93]. There, it is shown that if a map comes close to having a tangential intersection, then, it has periodic orbits which are not well ordered (sometimes called non-Birkhoff orbits). Then, a criterion of [BH87] – see also [LC00] – shows that there is no invariant circle.

2.5. **Dependence of stable manifolds on parameters.** One important ingredient in our formulation is the observation that the invariant manifolds of a periodic point depend in an analytic way on the map. This result can be found in [Mey75] and a more modern version (including more general results, explicit formulas for the derivatives and a discussion of finite regularities) can be found in [CFdlL02b].

Lemma 2.4. *Let T_0 be an analytic mapping of a neighborhood of* 0 *in* $\mathbb{R}^d$ *with* 0 *a hyperbolic fixed point.*

Assume that the dimension of the stable space of $DT_0(0)$ is s. Denote by B^s the unit ball in $\mathbb{C}^s$.

Let U be a sufficiently small neighborhood of T_0 in $\mathcal{X}$, a Banach space of analytic functions (endowed with the norm of the supremum in a complex extension).

Then, it is possible to find a analytic mapping $\mathcal{W} : X \times B^s \to M$ in such a way that

- $p_T \equiv \mathcal{W}(T, 0)$ *is a hyperbolic fixed point of the map T.*
- $W(T, B^s)$ *is contained in the stable manifold of p_T.*
- *The mapping $x \to \mathcal{W}(T, x)$ is a local diffeomorphism.*

Of course, analogous results hold for the unstable manifold as well as for periodic points rather than just fixed points. We refer to the references above for proofs and more precisions.

The importance for us of Lemma 2.4 is that we can follow the dependence on parameters of compact subsets of the stable and unstable manifolds.

Given two hyperbolic periodic points, we fix two closed balls of the stable manifold of the first fixed point and the unstable of the second one. The set of mappings for which the stable and unstable manifolds intersect transversally in the interior of these balls will contain an open set in the space of mappings.

The boundary of this set of transversal intersections in a specified ball will contain $\mathcal{T}$, the set of maps for which the stable and unstable manifolds have a tangency in the interior of the ball. The set $\mathcal{T}$ will be a codimension one manifold in the space of mappings.

Remark 2.5. We emphasize that we specify that the intersections occur in a ball in the intrinsic topology of the manifold. This will allow us to use implicit function theorems in infinite dimensional spaces etc.

Different choice of balls will lead to different possibilities of applying the obstruction criterion. This is perfectly reasonable since the obstruction gives a sufficient condition for non-existence of circles.

When we have an intersection in a ball, we have infinitely many other intersections of the stable manifolds, but these intersections will take place outside of the balls. In particular, we will obtain secondary intersections. As it is will known, the dynamics in the maps with a tangency – or those close to that – will be extremely complicated (this is exploited in [OS93]. Nevertheless, all these complications happen outside of the specified balls.

Given one parameter family of maps T_λ which crosses transversally the manifold $\mathcal{T}$ will have a critical value λ_0 such that for δ small enough $\lambda \in [\lambda_0 - \delta, \lambda_0)$ there is transverse intersection of the stable and unstable manifolds in the specified ball. When $\lambda \in (\lambda_0, \lambda_0 + \delta]$ there is no intersection in the specified ball.

Again, as in Remark 2.5 we emphasize that there is no claim about the global behavior of the map T_λ in $\lambda \in (\lambda_0, \lambda_0 + \delta]$. The only thing that we are claiming is that we do not have any intersection in the specified balls.

It is heuristically useful to think that we are discussing lower bounds on the complexity of the maps. On one side of $\mathcal{T}$ we assert that there is a transverse intersection of the stable and unstable manifolds. On the other side of the manifold $\mathcal{T}$, our bound on the complexity is trivial and we do not claim anything.

2.6. **Formulations of renormalization group.** In this section, we will discuss two formulations of the renormalization group that have been used both for numerical experiments and for some theorems.

There are other formulations of renormalization group for twist mappings (see [CJ02] for a discussion of many of them) but we point that in the formulation of the conjectures and in the derivation of the dynamical consequences we will only use a few properties of the renormalization groups. Hence, the results here can be verified for other formulations This properties Section 2.6.3.

This is natural since the main consequence (the optimality of the obstruction criterion) can be formulated independently of renormalization theory.

2.6.1. *MacKay Renormalization operator and commuting pairs.* In this section we describe briefly the formulation of renormalization group included in [McK82]. We do not aim to give a full description of the theory. We only cover the definitions that will allow us to formulate the conjectures precisely.

The basic idea of the renormalization group described below is that we select an small neighborhood of a point in the circle and look at the return map at this neighborhood. By scaling this neighborhood to a full size, we obtain a new twist map – the renormalized map – of the annulus. The number of iterates that takes to return to the neighborhood is given by the first term in the continued fraction of the continued expansion of the rotation number. If we perform the renormalization around an orbit of rotation number $\omega = [m, a_1, a_2, \ldots]$, the renormalized map will have at the origin an orbit of rotation number $\omega' = [a_1, a_2, \ldots]$. The orbit of the renormalized map will be an invariant circle with rotation number ω' if and only if the orbit of the the original map is a circle of rotation number ω.

More precisely, [McK82] defines the renormalization operator $\mathcal{N}_m$, for $m \in \mathbb{Z}$, acting on the pair (U, T) of maps, by

$$\mathcal{N}_m = \begin{cases} U' = & BTB^{-1} \quad , \\ T' = & BT^m UB^{-1} \quad . \end{cases} \tag{3}$$

where B is a coordinate change that has to be chosen so that the renormalization has a fixed point.

Remark 2.6. It is more natural from the mathematical point of view to chose B depending on the map so that the renormalized set of commuting pairs satisfies some normalization. Nevertheless, one often finds descriptions of the renormalization operator as including a fixed B.

The analysis of the two renormalization operators obtained is slightly different because, if we take B depending on the map, the derivative of the renormalization has a finite rank term extra compared with the derivative of the renormalization if we consider B as fixed.

As we will see, the analysis that we will describe works for both versions. In the discussion of numerical experiments, we will see that we impose certain normalizations, hence it is more appropriate to consider B as depending on the map, but since we will be working close to the fixed points this does not make a big difference in the numerics. In numerical applications, one often computes B in a preliminary run by imposing map dependent normalizations and, then, in other runs use the computed map.

Let $\{p_n, q_n\}$ be the convergents of ω such that its continued fraction expansion is $p_n/q_n = [m_0, m_1, m_2, \dots, m_n]$ and $\lim_{n\to\infty} p_n/q_n = \omega$. The renormalization of the invariant circle with rotation number ω is defined by

$$\mathcal{N}_{m_n} \dots \mathcal{N}_{m_0}(F, R) = (B_n F^{q_{n-1}} R^{p_{n-1}} B_n^{-1}, B_n F^{q_n} R^{p_n} B_n^{-1}) \quad . \tag{4}$$

The universal map F^* is defined as the limit of this process, such that $\mathcal{N}_{m_n} \dots \mathcal{N}_{m_0}(F, R)$ converges to a fixed point (U^*, T^*), with $T^* = F^*$, $U^* = BF^*B^{-1}$, where $B(y, x) = (\alpha y, \beta x)$.

Given a one parameter family of maps F_μ, we renormalize this family with a reparameterization of μ, then the one parameter families

$$B_n F^{q_n}_{\mu\delta^{-n}} R^{p_n} B_n^{-1} \tag{5}$$

converge to a universal one parameter family F^*_μ.

In [McK82] one can find very strong numerical evidence that the operator $\mathcal{N}_1$ has a fixed point which is hyperbolic with an unstable manifold of dimension 1 and a stable manifold of codimension 1. This fixed point controls the disappearance of all the the invariant circles whose rotation number has a continued fraction which consists of 1 except for a finite number of terms (these numbers are commonly called *"noble"* numbers).

To the best of our knowledge, in the literature there are no direct calculations of the fixed points for operators $\mathcal{N}_n$. Nevertheless, there is strong indirect evidence of their existence. It has been repeatedly observed that the destruction of invariant circles of the corresponding rotation numbers exhibits scaling properties and that the scaling coefficients are universal. Indeed in Section 5 we will present some of this evidence.

The precise mathematical formulation of the renormalization group in terms of commuting pairs has some technical difficulties. Notably, it is somewhat involved to make precise what is the nature of the space of commuting pairs. From the numerical point of view, one has to impose the constraint of area preservation as well as commutation. These inconveniences can be solved. In particular, [Sti97] contains a precise mathematical formulation and proves the existence of a non-trivial function which is a fixed point of a power of the renormalization group (the proof in [Sti97] does not establish properties such as area preserving. Numerical evidence suggests very strongly that this object is a fixed point and that it satisfies the other properties.)

2.6.2. *The Koch formulation or renormalization group.* In the formulation of [Koc99], the mappings considered are the time one maps of a periodic Hamiltonian system or as sections of a vector field. The results should be the same as the consideration of maps since all maps can be generated in this way. (For mappings of class C^r, $r \in \mathbb{N} \cup \{\infty\}$, this is done in [Mos86], in the analytic case [KP94].)

For simplicity we just describe the situation of one degree of freedom systems since it is the only situation where there are numerics suggesting a non-trivial fixed point.

As we will see, the transformation involves several choices. All of them have the same trivial fixed point (See Theorem 1.6 in [Koc99].) Nevertheless, it is possible that each of these transformations have different properties far from the trivial fixed point.

Given a Hamiltonian of the form $H(p, q)$, where p are real variables, q angle variables, p, q are symplectically conjugate variables, The definition of the renormalization in [Koc99] (1.11) p. 481 is obtained by setting basically

$$\mathcal{R}(H) = \frac{\theta}{\mu} H \circ \mathcal{U}_H \circ T_\mu$$

where

- T_μ is a transformation of the form $T_\mu(p, q) = (\mu(T^*)^{-1}p, Tq)$.

  That is T_μ consists on performing a transformation $T \in SL(2, \mathbb{Z})$ that makes one coordinate angle to be very close to a resonant combination. Of course, to keep the symplectic character, we need to perform another transformation on the action variables.

  The parameter μ is an scaling parameter that corresponds to looking just at an smaller region in the action variable. This scaling makes the transformation analyticity improving and, hence compact.
- The transformation $\mathcal{U}_H$ is a canonical transformation that eliminates non-resonant modes.

The full transformation requires to add some extra transformations that just preserve some normalizations

2.6.3. *Assumptions on renormalization group formulation.* The formulation of Conjectures 3.3, 3.5, 3.6 and the proof of their consequences can be done largely independent of the renormalization group formalism that we use.

There are very few properties of the renormalization operator $\mathcal{R}$ that we use.

On the analytic side, we will assume:

- It is possible to define fixed points, their linearization, stable and unstable manifolds.

- There are λ lemmas for the stable manifold under preimages of the renormalization and for the unstable manifold under iteration of the renormalization.

For the MacKay renormalization group, a rather satisfactory mathematical description is in [Sti97]. The paper [Koc99] includes a rather precise mathematical description.

Besides these rather analytical and technical properties, we use only a few geometric properties.

We denote by p_n/q_n the convergents of an irrational number. In the case of the golden mean, the convergents will be $p_n = F_n$, $q_n = F_{n+1}$ with F_n the Fibonacci numbers.

- If the map G has a periodic orbit o of type p_{n+1}/q_{n+1}, then the renormalized map $\mathcal{R}(G)$ has an orbit o' of type p_n/q_n.
- The orbit o' is hyperbolic or critical if and only if the orbit o is hyperbolic or critical.
- If the map G has two hyperbolic orbits o, p of type p_{n+1}/q_{n+1}, p_n/q_n respectively, whose local stable and unstable manifolds intersect, then, the corresponding orbits o', p' of the renormalized maps have local stable manifolds that intersect.
- The intersection of $W_o^{u,\mathrm{loc},(G)}$, $W_p^{s,\mathrm{loc},(G)}$ is transversal (resp. tangential) if and only if the intersection $W_{o'}^{u,\mathrm{loc},(\mathcal{R}(G))}$, $W_{p'}^{s,\mathrm{loc},(\mathcal{R}(G))}$ is transversal (resp. tangential).
- The map G has a golden mean non-trivial invariant circle if and only if the map $\mathcal{R}(G)$ does.

The verification that both the MacKay and the Koch formulation of renormalization groups satisfy the above properties is quite straightforward.

The above geometric properties seem also to be verified by other renormalization group formulations that have been proposed and that we will not discuss.

Analogous properties can be formulated for renormalization groups based on other rotation numbers.

Remark 2.7. We note that it is not in principle clear at all that different renormalization groups will have the same non-trivial fixed points and much less that they will have the same stable manifolds. Of course, this follows if one can give a dynamical characterization of these stable manifolds which is independent of the renormalization group. One consequence of Conjectures 3.3 and 3.5 is that the invariant manifolds of these renormalization groups can be characterized as the boundaries of the maps with a smooth invariant circle of the specified rotation number.

We think that it would be interesting to give a more direct proof of the identity of invariant manifolds of different renormalization groups.

3. Precise formulation of the conjectures

The main results of this section are the following Lemma 3.1 and the formulation of Conjectures 3.3, 3.5, 3.6. Later, in Section 4 we will establish some consequences of the conjectures. Some of the quantitative predictions of the conjectures will be verified numerically to high accuracy in Section 5. Other variants of the conjectures will be formulated in Section 5.4

The following Lemma 3.1 shows that we can define manifolds of maps with a tangency in a prescribed segment. The maps with a tangency in a specified segment are codimension one manifold in a space of mappings with we will denote by $\mathcal{T}$.

On one side of this submanifold $\mathcal{T}$, we know that there will be a transverse homoclinic intersection. Hence, we will be able to apply the obstruction criterion and conclude that there are no invariant circles. On the other side of the submanifold, we will not claim anything.

The gist of the conjectures is that the unstable manifold $\mathcal{T}$ at the nontrivial fixed point has a transverse intersection with some bifurcation manifolds. See Conjecture 3.3 more precise formulation of the main conjecture as well as Conjectures 3.5 and 3.6 for a more detailed conjectures that lead to more consequences.

Then, it will follow by the λ-lemma that $\mathcal{N}^{-n}(\mathcal{T})$ will accumulate on the stable manifold of the fixed point. We will show that if a family crosses $\mathcal{N}^{-n}(\mathcal{T})$, then, high period orbits have heteroclinic tangencies. This also implies that there is no invariant circle with golden mean rotation number. Therefore, by the accumulation, of $\mathcal{N}^{-n}(\mathcal{T})$, we see that, if T_λ crosses transversally the stable manifold of the fixed point at $\lambda = \lambda_0$, then, for any $\delta > 0$, we can find $\lambda \in (\lambda_0, \lambda_0 + \delta)$ so that the map T_λ does not have any invariant circle with rotation number golden mean.

Actually, a more quantitative statement about the intersections will imply the slightly stronger result that there is a $\delta_0 > 0$ such that $\lambda \in (\lambda_0, \lambda_0 + \delta_0)$, then, T_λ does not have a non-trivial invariant circle with golden mean rotation number.

Hence, putting together the results, we obtain that for a family F_λ in the universality class, there exists an analytic golden mean invariant circle for $\lambda \in (0, \lambda^*)$, there is a circle with a finite and universal degree of regularity for $\lambda = \lambda^*$ and for $\lambda > \lambda^*$ there is a hyperbolic Cantor set with rotation number golden mean.

3.1. **Tangencies of stable and unstable manifolds of periodic points.**

The first ingredient of the formulation of our conjectures is the following.

Lemma 3.1. *Let F be a C^ω area preserving diffeomorphism of $\mathbb{T}^1 \times \mathbb{R}$ (resp. area preserving reversible). Assume that*

i) *F has two monotone hyperbolic periodic orbits Z, Z' of rotation number m/n, m'/n' respectively.*

ii) *The local invariant manifolds* $W_Z^{S,L,(F)}$, $W_{Z'}^{U,L',(F)}$ *are regular for some* $L, L' > 0$.

iii) $W_Z^{S,L,(F)}$, $W_Z^{U,L',(F)}$ *have exactly one tangency in the interior of the segment considered. This tangency is quadratic.*

(In the case that F *is a reversible map, we assume furthermore that* $Z \in \mathcal{L}$, $Z' \in \mathcal{L}'$ *with* $\mathcal{L}, \mathcal{L}'$ *critical lines for* F.*)*

Then we can find a neighborhood U *of* F *in the space of area preserving (resp. reversible area preserving maps),* $\Delta > 0$ *and neighborhoods of* V, V' *of* Z, Z' *in* $\mathbb{T}' \times \mathbb{R}$ *(resp. in* $\mathcal{L}, \mathcal{L}'$*) such that:*

For every $G \in U$

a) *In* V *there is one and only one monotone hyperbolic periodic point* Z_G *of* G *of rotation number* m/n. *The mapping* $G \to Z_G$ *is analytic (similarly for* Z'_G*).*

b) $W_{Z_G}^{S,L-\Delta,(G)}$, $W_{Z'_G}^{U,L-\Delta,(G)}$ *are regular.*

The mappings

$$G \to W_{Z_G}^{S,L-\Delta,(G)} \quad , \quad G \to W_{Z'_G}^{U,L-\Delta,(G)}$$

are analytic when we give the space of segments the Banach manifold structure indicated before. Moreover, there is a codimension 1 Banach manifold Σ *contained in* U *such that:*

c) *If* $G \in \Sigma$, *then*

$$W_{Z_G}^{S,L-\Delta,(G)} \quad \text{and} \quad W_{Z'_G}^{S,L-\Delta,(G)}$$

have a quadratic tangency.

d) *if* F_λ *is a family such that* $F_0 \in \Sigma$, $\left(\frac{d}{d\lambda}F\right)_\lambda \neq 0$ *transversal to* Σ. *Then*

$$W_{Z_{F_\lambda}}^{S,L-\Delta,(F_\lambda)} \pitchfork W_{Z'_{F_\lambda}}^{U,L-\Delta,(F_\lambda)} \quad \text{for} \quad \lambda \in I_1 = (0, \Delta)$$

and

$$W_{Z_{F_\lambda}}^{S,L-\Delta,(F_\lambda)} \not\pitchfork W_{Z'_{F_\lambda}}^{U,L-\Delta,(F_\lambda)} \quad \text{for } \lambda = I_2 = (-\Delta, 0)$$

or we have the similar conclusion for $I_1 = (-\Delta, 0)$, $I_2 = (0, \Delta)$.

Which of the two alternatives above takes place depends on the sign of $\left\langle \Phi_{F_0}, \left(\frac{d}{d\lambda}F_\lambda\right)_{\lambda=0} \right\rangle$ *where* Φ_{F_0} *is a continuous functional (which we will describe later somewhat explicitly.*

Proof. Part a) is just the implicit function theorem for hyperbolic periodic fixed points. Part b) is the analytic dependence of the stable manifold that can be found in [Mey75], a more modern treatment in [CFdlL02a].

Part of the results of [CFdlL02a] are rather explicit formulas for the derivative of stable manifolds with respect to the map. They can be expressed as sums along the periodic orbit. (Sometimes this sums are called Melnikov functions).

In particular, this shows that if one has a tangent intersection and the first order perturbation does not vanish, then the family starts crossing. This establishes d).

□ □

Let F_n denote the sequence of Fibonacci numbers.

Of particular interest for us will be the manifolds $\Sigma_{n,B}$ which correspond to the tangent intersection of the stable manifold of a Birkhoff orbit (monotone periodic orbit) of type F_n/F_{n+1} and the unstable manifold of a Birkhoff orbit of type F_{n+1}/F_{n+2} taking place in a ball B.

As argued before, the manifolds $\Sigma_{n,B}$ are codimension 1 manifolds. These manifolds locally separate the space of maps in two components. The maps in one component – which we will denote $\Sigma^+_{n,B}$ – possess a transverse intersection of the stable manifolds of the orbit of type F_n/F_{n+1} and the unstable manifold of orbit of type F_{n+1}/F_{n+2} The maps in the other component – which we will denote $\Sigma^-_{n,B}$ – do not have such a transverse intersection in the ball B – even if they may have it in a larger ball. We note that the obstruction criterion applies to the maps in $\Sigma^+_{n,B}$ and for them, we can conclude that they do not have an invariant circle of rotation number golden mean. For the maps in $\Sigma^-_{n,B}$, we are not claiming anything.

As it is also standard, in the study of destruction of invariant circles, we define the manifolds Γ_n to be the set of points for which the orbit of type F_n/F_{n+1} in the critical line are such that they have an eigenvalue 1. These bifurcation surfaces played a very important role in the study in [Gre79].

It is important to note that all the formulations of the renormalization group have the property that if a map has a heteroclinic transverse intersection between an orbit of type F_n/F_{n+1} and another of type F_{n+1}/F_{n+2} occurring in a specified ball, then, the renormalized mapping has a heteroclinic transverse intersection between orbit of type F_{n-1}/F_n and another of type F_n/F_{n+1} occurring in another ball which is just as scaled version of the previous one.

Therefore, as an immediate consequence, we have:

Proposition 3.2. *With the above definitions of tangency manifolds Σ and the renormalization group $\mathcal{N}$, we have*

$$\mathcal{N}(\Sigma_{n,B}) \subset \Sigma_{n-1,B}$$
$$\mathcal{N}(\Gamma_n) \subset \Gamma_{n-1}$$

3.2. Formulation of the conjectures.

Conjecture 3.3. *Let $\mathcal{N}$ be the renormalization group for golden mean invariant circles.*

Let T^* be a non-trivial fixed point for the renormalization.

The operator $\mathcal{N}$ at T^* is hyperbolic. It has a one dimensional unstable manifold $W^u_{T^*}$ and a codimension 1 stable manifold $W^s_{T^*}$. Since the manifold $W^u_{T^*}$ is one-dimensional, $W^u_{T^*} - T^*$ has two connected components $W^{u,+}_{T^*}$,$W^{u,-}_{T^*}$.

(1) *For some $n_0 \in \mathbb{N}$ and some ball B, the manifold $W^{u,+}_{T^*}$ intersects transversally a manifold $\Sigma_{n_0,B}$ as described above.*
(2) *The manifold $W^{u,-}_{T^*}$ intersects transversally the manifold Γ_{n_0}.*
(3) *The manifold $W^{u,-}_{T^*}$ is contained in the basin of attraction of the trivial fixed point.*

Remark 3.4. Note that the conjecture involves an arbitrary choice of the sets B where the tangencies take place. As we will see, the consequences happen provided one suitable choice can be made. In the numerical section 5, we have detailed the choices that we have made and that seem to work. From the numerical point of view, it does not seem a hard task to find choices of the balls that work.

Moreover, it can be verified that these choices do not affect the numerical results appreciably. Of course, part of the predictions of the conjectures is that the results obtained with any of these choices will accumulate at an exponential rate (with a rather small constant).

Taking into account Proposition (3.2) and applying the λ-Lemma, we obtain that, Γ_n accumulate on $W^s_{T^*}$. Similarly, Σ_{n,B_n} accumulate on $W^s_{T^*}$. Both accumulations happen in the C^1 – or even stronger – sense and at a exponential rate (See [CEL80], [AK00] Theorem 1.9 for a proof of a λ-lemma adapted to this situation.)

In particular, for a one dimensional family T_λ sufficiently C^1 close to $W^u_{T^*}$, it will also cross the the manifolds Σ_n transversally.

Since on one side of each of the Σ_n we can apply the obstruction criterion, we conclude that a family family T_λ as above contains a sequence of semintervals $I_n = (\lambda_n, \lambda_n + \delta_n)$ such that $\lambda \in I_n$ implies that T_λ does not have an invariant circle with a golden mean rotation.

Furthermore, $\lambda_n \to \lambda_*$ where λ_* is determined by $T_{\lambda_*} \in W^s_{T^*}$. Note that, according to [Koc02a], T_{λ_*} possesses an invariant circle – albeit not very regular, see also [Sti93] –. For $\lambda \in (\lambda_* - \delta, \lambda_*)$, we will have that $\mathcal{N}^n(T_\lambda)$ will converge to the trivial fixed point and, therefore, according to [Koc02a] have an invariant circle of golden mean rotation number.

We see therefore that T_{λ_*} belongs to the boundary of the sets of maps that have an analytic invariant circle with golden mean rotation number.

The following Figure 3.2 depicts the geometry – in the infinite dimensional space of mappings – described in Conjecture 3.3.

Note that Conjecture 3.3 implies that in a semi-interval of $W^u_{T^*}$ whose end is $W^u_{T^*} \cap \Sigma_{n_0,B_{n_0}}$ there are no invariant circles with rotation number golden mean.

Since the argument applies to the

Conjecture 3.3 can be made somewhat stronger.

Conjecture 3.5. *With the notations of Conjecture 3.3.*

Besides the statements in Conjecture 3.3, we have

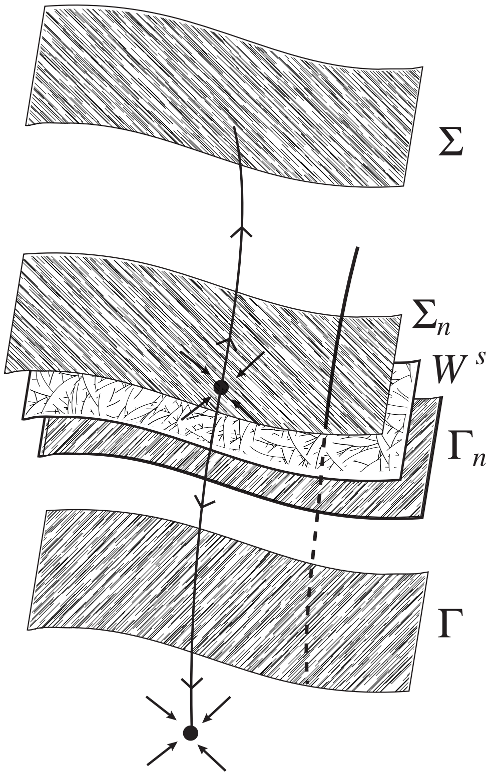

FIGURE 1. Pictorial representation of Conjecture 3.3

In a neighborhood of $W^u_{T^*}$, the unstable manifold of the fixed point under renormalization, we have for some $n_0 \in \mathbb{N}$ and B a ball,

$$\Sigma_{n_0+1,B_{n_0+1}} \cap W^u_{T^*} \in \Sigma^+_{n_0,B_{n_0}}$$

Note that Conjecture 3.3 implies that a segment of $W^u_{T^*}$ is contained in $\Sigma^+_{n,B}$. Conjecture 3.5 makes the quantitative assertion that this interval extends to reach the intersection of the next level.

If Conjecture 3.5 were true, then we see that the different obstructions *"play tag"*. That is, before the obstruction due to Σ_n ceases to apply, we can start applying the obstruction due to Σ_{n+1}. In this way, we can see that the intervals I_n described above overlap and that therefore $\cup_n I_n$ contains a full interval $(\lambda_*, \lambda_* + \delta)$ for some $\delta > 0$.

It seems from numerical experiments that the δ above can be reasonably big. Indeed, if Conjecture 3.5 were true, we could just select it by $T_{\lambda^*+\delta} \in \Sigma_{n_0}$.

Once we have used the combination of the obstruction and renormalization to exclude existence of invariant circles in the initial interval $(\lambda^*, \lambda^* + \delta]$ it becomes possible to use other criteria to exclude existence of invariant circles in the intervals $[\lambda^* + \delta, \infty)$. For example, in the numerical studies of the standard family, which we will discuss later, it seems that one can take $\delta \approx 1$. There exist proofs that the standard family does not have invariant circles for $\lambda > 4/3$ – see [Mat88] – and computer-assisted proofs that get much closer to the numerically conjectured value can be found in [MP85], [Jun91].

Hence, assuming Conjecture 3.5 – as well as some very mild bounds on the value of n_0 – and that the standard family crosses transversally $W^u_{T^*}$ sufficiently close to T^*, we would conclude that, for the standard family, there is a golden mean invariant circle precisely for $\lambda \in [-\lambda_*, \lambda_*]$.

When $\lambda \in (-\lambda_*, \lambda_*)$, the circle is analytic. When $\lambda = \pm\lambda_*$, the circle is finitely differentiable according to [Koc02a]. The paper [Sti93] contains an argument showing that some properties of the fixed point of renormalization imply that the critical circle is not more regular than a certain very explicit number. It looks like the argument in [Koc02a] could also make similar explicit numerical predictions or that the argument of [Sti93] could be adapted to the renormalization formalism in [Koc02a].

Recent work of the authors with N. Petrov extending the methods of [dlLN02] measures directly the regularity of the critical invariant circles and obtains a good agreement with the numerical values of [Sti93].

We certainly believe that Conjecture 3.5 is true and we will present numerical evidence for its validity in Section 5. Clearly, Conjecture 3.5 implies Conjecture 3.3.

We will end this section by formulating Conjecture 3.6, which is clearly implied by Conjecture 3.3. The reason for formulating such a weak conjecture is that it implies many of the consequences of the sharper conjectures but it will be easier to verify in a computer assisted proof. Certainly, it is so weak that it is hard to believe it could fail.

Conjecture 3.6. *In the segment $W^{u,+}_{T^*}$ there is a map T_{λ^*} with a heteroclinic transverse connection between an orbit of type F_n/F_{n+1} and an orbit of type F_{n-1}/F_n.*

Note that the existence of a transverse heteroclinic connection for T_{λ^*} implies the existence of a similar connection for all the maps in a neighborhood of $\mathcal{U}$ of T_{λ^*}.

Using the partial linearization established in [CEL80] or the λ-lemma of [AK00], we have that given any neighborhood $\mathcal{V}$ of a point $T \in W^s_{T^*}$, we can find a $k \in \mathbb{N}$ such that $\mathcal{N}^{-k}(\mathcal{V}) \cap \mathcal{U} \neq \emptyset$.

By the same argument as in Proposition 3.2, we have that if $\tilde{T} \in \mathcal{N}^{-k}(\mathcal{V}) \cap \mathcal{U}$, $T = \mathcal{N}^k(\mathcal{N}^{-k}(T))$ has a transverse heteroclinic intersection between an orbit of type F_{n+k}/F_{n+k+1} and an orbit of type F_{n+k-1}/F_{n+k}.

Therefore, Conjecture 3.6 implies that $W^s_{T^*}$ is part of the boundary of the points with an analytic invariant circle of rotation golden mean.

4. Some consequences of the conjectures

Lemma 4.1. *Under Conjecture 3.3 given a universal family of twist maps T_λ, then (by switching perhaps to $\tilde{T}_\lambda \equiv T_{-\lambda}$ to ensure a proper orientation of the parameter) we can find λ_0, $\delta > 0$ such that:*

- *When $\lambda_0 - \delta < \lambda < \lambda_0$, then T_λ has a smooth invariant circle with rotation golden mean.*
- *When $\lambda_0 < \lambda < \lambda_0 + \delta$, for all N sufficiently large, there are hyperbolic orbits of type F_N/F_{N+1}, F_{N+1}/F_{N+2}, such that their stable and unstable manifolds intersect transversally.*

Remark 4.2. Actually, the above results can be made somewhat more precise in the case that the map is reversible, and, as it is well known, we can prove that the hyperbolic orbits with the transversal intersection lie on the critical lines. See Subsection 2.2 for more details and definitions. For the moment, we just point out that the fact that the orbits lie on critical lines reduces the numerical calculation of the periodic orbits to a one-dimensional search.

Remark 4.3. When that the map is reversible, then the periodic orbits that satisfy (7) can be made to lie on the critical lines.

Note that a consequence of Lemma 4.1 is that, as soon as we are in the second alternative of Lemma 4.1, we have that for the values of parameters that there is not an invariant circle with golden mean rotation number, then, we can find a hyperbolic orbit with rotation number golden mean.

An argument in [FdlL92b], shows that, when Lemma 4.1 applies, we have that for $\lambda \in (\lambda_0, \lambda + \delta)$ there exists a sequence of points $\{p_N\}$ which are periodic of type F_N/F_{N+1} for the map T_λ and such that

$$\lim_{N\to\infty} |\operatorname{Tr}(DT_\lambda^{F_{N+1}}(p_N)) - 2|^{1/F_{N+1}} = \mu_\lambda > 0 \tag{6}$$

The above result is closely related to Greene's criterion [Gre79] Assertion VI. that an invariant circle exists if and only if the limit in (6), (7) is zero.

Hence, under Conjecture 3.5 we have that, for universal families, we have that if there is no circle, then the mean residue is positive.

The other side of the implication in [Gre79] is established rigorously independently of renormalization. In the papers [FdlL92b], [Mac92] (See also [McK82] 1.3.2.4) we can find proofs when there is an smooth KAM torus, we have

$$\lim_{N\to\infty} |\operatorname{Tr}(DT_\lambda^{F_{N+1}}(p_N)) - 2|^{1/F_{N+1}} = 0 \tag{7}$$

Hence, we have that

Lemma 4.4. *If Conjecture 3.5 is true, Greene's criterion applies for universal functions. More explicitly, There is an invariant circle if and only if*

$$\lim_{N\to\infty} |\operatorname{Tr}(DT_\lambda^{F_{N+1}}(p_N)) - 2|^{1/F_{N+1}} = 0 \tag{8}$$

We will also establish another consequence of Conjecture 3.3 is that the long made observation that the stable manifold of the fixed point is indeed the boundary between the existence and non-existence of invariant circles.

Therefore, we could think of Conjecture 3.3 as a keystone that holds together the renormalization and the obstruction criteria. It allows to use the obstruction criteria to show that renormalization is optimal and renormalization to show that obstruction is optimal.

Remark 4.5. The renormalization group has been studied mainly in spaces of analytic functions.

Hence, it is perhaps interesting to discuss the prospects of adapting the ideas to other spaces. One motivation for this study is that, according to the study here, this manifold is the validity among stable and unstable manifolds.

One first obstacle that has to be overcome is whether the renormalization operators define an differentiable operator in the spaces of finite differentiable functions. The renormalization of [McK82] has the difficulty that it involves composition, which is not a differentiable operator in spaces of finite differentiable functions (See e.g. [dlLO99].) The differentiability properties of the operator in [Koc99] have not been investigated.

On the other hand, we note that the theory of stable manifolds for renormalization group operators in spaces of finitely differentiable functions have been investigated in [Dav96], [CE99].

Noting that the analytic topology is stronger than the finite differentiability one shows that the boundary in analytic spaces is part of the boundary in the spaces of finitely differentiable mappings.

This observation can be made significantly more precise. The KAM theory shows that if some map has an analytic invariant circle, then in a neighborhood in the space of smooth mappings we have also an invariant circle. Hence, the interior in the analytic topology is contained in the interior in a smooth topology. Conversely, under the hypothesis that Conjectures 3.3 and 3.5 are true, we obtain that the places where the KAM tori with a given rotation number do not exist is characterized by the existence of transverse heteroclinic intersections. Those persist under small smooth perturbations.

Remark 4.6. Note that Conjectures 3.3 and 3.5 can be verified with a finite calculation.

It suffices to know with a finite precision the stable manifold and to verify that it intersects transversally the manifold Σ.

Similar proofs have been obtained in the literature for similar problems. For example in [EW87] one can find a rigorous computer assisted proof which establishes that the stable manifold for the renormalization of period doubling maps crosses transversally the period doubling bifurcation manifold.

For the case considered here, if one starts from the fixed point established in [Koc02b] and one wants to implement a strategy similar to that of [EW87] it will be necessary to implement operations acting on families of mappings.

For the case at hand, it seems that a verification with the manifold Σ_0 could require a rather extensive calculation. On the other hand, the verification for Σ_{n_0} – with n_0 perhaps around 6 – does not seem to require more than one or two terms in the expansion.

We also note that the bulk of the consequences of the conjectures – e.g. that $W^u_{T^0}$ is part of the boundary between the maps with an analytic invariant tori with rotation golden mean and those with a hyperbolic Aubry-Mather set with rotation golden mean – can be established by establishing the much weaker Conjecture 3.6 (later on, we introduce the even weaker Conjecture 5.2). The verification of this conjecture seems to us to be of comparable difficulty with the work already done in [Koc02b]. The only ingredient not already present in [Koc02b] the verification that points in the unstable manifold of the fixed points admits periodic orbits with transversal intersections. Note that the existence of transversal intersection of the periodic orbits is not a delicate property, hence, only a moderate accuracy for the points in the unstable manifold.

The work of [Koc02b] already gives good estimates of the unstable eigenvalue and estimates on the second derivative. Hence, one obtains a reasonable approximation for the unstable derivative.

There are already in the literature several implementations of the study of the invariant manifolds which provide computer assisted proofs of the transverse intersection of the manifolds. For example [FR81], [FG99]. (A more systematic theory of these methods is developed in [CFdlL02a].) The maps considered in these references are polynomials, but the methods developed for the non-rigorous computations in this paper are very similar to the calculations that could be the basis of a computer assisted proof.

Note also that, even if, for the sake of simplicity, we have formulated the conjectures and their consequences only for the golden mean renormalization, they are quite trivial to adapt for other fixed points corresponding to other rotation numbers of periodic continued number expansion. We leave the straightforward formulation to the interested reader.

5. Numerical experiments.

In this section we present some numerical evidences that provide support for the conjectures formulated in in Section 3.

Roughly speaking, we will present numerical evidence verifying the scaling relations predicted by the conjectures. We will provide detailed calculations of scaling relations related to the scaling relations depicted in Figure 3.2. Indeed, we will see that there are quite quantitative predictions in Figure 3.2 – or in the modification depicted in Figure 5.4. We note that the quantitative implications will be verified with several figures of precision and they will agree with precision in several mappings. This provides with strong – albeit circumstantial – evidence of the truth of the conjectures. We think that the verifications presented here, will provide motivation for a computer assisted proof that, as we discussed in Remark 4.6 seems within striking distance of modern technology.

5.1. **Description of the experiments.** All our experiments will be performed on maps of the form (2) where the map f will depend on a parameter K in the form $f(x) = Kf_1(x)$.

We will consider the scaling behavior not only of golden mean circles but also of other numbers with relatively simple – in particular periodic – continued fraction expansions such as $[2, 2, 2, \ldots]$, $[1, 2, 1, 2, \ldots]$. These numbers will be constant type, in particular Diophantine.

Given one of these numbers ω, we consider the sequence p_i/q_i of approximants – expressed in lowest common denominator – and obtain a sequence x_i of for hyperbolic orbits of type p_i/q_i along the critical lines.

Given orbits x_i and x_{i+1} we obtain segments of their stable and unstable manifolds (See Section 5.2.2).

If these manifolds intersect transversally in the specified segment we can compute efficiently the area of the lobe enclosed by two consecutive intersections (See Section 5.2.3).

See Figure 2. Note that, by the preservation of area, the value of the area of the lobe is independent of the iterate and it is also an invariant under symplectic changes of variables. Moreover, as we will see it is related to Melnikov functions. Hence, the area of the lobes has been often used as a good measure of the size of the intersections of stable and unstable manifolds.

We, thus obtain a function $A_i(K)$ that gives the value of the area as a function of the parameter of the map. The smooth dependence of manifolds on parameters implies that the function will be defined and smooth for an open set of parameters. The paper [VT91] and [Aub92] show that for high enough K there are transverse homoclinic intersections and that therefore the functions A_i are defined – at least – in domains of the form $[K_i^*, \infty)$.

More generally, we note that the set of maps for which A_i takes an specific value contains a codimension 1 manifold in the space of maps.

We denote by Λ_i^ρ the manifold in the space of maps where the function A_i takes the value ρ.

Since the MacKay renormalization group just involves changes of variables – that do not change the A_i – and a scaling that changes A_i by multiplying it by a fixed factor δ ($\delta \approx 4.3391438$ in the case of the golden mean. See later for description of other scaling values.)

Therefore, the renormalization operator $\mathcal{R}$ has the property that $\mathcal{R}(\Lambda_i^\rho) = \Lambda_{i-1}^{\delta\rho}$.

We note that if Conjecture 3.3 holds, the manifolds Λ_i^ρ are interleaved with the manifolds Σ_i of homoclinic tangencies. See Figure 5.4 for a pictorial description of the situation. Hence, by the λ-lemma, $\Lambda_n^{A_0\delta^n} = \mathcal{R}^{-n}(\Lambda_0^{A_0} = \mathcal{R}^{-n}$ will accumulate on the unstable manifold of the fixed point.

Our experiment, therefore is to find the numbers K_i for which the map T_{K_i} intersects transversally the manifold $\Lambda_i^{A_i}$ where $A_i = A_0\delta^i$.

If the scaling δ is chosen appropriately – but the choice is not that critical – the K_i will converge to the value of K_c for which the family T_K crosses the stable manifold of the fixed point.

We note that one of the predictions of Conjecture 3.3 is that

$$K_i \approx K_c + D\sigma^{-i}$$

where $\sigma > 1$ is the unstable eigenvalue of the derivative of the renormalization at the non-trivial fixed point. In particular, the rate of accumulation is universal and, it is the same as that of accumulation of the parameters at which the periodic orbits loose stability. The number D, of course, depends on the family.

This will be the main goal of our numerics. To verify the exponential convergence of the K_i, and the universality of the rate. We will do this verification for several rotation numbers with easy continued fraction expansions.

We note that the papers [OS87a],[OS87b] contain direct calculations of the parameter where the tangent intersections happen. In this paper, we have chosen to do calculations for a fixed value of the area of the lobe. This seems to be more efficient numerically. In general, it is much more efficient to determine the place where a function reaches a regular point than the points where a function reaches a critical point. Precisely at the critical point the values of the functions the dependence on the parameters is weakest.

For similar reasons [Gre79], one prefers to find the values of parameter for which the periodic orbits have a fixed residue of 1/2 rather than the values for which the intersection is zero.

This does not affect the scaling results since the corresponding manifolds interleave. See Figure 5.4.

5.2. **Some indication on the algorithms used in the computations.** All our computations have been carried out in generalized standard maps of the form (2).

Many of them have been done in the standard map which corresponds to

$$f(x) = \frac{K}{2\pi}\sin(2\pi x) \tag{9}$$

Nevertheless, we note that the function (9) is very special. In particular, it contains only one harmonic. Since one of the important predictions of renormalization group is the universality of the results, to confirm the universality we have also used

$$f(x) = \frac{K}{2\pi}\left(\sin(2\pi x) + 0.05\sin(6\pi x)\right) \tag{10}$$

Remark 5.1. It would certainly seem desirable to extend the calculations to functions with more harmonics or so that the different harmonics are more dominant. One important question that remains unsettled is to obtain a description of the domain of universality described by the non-trivial fixed point. Even if for some one-dimensional renormalizations there are global theories for circle maps and for unimodal maps, there are indications that the renormalization group description for circle maps is only local. See [FdlL92a] for some evidence and for references to previous literature.

5.2.1. *Calculation of periodic orbits.* For a fixed value of K, we compute the hyperbolic periodic orbits of an specified type by using the method described in [Gre79].

We recall that maps of the form (2) are reversible (see 2.2). That is, they can be written as the product of two involutions. $F = T \circ S$ with $S \circ S = \mathrm{Id}$, $T \circ T = \mathrm{Id}$.

$$S : \left\{ \begin{array}{ll} \bar{y} & = y - f(x) \\ \bar{x} & = -x \end{array} \right. \qquad T : \left\{ \begin{array}{ll} \bar{y} & = y \\ \bar{x} & = -x + y \end{array} \right. \tag{11}$$

The maps S and R have an invariant sets which correspond to four straight lines $x = 0$, $x = 1/2$, $x = y/2$ and $x = (y+1)/2$, they are called critical lines. It is well known that any reversible map F has a set of periodic orbits of type p/q such that two points of their orbits belong to the critical lines [Gre79]. This property allow us to search periodic orbits only in the four critical lines. Therefore we use a one dimensional root finder to obtain the periodic orbits, in that way we can find this kind of orbit with high accuracy and for a large period of the orbit.

5.2.2. *Local stable and unstable manifolds.* Any hyperbolic point of a two-dimensional area preserving map has two one dimensional invariant manifolds, one stable and one unstable.

A first linear approximation of these manifolds is given by the eigenvectors of the linear map. A possible algorithm is to select a segment along the eigendirections and then iterate it. This method is very inaccurate when the eigenvalues of the hyperbolic points are close to 1, which is the case of most interest to us.

Hence, the method that we will develop is to compute very accurate local approximations of the stable and unstable manifolds by computing a high order polynomial approximation. This will give enough accuracy so that only a few iterations are needed to reach the desired places where the intersections take place.

The standard theory of invariant manifolds shows that $\mathrm{W}^u_{\mathrm{loc}}(\bar{x})$ and $\mathrm{W}^s_{\mathrm{loc}}(\bar{x})$ can be represented as the graph of some analytic function. For the sake of simplicity in the notation, we assume that we have performed a change of variables so that the hyperbolic fixed points is at the origin.

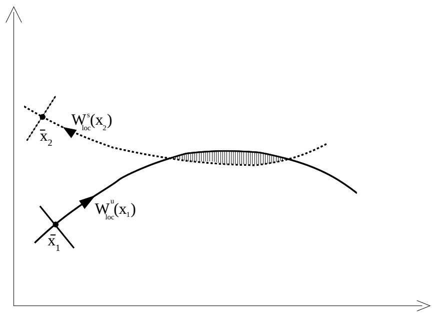

FIGURE 2. Heteroclinic intersection of two hyperbolic points.

We will take polynomial approximations $y(x) = \sum_{i=1}^{N} a_i x^i$ where the coefficients a_i are selected so that the graph of the function $y(x)$ is invariant up to order x^{N+1}. It is possible to see that the polynomials obtained in this way will be the Taylor polynomials of the function whose graph gives the invariant manifold. Hence, as N grows, the polynomial above will converge in a neighborhood – which in practice is rather large –. Hence, the procedure will provide with an extremely accurate approximation to the invariant manifolds.

Let F^q be the q-th iteration of our map, where q is the period of the hyperbolic periodic point, we can represent the map in this form,

$$F^q(x,y) = (G_q(x,y), H_q(x,y)) \quad .$$

where $G_q(x,y)$ and $H_q(x,y)$ are polynomials of order N. These functions are obtained using the recursion:

$$\begin{aligned} G_{i+1} = &\quad G_0\left(G_i(x,y), H_i(x,y)\right) \quad , \\ H_{i+1} = &\quad H_0\left(G_i(x,y), H_i(x,y)\right) \quad , \end{aligned}$$

where $i = 1, 2, \ldots, q-1$ and $(G_0(x,y), H_0(x,y))$ is the polynomial approximation of our map $F(x,y)$. The graph of the polynomial function $y(x)$ is close enough to the invariant manifold, therefore the image of the point $(x, y(x)) \in \mathrm{W}^u_{\mathrm{loc}}(\bar{x})$ belong to the same manifold:

$$y\left(G_q(x, y(x))\right) = H_q(x, y(x)) \quad .$$

Using a polynomial approximation of $G_q(x,y)$ and $H_q(x,y)$ up to order N, we can compute the unknown coefficients a_i. A more detailed description of this method can be found in [Sim90] and [Olv01].

5.2.3. *Computation of the area of the lobes.* For transverse heteroclinic intersection, the local manifolds $\mathrm{W}^u_{\mathrm{loc}}(\bar{x}_i)$ and $\mathrm{W}^s_{\mathrm{loc}}(\bar{x}_{i+1})$ form a lobe between the first heteroclinic point and the next point of intersection. Figure 3 shows the local manifold. If the manifold is chosen of appropriate length, we can order the intersection points. The first intersection point is denoted by a

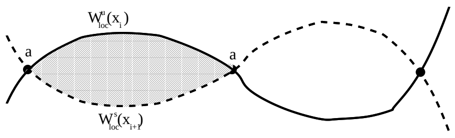

FIGURE 3. The lobe is bounded by the local invariant manifolds.

and the next by a', the local manifolds form the boundaries of the area of this lobe (we must remark that a' is not the image of a).

Using the polynomial approximation of $\mathrm{W}^u_{\mathrm{loc}}(\bar{x}_i)$ and $\mathrm{W}^s_{\mathrm{loc}}(\bar{x}_{i+1})$, we define a fundamental interval which correspond to a set of points x_j, $j = 1, \ldots, M$, such that $(x_1, y(x_1))$ is a point close to the hyperbolic point and $(x_M, y(x_M)) = F^q(x_1, y(x_1))$, in that form we have an initial partition of the fundamental interval.

The local invariant manifold is computed as the iteration of the basic interval. Provisionally, we consider the manifold as piece-wise linear, interpolating between consecutive points.

We compute approximations of the heteroclinic intersection of $\mathrm{W}^u_{\mathrm{loc}}(\bar{x}_i)$ and $\mathrm{W}^s_{\mathrm{loc}}(\bar{x}_{i+1})$ looking for the intersection of two straight segments of the corresponding manifolds. Once we have found an intersection, we refine the calculation by refining the partition that gets propagated till the intervals are smaller than a preset tolerance. Hence, we obtain the intersections with the precision of the polynomial discretizations. In that form we can compute these points which are the left and right ends of the lobe.

To compute the area of this lobe, we compute a set of points on the invariant manifold which lie in the border of the lobe, (x_i, y_i) for $i = 1, \ldots, s$. Now, we can compute the lobe's area using the following Green's identity: $A = \int_C x dy$, where C is the boundary of the lobe. To compute this integral we use the Simpson rule.

5.3. **Results for golden mean circle.** In this numerical experiment we chose the golden mean number γ_1 to be the rotation number of the invariant circle. We carried out calculations on transversal intersections of hyperbolic orbits as well as calculations for the elliptic orbits. That is, we have explored both sides of the stable manifold.

The sequence of approximants is γ_1 is $\{F_{i+1}/F_i\}$, where F_i are the Fibonacci numbers.

We try to find a sequence of parameter values K_i for which the lobe area is $A_i = \delta_1 A_{i-1}$ where $\delta_1 = 4.3391438$ is the scaling factor of the fixed point [McK82].

The starting point of the calculations was the heteroclinic intersection between 34/55 and 55/89 for which we fixed the value of the area $A_8 = 5 \times 10^{-7}$ value of the parameter $K_8 = 1.0018423 \pm 10^{-8}$.

The results of this experiment are displayed in table T1, the information that is showed in this table is the rotation numbers of the hyperbolic points p_i/q_i and p_{i+1}/q_{i+1}, the value of the parameter K_i, the ratio of the eigenvalues of the hyperbolic points λ_i/λ_{i+1} (we chose the expanding eigenvalue of the hyperbolic points) and the last element of the table correspond to the value of the lobe's area, A_i.

p_i/q_i	p_{i+1}/q_{i+1}	K_i	λ_i/λ_{i+1}	A_i
8/13	13/21	$1.10872 \pm 3 \times 10^{-6}$	0.3504	4.084906×10^{-5}
13/21	21/34	$1.053727 \pm 2 \times 10^{-7}$	0.359044	9.414084×10^{-6}
21/34	34/55	$1.02128 \pm 1 \times 10^{-8}$	0.35245	2.16957×10^{-6}
34/55	55/89	$1.001842 \pm 5 \times 10^{-8}$	0.35636	0.5×10^{-7}
55/89	89/144	$0.9900824 \pm 6 \times 10^{-7}$	0.35374	1.1523×10^{-7}
89/144	144/233	$0.9829137 \pm 3 \times 10^{-7}$	0.356	2.6558×10^{-8}
144/233	233/377	$0.9785662 \pm 2 \times 10^{-8}$	0.353444	6.12009×10^{-9}
233/377	377/610	$0.97584678 \pm 1 \times 10^{-8}$	0.35878	1.41044×10^{-9}
377/610	610/987	$0.97408 \pm 1 \times 10^{-5}$	0.387	3.2505×10^{-10}
610/987	987/1597	$0.973235134 \pm 1 \times 10^{-9}$	0.35459	7.49110×10^{-11}
987/1597	1597/2584	$0.9726175 \pm 1 \times 10^{-7}$	0.3545	1.726402×10^{-11}
1597/2584	2584/4181	$0.97223865 \pm 1 \times 10^{-8}$	0.35453	3.97867×10^{-12}
2584/4181	4181/6765	$0.9720060 \pm 1 \times 10^{-7}$	0.3543	9.16925×10^{-13}
4181/6765	6765/10946	$0.97186295 \pm 1 \times 10^{-8}$	0.3545	2.113114×10^{-13}
6765/10946	10946/17711	$0.97177520 \pm 1 \times 10^{-8}$	0.35450	4.86996×10^{-11}

Table T1

5.4. **The ratio of eigenvalues. Some further conjectures.** One observation on the results of Table T_1 is that the observable λ_i/λ_{i+1} remained more or less constant under the iteration.

It is interesting to compare the results of Table 1 with the computations in [OS87a] for heteroclinic tangency.

In the case of heteroclinic tangencies, the value of the area of the lobe is zero, but λ_i/λ_{i+1} took a value 0.6641 (which is very approximate to γ_1) for all the values of the intersection.

Note that the values of the eigenvalues are invariant under changes of variables and that, under iteration, they behave well. Therefore, the observable is invariant under renormalization.

Note also that the derivative of this observable with respect to the map, clearly does not vanish since we can make easily perturbations for which a periodic orbit changes more than the other.

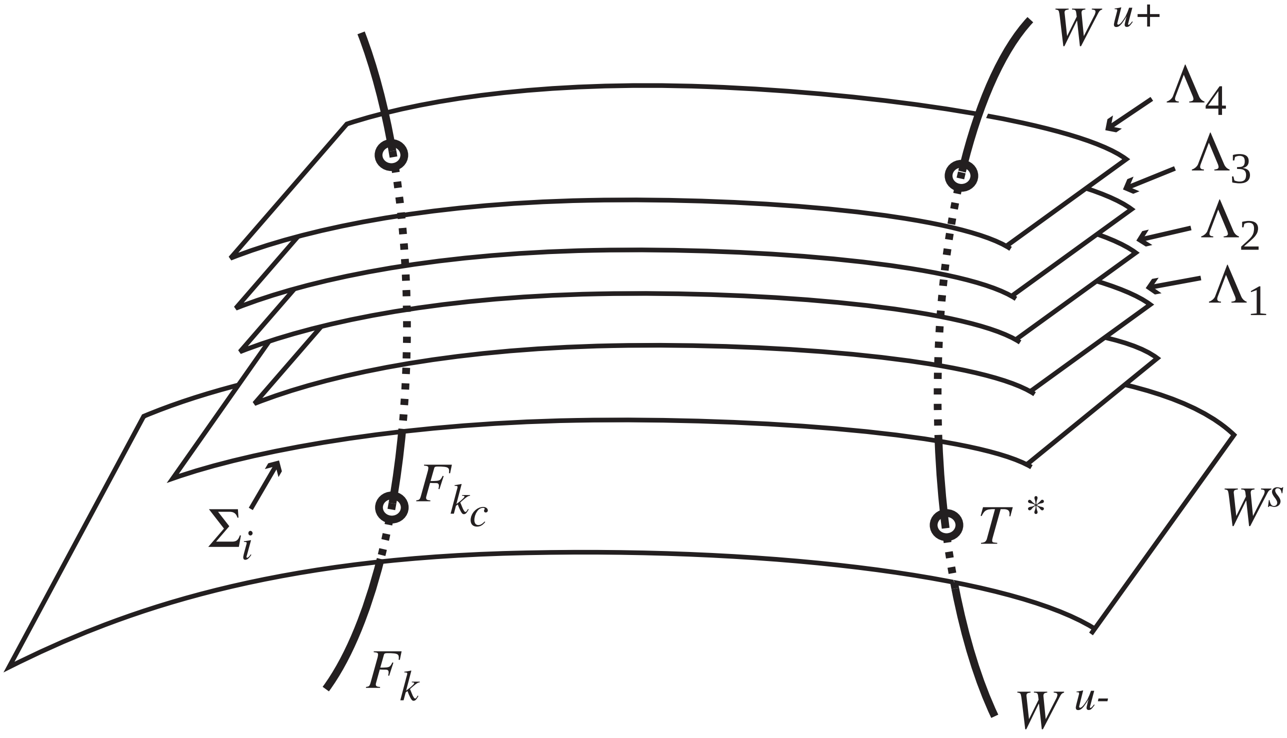

FIGURE 4. Pictorial representation of Conjecture 5.2

Therefore, the level sets of observable λ_i/λ_{i+1} define a codimension one foliation. We denote the leaves of such a foliation by $\Lambda^i_\beta = \{F|\lambda_i/\lambda_{i+1}(F) = \beta\}$.

Because of the invariance of λ_i/λ_{i+1} under renormalization, the leaves of the foliation Λ^i are mapped by renormalization into the leaves of the foliation Λ^{i-1}

The numerical experiments suggest that this foliation is transversal to the unstable manifold of the non-trivial fixed point. Hence the level sets of these foliations can play the same role than the other foliations we have discussed.

This suggests that we can take the observable λ_i/λ_{i+1} as an alternative for the area of the lobe to formulate our conjectures.

Conjecture 5.2. *In a neighborhood of the non-trivial fixed point, the stable and unstable manifold is transversal to the level sets of λ_i/λ_{i+1}, which define a local foliation.*

The following Figure 5.4 depicts the geometry described in Conjecture 5.2.

Note that the calculations of the observable λ_i/λ_{i+1} are significantly faster than those of the lobe area. Moreover, since λ_i/λ_{i+1} is kept constant we do not need to compute the scaling factors.

Hence, the verifications that we will do in subsequent calculations consist in computing intersections of the families of maps with Λ^i_β and verify that, indeed the manifolds have heteroclinic intersections and that the area of the lobes scale.

That is, we will compute values K_i for which $F_{K_i} \in \Lambda^i_\beta$, report A_i and verify that $A_i/A_{i+1} \to \delta_1$.

We performed twelve experiment where the parameter Λ took its value from the interval $[0.35, 0.625]$, the value $\Lambda = 0.664158$ correspond to the case $A_i = 0$ [OS87a]. Figure 5 shows the result the vertical axis represent the distance of the ratio of lobe's areas respect to Mackay's scaling factor, $\log(|A_i/A_{i+1} - \delta_1|)$, the horizontal axis has the value of $\log(q_i)$, where q_i is the period of the hyperbolic periodic orbit. Each line represent a sequence of periodic orbits which eigenvalue's ratio Λ is constant. From this picture we can estimate the speed of convergence of the scaling factor,

$$|A_i/A_{i+1} - \delta_1| = C(\Lambda)/q_i \quad , \tag{12}$$

where C is a constant which is function of Λ.

In order to check universality of the results, we repeated the experiment for the map corresponding to the function (10). The results are shown in Figure 6. We can see that Figures 5 and 6, indeed look very similar. Hence, we consider this as a confirmation of the conjecture that equation (12) is valid for all maps in a universality class. We verified that heteroclinic intersection, when $A_i = 0$, takes place for the same value of $\Lambda = 0.664158$ that we found for the standard map [OS87b].

In Figure 5, we summarize the results for the standard map.

The vertical axis is $\log(|A_i/A_{i+1} - \delta_1|)$ and the horizontal axis is $\log(q_i)$.

Each line corresponds to a sequence of periodic orbits whose rotation numbers are F_i/F_{i+1} and F_{i+1}/F_{i+2}.

The value of $\Lambda = \lambda_i/\lambda_{i+1}$ is: (a) 0.625, (b) 0.6, (c) 0.575, (d) 0.55, (e) 0.525, (f) 0.5, (g) 0.475, (h) 0.45, (i) 0.425, (j) 0.4 and (k) 0.375

Finally, we want to note that the conjecture 5.2 allows us to estimate the scale factor of any sequence of hyperbolic periodic orbits. The value of the scale factor δ_1 related to the critical circle with rotation number equal to the golden mean was computed by MacKay [McK82].

5.5. **Invariance of Λ and renormalization.** The numerical experiment described in the previous subsection shows that any foliation defined by the observable $\Lambda^i_\beta = \lambda_i/\lambda_{i+1} = \beta$ is asymptotically close to some foliation Σ. We have shown the numerical evidence that this transformation, which map Λ^i_β to Λ^{i-1}_β preserve, in an asymptotical way, the MacKay's scaling factor δ_1 (see figure 5). Relation 12 is then a good indication that this transformation coincide with the renormalization operator $\mathcal{N}$. Also, we can conjecture that

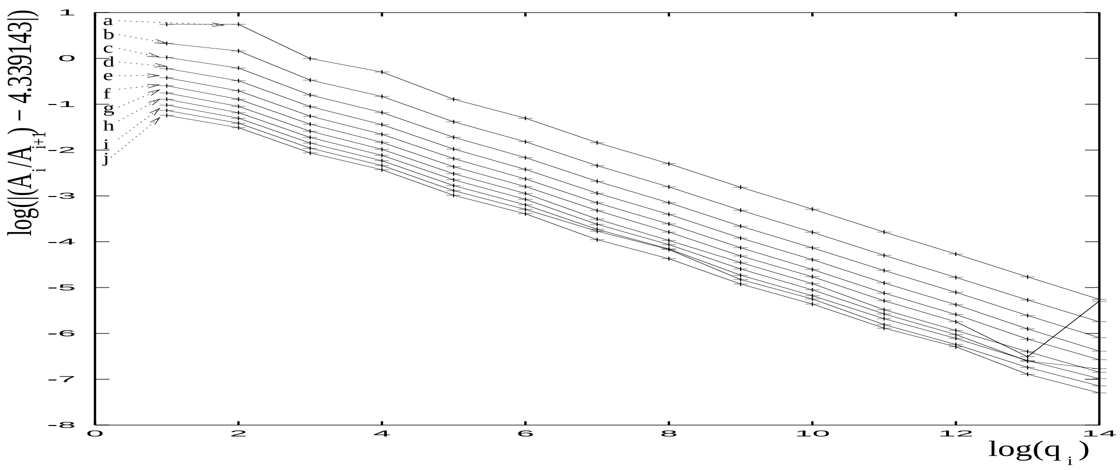

FIGURE 5. Asymptotic convergence of A_i/A_{i+1} to the limit $\delta_1 = 4.3391438$.

this is an universal property because our numerical experiments where tested using two different family of maps (figure 6).

We can conclude that the foliations Σ and Λ are indistinguishable in the numerical point of view. Therefore, we can conjecture that any foliation Λ can be replaced by some foliation Σ and viceversa. For example, the obstruction method can be defined in two forms: by the heteroclinic tangency of the invariant manifolds of two periodic orbits, or by the value of $\Lambda = 0.664158$ [OS87b].

5.6. **Results close to other circles.** We are interested in studying the renormalization of other invariant circles with rotation numbers whose continued fraction expansions are periodic with period one, $\gamma_n = [\bar{n}]$ where

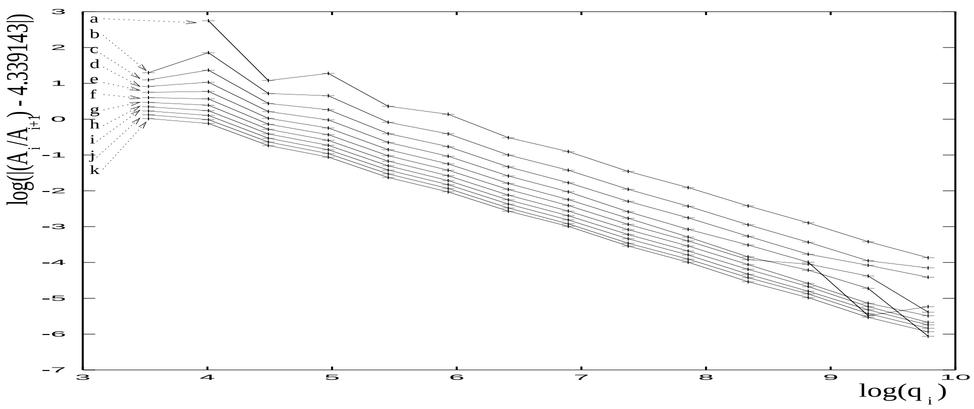

FIGURE 6. Similar to Figure 5. In this case, the perturbation function is: $-\frac{K}{2\pi}(\sin(2\pi x) + 0.05\sin(6\pi x))$.

$n = 2, 3, \ldots$. The set up of these experiments is similar to our first experiment, but now we are interested in performing these numerical experiment using $\gamma_2 = \sqrt{2} - 1$ and $\gamma_3 = (\sqrt{13} - 2)/2$. We did not know a priori the value of the corresponding scale factors δ_n, nevertheless we can use the algorithm described in Section 5.4.

Our first numerical experiment corresponds to study the sequence of hyperbolic orbits whose rotation numbers converge to $\gamma_2 = \sqrt{2} - 1$. The rotation number of these orbits are obtained by truncating the continued fraction expansion of $\gamma_2 = [2, 2, 2, \ldots]$. Our starting orbits were $p_3/q_3 = 29/70$ and $p_4/q_4 = 70/169$, we set the value of the parameter $K_3 = 0.9698 \pm 2 \times 10^{-4}$. The invariant manifolds of these hyperbolic periodic orbits had transverse heteroclinic intersection. The local invariant manifold formed a lobe between two consecutive heteroclinic points, the lobe's area was $A_3 = 2 \times 10^{-8}$, the corresponding eigenvalues of the hyperbolic orbits had a ratio $\lambda_3/\lambda_4 = 4.176/11.950 = 0.349$. Using our conjecture we could compute the value of the perturbation parameter K_4 for the hyperbolic orbits of type $p_4/q_4 = 70/169$ and $p_5/q_5 = 169/408$, we looked for these hyperbolic orbits such that the ratio of their eigenvalues was close enough to 0.349, in this case $K_4 = 0.962535$ and the corresponding lobe's area was $A_4 = 1.369 \times 10^{-9}$. Then, our first estimation of the scale factor was 14.6092. Table T2 shows the value of the sequence of hyperbolic orbits whose rotation numbers converge to γ_2.

p_i/q_i	p_{i+1}/q_{i+1}	K_i	λ_i/λ_{i+1}	A_i
5/12	12/29	$1.032676 \pm \times 10^{-7}$	0.35045	4.263×10^{-6}
12/29	29/70	$0.987999 \pm \times 10^{-7}$	0.339	2.92×10^{-7}
29/70	70/169	$0.9698 \pm \times 10^{-5}$	0.342	2×10^{-8}
70/169	169/408	$0.962535 \pm \times 10^{-7}$	0.3360	1.369×10^{-9}
169/408	408/985	$0.9595336 \pm \times 10^{-8}$	0.33608	9.3826×10^{-11}
408/985	985/2378	$0.95830144 \pm \times 10^{-9}$	0.33520	6.4264×10^{-12}
985/2378	2378/5741	$0.957796 \pm \times 10^{-7}$	0.333	4.4016×10^{-13}
2378/5741	5741/13860	$0.95759028 \pm \times 10^{-9}$	0.3304	3.01485×10^{-14}

Table T2

The value of the scale factor that we could estimate from this experiment was $\delta_{\gamma_2} \sim 14.60$, this factor is the limit of A_i/A_{i+1}.

Our next experiment was done with the value of the rotation number of the critical circle $\gamma_3 = (\sqrt{13}-2)/2$. The value of the rational approximants of γ_3 was obtained from the truncation of its continued fraction expansion $\gamma_3 = [3,3,3,3,\dots]$. The initial orbits was hyperbolic orbits of type $p_1/q_1 = 10/33$ and $p_2/q_2 = 33/109$, the value of the perturbation parameter was $K_1 = 0.906678$, the corresponding value of the eigenvalue's ratio was $\lambda_1/\lambda_2 = 0.38178$ and the lobe's area was $A_1 = 5\times 10^{-9}$. Table T3 shows the sequence of values of the approximants to γ_3, these values was computed with the help of our conjecture, the next hyperbolic orbits were computed fixing the value of $\lambda_i/\lambda_{i+1} \sim 0.3815$. The scale factor obtained from this procedure was $\delta_{\gamma_3} = 40.12$.

p_i/q_i	p_{i+1}/q_{i+1}	K_i	λ_i/λ_{i+1}	A_i
10/33	33/109	$0.906678 \pm \times 10^{-7}$	0.38178	5×10^{-9}
33/109	109/360	$0.895563 \pm \times 10^{-7}$	0.38172	1.2462×10^{-10}
109/360	360/1189	$0.892266 \pm \times 10^{-7}$	0.3817	3.1064×10^{-12}
360/1189	1189/3927	$0.891283 \pm \times 10^{-7}$	0.3812	7.7429×10^{-14}
1189/3927	3927/12970	$0.8909888 \pm \times 10^{-8}$	0.3815	1.929×10^{-15}

Table T3

Finally, we have computed the renormalization process of invariant circles whose rotation numbers have the form $[\overline{a_1,a_2}]$ and $[\overline{a_1,a_2,a_3}]$. We used the same numerical method to compute the corresponding lobe's areas. These results are going to be reported in our next paper.

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