On complex resonance vibration systems calculation

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This paper is one of a large cycle of papers on obtaining the exact analytical solutions for vibrant elastic systems. This was published first in the electronic journal of laboratory SELF "SELF Transactions", vol.2, issue 1, pp.48-59. Visiting this journal for free at http://angelfire.lycos.com/la3/selftrans you will find many other important results both on this subject and on many applications of this new method, as well as on many other subjects of mathematical physics and related themes. You will also find there that we check our results experimentally and yield the full coincidence of the calculation and experimental curves. And the main, this method gives exact solutions, and this is non-matrix. Already now we have solved many practically important problems which cannot be even approached properly by conventional methods. This is the future of dynamics out of which any development in any area of basic or applied wave physics is impossible.

You are welcomed to possess this wealth and contact us for discussions and joint further development.

Abstract

Basing on exact analytical solutions obtained for semi-finite elastic lines with resonance subsystems having the form of linear elastic lines with rigidly connected end elements, we will analyse the vibration pattern in systems having such structure. We will find that between the first boundary frequency for the system as a whole and that for the subsystem, the resonance peaks arise, and their number is equal to the integer part of $[(n-1)/2]$ *, where n is the number of subsystem elements. These resonance peaks arise at the bound between the aperiodical and complex aperiodical vibration regimes. This last regime is inherent namely in elastic systems having resonance subsystems and impossible in simple elastic lines. We will explain the reasons of resonance peaks bifurcation. We will show that the phenomenon of negative measure of subsystems inertia arising in such type of lines agrees with the conservation laws. So we will corroborate and substantiate Professor Skudrzyk's concept.*

We will obtain a good qualitative agreement of our theoretical results with Professor Skudrzyk's experimental results.

Keywords: Mathematical physics, Wave physics, Many-body theory, Complex resonance systems, ODE, nonlinear dynamics, resonance subsystems.

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1. Introduction

“The classical vibration theory is based on solving the differential equations and on joining the solutions for different parts of a system regarding continuity conditions. Any negligible change of the system form makes necessary to calculate it all anew. But out of any relation to the calculation difficulty, one should note that the high accuracy of classical theory is illusory. Materials never are absolutely homogeneous or isotropic, and natural frequencies and vibration distributions usually perceptibly differ from those what the theory gives, especially at high frequencies" [1, p.317].

At the same time, "multifrequent resonance systems are interesting by their applications to analytical and celestial mechanics, to Hamiltonian dynamics, theoretical and mathematical physics" [2, p.173]. Some of these problems are the problem of discrete-continual elastic system [3], of long molecular chains vibrations [1], of molecules vibration level [4], of lattice oscillations [5], [6], [7], molecular acoustics [8], quantum systems statistical mechanics [9], control problems [10] and so on.

Among the multitude of approaches to the solution of these problems, one can mark out "such well-known methods of vibration theory as perturbation theory methods, averaging method, analytical methods of slow and fast motions separation etc." [10, p.25]. Each of them has an ample literary basis. Particularly, the investigation by Tong Kin [11] is devoted to pure matrix methods; by Kukhta and others [3] – to finding the recursive relationships; by Atkinson [12] – to differential methods; by Palis and de Melo [13] – to geometrical methods, by Reiscig and others [14] – to qualitative theory, etc. Mitropolsky and Homa [15] and Cherepennikov [16] gave good surveys of solutions obtained with asymptotic methods. Methods based on perturbation theory are well stated by Giacagrilia [17] and Dymentberg [18]. Approaches based on an elastic model presented by mechanical resonance circuits were described quite completely by Skudrzyk [1].

Despite a broad spectrum of approaches, all these methods are qualitative, approximate or numerical. "The presence of singular boundaries in most of practical problems does not offer us to construct the analytical solution of differential equations, and numerical methods became the only possible means to yield quite exact and detailed results" [19, p.12]. "Even for a simplest case of hydrogen molecule H_2 , the exact quantum-mechanical calculation of constant quasi-elastic force is a laborious mathematical problem, and for more complicated cases the force constants calculation is practically unrealisable by means of sequential quantum-mechanical techniques" [4, p.12]. "The other difficulty connected with the collective motions method is, it gives no possibility to determine the collective motion nature, proceeding from the form of Hamiltonian. We have to guess suitable collective variables and then to check, whether the Hamiltonian divides into collective and interior parts" [9, p.120].

Giacagrilia [17], Reiscig [14] and Cherepennikov [16] gave quite complete analysis of problems arising with the conventional approaches to the multiresonance models investigation. Particularly, "the old problem is still open. Up to now no available 'modern' methods make possible to calculate the real frequencies of a nonlinear system. This problem stays unsolved for applications, because in approximations by series, converging or only formal, only finite and, generally speaking, little number of terms can be calculated. We still cannot find a way to express the common term and the sum of these series" [17, p.305]. Furthermore, "to make the series converging, sometimes we have to presume that the differential equations parameters determining the degree of nonlinearity have quite small module. By this reason the indirect technique is often applicable only in the narrow edge domain of nonlinear mechanics. The other demerit of these techniques is, they enable to obtain quite accurate information about the separate solutions, but give no idea about the structure of solutions family as a whole" [14, p.12]. Giacagrilia confirms this last: "The other problem of a great interest is the better understanding of solution 'in the near, in the far and at resonance conditions'. When we have a real process of resonance lock-in, and which definition of the system resonance is preferable?" [17, p.309]. "Exact analytical methods are preferable in the analysis, however obtaining the analytical formulas of solution even for comparatively simple differential equations entails great difficulties sometimes" [16, p.10].

In the light of indicated demerits of conventional methods, Skudrzyk has presented the most exact qualitative pattern. According to his approach, "any homogeneous system, either monolithic or consisting of homogeneous parts and loading masses, can be rigorously presented in the form of canonical scheme, specifically, of infinite number of sequential (mechanical) circuits connected in parallel, one for each form of natural vibrations" [1, p.317]. However Skudrzyk's application of matrix methods to solve the systems of differential equations for the systems he modelled disabled him to describe the pattern of processes analytically, since, as is known, for complex elastic systems the matrix method offers only numerical solutions. The analysis in matrix writing of vibration is practically impossible in analytical form. This demerit inherent in the most of conventional methods did not offer Skudrzyk to develop the introduced concept for the case of multiresonance elastic subsystems, in which the aggregate of subsystem resonance frequencies is determined not by the ensemble of mechanical resonance circuits, but by the integral multiresonance mechanical subsystem that forms all the gamut of subsystem resonances.

Now having the exact analytical solutions presented in [20] – [23], we have a scope to get over a number of problems in the resonance circuits method and to determine exact analytical solutions for some elastic mechanical systems having multiresonance subsystems.

In this paper we will consider the simplest case – a semi-finite homogeneous 1D system with rigidly fixed end elements of resonance subsystems. Though this problem is quite particular, it is used quite often in the engineering practice. Specifically, the problems of vibrant elastically connected rigid blocks containing some substructure of elements elastically connected between themselves and with the block are reduced to this case. Furthermore, we will suppose that the described method may be extended to the finite and heterogeneous elastic lines with resonance subsystems. The only, we will complicate the subsystem structure, presenting it as an elastic finite line with *n* masses equivalent to *n* circuits. Again, we will suppose that this method is easily extended to the case of a number of aforesaid type subsystems connected in parallel. In this way we will reduce the model in its generality to that investigated by Skudrzyk, but with the higher level of resonance subsystem structure.

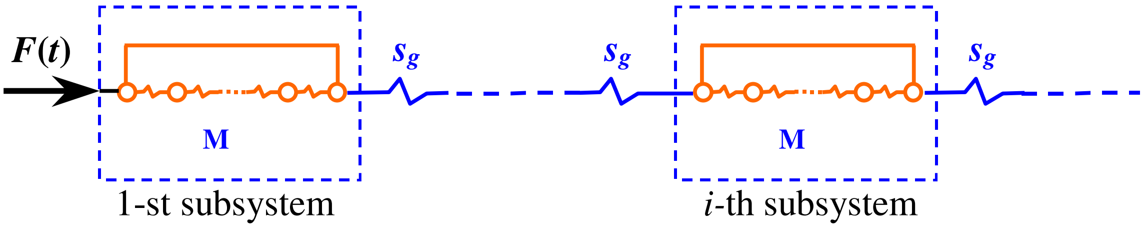


Fig. 1. 1D semi-finite homogeneous elastic system having resonance subsystems

2. The method to find the exact analytical solution

In Fig. 1 there is presented the studied semi-finite 1D elastic system with lumped masses *M* and elastic constraints s_g on whose start the harmonic force $F(t)$ affects. As it was said in the introduction, the first and the last elements of its subsystems are connected rigidly.

Before we begin studying the system as a whole, consider a separate subsystem under some harmonic force $F_i(t)$ acting on its start. This line consisting of *n* masses connected by constraints having stiffness s_s is shown in Fig. 2a. It is easy to see that this system can be presented by an

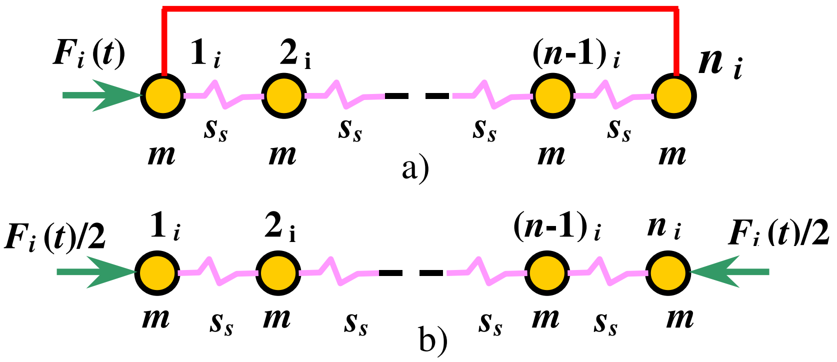


Fig. 2. The diagram of an elastic line resonance subsystem (a) and its calculation equivalent diagram (b)

equivalent scheme shown in Fig. 2b, where the rigid constraint is substituted by the second force acting on the end of elastic line. Note that such substitution is possible because the subsystem is perfectly symmetrical and the end elements are rigidly connected. In case of a heterogeneous subsystem or imperfectly rigid constraints of the end elements, the subsystem with its main system can be conveniently considered as a heterogeneous line. It naturally requires another technique being surplus for the present problem, so we will use the equivalent diagram, Fig. 2b.

To find the exact analytical solution for the diagram shown in Fig. 2b, it is convenient to use the results obtained in [22] for a homogeneous finite elastic line on whose interior element an harmonic force *F*(*t*) acts. In [22] three solutions were presented for forced vibrations according to the relationship between the parameter $\beta_s = \sqrt{\omega^2 m / 4s_s}$ and the unity. This regularity has the following form:

for the periodical regime, $\beta_s < 1$

$$\Delta_p = -\frac{F_{0i}\, e^{j\omega t}}{\omega\sqrt{ms_s}\cos\tau_s \ \sin 2n\tau_s} \times \begin{cases} \cos(2p-1)\tau_s \ \ \cos[2(n-k)+1]\tau_s;\ 1 \le p \le k; \\ \cos(2k-1)\tau_s \ \ \cos[2(n-p)+1]\tau_s;\ k \le p \le n, \end{cases} \tag{1}$$

for the aperiodical regime, $\beta_s > 1$

$$\Delta_p = (-1)^{(p-k+1)} \frac{F_{0i}\, e^{j\omega t}}{\omega\sqrt{ms_s}\,(\gamma_{s+} - \gamma_{s-})(\gamma_{s+}^{2n} - \gamma_{s-}^{2n})} \times$$

$$\times \begin{cases} (\gamma_{s+}^{(2p-1)} - \gamma_{s-}^{(2p-1)})(\gamma_{s+}^{[2(n-k)+1]} - \gamma_{s-}^{[2(n-k)+1]});\ 1 \le p \le k; \\ (\gamma_{s+}^{(2k-1)} - \gamma_{s-}^{(2k-1)})(\gamma_{s+}^{[2(n-p)+1]} - \gamma_{s-}^{[2(n-p)+1]});\ k \le p \le n \end{cases} \tag{2}$$

and for the critical regime, $\beta_s = 1$

$$\Delta_p = (-1)^{(p-k+1)} \frac{F_{0i}\, e^{j\omega t}}{4s_s n} \times \begin{cases} (2p-1)[2(n-k)+1];\ 1 \le p \le k; \\ (2k-1)[2(n-p)+1];\ k \le p \le n, \end{cases} \tag{3}$$

where $\tau_s = \arcsin\beta_s$; $\gamma_{s+} = \beta_s + \sqrt{\beta_s^2 - 1}$; $\gamma_{s-} = \beta_s - \sqrt{\beta_s^2 - 1}$; *k* is the number of element to which the external force is applied; *p* is the studied line element number and Δ_p is the momentary displacement of the *p*th element of subsystem.

Transforming (1) – (3) for the cases of our interest *k* = 1 and *k* = *n* and summing the results with regard to resulting displacement equal to the half-sum of displacements caused by each force, we yield:

for the periodical regime, $\beta_s < 1$

$$\Delta_p = -\frac{F_{0i}\cos[n-2p+1]\tau_s}{2\omega\sqrt{ms_s}\sin n\tau_s} e^{j\omega t}, \tag{4}$$

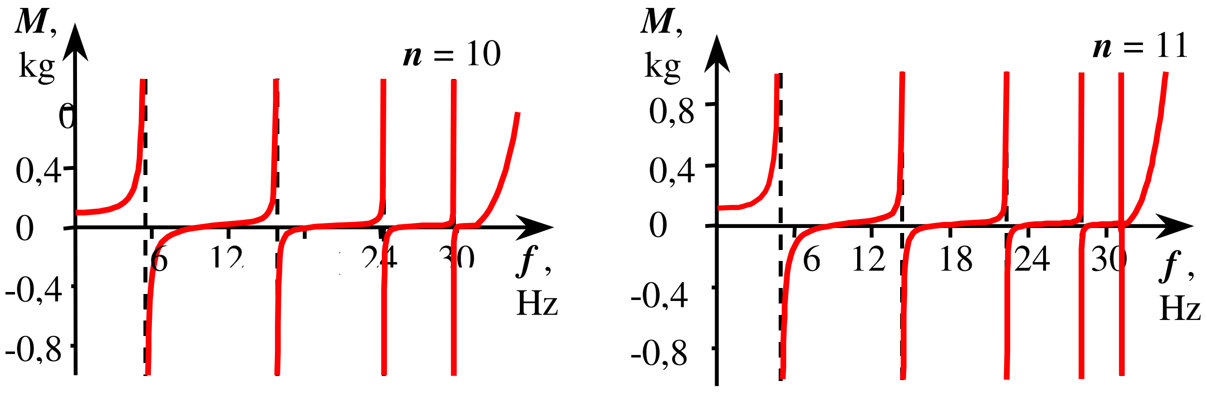
for the aperiodical regime, $\beta_s > 1$

$$\Delta_p = (-1)^{(p-1)} \frac{F_{0i}\left[\gamma_{s+}^{[n-2p+1]} + (-1)^{(n-1)}\gamma_{s-}^{[n-2p+1]}\right]}{2\omega\sqrt{ms_s}\left[\gamma_{s+}^{n} - (-1)^{n}\gamma_{s-}^{n}\right]} e^{j\omega t} \tag{5}$$

and for that critical, $\beta_s = 1$

$$\Delta_p = \frac{F_{0i}\left\{(2p-1)(-1)^{p-n+1} + [2(n-p)+1](-1)^{p}\right\}}{8s_s n} e^{j\omega t}. \tag{6}$$

In (4) – (6) the full symmetry of the studied subsystem reflects, hence the boundary elements displacements will be synchronous in all the vibration regimes. So we can present the subsystem as some rigid body having the measure of inertia *M* as



a) b)

Fig. 3. The measure of subsystem inertia M against the frequency f with different element numbers n. The subsystem parameters: $m_s = 0{,}01$ kg; $s_s = 100$ N/m

$$M = \begin{cases} \dfrac{2\sqrt{ms_s}\,\sin n\tau_s}{\omega\cos[n-1]\tau_s}; & \beta < 1, \\ \dfrac{2\sqrt{ms_s}\left[\gamma_{s+}^{n} - (-1)^{n}\gamma_{s-}^{n}\right]}{\omega\left[\gamma_{s+}^{[n-1]} + (-1)^{(n-1)}\gamma_{s-}^{[n-1]}\right]}; & \beta > 1, \\ \dfrac{8s_s n}{\omega^2\left\{(-1)^{n-1} + [2n-1]\right\}}; & \beta = 1. \end{cases} \qquad (7)$$

We see from (7) that when the substructure is resonance, the measure of inertia of the main line elements depends on frequency nonlinearly. When $n\tau_s = r\pi$ at $r = 0, 1, \ldots [n/2]$ or $(n-1)\tau_s = (2r+1)r\pi$ at $r = 0, 1, \ldots [(n-2)/2]$ were true, zeroes and poles correspondingly appear in the plot $M(\omega)$, and when conditions

$$\sin n\tau_s > 0;\ \cos(n-1)\tau_s < 0 \quad \text{or} \quad \sin n\tau_s < 0;\ \cos(n-1)\tau_s > 0 \qquad (8)$$

were true, the measure of subsystem inertia becomes negative. Skudrzyk also mentioned that the negative values of mechanical subsystem parameters appear possible: “In case of transient (mechanical) conductivity, it is not necessary, all the equivalent scheme elements to be positive. In the canonical scheme having negative elements, the vibration phase against the excitation force is opposite to the vibration phase in the scheme having positive elements… The energetic principle does not contradict it, because the excitation force work does not relate to the vibration velocity at any other points except the excitation point. In the canonical scheme for transition (mechanical) conductivity, some shunting circuits will consist of negative elements, and due to it, mutual wave suppression will be possible” [1, p.320]. And vice versa, “the reason of limited pattern of conductivity variation at the excitation point is the requirement, all the elements of equivalent canonical scheme to be positive” [1, p.320].

The typical form of regularity $M(\omega)$ plotted for odd and even numbers of subsystem elements n on the basis (7) is shown in Fig. 3. It fully corroborates the above analysis. We can see from the construction that this regularity is the succession of resonance peaks whose density increases with frequency, and the peaks width decreases with it. With the transition to the aperiodical regime, the subsystem inertia gains the monotonously increasing pattern with respect to frequency. We should especially note the pattern of this regularity at low frequencies. In the region preceding the first resonance peak, the subsystem inertia increases monotonously. In the considerable part of band it is approximately equal to the subsystem elements total mass, and at $\omega \to 0$ the value $M \to nm$. It is easy to check it, finding the limiting value for the first expression of (7), noting the values τ_s and β_s.

On the basis of subsystem inertia value, we can easy determine the exact analytical solution for the whole elastic system shown in Fig. 1. We will use the fact that the subsystem inertia in (7) depends not on time but only on the external action frequency. So, under external harmonic force action, the measure of subsystems inertia may be regarded as constant for each frequency, and the features caused by the subsystems resonances reveal only in the external action frequency variation.

Thus we can use directly the solutions presented in [20] for semi-finite elastic lumped line. The same as in case of solutions for finite elastic line, their form depends on the relationship between the parameter $\beta_g = \sqrt{\omega^2 M / 4s_g}$ and the unity. We should especially mark that despite the vibration pattern of the elements of elastic line as a whole and of subsystem elements depends on parameters β_g and β_s having similar functional regularity, they behave essentially different with respect to frequency. The parameter β_s depends on the subsystem elements masses that are constant for a specific line, while β_g depends on the measure of inertia of subsystem (7) which depends on frequency nonlinearly, and at definite values β_s becomes negative. With it β_g becomes complex, which is impossible for β_s. In this connection, the features appear in the vibration of elements within the subsystem and of the subsystems as elements of general system. For the subsystems, the same as for simple elastic lines, it is typical a clear division of the range into the periodical, aperiodical and critical vibration regimes with the single boundary frequency corresponding to the critical vibration regime. For the elastic line as a whole it is typical some other range division. At low frequencies, before the first resonance peak of subsystem, the considered elastic line behaves the same as a simple elastic line without any resonance subsystems. The boundary of this range is the first boundary frequency ω_{0g}, which is close to the similar frequency of a simple elastic line whose element masses are equal to the total static mass of the subsystem. Naturally, this boundary frequency is lower than that of subsystem ω_{0s}. Higher than ω_{0g}, in the simple elastic line there takes place the aperiodical regime of antiphase vibrations, damping along the line. In the line having the resonance subsystems, there reveals the influence of subsystem's measure of inertia dependently on frequency, which determines the vibration pattern up to the critical frequency ω_{0s} for the resonance subsystem. Due to this feature, further we will distinguish the concepts of subsystem element and the elastic line element being the resonance subsystem.

To analyse the vibration process arising in the elastic line having the resonance subsystems which is presented in Fig. 1, write the solutions in their general form. In case of forced vibrations they have the following form:

for the periodical regime at $\beta_g < 1$

$$\Delta_i = -\, j \frac{F_0}{\omega\sqrt{Ms_g}} e^{j\left[\omega t-(2i-1)\tau_g+\varphi_0\right]}, \tag{9}$$

for the aperiodical regime at $\beta_g > 1$

$$\Delta_i = \left(-1\right)^i \frac{F_0}{2s_g\beta_g} \gamma_{g-}^{2i-1}\, e^{j(\omega t+\varphi_0)} \tag{10}$$

and the for critical regime at $\beta_g = 1$

$$\Delta_i = \left(-1\right)^i \frac{F_0}{2s_g} e^{j(\omega t+\varphi_0)}, \tag{11}$$

where $\tau_g = \arcsin\beta_g$, $\gamma_{g+} = \beta_g + \sqrt{\beta_g^2 - 1}$, $\gamma_{g-} = \beta_g - \sqrt{\beta_g^2 - 1}$; Δ_i is the momentary displacement of the ith subsystem and φ_0 is the initial phase of external action.

The presence of the resonance pattern of subsystems naturally leads to the features appearing in solutions (9) – (11). First of all, the presence of resonance peaks in the regularity $M(\omega)$ makes impossible to divide exactly the frequency band into subcritical and overcritical regimes. The elastic line will multiply undergo one of these regimes dependently on the value M. The more, the possibility of negative M causes the additional, fourth vibration regime corresponding to the complex value of β_g.

The solution for this regime, which we will name the complex aperiodical regime, will be the following:

$$\Delta_i = -\frac{F_0}{2s_g\left|\beta_g\right|}\gamma_{g(+)}^{2i-1}e^{j(\omega t+\varphi_0)}, \tag{12}$$

where $\gamma_{g(+)} = \left|\beta_g\right| + \sqrt{\left|\beta_g\right|^2+1}$, $\left|\beta_g\right| = \sqrt{\omega^2\left|M\right|/4s_g}$. As we can see from (12), in the complex aperiodical regime the vibration amplitude

$$\Delta_{i\max} = \frac{F_0}{2s_g\left|\beta_g\right|}\gamma_{g(+)}^{2i-1} \tag{13}$$

varies differently, dependently on the subsystem number *i*. For the first subsystem ($i = 1$) the amplitude varies from F_0 / s_g at $\left|\beta_g\right| \to \infty$ to infinity at $\left|\beta_g\right| \to 0$. For all the following subsystems ($i > 1$) this variation takes place in limits from infinity to infinity with the minimum at

$$\left|\beta_g\right|_{\min} = \sqrt{\frac{1+\sqrt{1+(2i-1)^2}}{2}}\ . \tag{14}$$

It means, with the growing subsystem number, the value $\left|\beta_g\right|_{\min}$, at which the minimal amplitude is achieved, increases. Skudrzyk indicated the resonance peaks bifurcation in complex vibrant systems: “Due to damping, some resonances seem single, but if we cool the studied specimen in order to diminish the damping, then two maximums are clearly seen” [1, p.244]. He explained this phenomenon so: “If two resonance frequencies are closely spaced and the shape parameters of natural frequencies are opposite in sign, then the transfer impedance of homogeneous systems nearby some of their resonances can resemble the bound circuits. In this case the frequency regularity is alike the band filter characteristic” [1, p.323]. However, as the exact analytical solutions show, the reason of resonance peaks bifurcation are the complex aperiodical regime features, not the circuits connectivity. We should additionally mark that the existence of complex aperiodical regime is new in the view of elastic systems vibration. In simple elastic lines without resonance subsystems this regime is impossible. And as we said in the introduction, complex systems have been currently studied not so well to reveal this regime.

Summing up the aforesaid, we see that the system of solutions (9) – (12) completely describes the vibration pattern in all the range. Thus we have achieved the aim of this item.

3. Analysis of obtained solutions and comparison with experimental results

To analyse the obtained solutions, conveniently use the following concept of elastic line transfer function

$$K = \frac{\Delta_{i+1}}{\Delta_i} \tag{15}$$

and of the input resistance which in case of external force action on the start of line is equal to

$$\rho_{input} = \frac{\Delta_1}{F(t)}\ . \tag{16}$$

We can easy determine from (9) – (12) that for the studied model

$$K = \begin{cases} e^{-j2\tau_g}, \text{ for } \beta_g < 1; \\ -\gamma_{g-}^2, \text{ for } \beta_g > 1; \\ \text{-1 for } \beta_g = 1; \\ \gamma_{g(+)}^2, \text{ for } \beta_g \text{ is a complex value;} \end{cases} \tag{17}$$

$$\rho_{input} = \begin{cases} -j\dfrac{F_0}{\omega\sqrt{Ms_g}}e^{-j\tau_g}, \text{ for } \beta_g < 1; \\ -\dfrac{F_0}{2s_g\beta_g}\gamma_{g-}, \text{ for } \beta_g > 1; \\ -\dfrac{F_0}{2s_g}, \text{ for } \beta_g = 1; \\ -\dfrac{F_0}{2s_g\left|\beta_g\right|}\gamma_{g(+)}, \text{ for } \beta_g \text{ is a complex value.} \end{cases} \tag{18}$$

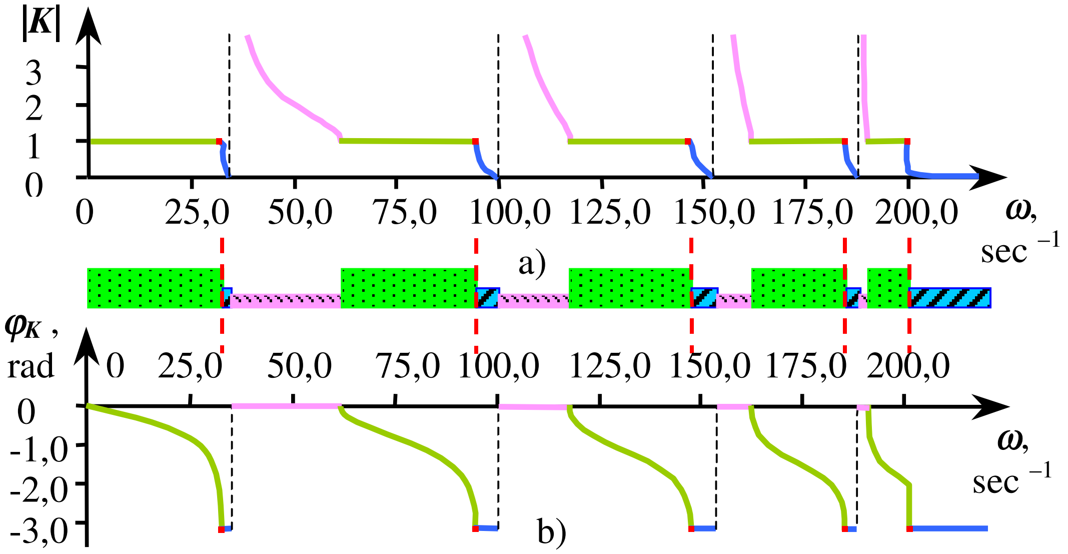


Fig. 4. The module |*K*| (a) and phase φ_K (b) of an elastic line transmission coefficient against the external force frequency, for a semi-finite elastic line having the subsystems consisting of 10 elements

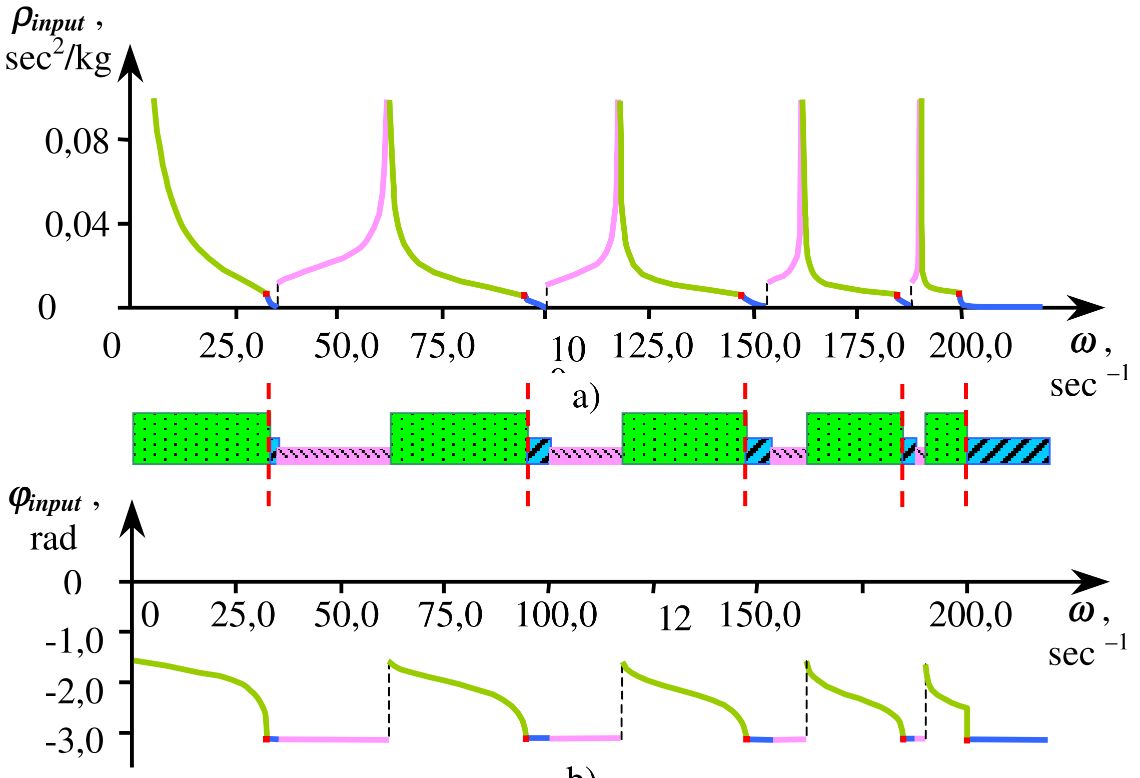
🟩 - periodical vibration regime

▨ - aperiodical vibration regime

- - - - - critical vibration regime

▒ - complex aperiodical vibration regime

We see from (17) and (18) that both parameters K and ρ_{input} have the amplitude $|K|$, ρ_{input} and phase φ_K , φ_{input} parts, and for different bands of β_g variation they differ. In Figures 4 and 5 the typical form of these parameters is shown for the case when the resonance subsystems contained 10 elastically connected bodies each. To visualise, under each plot we give the scale of regimes corresponding to the conditions (17) and (18). It is seen clearly that all these regularities consist of sequentially alternating regions of the above regimes. All the resonance peaks locate at frequencies higher than critical for an elastic system as a whole, i.e. higher than the frequency before which we can consider the system, disregarding the substructure of its elements. With variation ω from zero to



a)

b)

Fig. 5. The input resistance ρ_{imput} (a) and phase delay φ_{imput} (b) against the external force frequency, for a complex semi-finite elastic line having the subsystems consisting of 10 elements

the first maximum, the input resistance falls from $F_0/2s_g$ to zero. But practically, at this entire band the transfer function modulus is equal to unity. At frequencies exceeding the boundary frequency for the system as a whole, in simple models the aperiodical regime of antiphase damping vibrations is usually expected, but when taking the substructure into account, we see a resonance peaks succession. Each peak for the parameter K is formed at the joints of aperiodical and complex aperiodical vibration regimes, and for ρ_{input} – at the joints of complex aperiodical and periodical vibration regimes. The number of these peaks depends on the subsystem size and is equal to the integer part of the value $[(n-1)/2]$. This last is determined by the denominator of the first expression of (7).

The difference of resonance frequencies for $|K|(\omega)$ and $\rho_{input}(\omega)$ causes the double resonance peaks appearance in full accordance with Skudrzyk's experimental observations [1]. It is easy to calculate them on the basis of (9) – (12) with respect to (15) – (18). After a simple transformation we yield

$$\Delta_i = F(t)\,\rho_{input}K^{i-1}. \qquad (19)$$

The typical plot $\Delta_2(\omega)$ for $i = 2$ is presented in Fig. 6. We see that all pair maximums are formed in limits of complex aperiodical regime. With growing mode number, the minimum between the peaks moves towards the frequency increase; it fully corresponds to (14) and to the above analysis.

The phase characteristics of input impedance and transfer function also do not retain constant and nonlinearly depend on external force frequency. The input impedance is reactive at the entire range. The transfer function in at the entire range is also reactive, except the section corresponding to the complex aperiodical regime. At this section the transfer function is active, as its phase φ_K vanishes. We should especially mark that the pattern of displacement of system elements $\Delta_i(\omega)$, which we can measure in the course of experiment, will differ from $|K|(\omega)$, because, according to (19), in (17) we did not account the influence of $\rho_{input}(\omega)$. In the plots presented in Fig. 4 and Fig. 5 we can analyse the behaviour of $|K|(\omega)$ and $\rho_{input}(\omega)$ separately, and this is one of important merits of exact analytical solutions.

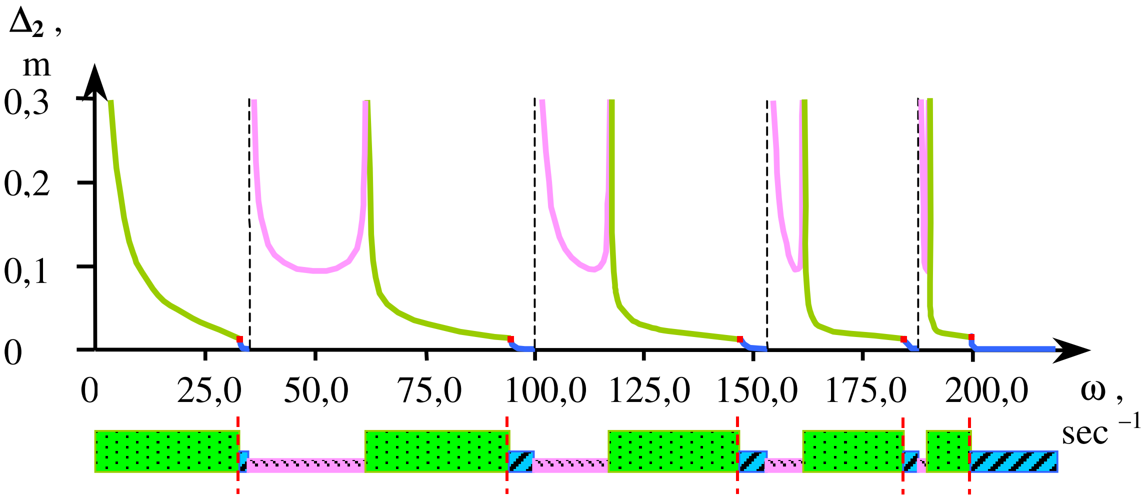


Fig. 6. The amplitude-frequency characteristic of an elastic line 2^{nd} element vibration

Basing on $\varphi_K(\omega)$ and on results presented in [22], we can determine the phase velocity v_f of the wave propagation along the ideal line

$$v_f = \frac{2\pi a}{\varphi_k}, \tag{20}$$

where a is the distance between the elastic line non-excited elements, and φ_K is the ith element phase delay in (17). The typical form of the phase velocity dependence on frequency is shown in Fig. 7.

We can see from Fig. 7 and (20) that in the complex aperiodical regime the phase velocity turns to infinity, as at these bands φ_K is zero (see Fig. 4). At the same time, the phase delay vanishing evidences that the standing wave has produced in the line, despite its limited length. This is the consequence of steady-state pattern of the studied wave process. If the line impedance was present, the phase φ_K does not vanish. Similarly, if along the line the wave process propagated whose spectrum is wider than one band of complex aperiodical regime, then the resulting phase delay also does not vanish. Thus, the obtained result is the idealisation too and can be corrected automatically when solving the practical models consisting the impedance.

Going on analysing (20), we see that the phase velocity achieves its minimal value in aperiodical regime, since at these bands φ_K is maximal. At these bands the phase velocity dependence on frequency is linear, as the phase delay is constant and equal to π. With it, all bands of aperiodical regime are located on one line coinciding with linear regularity, after which the phase velocity would increase in absence of resonance substructure. It corroborates additionally the above statement that the resonance peaks of the subsystem arise in the region between the periodical and aperiodical regimes

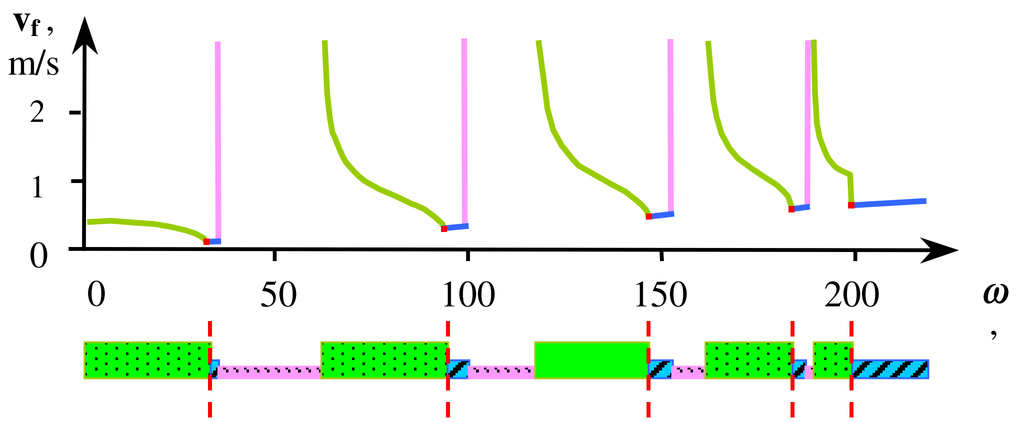


Fig. 7. The phase velocity of the wave propagation in a semi-finite elastic line having the resonance subsystems

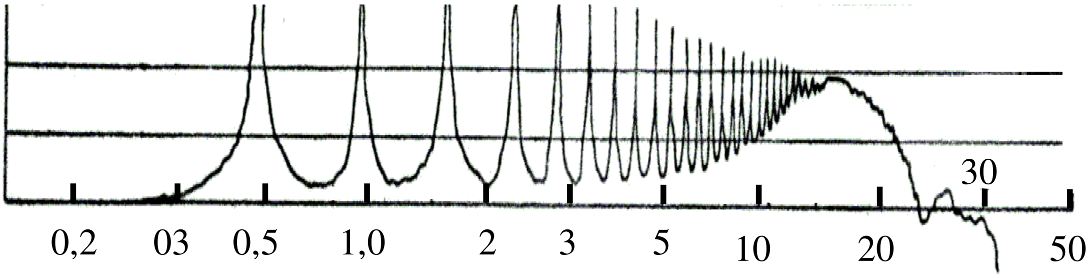


Fig. 8. The experimental plot of longitudinal vibration velocity amplitude against the vibration frequency in polystyrene rods [1, p.244]

for the system as a whole.

Furthermore, it is typical that despite the sections having negative measure of inertia appear, the transfer function phase retains delaying always, and this also is in full accordance with the above Skudrzyk's statement [1] that the negative measure of inertia of line elements fully corresponds to the conservation laws.

This negative measure of inertia, which we used to think strongly associated with the mass, does not mean a negative mass introduction. In this case, there reacts not a separate mass but a complex system of elastically connected masses being the parts of a general elastic system. So we have to identify just this reaction with the negative measure of inertia of the *subsystem*. We see that the pattern of subsystem reaction to the external action changes. With it the pattern of process also changes. And the phase with regard to the external action retains negative. Thus, introducing the idea of negative measure of inertia, we do not contradict the laws by Newton considered an accelerated body as an entire rigid system.

Another important characteristic of a wave process in a complex resistant line is the experimental measurement of vibration velocity of the end elements of lumped line or of the end of distributed line. Skudrzyk [1] has carried out this investigation. In Fig. 8 there is presented one of regularities of the rod end vibration velocity with respect to frequency taken from [1, p.244], and in Fig.9 – the similar theoretical regularity, obtained by way of (9) – (12) time differentiation. The difference between the theoretical and experimental results is only in the regions of lower spikes caused by the aperiodical vibration regime. This last is conditioned by the studied model ideality. With the line resistance appearance, the transition between the periodical and aperiodical regimes smoothes, and damping at the aperiodical band decreases, while at the periodical band it increases. The vibration pattern transformation itself with the resistance appearance is very interesting and can be studied on the basis of exact analytical solutions obtained here. However here we will limit ourselves with the above analysis, as this investigation far exceeds the frames of present task.

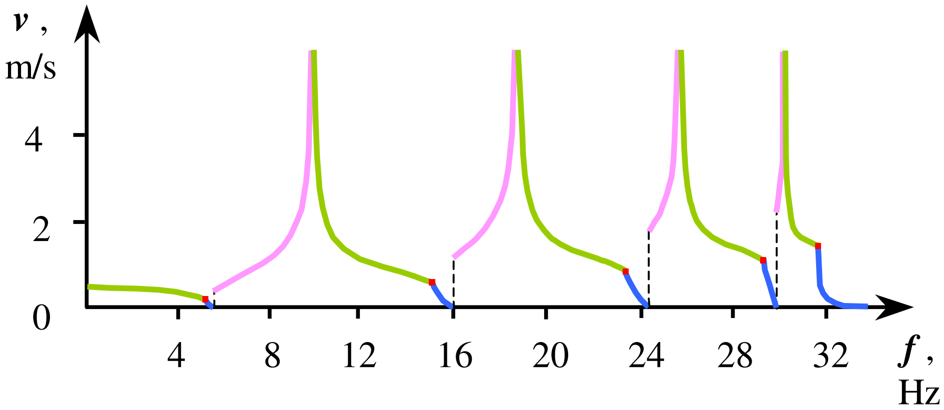


Fig. 9. The first element vibration velocity against the external force frequency, for a semi-finite elastic line having the resonance subsystems

Conclusions

Basing on exact analytical solutions obtained for semi-finite elastic line with resonance subsystems having the form of linear elastic lines with rigidly connected end elements, we have revealed that between the first boundary frequency for the elastic line as a whole and that for the subsystem the resonance peaks arise, and their quantity is equal to the integer part of $[(n-1)/2]$, where n is the number of elements in the subsystem. These resonance peaks arise at the boundary between the aperiodical and complex aperiodical vibration regimes. And this last regime is typical namely for elastic systems having resonance subsystems. In simple elastic lines its appearance is impossible.

Despite in complex aperiodical regime the measure of inertia of a resonance subsystem is negative, the phase delay of this process does not become leading. It fully accords with Skudrzyk's conclusion that the negative measure of inertia of resonance subsystem completely corresponds to the conservation law. In this regime, with the steady-state process in the line the standing waves form, and the wave propagation phase velocity turns to infinity for the ideal line.

We also have determined that for interior elements of an elastic line the resonance peaks bifurcate, and it fully corresponds to Skudrzyk's experimental results. However the reason of bifurcation is not the resonance circuits mismatch, as Skudrzyk supposed, but the features of resonance peaks formation for the input resistance and transfer function of an elastic line, which causes the resonance frequencies for these parameters to be not the same.

The theoretically obtained regularities for vibration velocity of a boundary element of a line are in good agreement with Skudrzyk's experimental results.

References

1. Skudrzyk, E. *Simple and complex vibratory systems*. Mir, Moscow, 1971, 557 pp. (Russian; original edition: E. Skudrzyk. *Simple and complex vibratory systems*. The Pennsylvania State University Press, University Park and London, 1968)
2. Grebennikov, E.A. *Introduction to the resonance systems theory*. Moscow State University, 1987, 174 pp. (Russian)
3. Kuhta, K.Ya., Boyko, A.G. and Garmash, N.Z. *Investigation of complex continouosly-discrete systems*. Naukova Dumka, Kyev, 1981, 270 pp. (Russian)
4. Volkenshtein, M.V., Eliashkevitch, M.A. and Stepanov B.I. *Molecules oscillations*, vol. 1: Geometry and mechanics of molecules oscillations. The State Publishing of Technical and Theoretical Literature, Moscow – Leningrad, 1949, 600 pp. (Russian)
5. Slater, J.C. *Insulators, semiconductors and metals*. Mir, Moscow, 1969, 647 pp. (Russian; original edition: Slater, J.C. Insulators, semiconductors and metals. In: *Quantum theory of molecules and solids*, vol.3. McGraw Hill Book Company, New York – St. Lous – Ssan Francisco – Toronto – London – Sydney, 1967).
6. Blakemore, J.S. *Solid state physics*. Metallurgia, Moscow, 1972, 488 pp. (Russian; original edition: Blakemore, J.S. *Solid state physics*. W.B. Saunders Company, Philadelphia – London – Toronto, 1970)
7. Born, M. and Hypper-Meyer, M. *Dynamical theory of lattice*. The Principal Editorial for Technical and Theoretical Literature, Moscow – Leningrad, 1938, 364 pp. (Russian, translated from German).
8. Mikhailov, I.G., Soloviev, V.A. and Sirnikov, Yu.P. *Foundations of molecular acoustics*. Nauka, Moscow, 1964, 514 pp. (Russian)

9. Thouless, D.J. *The quantum mechanics of many-body systems*. Publishing for Foreign Literature, Moscow, 1963 (Russian; original edition: Thouless, D.J. *The quantum mechanics of many-body systems*. Academic Press, New York – London,1961)
10. Chernous'ko, F.L., Akulenko, L.D. and Soloviev, B.N. *Control of oscillations*. Nauka, Moscow, 1980, 383 pp. (Russian).
11. Tong Kin, N. *Theory of mechanical vibrations*. Mashgiz, Moscow, 1963, 354 pp. (Russian)
12. Atkinson, F.V. *Discrete and continuous boundary problems*. Mir, Moscow, 1968, 750 pp. (Russian; original edition: Atkinson, F.V. *Discrete and continuous boundary problems*. Academic Press, new York – London, 1964)
13. Palis, J. and de Melo, J.W. *Geometric theory of dynamical systems*. Mir, Moscow, 1986, 302 pp. (Russian; original edition: J. Palis and J.W. de Melo. *Geometric theory of dynamical systems.* Springer-Verlag, New York – Heidelberg – Berlin, 1982) .
14. Reiscig, R., Sansone, G. and Conti, R. *Qualitative theory of nonlinear diffeerential equations*. Nauka, Moscow, 1974, 318 pp. (Russian; original edition: Reiscig, R., Sansone, G. and Conti, R. *Qualitative theorie nichtlinearer differentialleichungen*. Edizioni cremonese, Roma, 1963)
15. Mitropolsky, Yu.A. and Homa, G.P. *Mathematical background of nonlinear mechanics asymptotic methods*. Naukova Dumka, Kyev, 1983, 215 pp. (Russian)
16. Cherepennikov, V.B. *Functional parameters method in the ordinary differential equations theory*. Nauka, Novosibirsk, 1983 (Russian)
17. Giacagrilia, G.E.O. *Perturbation methods in non-linear systems*. Nauka, Moscow, 1979, 320 pp. (Russian; original edition: Giacagrilia, G.E.O. *Perturbation methods in non-linear systems.* Springer-Verlag, New York – Heidelberg – Berlin, 1972)
18. Dymentberg, M.F. *Nonlinear stochastic problems of mechanical vibrations*. Nauka, Moscow, 1980, 368 pp. (Russian)
19. Banerjee, P.K. and Butterfield, R. *Boundary element methods in engineering science*. Mir, Moscow, 1984, 494 pp. (Russian; original edition: Banerjee, P.K. andButterfield, R. *Boundary element methods in engineering science*. McGraw hill Book Company Ltd., London – New York – Sydney – Tokyo – Toronto, 1981)
20. Karavashkin, S.B. *Exact analytical solution on infinite one-dimensional elastic lumped-parameters line vibration*. Materials, Technologies, Tools. The Journal of National Academy of Sciences of Belarus. **4** (1999), 3, 15-23 (Russian)
21. Karavashkin, S.B. *Exact analytical solution on finite one-dimensional elastic lumped-parameters line vibration*. Materials, Technologies, Tools. The Journal of National Academy of Sciences of Belarus. **4** (1999), 4, 5-14 (Russian)
22. Karavashkin, S.B. *Some peculiarities of forced vibrations modelling in homogeneous elastic lumped-parameters lines*. Materials, Technologies, Tools. The Journal of National Academy of Sciences of Belarus, **5** (2000), 3, 15-23 (Russian)
23. Karavashkin, S.B. *Peculiarities of inclined force action upon one-dimensional homogeneous elastic lumped line.* arXiv, (Los Alamos) # math-ph/0006028

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