ORBITS OF ORBS:

sphere packing meets Penrose tilings

by

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Abstract

The venerable study of optimum density problems, such as packing spheres as efficiently as possible, has been revitalized by investigations on aperiodicity, a subject best known through the Penrose kite & dart tilings. The result is a generalization of the basic notion of geometric symmetry for patterns in space.

March 2003

Subject Classification: 52A40, 52C26, 52C23

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* Research supported in part by NSF Grant No. DMS-0071643 and Texas ARP Grant 003658-158

At the center of discussion will be the densest possible packings, of a region of infinite volume such as d-dimensional Euclidean space, $\mathbb{E}^d$, or hyperbolic space, $\mathbb{H}^d$, by finite-volume bodies such as balls or polyhedra. Giving a precise meaning to the "density" of such a packing will require some care and preparation, but intuitively it is appropriate to think of the density of a packing P as the limit, if it exists, of the fractional volume occupied by the bodies of P in a sphere as the sphere expands indefinitely about some common center; see Figure 1.

We begin by concentrating on Euclidean spaces, and will sharpen the discussion using the following definition.

Definition. By a "(densest) packing problem" we mean: Given a finite collection $\mathcal{B}$ of bodies in $\mathbb{E}^d$, find the densest packings of $\mathbb{E}^d$ by congruent copies of these bodies.

In practice it is very difficult to actually solve most densest packing problems. The main solved examples are:

a) For $\mathcal{B}$ consisting of a regular n-gon in $\mathbb{E}^2$, but essentially just for $n = 3, 4, 6$, namely the cases for which the optimal density is 1 (tilings);
b) For $\mathcal{B}$ consisting of a disk in $\mathbb{E}^2$, and (presumably) for $\mathcal{B}$ consisting of a ball in $\mathbb{E}^3$;
c) So-called "aperiodic" tilings of $\mathbb{E}^d$, such as the kites & darts in $\mathbb{E}^2$ [Gard]. (See Figure 2a.)

The proof by Fejes Tóth that the hexagonal packing (Figure 3) is densest for equal disks in $\mathbb{E}^2$ is beautiful and instructive; see [Feje]. It uses the notion of "Voronoi region" of a body b in a packing P, which is the set of all points in $\mathbb{E}^d$ which are as close, or closer, to b as to any other body of P. For every disk in the packing of Figure 3 the Voronoi region is a regular hexagon. What Fejes Tóth showed is that among all the Voronoi regions for any disk in any disk packing, the relative fraction, of the area of the Voronoi region that is filled by the disk, is maximized by any of the disks in Figure 3. For the disks in that packing the relative fraction, $\pi/2\sqrt{3}$, is thus the highest possible density of any packing by equal disks in $\mathbb{E}^2$: intuitively, the density has been maximized simultaneously in all local regions. The arguments by Hsiang [Hsia] and by Hales [Hale] for ball packings in $\mathbb{E}^3$ share this approach of simultaneous local optima, though the quantity being optimized is no longer simply the relative volume of the Voronoi region; see [Laga]. And we note that the other solved problems listed above, being tilings, share this feature of simultaneous local optimization.

Although densest packing problems are at the heart of our discussion, we will not be concerned with issues of actually solving such problems, but with broader questions such as the existence, and especially the qualitative features (symmetries), of their solutions. The reason is that although there are significant applications for the solution of a densest packing problem, especially for ball packings, the deeper interest for mathematics is the investigation such investigation spawns, particularly concerning subgroups of the isometry group $\mathcal{G}^d$ of $\mathbb{E}^d$ (or $\mathbb{H}^d$). The richness of this approach was recognized many years ago, for instance by Hilbert in his 18th problem: "Building up of space from congruent polyhedra" [Hilb], in which he posed 3 specific questions which we paraphrase as follows:

i) Are there only finitely many possible symmetry groups of densest packings of bodies in $\mathbb{E}^d$? (It was known there are infinitely many in $\mathbb{H}^d$.)
ii) Is it the case that for every polyhedron which can tile $\mathbb{E}^d$ or $\mathbb{H}^d$ there is (also) a tiling consisting precisely of the images of the polyhedron by the elements of a subgroup of rigid motions?
iii) What are the different structures (in particular symmetries) of the densest packings by bodies of $\mathbb{E}^d$ or $\mathbb{H}^d$?

The history of research on Hilbert's 18th problem is erratic. Bieberbach gave a celebrated solution of i) in 1910 [Bieb]. However a simple counterexample was given for ii) in 1928 by Reinhardt [Rein], and no significant progress was ever made on iii) (in the sense of general $\mathbb{E}^d$ or $\mathbb{H}^d$). Given this discouraging history the problem has generally passed from view.

There was a prescient note by Milnor, in his review of Hilbert's 18th problem [Miln], on the seminal paper by Robinson on aperiodic tilings [Robi], in which Milnor shows its connection with problem ii). For us its connection with problem iii) is more useful, as an indication of how symmetry may be manifested in packing problems. This article could be considered an analysis of the impact of the study of aperiodicity on Hilbert's problem.

Consider again the simplest interesting solved example of a packing problem, the densest packings of $\mathbb{E}^2$ by equal disks. We noted above that the hexagonal packing is densest (by any reasonable definition), but one might inquire whether there are other packings, not congruent to this one, which have the same density. (This has been considered, for many years, a minor difficulty by those analyzing densest packings; see [CoGS].) Clearly one could remove a finite number of disks from the hexagonal packing without affecting the density. Furthermore, one could actually remove certain infinite subcollections, such as the disks with centers on a line, without affecting the density, or we could (carefully) separate the disks so no two are touching, without affecting the density. However using a recent definition of densest packings (which adds a criterion that the packing be "nice" in a certain sense, as discussed further below) it has been shown [BHRS] that indeed the hexagonal packing is the unique densest packing for unit disks, unique up to rigid motion of course. This is significant since, being chiefly interested in understanding the symmetries of densest packings, we could hardly proceed if, for examples such as equal disks in the plane, we could not effectively distinguish between a packing and one obtained from it by removing a few bodies, thereby destroying symmetries.

Some packing problems do not show the uniqueness exhibited by unit disks. The problem for unit balls in $\mathbb{E}^3$ definitely does not, by the following argument (which is independent of the work of Hales and Hsiang). We first note the following recent result, which we prove later, connecting uniqueness and symmetry.

Theorem 1 [BHRS]. If a packing problem has a unique solution P up to rigid motion, then the symmetry group of P must be cocompact in the Euclidean group.

Note: A subgroup $H < G$ among topological groups is "cocompact" (resp. "cofinite") if the left coset space G/H is compact in the quotient topology (resp. of finite volume for the quotient of Haar measure) – the quotient topology being the finest topology for which the projection $G \to G/H$ is continuous. (For the isometry group $\mathcal{G}^d$ of $\mathbb{E}^d$ or $\mathbb{H}^d$, a discrete subgroup is cocompact if and only if its fundamental domain is compact [Kato].)

Continuing our discussion of ball packings in $\mathbb{E}^3$, it is known that *among packings with cocompact symmetry* the face centered cubic packing has the highest density. Now if this packing were densest in the above sense then so would be many others such as the hexagonal close packed structure, and thus the packing problem for unit balls does not have a solution unique up to rigid motion.

The packing problem for kites & darts (see Figure 2b) also does not exhibit the form of uniqueness shown by the problem for unit disks. In fact we note here two features of the kite & dart tilings: there are uncountably many pairwise noncongruent such tilings; yet they form a natural family, all closely related, in the sense that any finite region of one kite & dart tiling has a congruent copy in any other kite & dart tiling. In other words, although there is not just one optimally dense packing (tiling) of $\mathbb{E}^2$ by kites & darts, up to rigid motion, the situation is, in an intuitive sense, not as bad as the problem for unit balls in $\mathbb{E}^3$, where (presumably) the optimal packings are not all locally congruent. The packing problem for kites & darts is called "aperiodic" since *none* of its densest packings – in this case tilings – has cocompact symmetry group.

We will be mainly concerned with the symmetries of densest packings, and it will be convenient to translate the traditional classification of symmetry groups into other terms. But first we need the following basic notion of conjugacy of group actions.

Definition. Two actions of a group G, π_1 on a space X_1 and π_2 on a space X_2, are called conjugate if there is a bijection ϕ of X_1 onto X_2 such that $\phi[\pi_1(g)x] = \pi_2(g)\phi[x]$ for all $x \in X_1$ and $g \in G$.

Theorem 2. Two subgroups H_1 and H_2 of a group G are inner isomorphic if and only if the dynamical systems, of left actions of G on the coset spaces G/H_1 and G/H_2, are conjugate.

Proof. Suppose $\phi : G/H_1 \to G/H_2$ is a conjugacy. Let $k \in G$ be any element in $\phi([e]_1)$, where $e \in G$ is the unit and $[\cdot]_j$ denotes an element of G/H_j, i.e. $[g]_j = gH_j$. Then $\phi([g]_1) = \phi([ge]_1) = \phi(g[e]_1) = g\phi([e]_1) = g[k]_2 = [gk]_2$. Also, $g \in H_1 \Leftrightarrow [g]_1 = [e]_1 \Leftrightarrow [gk]_2 = [k]_2 \Leftrightarrow (gk)^{-1}k \in H_2 \Leftrightarrow k^{-1}g^{-1}k \in H_2 \Leftrightarrow k^{-1}H_1k = H_2$. ■

Consider a packing P of $\mathbb{E}^d$ with cocompact symmetry group Γ_P, and consider the action π of the Euclidean group $\mathcal{G}^d$ on the left quotient space $\mathcal{G}^d/\Gamma_P$, namely $\pi(g)[h] = [gh]$, where $[h]$ is the equivalence class $h\Gamma_P$ in $\mathcal{G}^d/\Gamma_P$ of $h \in \mathcal{G}^d$. Theorems 1 and 2 show that, among those packing problems with unique solution, the classification of those problems by the symmetry group Γ_P of their solutions P is *equivalent* to the classification by the conjugacy classes of the actions of $\mathcal{G}^d$ on the coset spaces $\mathcal{G}^d/\Gamma_P$.

This can be made a little neater by noticing the natural bijection between $\mathcal{G}^d/\Gamma_P$ and the orbit $O(P)$ of the packing P under the natural action of $\mathcal{G}^d$, namely $g\Gamma_P \leftrightarrow gp$ for $g \in \mathcal{G}^d$, which is well-defined since $g\Gamma_P = g'\Gamma_P \Leftrightarrow g^{-1}g' \in \Gamma_P \Leftrightarrow gp = g'p$. In fact it is easy to see that this bijection provides a conjugacy.

Combining this with our previous use of Theorems 1 and 2, we see that the classification of those packing problems with unique solution up to rigid motion by the inner isomorphism classes of the (cocompact) symmetry groups Γ_P of their (unique) solutions P is *equivalent* to the classification of those packing problems by the conjugacy classes of the actions of $\mathcal{G}^d$ on the orbits $O(P)$ of the solutions P.

This completes our discussion of those packing problems with unique solution, and we now turn to the rest of the packing problems. The first step is to generalize to these problems the classification scheme we applied to the simpler packing problems, and we choose to generalize the dynamical version. We begin by letting $\mathcal{P}_{\mathcal{B}}$ be the the set of all possible packings for the given set $\mathcal{B}$ of bodies, and defining a metric on $\mathcal{P}_{\mathcal{B}}$ such that a sequence $\mathcal{P}_j$ of packings converges to P in the metric if they converge uniformly on compact subsets of $\mathbb{E}^d$. (It is not hard to write a formula for the metric but it is neither canonical nor particularly illuminating; the curious may find it in [Radi].) Such a metric makes $\mathcal{P}_{\mathcal{B}}$ compact. (If this seems surprising think of the product topology on an infinite product of 2-point sets.) Getting back to our packing problems with not-necessarily-unique solution, determined by the sets $\mathcal{B}$ of bodies, we define $\tilde{\mathcal{P}}_{\mathcal{B}}$ as the closure in $\mathcal{P}_{\mathcal{B}}$ of the set of densest packings. For this general situation we seek to classify the dynamical systems, consisting of the natural action of $\mathcal{G}^d$ on $\tilde{\mathcal{P}}_{\mathcal{B}}$, up to topological conjugacy. (A conjugacy is a topological conjugacy if the spaces are topological and the bijection is bicontinuous.)

Effectively we are defining the symmetry of the solutions of the packing problem associated with $\mathcal{B}$ as the conjugacy class of the dynamical system of $\mathcal{G}^d$ on $\tilde{\mathcal{P}}_{\mathcal{B}}$. We note again that for the special subclass of those packing problems with unique solution the complete set of invariants is the inner isomorphism classes of the cocompact subgroups of $\mathcal{G}^d$, the possible symmetry groups of the solutions.

A good example of the need for a more general notion of symmetry for the packing problems with nonunique solution is the case of the kite & dart tilings. These tilings attracted increased attention in the mid 80's when physicists used three dimensional analogues to model materials showing unexpected behavior. These "quasicrystals" generated diffraction patterns with symmetries which were incompatible with the expected internal structure of the material. Eventually it was shown that the diffraction pattern could be directly related to the dynamical system associated with the tilings of the model [Dwor], and more specifically, the rotational symmetry of the diffraction patterns could be precisely understood as the rotational symmetry of all the translation invariant measures on the space of tilings [Radi]. In other words, for these tilings rotational symmetry was being exhibited not as symmetries of the tilings themselves, but as *symmetries of measures on the tilings.* (Intuitively, lifting the operation of symmetries from packings to measures on packings is similar to the employment of weak solutions of differential equations, as is necessary in some situations, for instance quantum field theories.)

As for kite & dart tilings, in practice one obtains invariants for packing problems with nonunique solution using subgroups of $\mathcal{G}^d$ which are *not cocompact.* Another example arose in the classification of certain tilings in $\mathbb{E}^3$ [CoRa], which led to the analysis of the family of (typically dense) subgroups of SO(3) generated by a pair of rotations, by finite order rotations, about axes which were separated by a "geodetic" angle, that is, an angle α for which $\sin^2(\alpha)$ is rational. A series of papers resulted ([RaS1], [RaS2], [CRS1], [CRS2]) in which such groups were almost completely classified, with unexpected consequences. This is the sort of result one expects from such a program: the search for conjugacy invariants of the dynamical systems leads to investigation of certain non-cocompact subgroups of $\mathcal{G}^d$, and generates results on them of independent interest. Other examples will be given below when we discuss packings in hyperbolic spaces.

It is now time to give the full technical definition of densest packings.

Definition. A packing P, in the support of the $\mathcal{G}^d$-invariant Borel probability measure μ on $\mathcal{P}_{\mathcal{B}}$, is "generic" for μ if for every continuous function f on $\mathcal{P}_{\mathcal{B}}$:

$$\int_{\mathcal{P}_{\mathcal{B}}} f(P')\,d\mu(P') = \lim_{n\to\infty} \frac{\int_{\mathcal{G}^d_n} f(gP)\,d\nu(g)}{\int_{\mathcal{G}^d_n} 1\;d\nu(g)}$$

where $\mathcal{G}^d_n = \{g \in \mathcal{G}^d \,|\, d(O, gO) \leq n\}$ and ν is Haar measure on $\mathcal{G}^d$.

In other words, the integral with respect to μ of any continuous function on the space of packings can be reproduced by averaging over the orbit of any packing which is generic for μ. The main fact about generic points is the Pointwise Ergodic Theorem of G.D. Birkhoff, which proves the existence, indeed the abundance, of such generic points.

Theorem 3 (Birkhoff) [Walt]. If $\mathcal{G}^d$ acts continuously on a compact metric space X, and the probability measure μ on X is ergodic under $\mathcal{G}^d$, the set of generic points for μ has full μ-measure.

(A probability measure invariant under the action of a group is "ergodic" if it is a convex extreme point among such invariant measures.) We first use this basic theorem to foliate $\mathcal{P}_{\mathcal{B}}$.

Theorem 4 [BHRS]. If the probability measure μ on $\mathcal{P}_{\mathcal{B}}$ is ergodic under $\mathcal{G}^d$, then for every generic packing P of μ, and $O \in \mathbb{E}^d$, the "density"

$$\lim_{n\to\infty} \frac{\mathrm{vol}[P \cap B_n(O)]}{\mathrm{vol}[B_n(O)]}$$

exists and has value $\mu(A_{\mathcal{B}})$, where $B_n(O)$ is the ball of radius n centered at O and $A_{\mathcal{B}} \equiv \{P \in \mathcal{P}_{\mathcal{B}} \,|\, O \text{ is in a body of } P\}$.

This shows that each ergodic measure μ singles out packings with a given fixed density, thus foliating $\mathcal{P}_{\mathcal{B}}$ by density. It is now natural to define optimally dense packings as follows.

Definition. P is a "densest packing" for the packing problem of given $\mathcal{B}$ if it is generic for an ergodic measure μ for which $\mu(A_{\mathcal{B}}) = \sup_{\mu'} \mu'(A_{\mathcal{B}})$.

The fundamental question of the existence of densest packings now follows rather easily; it can be proven that for any $\mathcal{B}$ there exists an ergodic measure

μ satisfying $\mu(A_{\mathcal{B}}) = \sup_{\mu'} \mu'(A_{\mathcal{B}})$ [BoR1]. Also, using this definition of densest packing one can prove, as stated earlier, that the packing problem for equal disks in $\mathbb{E}^2$ has a unique solution up to rigid motion [BoR1].

The proof of Theorem 1 also now follows easily. By assumption there is an invariant probability measure μ on the orbit $\mathcal{O}(P)$ of the unique P. But, as shown earlier, $\mathcal{O}(P)$ can be identified with $\mathcal{G}^d/\Gamma_{_P}$, so, by the uniqueness of Haar measure, μ must be the projection of Haar measure from $\mathcal{G}^d$ to $\mathcal{G}^d/\Gamma_{_P}$, that is, $\Gamma_{_P}$ is cofinite, which implies $\Gamma_{_P}$ is cocompact.

Packing in hyperbolic space $\mathbb{H}^d$

We begin with a short introduction to the upper half plane model of $\mathbb{H}^2$. The space is identified with the subset $\{x + iy \,|\, y > 0\}$ of the complex plane, with the metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ and area $dA = \dfrac{dxdy}{y^2}$. It is easy to show that the geodesics are the circular arcs which approach the x-axis perpendicularly, including vertical lines as special cases. The rigid motions, that is, those motions continuously connected to the identity and preserving the metric, are identifiable with the matrix group $\mathcal{G}^2$ =PSL(2,$\mathbb{R}$), acting by:

$$z \to \frac{az + b}{cz + d}.$$

Special cases are horizontal translation, $z \to z + b$, for $b \in \mathbb{R}$, and multiplication $z \to mz$, for $m \in \mathbb{R}_+$.

A very interesting example of a packing problem was considered by Penrose in 1978 [Penr], with $\mathcal{B}$ consisting a single body, the "binary tile", as shown in Figure 4b. From this body horizontal motions fill out a strip, and then multiplication by 4^j for $j \in \mathbb{Z}$ yields a tiling of the plane (Figure 4a). It is not the only tiling by copies of the binary tile. Once a tile is placed in the plane, for instance as in Figure 4b, a "horozontal" strip is forced. The two horizontal edges of that strip are horocycles, bounding horoballs, and the tiling of the region outside ("below") the larger of the two horoballs is also forced. But there are (uncountably) many ways to fill the horoballs. Penrose gave a beautiful argument showing that no such tiling could have cocompact symmetry group; for if it did it would tile a compact fundamental domain, necessarily using only finitely many tiles to do this. But each tile has more outgoing bumps than ingoing bumps, and this "imbalance" produces a contradiction. (This has been nicely generalized in [BlWe] and [MaMo].)

Unfortunately there are no $\mathcal{G}^2$-invariant probability measures on $\mathcal{P}_{\mathcal{B}}$ giving these tilings positive measure. (This follows easily from the identification of a boundary for $\mathbb{H}^2$ and the fact that PSL(2,$\mathbb{R}$) extends continuously to the boundary [BoR2].) In fact these tilings (but without the bumps) had figured in an earlier study, the density of certain disk packings, by Böröczky [Bor1]. Consider the disk packing indicated in Figure 5. In that packing each copy of a binary tile has one disk in it, suggesting a certain value for the "density" of the disk packing. However, if you use the rigid motion $z \to 2z$ on the disks you get the result in Figure 6, in which every binary tile has two disks in it, suggesting a density twice the original value! This contradiction has had a profound influence on those studying the density of packings in hyperbolic spaces; see [FeKu]. In fact it was to overcome this contradiction that the density formalism discussed in this article was introduced in [BoR1], [BoR2].

As suggested, we can easily adapt the formalism used in the first part of this article to packings in hyperbolic spaces, the only nontrivial new step being the appropriate generalization of the Birkhoff theorem to apply to actions of PSL(2,$\mathbb{R}$) (or more generally to isometry groups of $\mathbb{H}^d$), which was only proven by Nevo et al in the last decade [Nevo], [NeSt].

In order to identify densest packings using this formalism we need to find appropriate invariant measures. This can be done with modifications of Penrose's tile, as with the simple example shown in Figure 7. In this example certain of the bumps have been altered, and the basic shape stretched, for the following reason. Consider the packing of the plane by horoballs indicated in Figure 8, which has symmetry group PSL(2,$\mathbb{Z}$). As noted above, bodies like the Penrose binary "naturally" tiles horoballs, with great diversity. We can thus use them to tile each of the horoballs in Figure 8, the only complication being the bumps that stick out of the horoballs. This is the reason for the modification of the body, so these bumps can precisely fill up the regions between the horoballs in Figure 8.

Other modifications can be made, in at least the following two ways. First, one could start with a horoball packing where n horoballs meet leaving empty "n-gonal" regions, generalizing the case $n = 3$ considered above; see Figure 9. (The symmetry groups of such horoball packings are the Hecke groups $\Gamma[2\cos(\pi/n)]$ [Kato], which are cofinite but not cocompact in PSL(2,$\mathbb{R}$).) Also, instead of having only two outgoing bumps on the body, one can have any number $k \geq 2$, as in Figure 10. These variations yield a two parameter family of packing problems, and for each of these the underlying horoball packing can be used to prove the existence of a $\mathcal{G}^2$-invariant probability measure giving full measure to the tilings, thus proving that the generic tilings are actually "densest packings" in the sense of the formalism. (Those tilings analogous to the original Penrose tilings of Figure 4a are not generic points.)

As was the case for Euclidean packings with nonunique solution, to classify the above family of tilings through their dynamical systems under the action of $\mathcal{G}^2$ =PSL(2,$\mathbb{R}$) one is lead to investigate *non-cocompact* subgroups of $\mathcal{G}^2$, in this case the Hecke groups noted above. Results were obtained on general properties of actions of such a group, through investigation of the cusp point set of the corrsponding horoball packing [BHRS]. Another result in this direction was recently obtained by Bowen [Bowe]. It is a simple fact that a densest packing P in Euclidean space can always be well approximated by packings P_n with cocompact symmetry – the density of P_n approaching that of P. This is not so obvious for densest packings in hyperbolic space, but has recently been proven to hold at least for packings by equal disks in $\mathbb{H}^2$. This can be interpreted as, in some sense, proof of a kind of denseness, of the cocompact subgroups among the subgroups of PSL(2,$\mathbb{R}$).

The above examples of modifications of Penrose's binary tile are the only packing problems that have been solved in a hyperbolic space, aside from those with unique solution (and therefore cocompact symmetry), such as the packings of $\mathbb{H}^2$ by disks with special values of the radius (called "tight" in [BoR1]) for which a proof like that of Fejes Tóth was given by Böröczky [Bor2]. However, there is some symmetry information known about ball packings in $\mathbb{H}^d$, in fact more than is known for $\mathbb{E}^d$. Specifically, if $\mathcal{B}$ is a ball of radius R then for all but countably many R there are *no* densest packings with cocompact symmetry [BoR1]. This sort of behavior may well be common for packing problems in $\mathbb{E}^d$ but so far it has only been found there for polyhedra with complicated bumps on the faces, such as the kite & dart tilings, not for such natural bodies as balls.

In conclusion, we have tried to illustrate how the discovery of aperiodic tilings by Berger in 1966 [Berg], and their use in modeling quasicrystals, has led to new techniques and renewed interest in the study of the qualitative (symmetry) properties of densest packings, conventionally understood as Hilbert's 18th problem.

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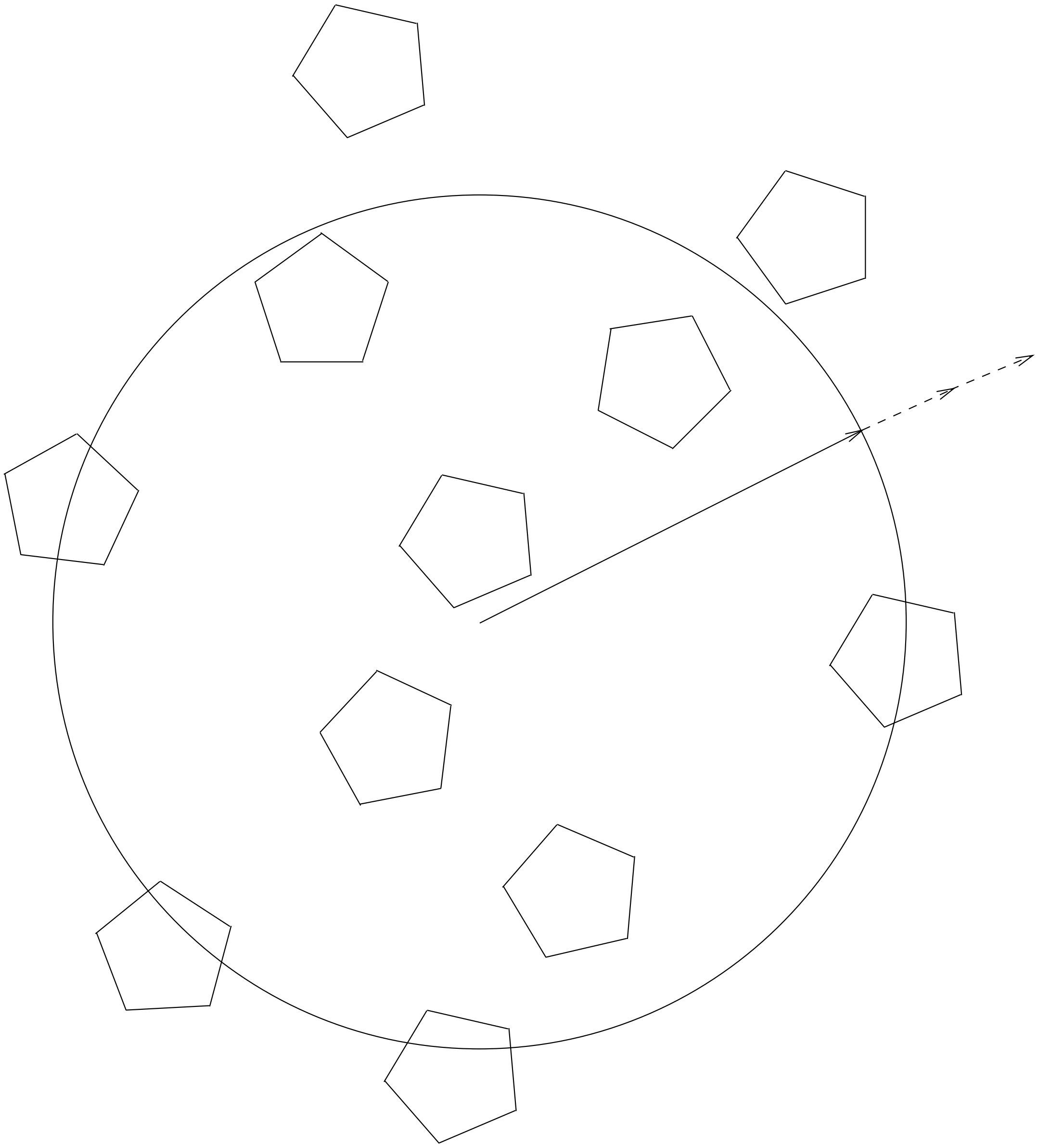

Figure 1. Expanding circle in a pentagonal packing

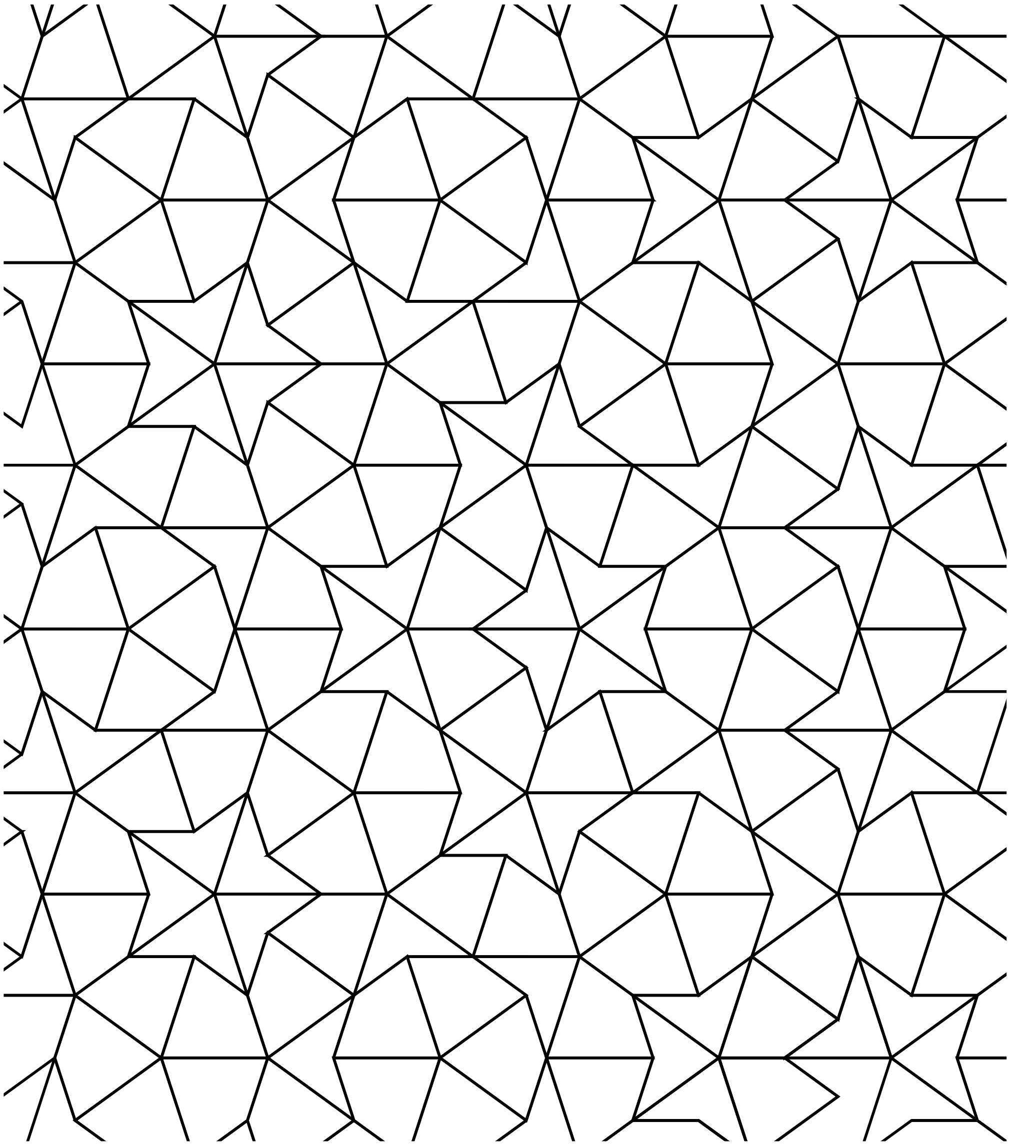

Figure 2a. A kite and dart tiling

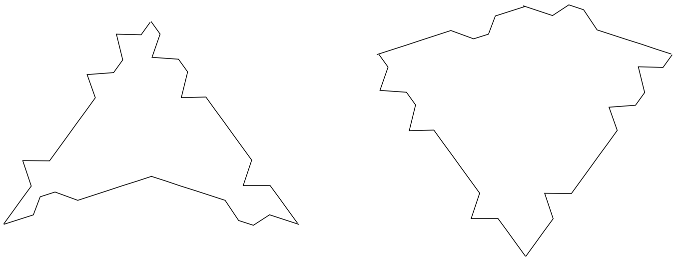

Figure 2b. The kite and dart tiles

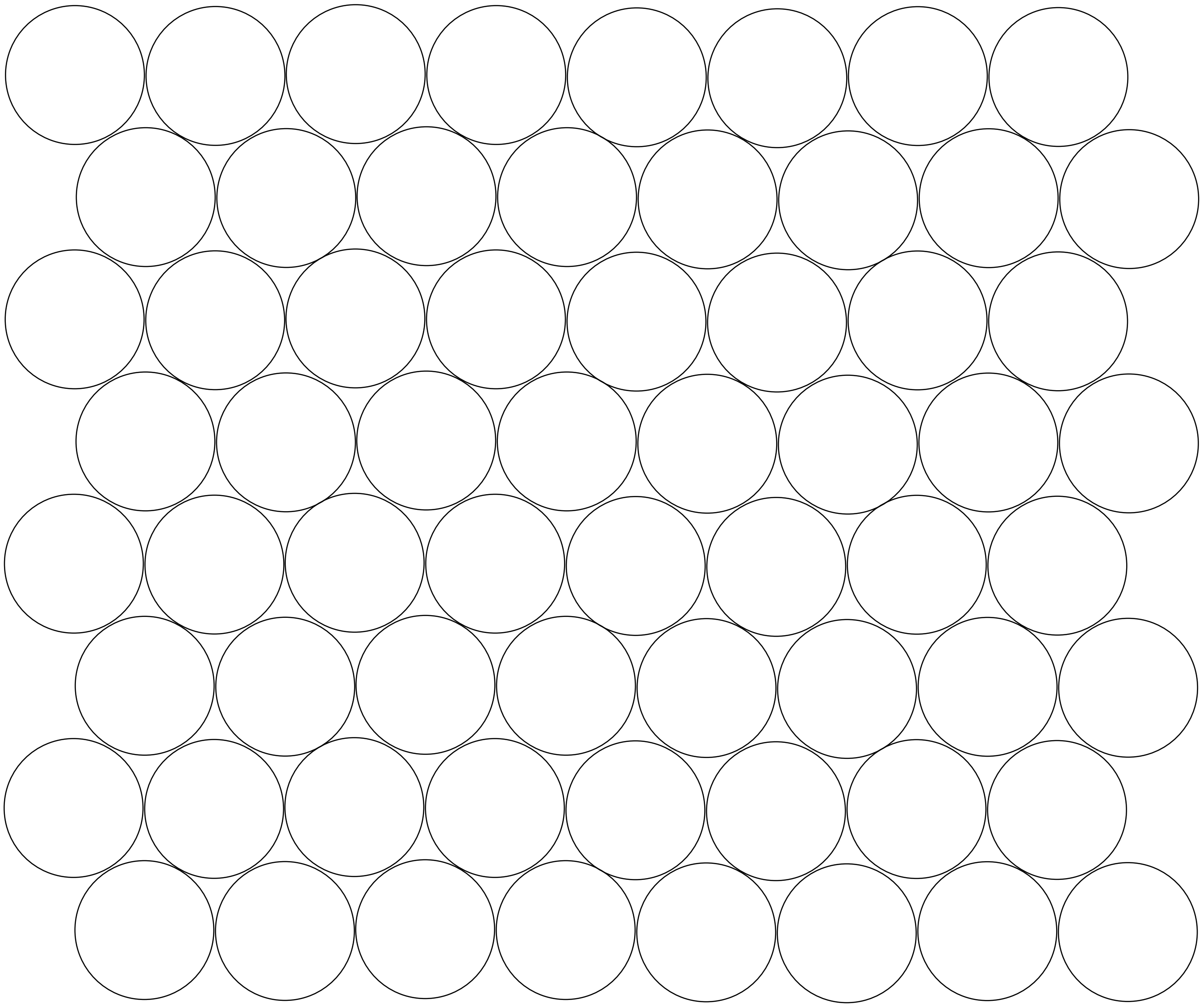

Figure 3. The optimal packing of disks in the Euclidean plane

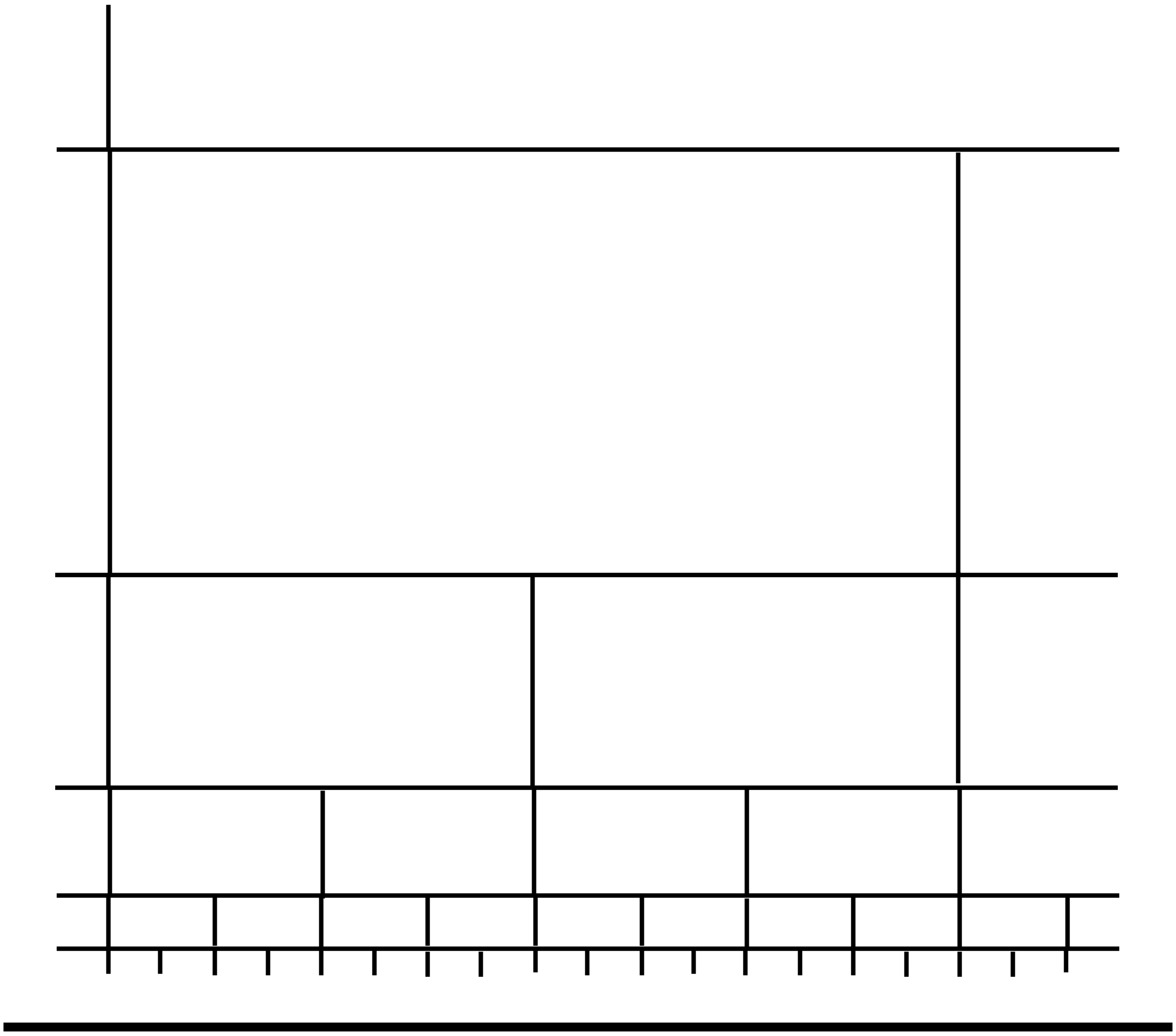

Figure 4a. The binary tiling of the hyperbolic plane

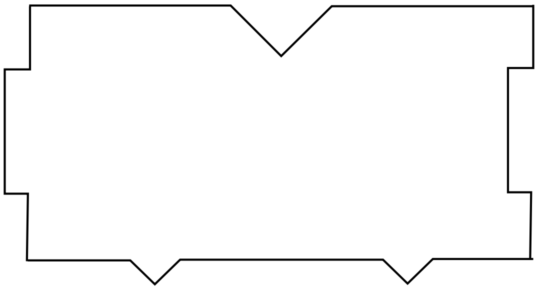

Figure 4b. The simple binary tile

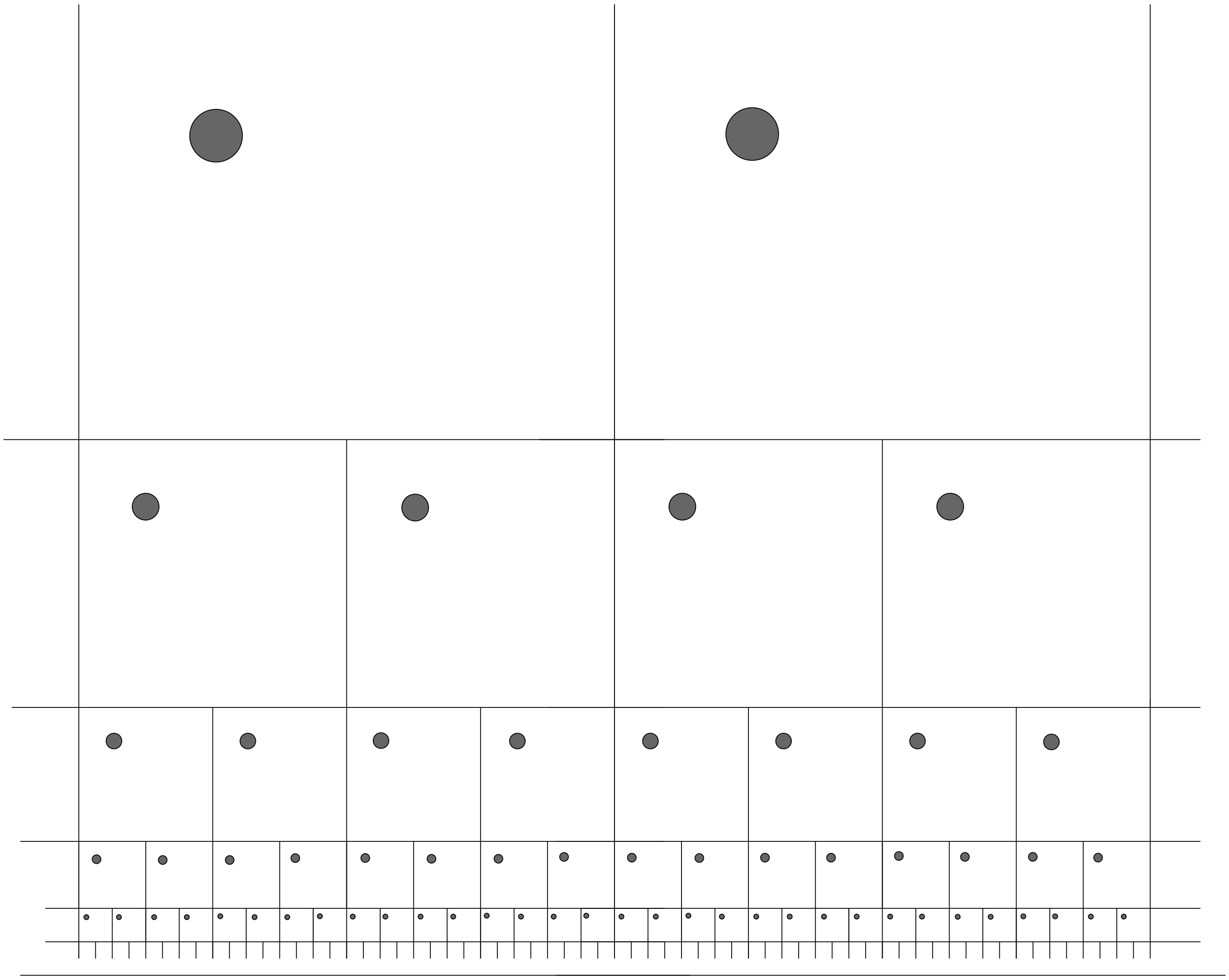

Figure 5. The binary tiling with a disk packing superimposed

Primary tile vertices $i, i+4, 4i, 4i+4$; disk at $3i+2$

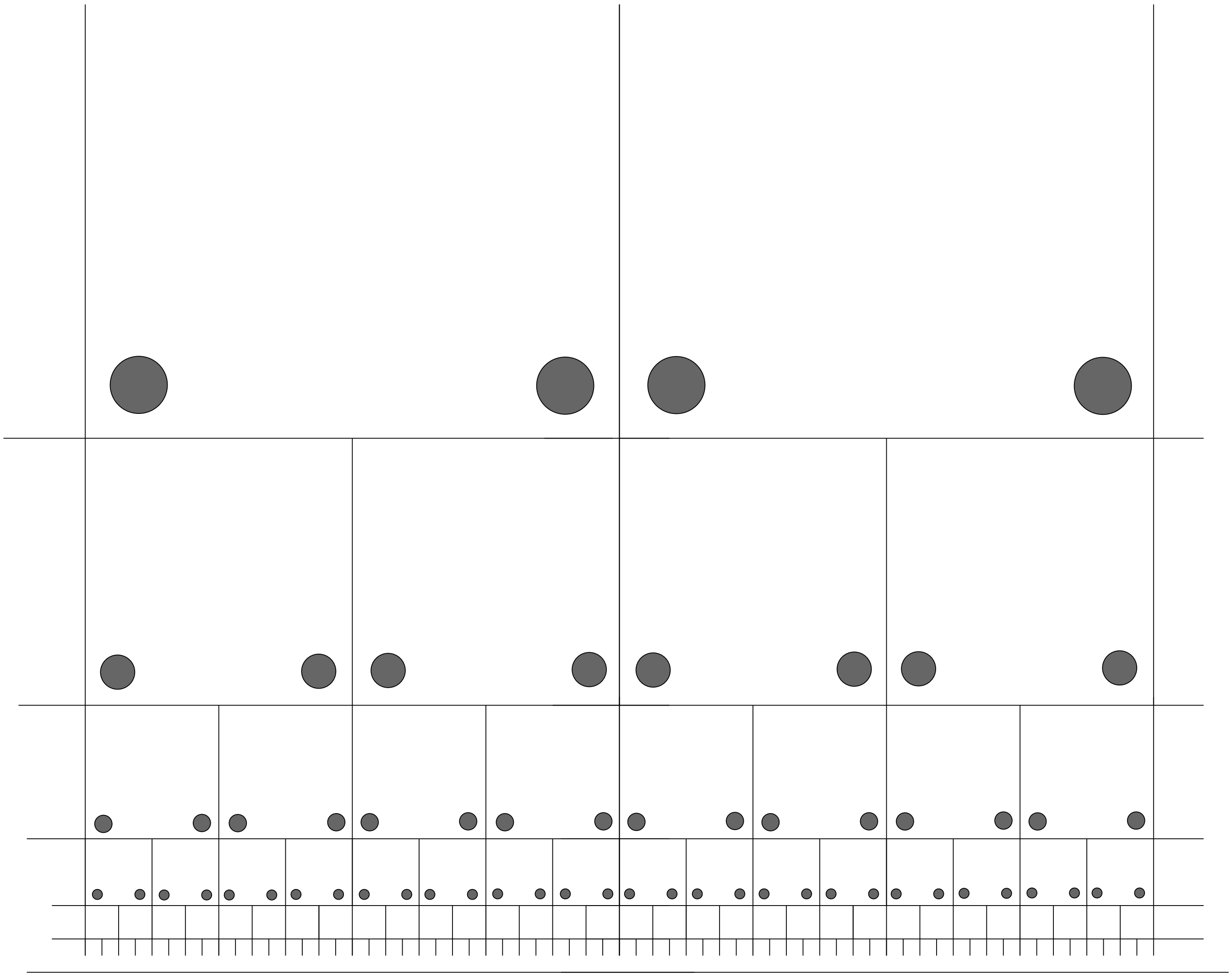

Figure 6. The binary tiling with a disk packing superimposed

Primary tile vertices $i, i+4, 4i, 4i+4$

Disks at $\frac{3}{2}i+1,\ \frac{3}{2}i+3$.

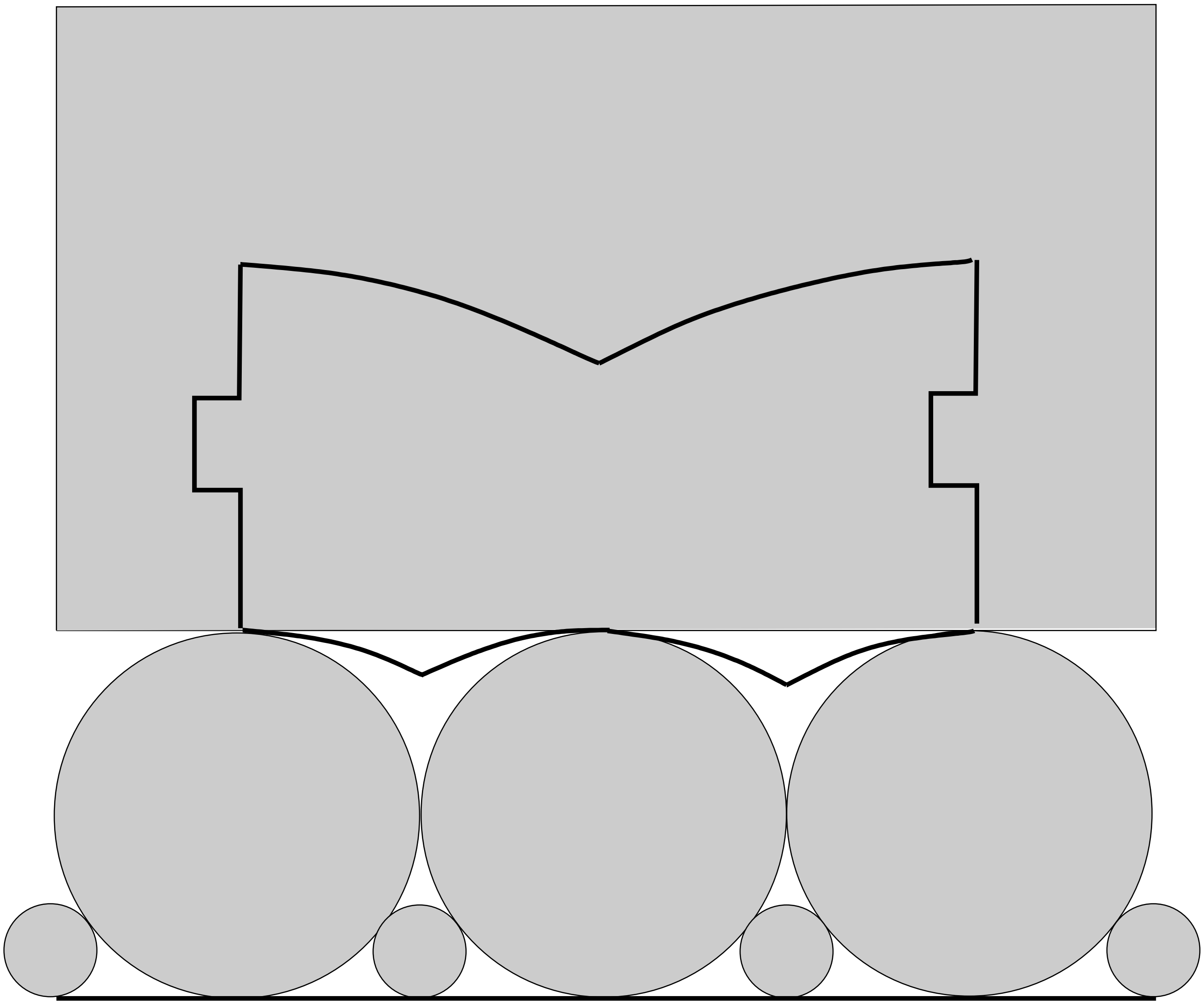

Figure 7. The modified binary tile in the upper half plane

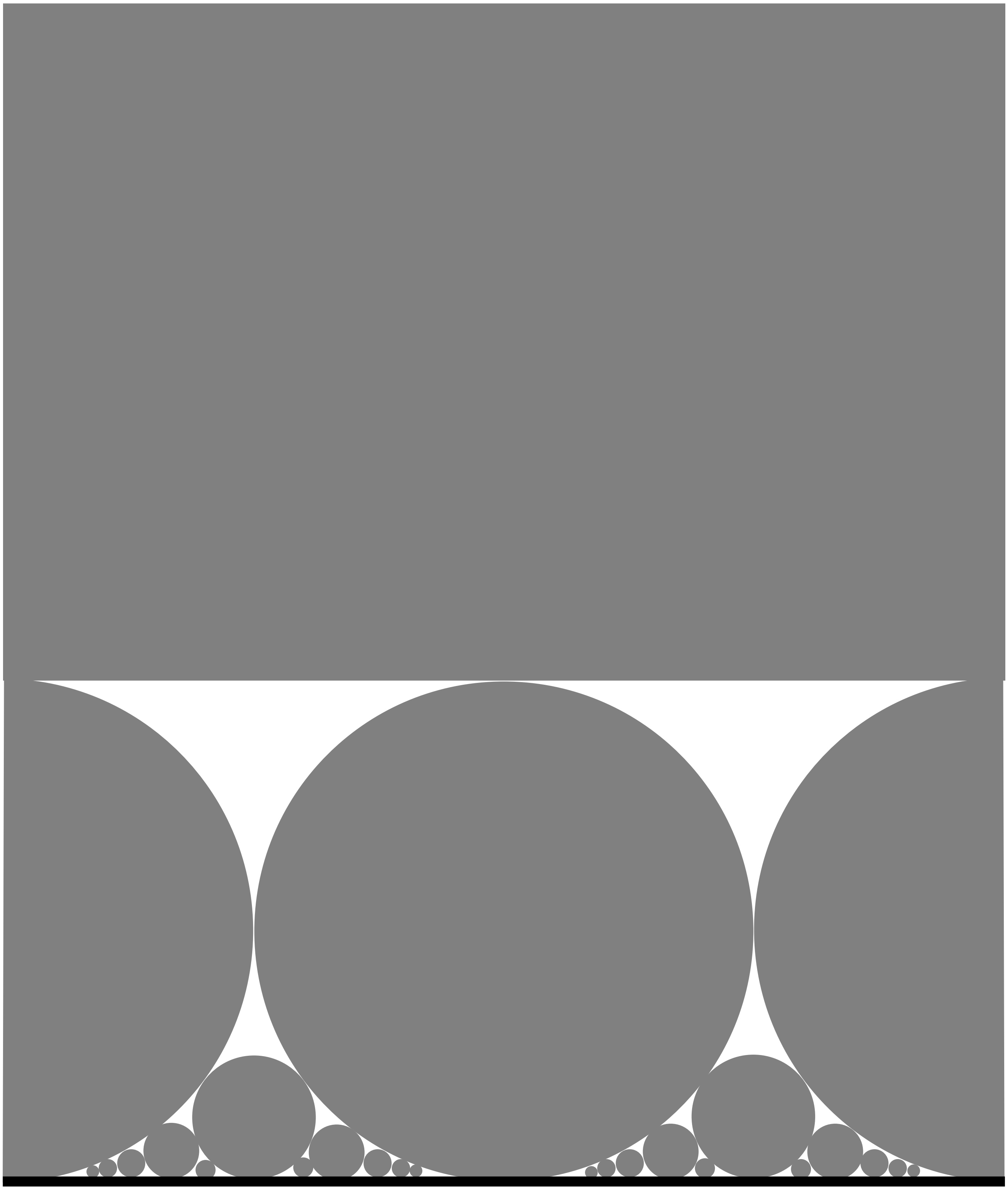

Figure 8. A packing of horoballs

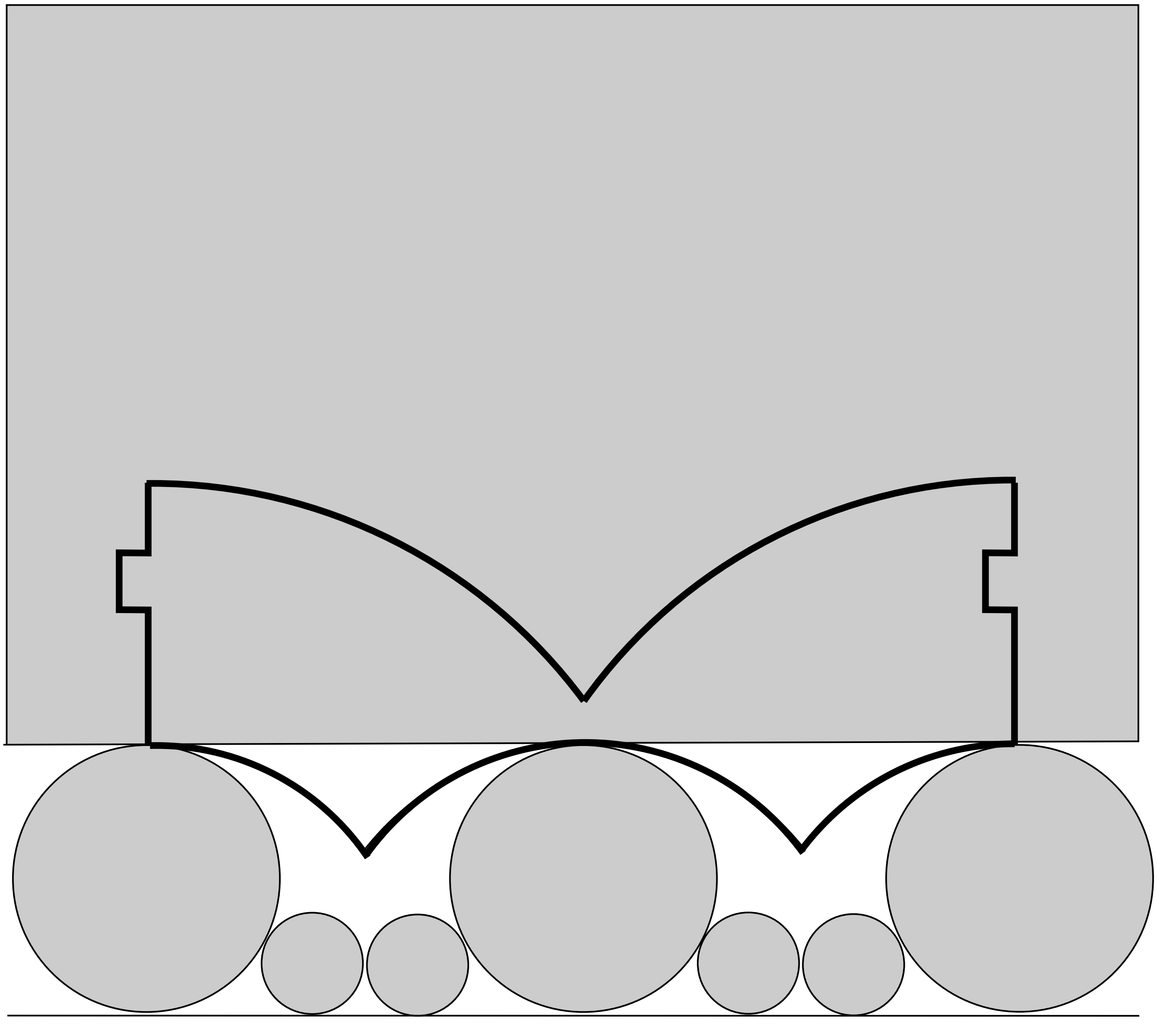

Figure 9. A “pentagonal” binary tile

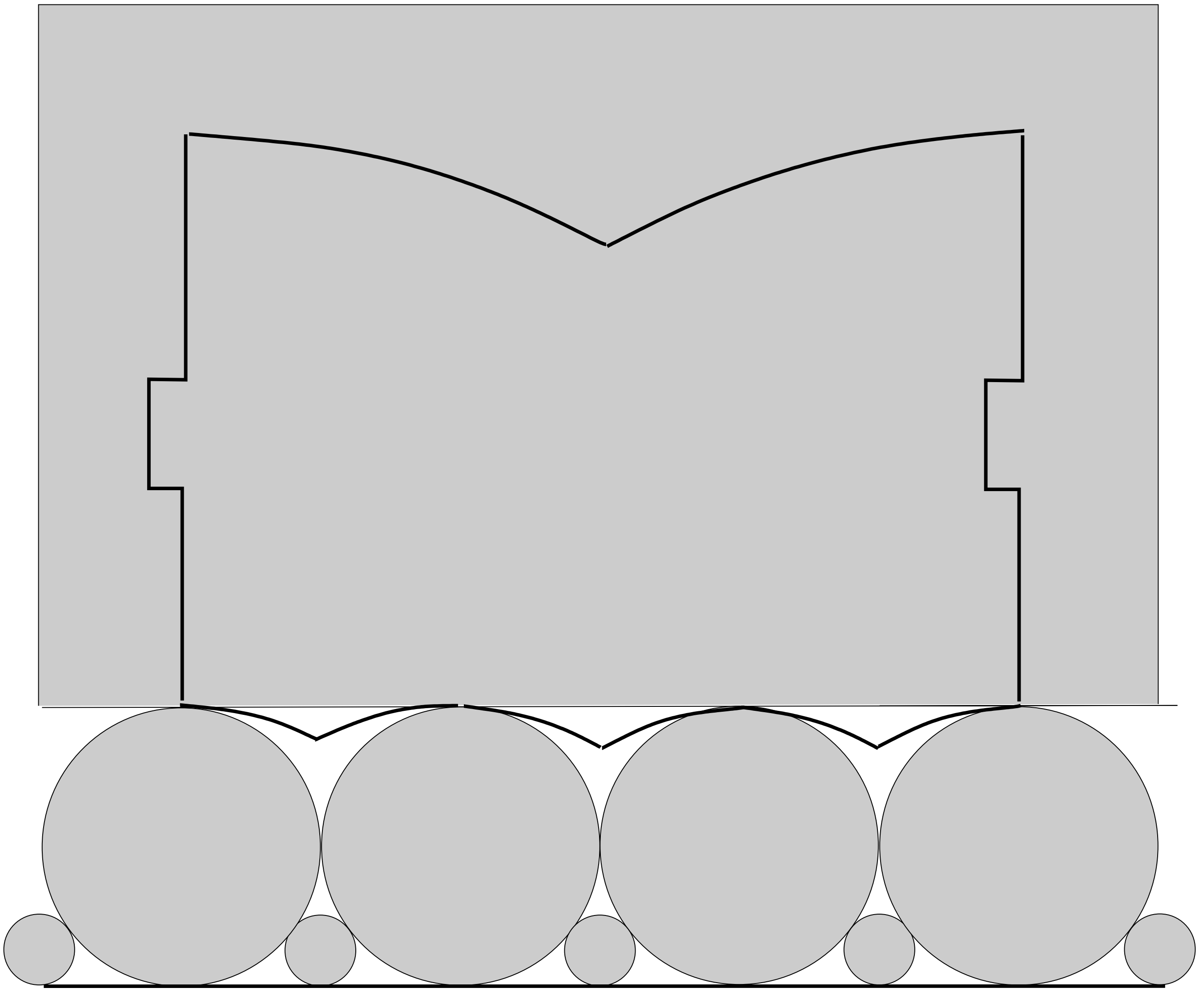

Figure 10. A triangular tile with 3 prongs