Semi-classical analysis of a Dirac equation without adiabatic decoupling

Clotilde FERMANIAN KAMMERER
Université de Cergy-Pontoise

Abstract. *We study adiabatic decoupling for Dirac equation with some scaling which yields that the mass appears with a coefficient ε^α where ε is the semi-classical parameter and $\alpha > 0$. Therefore, the system presents an avoided crossing. The scale $\alpha = 1/2$ is critical: adiabatic decoupling holds for $\alpha \in (0, 1/2)$ while for $\alpha \geq 1/2$, there is energy transfer at leading order between the two modes. We describe this transfer in terms of two-scale Wigner measures by means of Landau-Zener formula which takes into account the change of polarization of the measures after the crossing.*

Key words. Avoided crossings, Landau-Zener transitions, Semi-classical measures (Wigner measures), multi-scale convergence.

1 Introduction

Let us consider the Dirac equation

$$\left(\sum_{\mu=0}^{3} \gamma^\mu (i\hbar\partial_\mu + e\tilde{A}_\mu) - Mc\right)\phi = 0, \tag{1}$$

where ϕ denotes the "spinorfield", $\phi = \phi(y) \in \mathbf{C}^4$, $y = (y_0, y') \in \mathbf{R}^4$ with y_0 the time variable and $y' = (y_1, y_2, y_3)$ the space variable and where γ^μ, $0 \leq \mu \leq 3$, are the 4×4 Dirac matrices which satisfy

$$(\gamma^0)^* = \gamma^0, \;\; (\gamma^k)^* = -\gamma^k, \;\; 1 \leq k \leq 3 \;\; (\gamma^0\gamma^k)^* = \gamma^0\gamma^k \tag{2}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \;\; \mu \neq \nu, \;\; (\gamma^0)^2 = Id, \;\; (\gamma^k)^2 = -Id. \tag{3}$$

We use the standard representation of Dirac matrix : the Dirac-Pauli one (see [31] p. 36). In this representation, the 4×4 matrix γ^μ are functions of the 2×2 Pauli matrices σ_k, $1 \leq k \leq 3$. More precisely, we have

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

The functions $\tilde{A}_\mu = \tilde{A}_\mu(y)$ for $0 \leq \mu \leq 3$ are the components of the electromagnetic potential, in particular $\tilde{A}_0$ is the electric potential and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ is the magnetic potential vector. Hence the electric field is $E = \partial_{y_0}\tilde{A} - \nabla_{y'}\tilde{A}_0$ and the magnetic field $B = \mathrm{curl}_{y'}\tilde{A}$. The physical constants M and e are the electron's rest mass and its charge, c is the velocity of light and $\hbar$ the Planck constant. The matrix

$$P = \sum_{\mu=1}^{3} \gamma^0\gamma^\mu(\hbar\xi - e\tilde{A}_\mu) + \gamma^0 Mc$$

has two eigenvalues $l^\pm = \pm\sqrt{M^2c^2 + |\hbar\xi - e\tilde{A}_\mu|^2}$ which are of multiplicity 2.

Our aim is to study adiabatic decoupling for this equation in some special situation which is slightly different that the usual one. Several works have been devoted to the study of Dirac equation with slowly varying external potentials. One supposes that $\tilde{A} = A(\varepsilon y)$ and one studies ϕ on the macroscopic scale, i.e. one studies the evolution as ε goes to 0 of $\psi^\varepsilon(t,x) = \phi(y)$. Thus, one analyzes system

$$i\hbar\varepsilon\partial_t\psi^\varepsilon = \sum_1^3 \gamma^0\gamma^\mu(\frac{\hbar\varepsilon}{i}\partial_\mu - eA_\mu(t,x))\psi^\varepsilon + \gamma^0 Mc\psi^\varepsilon.$$

Any solution of this system can be decomposed on the two modes associated to each eigenvalue $l^\pm$ and these two components evolve independently. Such phenomenon is called adiabatic decoupling (from "*adiabatos*"= impassable). The reader can refer to [30] where he will find a presentation of adiabatic theory and its application to Dirac equation with slowly variable coefficients or to [16] where adiabatic decoupling is proved in terms of Wigner measures.

We focus here on the case where the oscillations of $\tilde{A}$ are not so small. More precisely, we set

$$\tilde{A}(y) = A(\Lambda y), \ \ \Lambda y = (t,x) = (t, x_1, x_2, x_3) \in \mathbf{R} \times \mathbf{R}^3,$$

and we suppose that $|A| \sim 1$. We study ϕ on the macroscopic scale: we set $\psi(t,x) = \phi(y)$. Then, ψ satisfies

$$i\hbar\Lambda\partial_t\psi = \sum_1^3 \gamma^0\gamma^\mu(\frac{\hbar\Lambda}{i}\partial_\mu - eA_\mu(t,x))\psi + \gamma^0 Mc\psi.$$

We suppose that $\varepsilon = \frac{\hbar\Lambda}{e} \ll 1$, but that $\frac{Mc}{e}$ and $\frac{\hbar\Lambda}{e}$ are comparable, namely that there exists $m > 0$ and $\beta > 0$ such that

$$\frac{Mc}{e} = m\varepsilon^\alpha.$$

We are concerned with the limit $\varepsilon \to 0$ and we prove, with Wigner measures approach, that the adiabatic decoupling becomes false as soon as $\alpha \geq 1/2$. We calculate the transfer of energy between the two modes by means of two-scale Wigner measures. Thus, we focus on system

$$\left\{ \begin{array}{l} i\varepsilon\partial_t\psi^\varepsilon = P_\varepsilon(t,x,\varepsilon D_x)\psi^\varepsilon, \\ \psi^\varepsilon \mid_{t=0} = \psi_0^\varepsilon \end{array} \right. \tag{6}$$

where P_ε is the matrix-valued symbol

$$P_\varepsilon(t,x,\xi) = \sum_{k=1}^{3} \gamma^0\gamma^k\left(\xi_k - A_k(t,x)\right) + V(t,x) + \varepsilon^\alpha m\gamma^0 \tag{7}$$

with $\alpha > 0$, $m > 0$, $A = (A_1, A_2, A_3)$ a $\mathcal{C}^\infty$ vector field (the magnetic potential vector) and V a $\mathcal{C}^\infty$ function (the electric potential).

The interest of Wigner measures approach is that the limit of physically meaningful quantities can be simply expressed in terms of Wigner measures. For example, in [16] (see also [29] and [1]), the authors calculate the weak limit of the relativistic current density J_k, $1 \leq k \leq 3$,

$$J_k^\varepsilon(t,x) = \gamma^0\gamma^k\psi^\varepsilon(t,x)\cdot\overline{\psi}^\varepsilon(t,x)$$

and of the relativistic position density n

$$n^\varepsilon(t,x) = \mid \psi^\varepsilon(t,x) \mid^2 .$$

Indeed, a semi-classical measure μ of the family (ψ^ε) is a positive Radon measure on $T^*\mathbf{R}^4$ which satisfy – up to some subsequence – that, for all the observable $a \in \mathcal{C}_0^\infty(\mathbf{R}^8)$,

$$(\mathrm{op}_\varepsilon(a)\psi^\varepsilon \mid \psi^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} < a\ , \mu >,$$

where $\mathrm{op}_\varepsilon(a)$ denotes the semi-classical pseudo-differential operator of symbol a; the kernel of $\mathrm{op}_\varepsilon\ (a)$ is, with Weyl quantification,

$$k(x,y) = \int a(\frac{x+y}{2},\xi)\mathrm{e}^{\frac{i(x-y)\cdot\xi}{\varepsilon}}\frac{d\xi}{(2\pi)^4}.$$

Let us assume that (ψ_0^ε) satisfies some assumption called ε-oscillation which implies that, roughly speaking, the oscillations of (ψ_0^ε) are not greater than $\frac{1}{\varepsilon}$, (see [15]), namely

$$\limsup_{\varepsilon\to 0}\int_{|\xi|\geq R/\varepsilon} \mid \widehat{\psi_0^\varepsilon}(\xi) \mid^2 \mathrm{d}\xi \underset{R\to+\infty}{\longrightarrow} 0.$$

Then, the weak limits of n^ε and J_k^ε express as

$$\begin{aligned} \mathrm{w}-\lim_{\varepsilon\to 0} n^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_{\tau,\xi} \mathrm{tr}\,\mu(t,x,\mathrm{d}\tau,\mathrm{d}\xi), \\ \mathrm{w}-\lim_{\varepsilon\to 0} J_k^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_{\tau,\xi} \mathrm{tr}\left(\gamma^0\gamma^k\mu(t,x,\mathrm{d}\tau,\mathrm{d}\xi)\right). \end{aligned}$$

In [16], P. Gerard, P. Markowich, N. Mauser and P. Poupaud describe the semi-classical measures of (ψ_ε) in the case $\alpha = 0$. The fact that $\alpha \neq 0$ induces serious difficulties that our contribution aims to deal with. Actually, the analysis performed in [16] crucially uses the fact that as $\alpha = 0$, the eigenvalues of matrix P_ε

$$\lambda_\varepsilon^{\pm}(t,x,\xi) = \pm\sqrt{\mid \xi - A(t,x) \mid^2 + \varepsilon^{2\alpha} m^2} + V(t,x) = \pm\sigma_\varepsilon(t,x,\xi) + V(t,x). \tag{8}$$

do not depend on ε and are distinct and of constant multiplicity 2
For $\alpha \neq 0$,

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon^{+}(t,x,A(t,x)) = \lim_{\varepsilon\to 0} \lambda_\varepsilon^{-}(t,x,A(t,x));$$

the system (6) displays an avoided crossing. Crossings are usually characterized by the codimension of the singular set. The crossing here is of codimension 3 since

$$(t,x,\tau,\xi) \mapsto \xi - A(t,x)$$

is of rank 3.

The works of Hagedorn and Joye ([18], [20], [21], [23]) and those of Colin de Verdière, Lombardi and Pollet ([6]), show that avoided crossings yield transfer of energy at leading order between the two modes and thus, that there is no adiabatic decoupling. This so-called *Landau-Zener* phenomenon has been first described independently and simultaneously by Landau and Zener in the 30's (see [24] and [32]) . We shall discuss this transfer of energy in terms of Wigner measures measures in the same spirit than the works of P. Gérard and the author for true crossings (see [10] – [12], [9]). Our purpose here is to apply to this avoided crossing the method introduced in [12] for eigenvalue of multiplicity one with the developments performed in [9] for higher multiplicity.

1.1 Wigner measures for Dirac equation

Let us introduce some notations.
We endow $T^*\mathbf{R}^4$ with the symplectic form $\sigma = \mathrm{d}\tau \wedge \mathrm{d}t + \mathrm{d}\xi \wedge \mathrm{d}x$, and we denote by $\{f,g\}$ the Poisson bracket of functions f and g, $\{f,g\} = \nabla_{\tau,\xi} f \cdot \nabla_{t,x} g - \nabla_{t,x} f \cdot \nabla_{\tau,\xi} g$. The vector field H_f is the hamiltonian vector field associated with the function f, thus we have $\mathrm{d}g\ H_f = \{f\ ,\ g\} = \sigma(H_f, H_g)$. We set

$$Q_\varepsilon = \sum_{k=1}^{3} \gamma^0\gamma^k\left(\xi_k - A_k(t,x)\right) + m\varepsilon^\alpha,\quad \sigma_\varepsilon = \sqrt{\varepsilon^{2\alpha} m^2 + \mid \xi - A \mid^2},$$

so that we have $P_\varepsilon = Q_\varepsilon + V(t,x) = Q_0 + m\varepsilon^\alpha\gamma^0 + V(t,x)$ and that the spectral projectors $\Pi_\varepsilon^{\pm}$ associated with $\lambda_\varepsilon^{\pm}$ are

$$\Pi_\varepsilon^{\pm}(t,x,\xi) = \frac{1}{2}\left(\mathrm{Id} \pm \frac{1}{\sigma_\varepsilon} Q_\varepsilon\right).$$

According to [16] and [17], any semi-classical measure μ of (ψ^ε) satisfies out of $\{\xi = A(t,x)\}$

$$\mu = \mu^+ + \mu^-, \text{ with } \mu^\pm = \mu^\pm \Pi_0^\pm = \Pi_0^\pm \mu^\pm, \ \operatorname{Supp} \mu^\pm \subset \{\tau + V \pm \mid \xi - A \mid = 0\}. \tag{9}$$

Moreover, μ^+ and μ^- satisfy outside $\{\xi = A(t,x)\}$ the transport equations,

$$\{\tau + \lambda_0^\pm \ , \ \mu^\pm\} = [\mu^\pm, F^\pm], \ \text{ outside } \ S,$$

with

$$\begin{aligned} F^\pm &= \left[\Pi_0^\pm \, , \, \{\tau + V \pm |\xi - A| \ , \ \Pi_0^\pm\}\right] + |\xi - A| \, \Pi_0^\pm \, \{\Pi_0^\mp \, , \, \Pi_0^\mp\} \, \Pi_0^\pm \\ &= \frac{i}{2|\xi - A|} \gamma_5 \sum_{k=1}^{3} \gamma_0 \gamma_k \left(\frac{\xi - A}{|\xi - A|} \times \left(E \pm B \times \frac{\xi - A}{|\xi - A|} \right) \right)_k \\ &\qquad - \frac{i}{2|\xi - A|} \left(\frac{\xi - A}{|\xi - A|} \cdot B \right) \Pi_0^\pm . \end{aligned} \tag{10}$$

This latter formula comes from [29] (see also [17]) and tedious but straightforward computation.

Outside

$$S = \{\xi = A, \ (\tau + V)^2 = \mid \xi - A \mid^2\},$$

measures $\mu^\pm$ propagate along the classical trajectories associated with the hamiltonian $\tau + \lambda_0^\pm = \tau + V \pm \mid \xi - A \mid$. Consider

$$\Omega = \{E \cdot B \neq 0 \ \text{ or } \ (E \cdot B = 0 \ \text{ and } \ \mid E \mid > \mid B \mid)\} .$$

Let us denote by $\rho_s^\pm = (t_s^\pm, x_s^\pm, \tau_s^\pm, \xi_s^\pm)$, the hamiltonian curves of $\lambda_0^\pm$ passing through S at $\rho_0 \in S \cap \Omega$, i.e. the curves satisfying

$$\dot\rho_s^\pm = H_{\lambda_0^\pm}(\rho_s^\pm), \ \ \rho^\pm_{|s=0} = \rho_0.$$

According to Proposition 1 in [12], the points $\rho_0 \in S \cap \Omega$ are reached by a unique classical trajectory of $\lambda_0^\pm$. Moreover, the curves $(\rho_s^\pm)_{s<0}$ are smoothly continued by $(\rho_s^\mp)_{s>0}$.

The existence of such trajectories yield that the transport equations outside S stated above are not enough to determine the semi-classical measure μ. Besides, we prove in the Annex that

$$\mu(S \cap \Omega) = 0.$$

Therefore, the energy is not trapped by the crossing: the energy carried out by the ingoing trajectories parts between the outgoing trajectories. We want to describe the branching of energy above such crossing points. We prove that it depends on α and that it is determined by the way (ψ^ε) concentrates at the scale $\sqrt{\varepsilon}$ on the set consisting in all the classical trajectories entering in S. Consider $\rho_0 \in \Omega \cap S$ and $\mathcal{V}$ some neighborhood of ρ_0 in Ω. Let us denote by $J^{\pm,in}$ (resp. $J^{\pm,out}$) the sets of all curves $\rho_s^\pm$ coming into (resp. out of) some point ρ in $S\cap\mathcal{V}$. The connection between $(\rho_s^\pm)_{s<0}$ and $(\rho_s^\mp)_{s>0}$ at $s=0$ yield that the sets

$$J = J^{+,in}\cup J^{-,out} \text{ and } J' = J^{-,in}\cup J^{+,out}$$

are smooth codimension 3 submanifolds of $T^*\mathbf{R}^4$. Moreover, according to Proposition 3 in [12], there exist two smooth vector-valued functions u, $u'\in\mathcal{C}^\infty(\mathbf{R}^8,\mathbf{S}^2)$ such that

$$J=\{\xi=A+(\tau+V)u\},\ \ J'=\{\xi=A+(\tau+V)u'\}.$$

For these vectors u and u', there exists $r>0$ such that

$$ru_{|S}=E+B\times u_{|S},\ \ -ru'_{|S}=E+B\times u'_{|S}. \tag{11}$$

In particular, if $E\cdot B=0$,

$$u=\frac{r}{|E|^2}E-\frac{E\times B}{|E|^2},\ \ u'=-\frac{r}{|E|^2}E-\frac{E\times B}{|E|^2},\ \ r=\sqrt{|E|^2-|B|^2}. \tag{12}$$

If $E\cdot B=0$ in $\mathcal{V}$, J and J' are involutive submanifolds of $T^*\mathbf{R}^4$, which means that for any $\rho\in J$, the orthogonal of $TJ_{|\rho}$ for the symplectic structure is included in $TJ_{|\rho}$. The situation is different if $E\cdot B\neq 0$. These facts are proved in [11] in a general context and in [8] for Dirac equation but in the case of constant electromagnetic vector fields. The situation $E\cdot B\neq 0$ in the case of constant electromagnetic fields have been studied in [8]. In this paper, we focus on the case where $E\cdot B=0$ near some $\rho_0\in S$. Thus, throughout the paper, we will assume that $E\cdot B=0$ so that we are in the geometric setting of [11]. Our aim is to study the concentration of (ψ^ε) above J and J' at the scale $\sqrt{\varepsilon}$ by means of two-scale Wigner measure for involutive submanifolds which have been introduced by L. Miller in [26] and developed in [10].

1.2 Two-scale semi-classical measures

We consider for a while a more general setting and recall results of [10]. Let I be a codimension m involutive submanifold of the cotangent space $T^*\mathbf{R}^D$. We suppose that I is given by some system of equations $f=0$ where $f=(f_1,\cdots,f_m)\in\mathcal{C}^\infty(\mathbf{R}^{2D},\mathbf{R}^m)$, Rank$(df)=m$ on $f=0$ and $\{f_j,f_k\}=0$.
Let us denote by $\overline{\mathbf{R}}^m$, the ball obtained by adding a sphere at infinity to $\mathbf{R}^m$. We consider the set $\mathcal{A}$ of symbols $a=a(z,\zeta,\eta)\in\mathcal{C}^\infty(\mathbf{R}^D\times\mathbf{R}^D\times\mathbf{R}^m)$ which are uniformly compactly supported in the variables (z,ζ) with respect to η and which can be extended as a function of $\mathcal{C}^\infty(\mathbf{R}^D\times\mathbf{R}^D\times\overline{\mathbf{R}}^m)$ by

$$a(z,\zeta,\infty\,\omega)=\lim_{R\to+\infty}a(z,\zeta,R\omega),\ \ \forall\omega\in\mathbf{S}^{m-1}.$$

With any $a\in\mathcal{A}$, we associate the two–scaled pseudo-differential operator,

$$\mathrm{op}_\varepsilon{}^I\,(a):=\mathrm{op}_h\left(a(z,\zeta,\frac{f(z,\zeta)}{\sqrt{\varepsilon}})\right).$$

By the Calderon-Vaillancourt theorem, the family of operators $\mathrm{op}_\varepsilon{}^I\,(a)$ is a bounded family of bounded operators in $L^2(\mathbf{R}^D)$. Let (ϕ^ε) be a bounded family in $L^2(\mathbf{R}^D, \mathbf{C}^N)$, $N \in \mathbf{N}$, we study the evolution as ε goes to 0 of

$$K_\varepsilon(a) := \left(\mathrm{op}_\varepsilon{}^I\,(a)\phi^\varepsilon \;\mid\; \phi^\varepsilon\right).$$

The limit of K_ε is described by a positive Radon measure on the compactified normal bundle to I, $\overline{N}(I)$.
Let us explain which is the bundle $\overline{N}(I)$. We associate with I its tangent bundle TI. Taking the quotient of the tangent space $T(\mathbf{R}^D)\mid_\rho$ above some point ρ of I by $TI\mid_\rho$, we obtain the fiber above ρ of the normal bundle to I, $N(I)$. Then, $\overline{N}(I)_{|\rho}$ is the closed m-dimensional ball obtained by adding a sphere at infinity to $N(I)\mid_\rho$. The choice of the equation f yields local coordinates on $\overline{N}(I)_{|\rho}$ given by the continuation $\overline{\chi}$ of the isomorphism χ,

$$\chi : [\delta\rho] \in N(I)_{|\rho} \mapsto \eta = df(\rho)\delta\rho \in \mathbf{R}^m.$$

If ν is a measure on $\overline{N}(I)$, we denote by ν_f the measure on $\overline{\mathbf{R}^m}$ which is the image of ν by $\overline{\chi}$. Let us come back to the limit of $K_\varepsilon(a)$.

There exists a sequence $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$, *a positive Radon measure* ν *on* $\overline{N}(I)$ *such that for all* $a \in \mathcal{A}$,

$$K_{\varepsilon_k}(a) \underset{k\to+\infty}{\longrightarrow} \mathrm{tr}\left(\int_{\overline{\mathbf{R}}^m} a\,\mathrm{d}\nu_f\right) + \mathrm{tr}\left(\int_{f\neq 0} a\left(x,\xi,\frac{f(x,\xi)}{\mid f(x,\xi)\mid}\infty\right)\mathrm{d}\mu\right),$$

where μ *is a semi-classical measure of* (ϕ^ε).

We point out that ν determines μ above I by

$$\mu \mathbf{1}_I = \int_I \nu(x,\xi,\mathrm{d}\eta).$$

Our aim in the following is to calculate two–scale Wigner measures associated with a family (ψ^ε) solution to (6), and with the involutive submanifolds J and J'.

1.3 The branching of the energy : matrix-valued Landau-Zener formula

We denote by $\overline{\nu}$ (resp. $\overline{\nu}'$) the measures associated with (ψ^ε) and J (resp. J'). We denote by Σ^+, Σ^-, $\dot{J}^{\pm,in}$ and $\dot{J}^{\pm,out}$ the sets

$$\Sigma^\pm = \{\lambda^\pm = 0\},\;\; \dot{J}^{\pm,in} = J^{\pm,in}\backslash S,\;\; \dot{J}^{\pm,out} = J^{\pm,out}\backslash S.$$

We consider the bundles above $\Sigma^\pm$, $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,in})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,out})$ obtained respectively by adding a sphere at infinity to the fibers of $T\Sigma^\pm/T(\dot{J}^{\pm,in})$ and of $T\Sigma^\pm/T(\dot{J}^{\pm,out})$. Because of the properties of localization of semi-classical measures, there exist scalar positive Radon measures $\nu^{\pm,in}$ and $\nu^{\pm,out}$ supported on $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,in})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,out})$ respectively such that

$$\overline{\nu} = \nu^{+,in} + \nu^{-,out}, \;\; \overline{\nu}' = \nu^{-,in} + \nu^{+,out},$$
$$\Pi^\pm \nu^{\pm,in} \Pi^\pm = \nu^{\pm,in}, \;\; \Pi^\pm \nu^{\pm,out} \Pi^\pm = \nu^{\pm,out}.$$

Moreover, if $\mathcal{L}^{\pm,in}(H_{\lambda^\pm})$ (resp. $\mathcal{L}^{\pm,out}(H_{\lambda^\pm})$) is the linearized hamiltonian flow of $\lambda^\pm$ transversally to $\dot{J}^{\pm,in}$ (resp. $\dot{J}^{\pm,out}$), in $\Sigma^\pm$, the measures $\nu^{\pm,in}$ (resp. $\nu^{\pm,out}$) satisfy

$$\mathcal{L}^{\pm,in}(H_{\lambda^\pm})\nu^{\pm,in} = [\nu^{\pm,in} \,,\; F^{\pm,j}] \;\text{ on }\; \dot{J}^{\pm,in}$$
$$\mathcal{L}^{\pm,out}(H_{\lambda^\pm})\nu^{\pm,out} = [\nu^{\pm,out} \,,\; F^{\pm,j}] \;\text{ on }\; \dot{J}^{\pm,out}.$$

These propagation properties result from [16] and [10]. Using (10), (11), the equations of J and J' and the fact that $|\xi - A| = \mp(\tau + V)$ on $J^{\pm,in}$ and $J^{\pm,out}$, we obtain that

$$F^\pm_{|\dot{J}^{\pm,in}} = O(1), \;\; F^\pm_{|\dot{J}^{\pm,out}} = O(1) \;\text{ near }\; S.$$

Therefore, the transversality of the hamiltonian flows to S yields that, in the set of distributions, measures $\nu^{\pm,in}$ and $\nu^{\pm,out}$ have traces on S that we denote by $\nu_S^{\pm,in}$ and by $\nu_S^{\pm,out}$. These four traces can be identified to measures on one set in which we can study the existing link between $\nu_S^{\pm,out}$ and $\nu_S^{\pm,in}$.

Lemma 1 . *For $\rho_0 \in S$, the map*

$$\eta : T(T^*(\mathbf{R}^4))_{|\rho_0} \to \mathbf{R}^3$$
$$\delta\rho \mapsto \mathrm{d}(\tau + V)\delta\rho \, B + \mathrm{d}(\xi - A)\,\delta\rho \times E$$

induces some isomorphism between the limits of the fibres of $N_{\Sigma^\pm}(\dot{J}^{\pm,in})$ and $N_{\Sigma^\pm}(\dot{J}^{\pm,out})$ above some point ρ which goes to ρ_0 and the hyperplane normal to E.

This lemma is proved in the Annex.
We extend the map η to $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,in})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,out})$ which are now identified. Provided this identification, the connection between $\nu_S^{\pm,in}$ and $\nu_S^{\pm,out}$ near the point ρ_0 is described by the following Theorem where we denote by $p(\xi)$ the matrix

$$p(\xi) = \sum_{j=1}^{3} \xi_k \sigma_k = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix}.$$

Theorem 1 . *We assume that $\nu_S^{+,in}$ and $\nu_S^{-,in}$ are singular.*
1) **The adiabatic regime***: $\alpha \in]0, \frac{1}{2}[$,*

$$\nu_S^{+,out} = \frac{|E|^2}{|E|^2 - |B|^2} \mathcal{R}_a \nu_S^{+,in} \mathcal{R}_a^*, \;\; \nu_S^{-,out} = \frac{|E|^2}{|E|^2 - |B|^2} \mathcal{R}_a \nu_S^{-,in} \mathcal{R}_a^*,$$

with

$$\mathcal{R}_a = \begin{pmatrix} \mathbf{1} & -p\left(\frac{E\times B}{|E|^2}\right) \\ p\left(\frac{E\times B}{|E|^2}\right) & -\mathbf{1} \end{pmatrix}.$$

2) **The non-adiabatic regime***:* $\alpha \geq \frac{1}{2}$*,*

$$\begin{aligned} \nu_S^{+,out} &= T\,\nu_S^{-,in} + (1-T)\,\mathcal{R}\,\nu_S^{+,in}\,\mathcal{R}^* \\ \nu_S^{-,out} &= T\,\nu_S^{+,in} + (1-T)\,\mathcal{R}\,\nu_S^{-,in}\,\mathcal{R}^* \end{aligned} \tag{13}$$

with if $\alpha = 1/2$*,*

$$\begin{aligned} T &= \exp\left(-\frac{\pi}{(|E|^2-|B|^2)^{3/2}}\Phi(\eta,m)^2\right), \\ \mathcal{R} &= \frac{|E|}{\Phi(\eta,m)}\left(\mathcal{R}_{na}(\eta,m) + m\mathcal{R}_a\right), \end{aligned}$$

if $\alpha > 1/2$*,*

$$\begin{aligned} T &= \exp\left(-\frac{\pi}{(|E|^2-|B|^2)^{3/2}}\Phi(\eta,0)^2\right), \\ \mathcal{R} &= \frac{|E|}{\Phi(\eta,0)}\mathcal{R}_{na}(\eta,0), \end{aligned}$$

where

$$\Phi(\eta,m) = \left[|\eta|^2 - \left(\frac{E\times B}{|E|^2}\cdot\eta\right)^2 + (|E|^2-|B|^2)m^2\right]^{1/2},$$

and

$$\mathcal{R}_{na}(\eta) = \begin{pmatrix} \mathbf{0} & p(\frac{\eta\times E}{|E|^2}) \\ p(\frac{\eta\times\eta}{|E|^2}) & 0 \end{pmatrix} + i\,\eta\cdot\frac{E\times B}{|E|^3}\begin{pmatrix} p\left(\frac{E}{|E|}\right) & 0 \\ 0 & p\left(\frac{E}{|E|}\right) \end{pmatrix}.$$

Remarks 1) The matrix $\mathcal{R}$ describes the change of polarization at the crossing. Observe that if $B = 0$, the transfer coefficient T and the polarization matrix $\mathcal{R}$ are rather simple since we have

$$T = \exp\left(-\frac{|\eta|^2}{|E|^3}\right), \quad \mathcal{R}_{na} = \begin{pmatrix} \mathbf{0} & p(\eta) \\ p(\eta) & 0 \end{pmatrix}.$$

In that case, we are reduce to the situation studied in [9].
2) It has already been noticed in [6], [23] or [20] that the scale $\sqrt{\varepsilon}$ is critical. If $\alpha \in]0, \frac{1}{2}[$, the mass $\varepsilon^\alpha m$ is big enough so that the measure is propagated as if there were no crossing ($\alpha = 0$). If $\alpha > \frac{1}{2}$, the mass is so small that we have the same result as if $m = 0$, i.e. as if we had a true crossing as studied in [12] – [9]. Finally, in the critical case $\alpha = \frac{1}{2}$, the Landau-Zener formula depend on m. Observe too that the part of the measure localized on $\{|\eta| = +\infty\}$ propagates adiabatically.

3) It is likely that the adiabatic regime could be studied directly on the semi-classical scale and that the introduction of two-scale Wigner measure is not necessary to calculate the evolution of Wigner measures: the reflection of Wigner measure could be obtained by the same method as in Section 3 below.
4) In the case $E \cdot B \neq 0$, the two-scale Wigner measures which have to be considered are more complicated (see [8]) but similar results may be expected.
5) In the proofs below, we do not use the fact that $\xi - A$ is a linear function of ξ. Thus Theorem 1 can be generalized to systems of the same form that (6) where we turn $\xi - A$ into some function $l = l(t, x, \xi)$ and where V may depend on ξ with the conditions of [12]: $\mathrm{d}l$ of rank 3 on $\{l = 0\}$ and $E \cdot B = 0$, with

$$E = \{\tau + V, l\}, \;\; B = (\{l_3, l_2\}, \{l_1, l_3\}, \{l_2, l_1\})\,.$$

We will proceed as follows. Section 2 is devoted to the reduction of system (6) to a model system, via a Fourier integral operator and a canonical transform. Then, in Section 3, we reduce Theorem 1 to some equivalent statement on the solution of the model system. In Section 4 and 5, we treat separately the "adiabatic" cases – namely $\alpha \in (0, 1/2)$ and $(\alpha \geq 1/2, \mid \eta \mid = +\infty)$ – and the "non-adiabatic" ones: $\alpha \geq 1/2$ and $|\eta| < +\infty$. Finally, in the Appendix, we prove the restitution of the energy by the crossing and the geometric Lemma 1.

2 Reduction to a model problem

Let us describe first the Fourier integral operators we use. If κ is a canonical transform of $T^*\mathbf{R}^4$, there exists some semi-classical Fourier integral operator K that we said *associated with* κ, which satisfies

$$\forall f \in L^2(\mathbf{R}^4),\, \forall a \in \mathcal{C}_0^\infty(\mathbf{R}^8),\; K^* \,\mathrm{op}_h\,(a) K f = \mathrm{op}_h\,(a \circ \kappa) f + O(h^2) ||f||_{L^2},\; \text{in}\, L^2(\mathbf{R}^4). \quad (14)$$

The reader can refer to [28] for a complete study on Fourier Integral Operator or to [10] where this claim is proved.
This section is devoted to the proof of the following Proposition.

Proposition 1 . *Consider* $\rho_0 \in S$ *such that* $|E(\rho_0)| > |B(\rho_0)|$. *There exist some local canonical transform* κ *from a neighborhood of* ρ_0 *into a neighborhood* Ω *of* 0,

$$\kappa : (t, x, \tau, \xi) \mapsto (s, z, \sigma, \zeta), \;\; \kappa(\rho_0) = 0,$$

a matrix C *and a Fourier integral operator* K *associated with* κ *such that* $v^\varepsilon = K\,\mathrm{op}_\varepsilon\,(C)\psi^\varepsilon$ *satisfies for all* $a \in \mathcal{C}_0^\infty(\Omega)$,

$$\mathrm{op}_\varepsilon\,(a)\,\mathrm{op}_\varepsilon \begin{pmatrix} -\sigma + s & p(m_\varepsilon, \Gamma\tilde{\zeta}) \\ p(m_\varepsilon, \Gamma\tilde{\zeta}) & -\sigma - s \end{pmatrix} = O(\varepsilon), \quad (15)$$

where $\tilde{\zeta} = (\zeta_1, \zeta_2)$, $m_\varepsilon = k(s, z, \sigma, \zeta) m \varepsilon^\alpha$ *with*

$$k_{|S} = \Big(|E|^2 - |B|^2\Big)^{-1/4}, \tag{16}$$

and where the 2×2 *matrix-valued function* $\Gamma = \Gamma(s, z, \sigma, \zeta)$ *is smooth with* $\det(\Gamma) \neq 0$ *in* Ω. *Moreover in the coordinates* (s, z, σ, ζ),

$$J^{\pm, in} = \{\sigma \mp s = 0,\ \tilde{\zeta} = 0,\ \ s < 0\},\ \ J^{\pm, out} = \{\sigma \pm s = 0,\ \tilde{\zeta} = 0,\ \ s > 0\}, \tag{17}$$
$$J \cup J' = \{\tilde{\zeta} = 0\} \cap \Sigma = \{\sigma^2 = s^2\} \cap \{\tilde{\zeta} = 0\}.$$

The proof of Proposition 1 follows the same schedule than the proof of Theorem 2 in [12] through modifications required by the presence of $m\varepsilon^\alpha$. We proceed in two steps: first we transform the matrix P_ε by algebraic operations so that the equations of J and J' appear, then we define the canonical transform and conclude the proof.

2.1 An algebraic lemma

Let us first introduce some notations. We shall use the following matrices:

$$M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \ \ M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which have the following properties.

$$M_1^* = M_1^{-1} \qquad M_2 = M_2^{-1} = M_2^* \tag{18}$$
$$M_1 \begin{pmatrix} m & p(\xi) \\ p(\xi) & -m \end{pmatrix} M_1^* \ = \ \begin{pmatrix} -p(\xi) & m \\ m & p(\xi) \end{pmatrix}, \tag{19}$$
$$M_2 \begin{pmatrix} -p(\xi) & m \\ m & p(\xi) \end{pmatrix} M_2 \ = \ - \begin{pmatrix} \xi_3 & p(\xi_1, \xi_2, m) \\ p(\xi_1, \xi_2, m) & -\xi_3 \end{pmatrix} \tag{20}$$

We denote by L the submanifold of $T^*\mathbf{R}^4$ defined by

$$L = \{(\xi - A - (\tau + V)u) \times (u - u') = 0\}.$$

Then, we have $J \cup J' = \Sigma \cap L$. Moreover, we set

$$e_1 = e_1(t, x, \tau, \xi) := \frac{u - u'}{|\, u - u' \,|},$$

so that

$$(e_1)_{|S} = \frac{E}{|\, E \,|}. \tag{21}$$

We choose some smooth functions (e_2, e_3) so that (e_1, e_2, e_3) is a direct orthonormal basis of $\mathbf{R}^3$ with moreover

$$(e_2)_{|S} = \frac{B \times E}{\mid E \mid \mid B \mid}, \quad (e_3)_{|S} = \frac{B}{|B|}, \text{ if } B \neq 0. \tag{22}$$

Consider now the complex valued function $\theta = \theta(t, x, \tau, \xi)$ defined by

$$\theta = u \cdot e_2 + iu \cdot e_3.$$

Note that near ρ_0, $\mid \theta \mid < 1$. Actually, in view of (11) we obtain

$$\theta_{|S} \in \mathbf{R}, \text{ and } \theta_{|S} = \frac{\mid B \mid}{\mid E \mid} < 1. \tag{23}$$

Lemma 2 . *There exists some smooth invertible matrix $C_1 = C_1(t, x, \tau, \xi)$ such that*

$$\tau + P_\varepsilon = C_1^* H_1 C_1$$

with

$$H_1 = \tau + V + Y \cdot g - \begin{pmatrix} \frac{(\xi - A)\cdot e_1}{\sqrt{1-|\theta|^2}} & p(g, \tilde{m}_\varepsilon) \\ p(g, \tilde{m}_\varepsilon) & -\frac{(\xi - A)\cdot e_1}{\sqrt{1-|\theta|^2}} \end{pmatrix}, \tag{24}$$

where $\tilde{m}_\varepsilon = \tilde{k}(s, z, \sigma, \zeta) m\varepsilon^\alpha$ and where $g = (g_1, g_2)$ is an equation of L satisfying

$$\begin{cases} \mathrm{d}g_{1|S} = \frac{|E|^2}{r^2}\mathrm{d}\left[(\xi - A - (\tau + V)u) \cdot e_2\right], \\ \mathrm{d}g_{2|S} = \frac{|E|}{r}\mathrm{d}\left[(\xi - A - (\tau + V)u) \cdot e_3\right]. \end{cases} \tag{25}$$

The complex-valued vector $Y = Y(t, x, \tau, \xi) = (Y_1, Y_2)$ satisfies

$$\begin{cases} Y_{1|S} = -\frac{|B|}{|E|}, \\ Y_{2|S} = 0. \end{cases} \tag{26}$$

The real-valued function $\tilde{k}$ satisfies

$$\tilde{k}_{|S} = \frac{|E|}{\sqrt{|E|^2 - |B|^2}} \tag{27}$$

Proof of Lemma 2: We first use the quaternion structure of matrix $p(v)$. By Lemma 2 of [12], there exists a smooth unitary matrix $U = U(t, x, \tau, \xi)$ such that

$$Up(v)U^* = p(v.e_2, v.e_3, v.e_1). \tag{28}$$

Therefore, if $U' = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} M_1$, we have

$$U' Q_\varepsilon U'^* = \begin{pmatrix} -p((\xi - A).e_2, (\xi - A).e_3, (\xi - A).e_1) & m\varepsilon^\alpha \\ m\varepsilon^\alpha & p((\xi - A).e_2, (\xi - A).e_3, (\xi - A).e_1) \end{pmatrix}. \tag{29}$$

We aim now to introduce the equations of L given by the vector-valued function f,

$$\begin{aligned} f &= \Big((\xi - A - (\tau+V)u)\cdot e_2, (\xi - A - (\tau+V)u)\cdot e_3\Big) \\ &= \Big((\xi - A)\cdot e_2 - (\tau+V)\theta_1, (\xi - A)\cdot e_3 - (\tau+V)\theta_2\Big). \end{aligned}$$

Therefore,

$$U'(\tau+P_\varepsilon)U'^* = (\tau+V)\begin{pmatrix} S(-\theta) & \mathbf{0} \\ \mathbf{0} & S(\theta) \end{pmatrix} + \begin{pmatrix} -p(f,(\xi-A)\cdot e_1) & m\varepsilon^\alpha \\ m\varepsilon^\alpha & p(f,(\xi-A)\cdot e_1) \end{pmatrix}, \tag{30}$$

where $S(\theta) = \begin{pmatrix} 1 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & 1 \end{pmatrix}$.

Let us denote by $\mathcal{N}$ the set of such matrices

$$\mathcal{N} = \left\{ S(z) = \begin{pmatrix} 1 & \overline{z} \\ z & 1 \end{pmatrix}, \;\; z \in \mathbf{C} \right\}.$$

Lemma 3 . *Let $S = S(z) \in \mathcal{N}$, then for all $v = (v_1, v_2, v_3) = (v', v_3) \in \mathbf{R}^3$,*

$$S(z)p(v',v_3)S(z) = z\overline{v'} + \overline{z}v' + p\left(v' + z^2\overline{v'}, (1-|z|^2)v_3\right),$$

The proof is straightforward.
We use matrices $\sqrt{S(\theta)^{-1}}$ and $\sqrt{S(-\theta)}^{-1}$ to get rid of the matrix-valued coefficient of $(\tau+k)$ in (30). Notice that

$$S(\theta)^{-1} = \frac{S(-\theta)}{1-\mid\theta\mid^2}, \;\; \sqrt{S(\theta)}^{-1} = a\, S(-b\theta),$$

with

$$a = \frac{1}{2}(1-\mid\theta\mid^2)^{-1/2}\left(\sqrt{1+\mid\theta\mid} + \sqrt{1-\mid\theta\mid}\right),$$
$$b = \frac{1}{1+\sqrt{1-\mid\theta\mid^2}}.$$

Define

$$C_1 = M_2\begin{pmatrix} \sqrt{S(-\theta)} & \mathbf{0} \\ \mathbf{0} & \sqrt{S(\theta)} \end{pmatrix} U'. \tag{31}$$

Then, if $H_1 = (C_1^*)^{-1}(\tau + P_\varepsilon)C_1^{-1}$, Lemma 3 and (30) yield

$$H_1 = M_2\Bigg[\tau + k - a^2 b(\theta\overline{f} + f\overline{\theta}) + m\varepsilon^\alpha a^2(1-b^2|\theta|^2)\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} +$$
$$a^2\begin{pmatrix} -p\left(f + b^2\theta^2\overline{f}, (1-b^2\mid\theta\mid^2)(\xi-A)\cdot e_1\right) & \mathbf{0} \\ \mathbf{0} & p\left(f + b^2\theta^2\overline{f}, (1-b^2\mid\theta\mid^2)(\xi-A)\cdot e_1\right) \end{pmatrix}\Bigg] M_2.$$

We set $\tilde{k} = a^2(1 - b^2 \mid \theta \mid^2)$ and we check that $\tilde{k} = \frac{1}{\sqrt{1-|\theta|^2}}$.
Let $g = (g_1, g_2)$ be such that $g_1 + ig_2 = a^2(f + b^2\theta^2\overline{f})$. Then we get (24) through straightforward computations.

Moreover, we have $\mathrm{d}g_{|S} = \mathrm{d}f_{|S} + b^2\theta^2\mathrm{d}\overline{f}_{|S}$ and since $a^2(1 + b^2\theta^2) = (1 - |\theta|^2)^{-1}$, we have

$$\begin{aligned}
\mathrm{d}g_{1|S} &= a^2(1 + b^2\theta^2)_{|S}\mathrm{d}f_{1\,|S} = \frac{1}{1- \mid \theta \mid^2}_{|S}\mathrm{d}f_{1\,|S}, \\
\mathrm{d}g_{2|S} &= a^2(1 - b^2\theta^2)_{|S}\mathrm{d}f_{2\,|S} = \frac{1}{\sqrt{1- \mid \theta \mid^2}}_{|S}\mathrm{d}f_{2\,|S},
\end{aligned}$$

hence (25).
Moreover if $Y_1g_1 + Y_2g_2 = -a^2b(\theta\overline{f} + \overline{\theta}f) = -2a^2b\mathcal{R}e(\theta\overline{f})$, then we have

$$(Y_1\mathrm{d}g_1 + Y_2\mathrm{d}g_2)_{|S} = -2a^2b\,\theta\mathrm{d}\overline{f}_{|S}.$$

Therefore, $(Y_2)_{|S} = 0$ and $(Y_1)_{|S} = -2(1- \mid \theta \mid^2)a^2b\theta$ above S, hence (26).♢

2.2 The canonical transform

Observe that

$$\left\{\tau + V + Y \cdot g\ ,\ \frac{(\xi - A) \cdot e_1}{\sqrt{1 - |\theta|^2}}\right\} = \frac{|E|^2}{\sqrt{|E|^2 - |B|^2}} > 0.$$

Therefore, arguing as for Proposition 4 in [12], we get the following Proposition.

Proposition 2 . *There exists some function λ, $\lambda > 0$ near ρ_0, and some local canonical transform κ, $\kappa(\rho_0) = 0$,*

$$(t, x, \tau, \xi) \mapsto (s, z, \sigma, \zeta),$$

such that

$$\begin{gathered}
\sigma = \lambda\left(\tau + V + Y_1g_1 + Y_2g_2\right), \\
s = \lambda\,\frac{(\xi - A) \cdot e_1}{\sqrt{1- \mid \theta \mid^2}}, \\
\lambda g(t, x, \tau, \xi) = \left(\check{\gamma}^1(s, z, \sigma, \zeta) \cdot \tilde{\zeta}, \check{\gamma}^2(s, z, \sigma, \zeta) \cdot \tilde{\zeta}\right) + (\sigma^2 - s^2)\beta,
\end{gathered}$$

where $\tilde{\zeta} = (\zeta_1, \zeta_2)$ and where $\beta = (\beta_1, \beta_2)$, $\check{\gamma}^1$ and $\check{\gamma}_2$ are smooth functions valued in $\mathbf{R}^2$ with $\det(\check{\gamma}_1, \check{\gamma}_2) \neq 0$. *Moreover,*

$$\lambda_{|S}^2 = \mid E \mid^{-2} \sqrt{\mid E \mid^2 - \mid B \mid^2}. \tag{32}$$

Let us identify J and J' in these new coordinates. Remark that, if $\rho \in L$, $g(\rho) = 0$, thus

$$\lambda^2 \mid \tau + V \mid^2 = \sigma^2, \;\; \lambda^2 \mid \xi - A \mid^2 = s^2(1 - \mid \theta \mid^2) + \sigma^2 \mid \theta \mid^2 .$$

Therefore $L \cap \{s^2 = \sigma^2\} = L \cap \{|\xi - A|^2 = (\tau + V)^2\} = J \cup J'$. We study now the sign of σ which is $\mathrm{sgn}(\tau + V)$ i.e. $\mp$ on $J^{\pm,in} \cup J^{\pm,out}$, and the sign of s which is $\mathrm{sgn}((\xi - A) \cdot e_1)$ i.e. $+\mathrm{sgn}(\tau + V)$ on J and $-\mathrm{sgn}(\tau + V)$ on J' (since $u \cdot e_1 > 0$ and $u' \cdot e_1 < 0$). For example, we obtain $J = \{\sigma = s\} \cap \{g = 0\}$ and $J^{+,in} \subset \{s < 0, \;\; \sigma < 0\}$. Hence the equations of J, J', $J^{\pm,in}$ and $J^{\pm,out}$ stated in Proposition 1.

Let us conclude now the proof of Proposition 1. We set

$$C = \lambda^{-1/2} C_1. \tag{33}$$

In the following calculations, we shall denote by $O(\varepsilon)$ either a family (R_ε) of operators or a family (r_ε) of functions such that, for every $\phi \in C_0^\infty$,

$$\frac{1}{\varepsilon}\left[\| \mathrm{op}_\varepsilon(\phi) R_\varepsilon \|_{\mathcal{L}(L^2)} + \mid \mathrm{op}_\varepsilon(\phi) r_\varepsilon \mid_{L^2}\right] \;\text{ is bounded.}$$

We have

$$\mathrm{op}_\varepsilon\,(\lambda H_1) = \mathrm{op}_\varepsilon\,(C^{*-1})\,\mathrm{op}_\varepsilon\,(\tau + P_\varepsilon)\,\mathrm{op}_\varepsilon\,(C^{-1}) + O(\varepsilon).$$

Consider now some Fourier integral operator K associated with κ and

$$v^\varepsilon = K\,\mathrm{op}_\varepsilon\,(C)\psi^\varepsilon.$$

Then we have

$$\begin{aligned} \mathrm{op}_\varepsilon\left(\sigma - \begin{pmatrix} s & p(\tilde f, m_\varepsilon) \\ p(\tilde f, m_\varepsilon) & -s \end{pmatrix}\right)\tilde v^\varepsilon &= K\,\mathrm{op}_\varepsilon\,(\lambda H_1)\tilde v^\varepsilon \\ &= K\,\mathrm{op}_\varepsilon\,(C^{*-1})\,\mathrm{op}_\varepsilon\,(\tau + P_\varepsilon)\psi^\varepsilon + O(\varepsilon) \\ &= O(\varepsilon), \end{aligned}$$

where we have used (14) and (6) and where $m_\varepsilon = \lambda \tilde m_\varepsilon$ and

$$\tilde f = (\check\gamma^1 \cdot \tilde\zeta, \check\gamma^2 \cdot \tilde\zeta) + (\sigma^2 - s^2)\beta.$$

We rewrite this equation as

$$\mathrm{op}_\varepsilon\,(D - DBD)v^\varepsilon = \mathrm{op}_\varepsilon\,(F)v^\varepsilon + O(\varepsilon), \tag{34}$$

with

$$\begin{aligned} D &= \begin{pmatrix} \sigma - s & \mathbf{0} \\ \mathbf{0} & \sigma + s \end{pmatrix}, \; B = \begin{pmatrix} \mathbf{0} & p(0,\beta) \\ p(0,\beta) & \mathbf{0} \end{pmatrix}, \\ F &= \begin{pmatrix} \mathbf{0} & p(\check\gamma^1 \cdot \tilde\zeta, \check\gamma^2 \cdot \zeta_2, m_\varepsilon) \\ p(\check\gamma^1 \cdot \tilde\zeta, \check\gamma^2 \cdot \zeta_2, m_\varepsilon) & \mathbf{0} \end{pmatrix}. \end{aligned}$$

Using that $DBF = BFD$ and that $B_{|S} = 0$, $F_{|S} = 0$ we get by symbolic calculus

$$\mathrm{op}_\varepsilon\,(D)v^\varepsilon = \mathrm{op}_\varepsilon\,(P)\ \mathrm{op}_\varepsilon\,(F)v^\varepsilon + O(\varepsilon),$$

with $P_{|S} = \mathbf{1}$. Hence Proposition 1 with

$$\Gamma_{|S}\tilde{\zeta} = \left((\check{\gamma}_1)_{|S}\cdot\tilde{\zeta} + (\check{\gamma}_2)_{|S}\cdot\tilde{\zeta}\right). \tag{35}$$

Therefore matrix Γ is invertible near $S.\diamondsuit$

3 Consequences of the reduction to system (15)

Proposition 1 has two important consequences. The first one is that it states that there exists some involutive submanifold of $T^*\mathbf{R}^{d+1}$, $I = \{\tilde{\zeta} = 0\}$, such that

$$J \cup J' = \Sigma \cap I,$$

with transverse intersection. Thus, if ν is the two-scale Wigner measure of (ψ^ε) for I, we can identify measures $\overline{\nu}$, (resp. $\overline{\nu}'$) with ν above $\overline{N}(I)_{|J}$ (resp. $\overline{N}(I)_{|J'}$). Actually, if $\overline{N}_\Sigma(J)$ is the bundle above Σ obtained by adding a sphere at infinity of the fibers of $T\Sigma/TJ$, the canonical isomorphism from $T\Sigma/TJ$ onto $T(T^*\mathbf{R}^{d+1}_{|J})/TJ_J$ extends in some isomorphism

$$\theta_{\Sigma,I} : \overline{N}_\Sigma(J) \to \overline{N}(I),$$

that we use for identifying $\overline{\nu}$ and $\nu\,\mathbf{1}_{\overline{N}(I)_{|J}}$. The reader can refer to Lemma 4 in [10] for a proof of this fact. Because of this identification, we shall focus in calculating the two-scale Wigner measure of (ψ^h) for I.

The second consequence is that because of the invariance of two-scale Wigner measure through canonical transform (see Lemma 2 in [10]), it is equivalent to study the two-scale Wigner measure of (v^ε) for I, or of (ψ^ε) for the same set. Indeed, the two-scale Wigner measures $\tilde{\nu}$ of (v^ε) and ν of (ψ^ε) are linked by

$$\forall a \in \mathcal{A},\ < a\ ,\ \tilde{\nu} > = < a \circ \overline{N}(\kappa)\ ,\ C\nu C^* >, \tag{36}$$

where for $(\rho,\eta) \in \overline{N}(I)$, $\overline{N}(\kappa)(\rho,\eta) = (\kappa(\rho),\eta)$. We set

$$\tilde{\Pi}_0^\pm = \frac{1}{2}\left(1 \mp \frac{1}{\sqrt{s^2 + |\Gamma\tilde{\zeta}|^2}}\begin{pmatrix} s & p(\Gamma\tilde{\zeta},0) \\ p(\Gamma\tilde{\zeta},0) & -s \end{pmatrix}\right). \tag{37}$$

We decompose $\tilde{\nu}$ as $\tilde{\nu} = \tilde{\nu}^+ + \tilde{\nu}^-$ with the commutation's relations:

$$\tilde{\Pi}_0^\pm \tilde{\nu}^\pm \tilde{\Pi}_0^\pm = \tilde{\nu}^\pm, \tag{38}$$

and the localization property: $\tilde{\nu}^{\pm}$ is supported on $\{\sigma \pm |s|\} = J^{\pm,in} \cup J^{\pm,out}$. Let us denote by $\tilde{\nu}_S^{\pm,in}$ (resp. $\tilde{\nu}_S^{\pm,out}$) the traces of $\tilde{\nu}^{\pm}$ on $s = 0^-$ (resp. $s = 0^+$). In view of (36), we have

$$\tilde{\nu}_S^{\pm,out} \circ \overline{N}(\kappa)^{-1} = C_{|S}\, \nu_S^{\pm,out}\, C_{|S}^*, \;\; \tilde{\nu}_S^{\pm,in} \circ \overline{N}(\kappa)^{-1} = C_{|S}\, \nu_S^{\pm,in}\, C_{|S}^*. \tag{39}$$

In the coordinates (s, z, σ, ζ) we choose the equations of I, $\tilde{\zeta} = 0$. Let $\rho \in S$, this equation generates coordinates $\tilde{\eta} \in \overline{\mathbf{R}^4}$ on $\overline{N}(I)_{|\rho}$. In these coordinates the branching of the energy is described by the following theorem.

Theorem 2 . *Assume that $\tilde{\nu}^{+,in}$ and $\tilde{\nu}^{-,in}$ are mutually singular, then we have*

$$\begin{aligned} \tilde{\nu}_S^{+,out} &= \tilde{T}\, \tilde{\nu}_S^{-,in} + (1 - \tilde{T})\, \tilde{\mathcal{R}}\, \tilde{\nu}_S^{+,in}\, \tilde{\mathcal{R}} \\ \tilde{\nu}_S^{-,out} &= \tilde{T}\, \tilde{\nu}_S^{+,in} + (1 - \tilde{T})\, \tilde{\mathcal{R}}\, \tilde{\nu}_S^{-,in}\, \tilde{\mathcal{R}} \end{aligned} \tag{40}$$

with

1) If $\alpha \in]0, \frac{1}{2}[$,

$$\tilde{T} = 0, \;\; \tilde{\mathcal{R}} = \begin{pmatrix} \mathbf{0} & p(0,0,1) \\ p(0,0,1) & \mathbf{0} \end{pmatrix}.$$

2) If $\alpha = \frac{1}{2}$,

$$\tilde{T} = \exp\left[-\pi \left(|\Gamma_{|S}\tilde{\eta}|^2 + k_{|S}^2 m^2\right)\right],$$

$$\tilde{\mathcal{R}} = \left(|\Gamma_{|S}\tilde{\eta}|^2 + k_{|S}^2 m^2\right)^{-1/2} \begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta}, k_{|S}m) \\ p(\Gamma_{|S}\tilde{\eta}, k_{|S}m) & \mathbf{0} \end{pmatrix}.$$

3) If $\alpha > \frac{1}{2}$,

$$\tilde{T} = \exp\left[-\pi |\Gamma_{|S}\tilde{\eta}|^2\right], \;\; \tilde{\mathcal{R}} = \frac{1}{|\Gamma_{|S}\tilde{\eta}|} \begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta}, 0) \\ p(\Gamma_{|S}\tilde{\eta}, 0) & \mathbf{0} \end{pmatrix}.$$

Let us explain now why this theorem implies Theorem 1.
Proof of Theorem 1: We focus on the case $\alpha = 1/2$. Because of (39), we have

$$T = \tilde{T} \circ \overline{N}(\kappa)^{-1}, \;\; \mathcal{R} = \left(C_{|S}^{-1} \tilde{\mathcal{R}} C_{|S}\right) \circ \overline{N}(\kappa)^{-1}.$$

The first observation consists on studying the link between $\tilde{\eta}$ and the function η defined in the introduction. In view of (35) and of Proposition 2, we have

$$\Gamma_{|S}\, \mathrm{d}\tilde{\zeta} = \lambda\, \mathrm{d}g_{|S}.$$

Moreover,

$$\eta = [\mathrm{d}(\xi - A)\, \delta\rho - \mathrm{d}(\tau + V)\, \delta\rho\, u] \times E.$$

Using (25), (27) and (11), we obtain

$$\Gamma_{|S}\mathrm{d}\tilde{\eta} = \left(-\frac{\lambda_{|S}|E|}{r^2}\,\eta \cdot e_3, \frac{\lambda_{|S}}{r}\,\eta \cdot e_2 \right). \tag{41}$$

Hence, by (27), (12), (32) and (22), we get

$$|\Gamma_{|S}\mathrm{d}\tilde{\eta}|^2 + k_{|S}^2 m^2 = \left(|E|^2 - |B|^2 \right)^{-3/2} \Phi(\eta, m)^2.$$

This yields the value of T stated in Theorem 1.

It remains to cxalculate $C_{|S}^{-1}\tilde{\mathcal{R}}C_{|S}$. Notice that

$$\tilde{\mathcal{R}} = \Phi(\eta, m)^{-1} \left(|E|^2 - |B|^2 \right)^{3/4} \begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta}, mk_{|S}) \\ p(\Gamma_{|S}\tilde{\eta}, mk_{|S}) & \mathbf{0} \end{pmatrix}.$$

Moreover, because of (33), we have

$$C_{|S}^{-1}\tilde{\mathcal{R}}C_{|S} = C_{1|S}^{-1}\tilde{\mathcal{R}}C_{1|S}.$$

We shall use the following lemma.

Lemma 4 *For any* $v = (v_1, v_2, v_3) \in \mathbf{R}^3$,

$$(C_1)_{|S}^{-1} \begin{pmatrix} \mathbf{0} & p(v) \\ p(v) & \mathbf{0} \end{pmatrix} (C_1)_{|S} =$$
$$-\frac{v_3}{\sqrt{1-\theta^2}} \begin{pmatrix} \mathbf{1} & -p\left(\frac{E\times B}{|E|^2}\right) \\ p\left(\frac{E\times B}{|E|^2}\right) & -\mathbf{1} \end{pmatrix} + i\,\frac{\theta\, v_2}{\sqrt{1-\theta^2}} \begin{pmatrix} p(\frac{E}{|E|}) & \mathbf{0} \\ \mathbf{0} & p(\frac{E}{|E|}) \end{pmatrix}$$
$$-\frac{1}{\sqrt{1-\theta^2}} \begin{pmatrix} \mathbf{0} & U^* p(\sqrt{1-|\theta|^2}v_1, v_2, 0)U \\ U^* p(\sqrt{1-|\theta|^2}v_1, v_2, 0)U & \mathbf{0} \end{pmatrix}.$$

Proof: This lemma comes from simple computations and the use of

$$\sqrt{S(\theta_{|S})}p(v', 0)\sqrt{S(-\theta_{|S})} = p\left(v_1, \frac{v_2}{\sqrt{1-\theta^2}}, 0\right) + i\theta_{|S}p(0, 0, 1),$$
$$U^* p(\theta_{|S})U = -p\left(\frac{E \times B}{|E|^2}\right), \;\; U^* p(0, 0, 1)U = p\left(\frac{E}{|E|}\right).$$

These latter equations are consequences of the definition of U (see (28)), of (22) and (12).◊

By definition of $\Phi(\eta, m)$, we have

$$\mathcal{R} = \frac{|E|}{\Phi(\eta, m)} \left(|E|^2 - |B|^2 \right)^{1/4} C_1^{-1} \begin{pmatrix} 0 & p(v) \\ p(v) & \mathbf{0} \end{pmatrix} C_1,$$

with $v = \Big(\lambda_{|S}\Gamma_{|S}\tilde{\eta}, m\, k_{|S}\Big)$. By (41), we obtain

$$\begin{aligned} v_3 &= \Big(|E|^2 - |B|^2\Big)^{-1/4} m, \\ \theta v_2 &= -\Big(|E|^2 - |B|^2\Big)^{-1/4} \eta \cdot \frac{E \times B}{|E|^3}, \\ \Big(\sqrt{1-|\theta|^2}v_1, v_2\Big) &= \Big(|E|^2 - |B|^2\Big)^{-1/4} \Big((\frac{E}{|E|^2} \times \eta) \cdot e_2, (\frac{E}{|E|^2} \times \eta) \cdot e_3, (\frac{E}{|E|^2} \times \eta) \cdot e_1 \Big). \end{aligned}$$

In view of (28), we get the expression of $\mathcal{R}$ stated in Theorem 1.$\diamondsuit$

In the following sections, we drop the ˜ on $\tilde{\eta}$ and we focus on proving Theorem 2, for $\alpha \in (0, 1/2)$ or ($\alpha \geq 1/2$ and $|\eta| = \infty$) first, then for ($\alpha \geq 1/2$ and $\eta < +\infty$).

Before closing this section, let us state a last result concerning system (15). The dependance on the variable σ of the function γ_1 and γ_2 does not prevent from dealing with system (15) as a system of evolution equations. Actually, arguing as in [12], Proposition 5, we can prove the following hyperbolic estimate.

Proposition 3 *Consider* (v^ε) *a bounded family in* $L^2(\mathbf{R}^4_{s,z}, \mathbf{C}^4)$ *satisfying (15), consider* $\rho \in \mathcal{C}_0^\infty(\mathbf{R}^3)$ *such that* $\rho(0) = 1$*, then there exists* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that the family* $\left(\rho\left(\frac{(s, \varepsilon D_{z_1}, \varepsilon D_{z_2})}{\delta}\right) v^\varepsilon\right)_{\varepsilon > \varepsilon_0}$ *is bounded in* $L^\infty\left(\mathbf{R}_s, L^2(\mathbf{R}^3_z)\right)$.

Therefore, arguing as in [12] and [10], we can estimate $(\mathrm{op}_\varepsilon\,(a)v^\varepsilon \mid v^\varepsilon)$ for $a \in \mathcal{C}_0^\infty(K)$ where K is some compact subset of $\{|\tilde{\zeta}|^2 + s^2 \leq \delta\}$, for δ small enough; by Proposition 3 and Schurr lemma, we have

$$|\,(\mathrm{op}_\varepsilon\,(a)v^\varepsilon \mid v^\varepsilon)\,| \leq C \int_{-\infty}^{+\infty} \sup_{k+|\beta| \leq N} \; \sup_{(z,\sigma,\zeta) \in \mathbf{R}^7} |\, \partial_\sigma^k \, \partial_z^\beta a(s, z, \sigma, \zeta) \,|\; ds, \tag{42}$$

uniformly with respect to K.

4 The adiabatic cases: $(|\eta| = +\infty,\ \alpha \geq 1/2)$ or $\alpha \in (0, 1/2)$

In this section, we prove equations (40) for $\alpha \in (0, 1/2)$ and for ($\alpha \geq 1/2$, $|\,\eta\,| = +\infty$), i.e.

$$\tilde{\nu}^{+,out} = \tilde{\mathcal{R}}\, \tilde{\nu}^{+,in}\, \tilde{\mathcal{R}}, \ \ \tilde{\nu}^{-,out} = \tilde{\mathcal{R}}\, \tilde{\nu}^{-,in}\, \tilde{\mathcal{R}}.$$

We follow the method initiated in [10]– [12] and developed in [9] for eigenvalues of multiplicity higher than 1.
We shall focus on the plus mode, the proof for the minus mode is similar. Let us introduce first some notations. We set

$$\begin{aligned} \lambda^\varepsilon &= \sqrt{s^2 + m_\varepsilon^2 + |\Gamma\tilde{\zeta}|^2} = \sqrt{s^2 + m^2 k^2 \varepsilon^{2\alpha} + |\Gamma\tilde{\zeta}|^2}, \\ A^\varepsilon &= \begin{pmatrix} s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ p(\Gamma\tilde{\zeta}, m_\varepsilon) & -s \end{pmatrix}, \end{aligned}$$

and we denote by (f^ε) the family such that, locally near 0,

$$\frac{\varepsilon}{i}\partial_s v^\varepsilon = \mathrm{op}_\varepsilon\,(A_\varepsilon)v^\varepsilon + \varepsilon f^\varepsilon,$$

with $(\mathrm{op}_\varepsilon\,(\phi)f^\varepsilon)$ bounded in $L^2_{s,z}$ for any ϕ compactly supported in a neighborhood of 0 small enough. Then, for $\omega \in \mathbf{S}^1$, we denote by $X_\omega = X_\omega(s,\sigma,z,\zeta)$ the norm 1 eigenvector of A for the eigenvalue λ defined for $s \neq 0$ or $\tilde{\zeta} \neq 0$ by

$$X^\varepsilon_\omega = \frac{1}{\sqrt{2\lambda^\varepsilon(\lambda^\varepsilon - s)}}\left(p(\Gamma\tilde{\zeta}, m_\varepsilon)\omega, (\lambda^\varepsilon - s)\omega\right) = \frac{1}{\sqrt{2\lambda^\varepsilon(\lambda^\varepsilon - s)}}(\lambda^\varepsilon + A^\varepsilon)\begin{pmatrix} 0 \\ \omega \end{pmatrix}.$$

For ω, $\omega' \in \mathbf{S}^1$, we consider the matrix

$$\Pi^\varepsilon_{\omega,\omega'} = \Pi^\varepsilon_{\omega,\omega'}(s,z,\sigma,\zeta) := X^\varepsilon_\omega \otimes \overline{X}^\varepsilon_{\omega'}.$$

The function $\Pi^\varepsilon_{\omega,\omega'}$ depends on ε, thus, for $a \in \mathcal{A}$, the function

$$q : (s,z,\sigma,\zeta,\eta) \mapsto a(s,z,\sigma,\zeta,\eta)\Pi^\varepsilon_{\omega,\omega'}(s,z,\sigma,\zeta)$$

is not in $\mathcal{A}$. However, this function is smooth and one can consider the family of operators $\mathrm{op}_\varepsilon\,(q(s,z,\sigma,\zeta,\frac{\zeta}{\sqrt{\varepsilon}}))$ that we will denote by $\mathrm{op}_\varepsilon{}^I\,(a\Pi^\varepsilon_{\omega,\omega'})$. Consider the vector-valued function $X^\infty_\omega = X^\infty_\omega(s,z,\sigma,\zeta,\eta)$ defined by
for $\alpha > 1/2$,

$$X^\infty_\omega = \left(p\left(\frac{\Gamma\eta}{|\Gamma\eta|}, 0\right)\omega, 0\right) \text{ if } s > 0, \quad X^\infty_\omega = (0,\omega) \text{ if } s < 0,$$

for $\alpha = 1/2$,

$$X^\infty_\omega = \left(p\left(\frac{(\Gamma\eta, m\,k)}{\sqrt{|\Gamma\eta|^2 + m^2k^2}}\right)\omega, 0\right) \text{ if } s > 0, \quad X^\infty_\omega = (0,\omega) \text{ if } s < 0,$$

for $\alpha \in (0, 1/2)$,

$$X^\infty_\omega = (p(0,0,1)\omega, 0) \text{ if } s > 0, \quad X^\infty_\omega = (0,\omega) \text{ if } s < 0,$$

and the matrix-valued function $\Pi^\infty_{\omega,\omega'} = \Pi^\infty_{\omega,\omega'}(s,z,\sigma,\zeta,\eta)$ defined by

$$\Pi^\infty_{\omega,\omega'} := X^\infty_\omega \otimes \overline{X}^\infty_{\omega'},$$

Lemma 5 . *Consider scalar symbol $a \in \mathcal{A}$ compactly supported outside $s = 0$ and in the ball $\{s^2 + |\tilde{\zeta}|^2 < \delta^2\}$, for $\delta > 0$ as in Proposition 3. Then, we have,*

$$\left(\mathrm{op}_\varepsilon{}^I\,(a\,\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon \,|\, v^\varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{tr} < a\,\Pi^\infty_{\omega,\omega'}\,,\ \tilde{\nu}^+ > .$$

Proof: Let us suppose that $s > 0$ on $\mathrm{Supp}(a)$, the proof is similar in the other cases. We have

$$X_\omega^\varepsilon = \left(\frac{\sqrt{\lambda_\varepsilon + s}}{\sqrt{2\lambda_\varepsilon}} p \left(\frac{(\Gamma\tilde{\zeta}, m_\varepsilon)}{\sqrt{|\Gamma\tilde{\zeta}|^2 + m_\varepsilon^2}} \right) \omega, \, \frac{\sqrt{\lambda_\varepsilon - s}}{\sqrt{2\lambda_\varepsilon}}\, \omega \right).$$

The functions $(s, z, \sigma, \zeta) \mapsto \frac{\sqrt{\lambda_\varepsilon + s}}{\sqrt{2\lambda_\varepsilon}}$ and $(s, z, \sigma, \zeta) \mapsto \frac{\sqrt{\lambda_\varepsilon - s}}{\sqrt{2\lambda_\varepsilon}}$ are smooth functions on $\mathrm{Supp}(a)$. They go respectively to 1 and 0 as ε goes to 0. Therefore, we get

$$\left(\mathrm{op}_\varepsilon{}^I \, (a \Pi_{\omega,\omega'}^\varepsilon) v^\varepsilon \,|\, v^\varepsilon \right) \quad = \quad \left(\mathrm{op}_\varepsilon{}^I \, (a \Pi_{\omega,\omega'}^\infty) v^\varepsilon \,,\; v^\varepsilon \right) + o(1),$$

where we used (42). Hence the result.$\diamondsuit$

Since $\tilde{\nu}^+$ commutes with the projector on the eigensubspace associated with λ^ε, measure $\tilde{\nu}^+$ is utterly determined by the knowledge of $\mathrm{tr}\left(\Pi_{\omega,\omega'}^\infty \tilde{\nu}^+ \right)$ for all $\omega, \omega' \in \mathbf{S}^1$. Consider now $\rho \in \mathcal{C}_0^\infty(\mathbf{R})$, $\rho = 1$ near 0. Set

$$\Pi_{\omega,\omega'}^{in} = \lim_{s \to 0^+} \Pi_{\omega,\omega'}^\infty, \;\; \Pi_{\omega,\omega'}^{out} = \lim_{s \to 0^-} \Pi_{\omega,\omega'}^\infty.$$

We have

$$\tilde{\mathcal{R}} \, \Pi_{\omega,\omega'}^{in} \, \tilde{\mathcal{R}} = \Pi_{\omega,\omega'}^{out}.$$

Therefore, equation (40) for $|\eta| = \infty$ and $\alpha \geq 1/2$ is equivalent to the fact that for all symbol $a_0 \in \mathcal{C}_0^\infty(\mathbf{R}_{z,\sigma,\zeta}^{2d+1} \times \mathbf{S}^1)$ and for all $\omega, \omega' \in \mathbf{S}^1$,

$$\begin{aligned} & \lim_{\varepsilon \to 0} \; \mathrm{tr} \Big(\;\; < \;\; \tilde{\nu}^+ \, \mathbf{1}_{|\eta|=\infty} \,,\; a_0(z, \sigma, \zeta, \frac{\eta}{|\eta|}) \, \Pi_{\omega,\omega'}^{out} \, \rho(\frac{s}{\varepsilon}) \;\; > \Big) \\ & \qquad = \;\; \lim_{\varepsilon \to 0} \mathrm{tr} \Big(< \tilde{\nu}^+ \, \mathbf{1}_{|\eta|=\infty} \,,\; a_0(z, \sigma, \zeta, \frac{\eta}{|\eta|}) \, \Pi_{\omega,\omega'}^{in} \, \rho(\frac{s}{\varepsilon}) \;\; > \Big) \, . \end{aligned} \tag{43}$$

Similarly, if $\alpha \in (0, 1/2)$, (40) is equivalent to

$$\begin{aligned} & \lim_{\varepsilon \to 0} \; \mathrm{tr} \Big(\;\; < \;\; \tilde{\nu}^+ \;\,,\; a_0(z, \sigma, \zeta, \frac{\eta}{|\eta|}) \, \Pi_{\omega,\omega'}^{out} \, \rho(\frac{s}{\varepsilon}) \;\; > \Big) \\ & \qquad = \;\; \lim_{\varepsilon \to 0} \mathrm{tr} \Big(< \tilde{\nu}^+ \;\,,\; a_0(z, \sigma, \zeta, \frac{\eta}{|\eta|}) \, \Pi_{\omega,\omega'}^{in} \, \rho(\frac{s}{\varepsilon}) \;\; > \Big) \, , \end{aligned} \tag{44}$$

for all a_0, ω and ω' as above.

We shall use the two following facts:

- In the non adiabatic case: $\alpha \geq 1/2$, we consider a_0, δ and δ' as before and for $R > 0$, we define a as

$$a(s, z, \sigma, \zeta, \eta) = \rho\left(\frac{s}{\delta'} \right) a_0(z, \sigma, \zeta, \frac{\eta}{|\eta|}) \rho\left(\frac{|\tilde{\zeta}|^2}{\delta} \right) \left(1 - \rho\left(\frac{|\eta|}{R} \right) \right).$$

The function a is again in $\mathcal{A}$ and for $q^\varepsilon = a\Pi^\varepsilon_{\omega,\omega'}$, the operator $\mathrm{op}_\varepsilon\,(q^\varepsilon)$ satisfies

$$\left|\partial^\beta_{\sigma,z,\zeta_3}\partial^\gamma_{s,\tilde{\zeta}}\, q^\varepsilon\right| \leq C_{\beta,\gamma}(\delta',\delta)\left(\frac{1}{R\sqrt{\varepsilon}}\right)^{|\gamma|},$$

where we implicitly assumed $R\sqrt{\varepsilon} \leq 1$.
If we consider the Weyl-Hörmandes metric

$$g_\varepsilon = \mathrm{d}z^2 + \frac{\mathrm{d}s^2}{R^2\varepsilon} + \varepsilon^2(\mathrm{d}\sigma^2 + \mathrm{d}\zeta_3^2) + \varepsilon\frac{\mathrm{d}\tilde{\zeta}^2}{R^2},$$

we have $\frac{g_\varepsilon}{g_\varepsilon^\sigma} \leq \left(\frac{\sqrt{\varepsilon}}{R}\right)^2$; thus the gain of this symbolic calculus is $\frac{\sqrt{\varepsilon}}{R}$.
Equation (43) is equivalent to

$$\lim_{\delta'\to 0}\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\mathrm{op}_\varepsilon{}^I\,\left(\partial_s a\Pi_{\omega,\omega'}\right) v^\varepsilon \mid v^\varepsilon\right) = 0,$$

which can be proved as in [9], Section 3. We point out that this proof can be adapted to our case since we have the analogue of Lemma 4 in [9]: there exists some matrix-valued function ϕ, homogeneous of degree 0 in s, ε^α and $\tilde{\zeta}$, such that if (ω,ω') is some orthonormal basis of $\mathbf{R}^2$,

$$\partial_s X^\varepsilon_\omega = \frac{1}{2\lambda^\varepsilon}\left(\partial_s A^\varepsilon - \partial_s\lambda^\varepsilon\right) X^\varepsilon_\omega + (\phi\,\omega\cdot\omega')X^\varepsilon_{\omega'}.$$

- In the adiabatic case: $\alpha \in (0,1/2)$, for a_0 as before, $\delta < \delta_0$, (where δ_0 is defined in Proposition 3), $\delta' > 0$, we define a as

$$a(s,z,\sigma,\zeta,\eta) = \rho\left(\frac{s}{\delta'}\right) a_0(z,\sigma,\zeta,\frac{\eta}{|\eta|})\rho\left(\frac{|\tilde{\zeta}|^2}{\delta}\right).$$

We get that if $q_\varepsilon = a\Pi^\varepsilon_{\omega,\omega'}$,

$$\left|\partial^\beta_{\sigma,z,\zeta_3}\partial^\gamma_{s,\tilde{\zeta}}\, q^\varepsilon\right| \leq C_{\beta,\gamma}(\delta',\delta)\left(\frac{1}{\varepsilon^\alpha}\right)^{|\gamma|}.$$

This estimate allows us to use Weyl-Hörmandes symbolic calculus for which we refer to Sections 18.4, 18.5 and 18.6 in [22]. The symbol q^ε belongs to the class $S(1, g_\varepsilon)$ where g_ε is the metric

$$g_\varepsilon = \mathrm{d}z^2 + \frac{\mathrm{d}s^2}{\varepsilon^{2\alpha}} + \varepsilon^2(\mathrm{d}\sigma^2 + \mathrm{d}\zeta_3^2) + \varepsilon^{2(1-\alpha)}\mathrm{d}\tilde{\zeta}^2.$$

Since $\frac{g_\varepsilon}{g_\varepsilon^\sigma} \leq \varepsilon^{2(1-2\alpha)}$, the gain of this symbolic calculus is $\varepsilon^{1-2\alpha}$ which goes to 0 as ε goes to 0. Thus, we can argue again as in [9], and we obtain that

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\mathrm{op}_\varepsilon{}^I\,\left(\partial_s a\Pi_{\omega,\omega'}\right) v^\varepsilon \mid v^\varepsilon\right) = 0,$$

which is equivalent to (44).

5 The non-adiabatic case: $\alpha \geq 1/2$ and $|\eta| < +\infty$

In this section, we aim at proving Landau-Zener formula (40) in $\{|\eta| < +\infty\}$ for $\alpha \geq 1/2$. We proceed in two steps. We begin by stating a normal form which holds in any ball $B \subset \mathbf{R}^2_\eta$. Then, we are reduced to deal with some abstract scattering problem which can be solved explicitly. This allows to obtain Landau-Zener formula for measure $\tilde{\nu}$, thus for ν.

5.1 A Normal Form at Finite Distance

Proposition 4 . *For any ball $B \subset \mathbf{R}^2_\eta$, there exists a matrix $C \in \mathcal{C}_0^\infty(\mathbf{R}^{10}_{s,z,\sigma,\zeta,\eta})$ such that if $u^\varepsilon = \left(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(C)\right) v^\varepsilon$, then, in $L^2(\mathbf{R}^4)$,*

$$\forall a \in \mathcal{C}_0^\infty(\mathbf{R}^8 \times B),\ \ \mathrm{op}_\varepsilon{}^I\,(a)\,\mathrm{op}_\varepsilon \begin{pmatrix} -\sigma + s & p(\Gamma_{|S}\tilde{\zeta}, m_{\varepsilon|S}) \\ p(\Gamma_{|S}\tilde{\zeta}, m_{\varepsilon|S}) & -\sigma - s \end{pmatrix} u^\varepsilon = O(\varepsilon).$$

Remarque 1 . *Notice that (u^ε) and (v^ε) have the same two-scale Wigner measure for I.*

Proof: We set $J = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \in \mathbf{R}^{4,4}$, $\tilde{H}_2 = \begin{pmatrix} -\sigma + s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ p(\Gamma\tilde{\zeta}, m_\varepsilon) & -\sigma - s \end{pmatrix}$,

$$\Gamma^0 = \Gamma(0, z, \sigma, \zeta),\ \ \Gamma^{00} = \Gamma(0, z, 0, \zeta),$$
$$m_\varepsilon^0 = \varepsilon^\alpha m\, k(0, z, \sigma, \zeta),\ \ m_\varepsilon^{00} = \varepsilon^\alpha m\, k(0, z, 0, \zeta),$$
$$\tilde{H}_2^0 = \begin{pmatrix} -\sigma + s & p(\Gamma^0\tilde{\zeta}, m_\varepsilon^0) \\ p(\Gamma^0\tilde{\zeta}, m_\varepsilon^0) & -\sigma - s \end{pmatrix},\ \ \tilde{H}_2^{00} = \begin{pmatrix} -\sigma + s & p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00}) \\ p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00}) & -\sigma - s \end{pmatrix}.$$

We prove the following lemma.

Lemma 6 . *There exist four smooth matrices $C_j = C_j(s, z, \sigma, \zeta, \eta)$, $\tilde{C}_j = \tilde{C}_j(s, z, \sigma, \zeta, \eta)$, $j \in \{1, 2\}$, C_j, $\tilde{C}_j \in \mathcal{C}_0^\infty(\mathbf{R}^{10})$ such that for all $a \in \mathcal{C}_0^\infty(\mathbf{R}^{10} \times B)$,*

$$\Big|\Big|\mathrm{op}_\varepsilon{}^I\,(a)\ \Big[\ \left(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(J(C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1)J)\right)\mathrm{op}_\varepsilon\,(\tilde{H}_2) - \mathrm{op}_\varepsilon\,(\tilde{H}_2^0)\left(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1)\right)\Big]\Big|\Big|_{\mathcal{L}(L^2)} = O(\varepsilon), \tag{45}$$

$$\Big|\Big|\,\mathrm{op}_\varepsilon{}^I\,(a)\ \Big[\ \left(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(C_2 + \varepsilon^{\alpha-1/2}\tilde{C}_2)\right)\mathrm{op}_\varepsilon\,(\tilde{H}_2^0) - \mathrm{op}_\varepsilon\,(\tilde{H}_2^{00})\left(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(C_2 + \varepsilon^{\alpha-1/2}\tilde{C}_2)\right)\Big]\Big|\Big|_{\mathcal{L}(L^2)} = O(\varepsilon). \tag{46}$$

This lemma yields Proposition 4. Actually, if a is compactly supported in all the variables $s, z, \sigma, \zeta, \eta$, then, we have in $\mathcal{L}(L^2)$,

$$\mathrm{op}_\varepsilon{}^I\,(a)\,\mathrm{op}_\varepsilon\,(\zeta_j b) = \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I\,(a\eta_j b) + O(\sqrt{\varepsilon}).$$

Hence

$$||\,\mathrm{op}_\varepsilon{}^I\,(a)\,\mathrm{op}_\varepsilon\,(\zeta_j b)||_{\mathcal{L}(L^2)} = O(\sqrt{\varepsilon}), \tag{47}$$

for $j \in \{0,1,2,3\}$ and $b \in \mathcal{C}_0^\infty(\mathbf{R}^{10})$. Therefore, writing

$$\Gamma\tilde{\zeta} = \Gamma_{|S}\tilde{\zeta} + O(|\zeta|^2), \;\; m_\varepsilon^{00} = m_{\varepsilon|S} + \varepsilon^\alpha\, O(|\tilde{\zeta}|),$$

and using that $\alpha \geq 1/2$, we obtain that, for any a compactly supported in all the variables,

$$\left\| \mathrm{op}_\varepsilon{}^I\,(a)\;\mathrm{op}_\varepsilon\,\left(p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00})\right) - \mathrm{op}_\varepsilon{}^I\,(a)\;\mathrm{op}_\varepsilon\,\left(p(\Gamma_{|S}\tilde{\zeta}, m_{\varepsilon|S})\right)\right\|_{\mathcal{L}(L^2)} = O(\varepsilon).$$

Therefore, equations (45) and (46) yield Proposition 4. $\diamondsuit$

Proof of Lemma 6: Set $C_1^\varepsilon = C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1$, equation (45) is equivalent to

$$\left\| \mathrm{op}_\varepsilon{}^I\,(a)\left[\left(1+\sqrt{\varepsilon}\;\mathrm{op}_\varepsilon{}^I\,(C_1^\varepsilon)\right)\mathrm{op}_\varepsilon\,(J\tilde{H}_2) - \mathrm{op}_\varepsilon\,(J\tilde{H}_2^0)\left(1+\sqrt{\varepsilon}\;\mathrm{op}_\varepsilon{}^I\,(C_1^\varepsilon)\right)\right]\right\|_{\mathcal{L}(L^2)} = O(\varepsilon).$$

Note that $J\tilde{H}_2 = \begin{pmatrix} -\sigma+s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ -p(\Gamma\tilde{\zeta}, m_\varepsilon) & \sigma+s \end{pmatrix}$. We use symbolic calculus to expand in power of ε the left hand side. Because of (47), we obtain that C_1^ε must satisfy,

$$\sigma[J, C_1^\varepsilon] = \begin{pmatrix} \mathbf{0} & p\left((\Gamma^0-\Gamma)\eta, \varepsilon^{\alpha-1/2}m(k^0-k)\right) \\ -p\left((\Gamma^0-\Gamma)\eta, \varepsilon^{\alpha-1/2}m(k^0-k)\right) & \mathbf{0} \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} C_1 &= \frac{1}{2\sigma}\begin{pmatrix} \mathbf{0} & p\left((\Gamma^0-\Gamma)\eta, 0\right) \\ p\left((\Gamma^0-\Gamma)\eta, 0\right) & \mathbf{0} \end{pmatrix}\chi(\eta), \\ \tilde{C}_1 &= \frac{1}{2\sigma}\begin{pmatrix} \mathbf{0} & p\left(0, m(k^0-k)\right) \\ p\left(0, m(k^0-k)\right) & \mathbf{0} \end{pmatrix}\chi(\eta), \end{aligned}$$

for some function $\chi \in \mathcal{C}_0^\infty(\mathbf{R}^4_\eta)$ identically equal to 1 on B.
A similar proof determines C_2 and $\tilde{C}_2$, whence Lemma 6.$\diamondsuit$

5.2 Landau-Zener Formula

Because of Proposition 4, once given some ball $B \subset \mathbf{R}^2_\eta$, we are reduced to the study of the traces on $s = 0^+$ and $s = 0^-$ of the two-scale Wigner measure of a family (u^ε) satisfying

$$\forall a \in \mathcal{C}_0^\infty(\mathbf{R}^8 \times B),\;\; \mathrm{op}_\varepsilon{}^I\,(a)\;\mathrm{op}_\varepsilon\begin{pmatrix} -\sigma+s & p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) \\ p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) & -\sigma-s \end{pmatrix} u^\varepsilon = O(\varepsilon) \;\text{ in }\; L^2(\mathbf{R}^4).$$

Moreover, by applying a cut-off function, we may suppose that Γ and k are compactly supported and turn $\Gamma_{|S}$, $k_{|S}$ into $\phi(\eta)\Gamma_{|S}$ and $\phi(\eta)k_{|S}$ with ϕ compactly supported and identically equal to 1 on B. This way, our system is micro-localized in the ball B; which is enough to calculate the two-scale Wigner measures in B. We are left with a system of the form

$$\frac{\varepsilon}{i}\partial_s u^\varepsilon = \mathrm{op}_\varepsilon{}^I \begin{pmatrix} s & \phi(\eta)p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) \\ \phi(\eta)p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) & -s \end{pmatrix} u^\varepsilon + \varepsilon f^\varepsilon,$$

where $(\mathrm{op}_\varepsilon{}^I\ (a)f^\varepsilon)$ is uniformly bounded in $L^2_{s,z}$ for symbols a compactly supported in $\mathbf{R}^8 \times B$. However, f^ε does not contribute to the description of the traces on $s = 0^+$, $s = 0^-$ of the two-scale Wigner measure of (u^ε). Actually, if $S_\varepsilon(s, s')$ denotes the evolution operator associated with the free system

$$\frac{\varepsilon}{i}\partial_s \underline{u}^\varepsilon = \mathrm{op}_\varepsilon{}^I \begin{pmatrix} s & \phi(\eta)p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) \\ \phi(\eta)p(\Gamma_{|S}\,\tilde{\zeta}, m_{\varepsilon|S}) & -s \end{pmatrix} \underline{u}^\varepsilon, \quad \underline{u}^\varepsilon_{|s=0} = u^\varepsilon_{|s=0},$$

then we have

$$u^\varepsilon = \underline{u}^\varepsilon + i\int_0^s S_\varepsilon(0,t)f^\varepsilon(t)\mathrm{d}t.$$

Hence, $u^\varepsilon = \underline{u}^\varepsilon + O(\sqrt{|s|})$ in $L^2(\mathbf{R}^3_z)$. Therefore, the traces of the two-scale Wigner measures of (u^ε) and $(\underline{u}^\varepsilon)$ on $s = 0^\pm$ are the same. Let us denote by G the compact operator

$$G = \mathrm{op}_\varepsilon{}^I \left(\phi(\eta)p(\Gamma_{|S}\,\eta, \varepsilon^{\alpha-1/2}m\,k_{|S})\right).$$

The family $(\underline{u}^\varepsilon)$ satisfies

$$\frac{\varepsilon}{i}\partial_s \underline{u}^\varepsilon = \begin{pmatrix} s & \sqrt{\varepsilon}\,G \\ \sqrt{\varepsilon}\,G^* & -s \end{pmatrix}.$$

As a consequence of Proposition 7 in [12], we have the following Lemma.

Lemma 7 *There exist* $\alpha_j^\varepsilon = \alpha_j(z)$, $\omega_j^\varepsilon = \omega_j^\varepsilon(z)$, $j \in \{1,2\}$, *such that, as* ε *goes to* 0, *for any* $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$, $\chi(GG^*)\alpha_1^h$, $\chi(G^*G)\alpha_2^h$, $\chi(GG^*)\omega_1^h$ *and* $\chi(G^*G)\omega_2^h$ *are bounded families in* $L^2(\mathbf{R}^3_z, \mathbf{R}^2)$ *and*
for $s < 0$,

$$\begin{aligned} \chi(GG^*)\begin{pmatrix} u_1^\varepsilon(s,z) \\ u_2^\varepsilon(s,z) \end{pmatrix} &= \chi(GG^*)\mathrm{e}^{i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}}\alpha_1^\varepsilon + o(1), \\ \chi(G^*G)\begin{pmatrix} u_3^\varepsilon(s,z) \\ u_4^\varepsilon(s,z) \end{pmatrix} &= \chi(G^*G)\mathrm{e}^{-i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}}\alpha_2^\varepsilon + o(1), \end{aligned}$$

for $s > 0$,

$$\begin{aligned} \chi(GG^*)\begin{pmatrix} u_1^\varepsilon(s,z) \\ u_2^\varepsilon(s,z) \end{pmatrix} &= \chi(GG^*)\mathrm{e}^{i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}}\omega_1^\varepsilon + o(1), \\ \chi(G^*G)\begin{pmatrix} u_3^\varepsilon(s,z) \\ u_4^\varepsilon(s,z) \end{pmatrix} &= \chi(G^*G)\mathrm{e}^{-i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}}\omega_2^\varepsilon + o(1), \end{aligned}$$

Moreover

$$\begin{pmatrix} \omega_1^\varepsilon \\ \omega_2^\varepsilon \end{pmatrix} = S^\varepsilon \begin{pmatrix} \alpha_1^\varepsilon \\ \alpha_2^\varepsilon \end{pmatrix} \tag{48}$$

with

$$S^\varepsilon = \begin{pmatrix} a(GG^*) & -\overline{b}(GG^*)G \\ b(G^*G)G^* & a(G^*G) \end{pmatrix},$$

with $a(\lambda) = \mathrm{e}^{-\pi\frac{\lambda}{2}}$, $b(\lambda) = \frac{2i\mathrm{e}^{i\frac{\pi}{4}}}{\lambda\sqrt{\pi}} 2^{-i\lambda/2} \mathrm{e}^{-\pi\frac{\lambda}{4}}\, \Gamma(1+i\frac{\lambda}{2})\, \mathrm{sh}(\frac{\pi\lambda}{2})$, $\;a(\lambda)^2 = 1 - \lambda|b(\lambda)|^2$.

These formula allow to calculate two-scale Wigner measures thanks to Lemma 8 in [12] which states that for $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$ and for $\alpha = 1/2$,

$$\mathrm{op}_\varepsilon{}^I \left(\chi\left(|\,\phi(\frac{\eta}{R})|^2 \left(|\Gamma_{|S}\eta|^2 + m^2|k_{|S}|^2\right)\right)\right) = \chi(GG^*) + o(1) = \chi(G^*G) + o(1), \tag{49}$$

for $\alpha > 1/2$,

$$\mathrm{op}_\varepsilon{}^I \left(\chi\left(|\,\phi(\frac{\eta}{R})\Gamma_{|S}\eta\,|^2\right)\right) = \chi(GG^*) + o(1) = \chi(G^*G) + o(1). \tag{50}$$

In view of

$$\tilde{\Pi}_0^+ = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ on } J^{+,out},\; \tilde{\Pi}_0^- = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ on } J^{-,out},$$
$$\tilde{\Pi}_0^+ = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ on } J^{+,in},\; \tilde{\Pi}_0^- = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ on } J^{-,in},$$

we obtain that $\tilde{\nu}^+\,\mathbf{1}_{s<0}$ (resp. $\tilde{\nu}^-\,\mathbf{1}_{s<0}$) is the two-scale Wigner measure of the family $\mathrm{e}^{i\frac{s^2}{2\varepsilon}} \left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}} (\alpha_1^\varepsilon, 0)$ (resp. $\mathrm{e}^{-i\frac{s^2}{2\varepsilon}} \left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}} (0, \alpha_2^\varepsilon)$). Similarly, $\tilde{\nu}^+\,\mathbf{1}_{s>0}$ (resp. $\tilde{\nu}^-\,\mathbf{1}_{s>0}$) is the two-scale Wigner measure of the family $\mathrm{e}^{-i\frac{s^2}{2\varepsilon}} \left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}} (0, \omega_2^\varepsilon)$ (resp. $\mathrm{e}^{i\frac{s^2}{2\varepsilon}} \left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}} (\omega_1^\varepsilon, 0)$). We use Lemma 9 in [12]: for every $\chi \in C_0^\infty(\mathbf{R}^{10})$), we have in $\mathcal{L}(L^2)$,

$$\left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{GG^*}{2}} \mathrm{op}_\varepsilon{}^I\,(\chi) \left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}} = \left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}} \mathrm{op}_\varepsilon{}^I\,(\chi) \left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{G^*G}{2}} + o(1) = \mathrm{op}_\varepsilon{}^I\,(\chi) + o(1).$$

Therefore, $\tilde{\nu}^{+,out}$ (resp. $\tilde{\nu}^{-,out}$) is the two-scale Wigner measure of the family $(\alpha_1^\varepsilon, 0)$ (resp. $(0, \alpha_2^\varepsilon)$). Similarly, $\tilde{\nu}^{+,in}$ (resp. $\tilde{\nu}^{-,in}$) is the two-scale Wigner measure of the family $(0, \omega_2^\varepsilon)$ (resp. $(\omega_1^\varepsilon, 0)$). Observe that equation (48) yields,

$$\begin{pmatrix} \omega_1^\varepsilon(z) \\ 0 \end{pmatrix} = a(GG^*) \begin{pmatrix} \alpha_1^\varepsilon(z) \\ 0 \end{pmatrix} - \overline{b}(GG^*) \begin{pmatrix} \mathbf{0} & G \\ G^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_2^\varepsilon(z) \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ \omega_2^\varepsilon(z) \end{pmatrix} = a(G^*G) \begin{pmatrix} 0 \\ \alpha_2^\varepsilon(z) \end{pmatrix} + b(G^*G) \begin{pmatrix} \mathbf{0} & G \\ G^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \alpha_1^\varepsilon(z) \\ 0 \end{pmatrix}.$$

These relations yield (40) via (49) and (50), which completes the proof of Theorem 1.◇

6 Appendix

6.1 Restitution of the energy by the crossing

We denote by Ω the set

$$\Omega = \{E \cdot B = 0 \ \text{ or } \ (E \cdot B = 0 \ \text{ and } \ |E| > |B|)\}.$$

Proposition 5 . *If (ψ^ε) is a family of solutions to (6) and μ a semi-classical measure of (ψ^ε), then we have,*

$$\mu\left(\{\xi = A(t,x)\} \cap \Omega\right) = 0.$$

Proof of Proposition 5: We recall that

$$\begin{aligned} P_\varepsilon(x,\xi) = Q_\varepsilon(x,\xi) + V(t,x) = Q_0 + \varepsilon^\alpha m\gamma^0 + V(t,x) \\ = \sum_{k=1}^{3} \gamma^0\gamma^k \left(\xi_k - A_k(t,x)\right) + V(t,x) + \varepsilon^\alpha m\gamma^0. \end{aligned}$$

Note that $(\tau + V - Q_\varepsilon)(\tau + V + Q_\varepsilon) = (\tau + V)^2 - \mid \xi - A \mid^2 - \varepsilon^{2\alpha} m^2$. Therefore, we have

$$\begin{aligned} T &= \operatorname{op}_\varepsilon\left(\tau + V - Q_\varepsilon\right) \operatorname{op}_\varepsilon\left(\tau + V + Q_\varepsilon\right) \\ &= \operatorname{op}_\varepsilon\left((\tau+V)^2 - \mid \xi - A \mid^2 - \varepsilon^{2\alpha} m^2\right) + \frac{\varepsilon}{2i} \operatorname{op}_\varepsilon\left(\{\tau + V - Q_0 \ , \ \tau + V + Q_0\}\right) + o(\varepsilon). \end{aligned} \tag{51}$$

in $\mathcal{L}(L^2)$. Let us denote by Γ the operator

$$\begin{aligned} \Gamma &= \frac{1}{2}\{\tau + V - Q_0 \ , \ \tau + V + Q_0\} \\ &= \sum_{k=1}^{3} \gamma_0\gamma_k E_k - \gamma_2\gamma_3 B_1 - \gamma_3\gamma_1 B_2 - \gamma_1\gamma_2 B_3. \end{aligned}$$

Since $i\partial_t \psi^\varepsilon = \operatorname{op}_\varepsilon(P_\varepsilon)\psi^\varepsilon$, for any matrix-valued symbol $a \in \mathcal{C}_0^\infty(\mathbf{R}^8_{t,x,\tau,\xi})$ we have

$$\frac{1}{\varepsilon}\left(\operatorname{op}_\varepsilon(a)\, T\psi^\varepsilon \ \mid \ \psi^\varepsilon\right) - \frac{1}{\varepsilon}\left(\operatorname{op}_\varepsilon(a)\psi^\varepsilon \ \mid \ T\psi_\varepsilon\right) = 0.$$

Using (51), we obtain

$$\begin{aligned} &\frac{1}{\varepsilon}[\operatorname{op}_\varepsilon(a) \ , \ \operatorname{op}_\varepsilon\left((\tau+V)^2 - \mid \xi - A \mid^2\right)] \\ &\qquad + \frac{1}{i}\left(\left(\operatorname{op}_\varepsilon(a) \operatorname{op}_\varepsilon(\Gamma) + \operatorname{op}_\varepsilon(\Gamma^*) \operatorname{op}_\varepsilon(a)\right)\psi^\varepsilon \ \mid \ \psi^\varepsilon\right) = o(1), \end{aligned}$$

Whence $\{\mu \ , \ (\tau+V)^2 - \mid \xi - A \mid^2\} = \mu\,\Gamma^* + \Gamma\,\mu$. Consider $\Phi \in \mathcal{C}_0^\infty(\mathbf{R}^3)$ such that $0 \leq \Phi \leq 1$ with $\Phi(0) = 1$. Using test functions of the form $\Phi\left(\frac{\xi - A(t,x)}{\delta}\right)$ for some positive δ, then letting δ which go to 0, we get

$$\Gamma \mathbf{1}_{\xi = A(t,x)}\, \mu + \mathbf{1}_{\xi = A(t,x)}\, \mu \Gamma^* = 0.$$

Consider $\chi=\chi(t,x,\tau,\xi)$ a scalar test function and $M=<\chi\ ,\ \mu\,\mathbf{1}_{\{\{\xi=A\}\cap\Omega\}}>$. Matrix M is a positive hermitian matrix satisfying $M\Gamma^*+\Gamma M=0$. Let us prove that $M=0$.
Observe that

$$\Gamma=\begin{pmatrix}0 & p(E)\\ p(E) & 0\end{pmatrix}+i\begin{pmatrix}p(B) & 0\\ 0 & p(B)\end{pmatrix}.$$

Therefore,

$$\Gamma^2=\begin{pmatrix}|E|^2-|B|^2 & 2iB\cdot E\\ 2iB\cdot E & |E|^2-|B|^2\end{pmatrix}.$$

Since we have $\Gamma M\Gamma^*=-\Gamma^2 M$, the matrix $-\Gamma^2M$ is hermitian and positive. Thus, in the case $E\cdot B=0$, it is obvious that the fact that $|E|>|B|$ yields $M=0$. We focus now on the case $E\cdot B\neq 0$. Matrix M can be decomposed in 2×2 blocks

$$M=\begin{pmatrix}A & C\\ C^* & D\end{pmatrix},$$

where A and D are 2×2 hermitian positive matrices. Moreover, if $-\Gamma^2M$ is hermitian positive, we obtain that we must have $(E\cdot B)(A+D)=0$. Therefore, necessarily, $A=D=0$, which yields $C=0$ and $M=0$. This comes from the fact that the non-diagonal coefficient $\mu_{i,j}$ of measure-valued matrix μ is a measure absolutely continuous with respect to the diagonal measure-valued coefficients $\mu_{i,i}$ and $\mu_{j,j}$.$\diamondsuit$

6.2 Proof of Lemma 1

Let us study the limits of the fibers of $\overline{N}_{\Sigma^\pm}(j^{\pm,in})$ and $\overline{N}_{\Sigma^\pm}(j^{\pm,out})$ above some point ρ which tends to S. Consider H and H' the limits of the two hamiltonian vector fields along J and J' above S, we have

$$\lim_{s\to 0^-}H_{\lambda^+}(\rho_s^+)=\lim_{s\to 0^+}H_{\lambda^-}(\rho_s^-)=H_{\tau+V}(\rho_0)-u\cdot H_{\xi-A}(\rho_0):=H,$$
$$\lim_{s\to 0^+}H_{\lambda^+}(\rho_s^+)=\lim_{s\to 0^-}H_{\lambda^-}(\rho_s^-)=H_{\tau+V}(\rho_0)-u'\cdot H_{\xi-A}(\rho_0):=H'.$$

Let $H^\perp$ and $(H')^\perp$ be the orthogonal of H and H' respectively for the symplectic form on $T(T^*\mathbf{R}^4)$. The measures $\nu_S^{+,in}$ and $\nu_S^{-,out}$ are measures on the compactification of $H^\perp/TJ$, and similarly, the measures $\nu_S^{-,in}$ and $\nu_S^{+,out}$ live on the compactification of $(H')^\perp/TJ'$. Note that $TJ=TS\oplus\mathbf{R}H$ and $TJ'=TS\oplus\mathbf{R}H'$, therefore if we set

$$F=TJ+TJ'=TS\oplus\mathbf{R}H\oplus\mathbf{R}H',$$

the planes $H^\perp/TJ$ and $(H')^\perp/TJ'$ can be identified to $T(T^*\mathbf{R}^4)/F$.
Consider $\delta\rho\in T(T^*\mathbf{R}^d)_{|\rho}$, $\rho\in S$. We decompose $\delta\rho$ as

$$\delta\rho=lH+l'H'+\delta s+\delta\tilde\rho,$$

with l, $l' \in \mathbf{R}$ and $\delta s \in TS_{|\rho}$. The class of $\delta\rho$ modulo F is characterized by the class of $\delta\tilde{\rho}$ modulo J, i.e. by $\mathrm{d}(\xi - A)\delta\tilde{\rho} - \mathrm{d}(\tau + V)\delta\tilde{\rho}\, u$ (because $\xi - A = (\tau + V)u$ is an equation of J). Observe that

$$\mathrm{d}(\xi - A)H - \mathrm{d}(\tau + V)Hu = 0, \;\; \mathrm{d}(\xi - A)\delta s - \mathrm{d}(\tau + V)\delta s\, u = 0,$$
$$\mathrm{d}(\xi - A)H' - \mathrm{d}(\tau + V)H'\, u = -2r\frac{E}{|E|^2}.$$

Therefore, the class of $\delta\rho$ modulo F is utterly determined by the knowledge of

$$\begin{aligned}\eta &= (\mathrm{d}(\xi - A)\delta\rho - \mathrm{d}(\tau + V)\delta\rho\, u) \wedge E \\ &= \mathrm{d}(\xi - A)\delta\rho \wedge E + \mathrm{d}(\tau + V)\delta\rho\, B,\end{aligned}$$

where we have used (11). This concludes the proof of Lemma 1.◇

Acknowledgements: This article has been motivated by discussions with N. Mauser, F. Poupaud and P. Gérard in one hand, A. Iftimovici in the other. May they find here the acknowledgements of the author.

References

[1] P. Bechouche, N. Mauser, F. Poupaud: (Semi)-non relativistic limits of the Dirac equation with external time dependent electromagnetic field. *Comm. Math. Phys.* **197**, N^o 2, (1998), p. 405-425.

[2] A. P. Calderón, R. Vaillancourt: On the boundedness of pseudo-differential operators. *J. Math. Soc. Japan* **23**, N^o 2, (1971), p. 374-378.

[3] A. Cordoba, C. Fefferman: Wave packets and Fourier integral operators. *Comm.Part.Diff.Eq.*, **3**, (1978), p. 979-1005.

[4] Y. Colin de Verdière: Sur les singularités de Van Hove génériques, *Bull. Soc. Math. de France*, **119**, Mémoire 46, (1991), p. 99-109.

[5] Y. Colin de Verdière: The level crossing problem in semi-classical analysis I. the symmetric case. *To appear in the Proceedings of Frédéric Pham's congress.*

[6] Y. Colin de Verdière, M. Lombardi, J. Pollet: The microlocal Landau-Zener formula. *Ann. Inst. Henri Poincaré*, **71**, N^{o}1, (1999), p.95-127.

[7] C. Fermanian Kammerer: Mesures semi–classiques deux–microlocales, *C. R. Acad. Sci. Paris*, 331, Série 1, (2000), p.515-518.

[8] C. Fermanian Kammerer: A non commutative Landau-Zener formula. *Prépublication de l'Université Cergy-Pontoise*, (2002).

[9] C. Fermanian Kammerer: Wigner measures and molecular propagation through energy level crossings, prépublication de l'université de Cergy-Pontoise, (2002).

[10] C. Fermanian Kammerer, P. Gérard: Mesures semi-classiques et croisements de modes. *Bull. Soc. math. France*, **130**, N^{o}1, (2002), p.123–168.

[11] C. Fermanian Kammerer, P. Gérard: Une formule de Landau-Zener pour un croisement générique de codimension 2, Séminaire Equations aux Dérivées Partielles 2001–2002, Exposé N^{o}21, Ecole Polytechnique. `http://math.polytechnique.fr/seminaires/seminaires-edp/`

[12] C. Fermanian Kammerer, P. Gérard: A Landau-Zener formula for non-degenerated involutive codimension 3 crossings. *Prépublication de l'Université de Cergy-Pontoise*, Sept. 2002.

[13] P. Gérard: Microlocal defect measures, *Comm.Part.Diff.Eq.*, **16**, (1991), p.1761-1794.

[14] P. Gérard: Mesures semi-classiques et ondes de Bloch, Exposé de l'Ecole Polytechnique, E.D.P., Exposé N^{o}XVI, (1991).

[15] P. Gérard, E. Leichtnam: Ergodic Properties of Eigenfunctions for the Dirichlet Problem. *Duke Math. J.*, **71**, (1993), p.559-607.

[16] P. Gérard, P. A. Markowich, N. J. Mauser, F. Poupaud: Homogenization Limits and Wigner Transforms. *Comm. Pure Appl. Math.*, **50**, (1997), 4, p.323-379.

[17] P. Gérard, P. A. Markowich, N. J. Mauser, F. Poupaud: Erratum : Homogenization limits and Wigner transform. *Comm. Pure Appl. Math.*, **53**, (2000), p.280-281.

[18] G. A. Hagedorn: Proof of the Landau-Zener formula in an adiabatic limit with small eigenvalue gaps. *Commun. Math. Phys.* **136**, (1991), p. 433-449.

[19] G. A. Hagedorn: Molecular Propagation through Electron Energy Level Crossings. *Memoirs of the A. M. S.*, **111**, N^o 536, (1994).

[20] G. A. Hagedorn, A. Joye: Landau-Zener transitions through small electronic eigenvalue gaps in the Born-Oppenheimer approximation. *Ann. Inst. Henri Poincaré*, **68**, N^{o}1, (1998), p.85-134.

[21] G. A. Hagedorn, A. Joye: Molecular propagation through small avoided crossings of electron energy levels. *Rev. Math. Phys.*, **11**, N^o 1, (1999), p. 41-101.

[22] L. Hörmander: The analysis of linear Partial Differential Operators III. *Springer-Verlag*, (1985).

[23] A. Joye: Proof of the Landau-Zener formula. *Asymptotic Analysis* **9**, (1994), p.209-258.

[24] L. Landau: *Collected papers of L. Landau*, Pergamon Press, (1965).

[25] P-L. Lions, T. Paul: Sur les mesures de Wigner. *Revista Matemática Iberoamericana*, **9**, (1993), p.553-618.

[26] L. Miller: Propagation d'onde semi-classiques à travers une interface et mesures 2-microlocales. *Thèse de l'Ecole Polytechnique*, (1996).

[27] L. Miller: Refraction of high–frequency waves density by sharp interfaces and semiclassical measures at the boundary, *Journal de Mathématiques Pures et Appliquées*, **79**, (2000), p.227-269.

[28] D. Robert: Autour de l'approximation semi–classique, *Birkhaüser*, (1983).

[29] H. Spohn: Semiclassical limit of the Dirad equation and spin precession.

[30] S. Teufel: Adiabatic perturbation theory in quantum dynamics, *Habilitationsschrift Zentrum Mathematik Technische Universität München*, (2002).

[31] B. Thaller: The Dirac equation, *Springer-Verlag*, (1992)

[32] C. Zener: Non-adiabatic crossing of energy levels, *Proc. Roy. Soc. Lond.* **137**, (1932), p.696-702.

Adress: C. Fermanian Kammerer: Université de Cergy-Pontoise, Mathématiques, 2 avenue Adolphe Chauvin, BP 222, Pontoise, 95 302 Cergy-Pontoise cedex, France.
`e-mail: Clotilde.Fermanian@math.u-cergy.fr`