MULTIPLICITY OF RESONANCES EQUALS MULTIPLICITY OF THE SCATTERING MATRIX

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ABSTRACT. We present a easy proof of a result of Petkov and Zworski using dilation. We proove that the multiplicity of a resonance is equal to the multiplicity of the scattering matrix.

1. NOTATION

Let $r_0 > 0$. We denote the closed ball in $\mathbb{R}^n$ by $B(x, r_0) = \{y \in \mathbb{R}^n, |x-y| \leq r_0\}$. Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition :

$$\mathcal{H} = \mathcal{H}_{r_0} \oplus L^2(\mathbb{R}^n \setminus B(0, r_0)).$$

We assume that $P_0 : \mathcal{D} \to \mathcal{H}$ is a self adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$ satisfying :

$$1_{\mathbb{R}^n \setminus B(0,r_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, r_0)),$$

$$1_{\mathbb{R}^n \setminus B(0,r_0)} P_0(h) = -h^2 \Delta|_{\mathbb{R}^n \setminus B(0,r_0)}.$$

We assume that there exists $m_0 \in \mathbb{N}^\star$ such that $1_{B(0,r_0)}(P_0 + i)^{-m_0}$ is trace class, and that there exists $C \geq 0$ such that $P_0 \geq -C$. Let $r > r_0$. Let $\varepsilon > 0$, $a > 0$. Then there exists $b > r + a$ and a function f_θ on $\mathbb{R}^+$ such that

$$f_\theta(t) = t, \qquad t \leq r + a,$$
$$f_\theta(t) = e^{i\theta} t, \quad t \geq b.$$

$$\forall t \in \mathbb{R}^+, 0 \leq \mathrm{Arg}(f_\theta(t)) \leq \theta,$$
$$\partial_t f_\theta(t) \neq 0,$$
$$\mathrm{Arg}(f_\theta(t)) \leq \mathrm{Arg}(\partial_t f_\theta(t)) \leq \mathrm{Arg}(f_\theta(t)) + \varepsilon.$$

Let $\Gamma_\theta = v(\mathbb{R}^n) \in \mathbb{C}^n$ where $v(x) = f_\theta(|x|)\frac{x}{|x|}$.
Let $\chi \in \mathcal{C}_0^\infty(B(0, r))$, $\tilde{\chi} \in \mathcal{C}_0^\infty(B(0, r))$ be such that

$$\forall x \in B(0, r_0),\ \chi(x) = \tilde{\chi}(x) = 1,\ \tilde{\chi}\chi = \chi.$$

*Supported in part by ACI : Méthodes mathématiques en physique de la matiére condensée.

Denote by $P_1 = -h^2\Delta$. We denote by $P_{0,\theta}$ (res. $P_{1,\theta}$) the operator P_0 (res. P_1) on the space Γ_θ (see [6] for the definition). A bullet will denote either 0 or 1, so $P_\bullet$ is either P_0 or P_1. It is known that the resolvents $R_\bullet(z) = (P_\bullet - z)^{-1} : \mathcal{H} \to \mathcal{D}$ has a meromorphic extension from the upper half plane $\{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when n is odd, and to the logarithmic plane Λ when n is even. This continuation is an operator from $\mathcal{H}_{comp}$ to $\mathcal{D}_{loc}$ [4] and the poles are of finite rank. The poles of R_0 are called resonances of P_0. Denote by $R_{\bullet,\theta} = (P_{\bullet,\theta} - z)^{-1}$. We will denote the set of resonances of P_0 by $\mathrm{Res}(P_0)$ and will always include them according to their multiplicity $m_{R_0}(z)$, which for $z \neq 0$ are defined as

$$m_{R_0}(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_0(u)\, du\Big) = \mathrm{Tr}\left(-\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)} R_0(u)\, du\right),$$

where $\gamma_\varepsilon(z) = \{z + \varepsilon e^{it}, 0 \leq t \leq 2\pi\}$ for ε sufficiently small [6]. It is known that for $\theta > 0$,

$$m_{R_0}(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_{0,\theta}(u)\, du\Big) := m_{R_{0,\theta}(z)}.$$

It is known that the resonances of P_0 that belong to the real axe are eigenvalues of P_0.
The scattering matrix is defined just as in the usual potential scattering [4], [3]. We denote by $S(\lambda)$ the standard (relative) scattering matrix. The standard scattering matrix $S(\lambda)$ continues meromorphically in λ from $\{\lambda, \mathrm{Im}(\lambda) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when n is odd, and to the logarithmic plane Λ when n is even. The poles of its continuation correspond to resonances of P_0. They are of finite rank. We define the multiplicity of a pole of $\det S(z)$ as

$$m_S(z) = -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} \Big(S(u)^{-1}\frac{d}{du}S(u)\Big)\, du.$$

2. RESULT

This paper is devotes to the proove of the following theorem that we could find in [4] proposition 2-2

Theorem 2.1. *For z such that $\mathrm{Re}(z) > 0$,*

$$m_S(z) = m_{R_0}(z) - m_{R_0}(\overline{z}). \tag{2.1}$$

The proof is just based on basic computation on the dilate operator, unless the author use the generic symplicity of resonances and continuity propreties of the multiplicities.

3. Proof of 2.1

Assume $\mathrm{Re}(\lambda_0) > 0$ and $\lambda_0 \notin \mathrm{Res}(P_0)$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R})$. We write $g(\lambda) = (\lambda - \lambda_0)^m f(\lambda)$. Let $\tilde{g}$ be an almost analytic extension of g of compact support [2]. Using the Birman Krein formula [1] we get

$$
\begin{aligned}
&\mathrm{Tr}[f(P_0) - (1-\chi) f(P_1)(1-\tilde{\chi})] - \mathrm{Tr}[\chi f(P_1)] \\
=&\frac{1}{2i\pi} \int_{\mathbb{R}} \Big(\frac{df}{d\lambda} \log(\det S(\lambda)) \Big)\, d\lambda + \sum_{\lambda \in \mathrm{Res}(P_0) \cap \mathbb{R}} g(\lambda)(\lambda - \lambda_0)^{-m_0} \\
=&-\frac{1}{2i\pi} \int g(\lambda)(\lambda - \lambda_0)^{-m_0} \mathrm{Tr}(S' S^{-1}(\lambda))\, d\lambda \\
&+ \sum_{\lambda \in \mathrm{Res}(P_0) \cap \mathbb{R}} g(\lambda)(\lambda - \lambda_0)^{-m_0}.
\end{aligned} \tag{3.1}
$$

The expression of the functionnal calculus is [2]

$$
f(P_\bullet) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\mathrm{Im}(z)| > \varepsilon} \overline{\partial} \tilde{g}(z) (P_\bullet - \lambda_0)^{-m_0} R_\bullet(z)\, L(dz).
$$

Denote by

$$
\begin{aligned}
G(z) &= (P_0 - \lambda_0)^{-m_0} R_0(z) - (1-\chi)(P_1 - \lambda_0)^{-m_0} R_1(z)(1-\tilde{\chi}), \\
G_\theta(z) &= (P_{0,\theta} - \lambda_0)^{-m_0} R_{0,\theta}(z) - (1-\chi)(P_{1,\theta} - \lambda_0)^{-m_0} R_{1,\theta}(z)(1-\tilde{\chi}),
\end{aligned}
$$

$\langle z \rangle = (1+z^2)^{-\frac{1}{2}}$, and $\|A\|_{\mathrm{Tr}}$ the norm trace of A. Denote by

$$
f_\varepsilon = \frac{1}{\pi} \int_{|\mathrm{Im}(z)| > \varepsilon} \overline{\partial} \tilde{g}(z) G(z)\, L(dz).
$$

The estimates

$$
\|\overline{\partial} \tilde{g}(z)\| \leq C |\mathrm{Im}(z)|^3,
$$

and [1]

$$
\|G(z)\|_{\mathrm{Tr}} \leq \frac{C \langle z \rangle}{|\mathrm{Im}(z)|^2},
$$

give

$$
\|f_\varepsilon - f_{\varepsilon'}\|_{\mathrm{Tr}} \leq C \sup(\varepsilon, \varepsilon').
$$

Using the completeness of the trace class we get

$$
\begin{aligned}
&\mathrm{Tr}[f(P_0) - (1-\chi) f(P_1)(1-\tilde{\chi})] \\
=&\frac{1}{\pi} \lim_{\varepsilon \to 0} \mathrm{Tr}\Big[\int_{|\mathrm{Im}(z)| > \varepsilon} \overline{\partial} \tilde{g}(z) G(z) L(dz) \Big].
\end{aligned}
$$

So

$$\begin{aligned}&\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde\chi)]\\&=-\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\mathrm{Tr}\Big[\int_{\mathbb{R}}\tilde g(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde g(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\,d\lambda\Big].\end{aligned}$$

Lebesgue's theorem of dominated convergence gives

$$\begin{aligned}&\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde\chi)]\\&=-\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{\mathbb{R}}\mathrm{Tr}\Big[\tilde g(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde g(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\Big]\,d\lambda.\end{aligned}$$

For $\mathrm{Im}(z)>0$, the function $\theta\to\mathrm{Tr}\big(G_\theta(z)\big)$ is independant of θ for $\theta\geq 0$ [5]. This implies for $\theta\geq 0$

$$\begin{aligned}\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde\chi)]=&-\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathrm{Im}(z)=\varepsilon}\mathrm{Tr}\big(\tilde g(z)G_\theta(z)\big)\,dz\\&+\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathrm{Im}(z)=-\varepsilon}\mathrm{Tr}\big(\tilde g(z)G_{-\theta}(z)\big)\,dz.\end{aligned}$$

The function $z\to\mathrm{Tr}\big(\tilde g(z)G_\theta(z)\big)$ is analytic in the domain where $R_{0,\theta}(z)$ is defined, and is locally independant of θ. Denote by $D(x,\varepsilon)=\{z\in\mathbb{C},|z-x|=\varepsilon\}$, $d_x^+(\varepsilon)=D(x,\varepsilon)\cap\{z,\mathrm{Im}(z)\geq 0\}$ and $d_x^-(\varepsilon)=D(x,\varepsilon)\cap\{z,\mathrm{Im}(z)\leq 0\}$. We obtain for ε sufficiently small

(3.2)

$$\begin{aligned}&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathrm{Im}(z)=\pm\varepsilon}\mathrm{Tr}\big(\tilde g(z)G_{\pm\theta}(z)\big)\,dz\\=&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}\{z\in\mathbb{R},|z-z_j|\leq\varepsilon\}}\tilde g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\\&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{\pi}\int_{\{z,0\leq\pm\mathrm{Im}(z)\leq\varepsilon\}\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}B(z_j,\varepsilon)}\overline\partial\tilde g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,L(dz)\\&+\sum_{\{1,-1\}}\sum_{z_j\in\mathrm{Res}(P_0)}\frac{1}{2i\pi}\int_{d_{z_j}^{\pm}(\varepsilon)}\tilde g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz.\end{aligned}$$

We get

$$\begin{aligned}&\Big\|\sum_{\{1,-1\}}(\pm 1)\frac{1}{\pi}\int_{0\leq\pm\mathrm{Im}(z)\leq\varepsilon\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}B(z_j,\varepsilon)}\overline\partial\tilde g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,L(dz)\Big\|\\&\leq 2\frac{1}{\pi}\int_{0\leq|\mathrm{Im}(z)|\leq\varepsilon\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}\{z,|z-z_j|\leq\varepsilon\}}\big\|\overline\partial\tilde g(z)\big\|\;\|\mathrm{Tr}(G(z))\|\;L(dz)\\&\leq C|\varepsilon|.\end{aligned}$$

For $z_j \in \mathbb{R} \cap \mathrm{Res}(P_0)$, the function $z \to (z - z_j)G_\theta(z)$ is analytic. With a control on the trace norme we obtain that the function $z \to (z - z_j)\mathrm{Tr}\big(G_\theta(z)\big)$ is analytic. Therefore

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\sum_{z_j\in\mathrm{Res}(P_0)}\sum_{\{1,-1\}}\frac{1}{2i\pi}\int_{d^{\pm}_{z_j}(\varepsilon)}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\Big)\\
=&\lim_{\varepsilon\to 0}\Big(\sum_{z_j\in\mathrm{Res}(P_0)}\sum_{\{1,-1\}}\frac{1}{2i\pi}g(z_j)\int_{d^{\pm}_{z_j}(\varepsilon)}\mathrm{Tr}(G_{\pm\theta}(z))\,dz\Big)\\
=:&\sum_{z_j\in\mathrm{Res}(P_0)}g(z_j)H_j(z_j-\lambda_0)^{-m_0}.
\end{aligned}$$

The limit $\varepsilon \to 0$ gives

$$\begin{aligned}
&\mathrm{Tr}(f(P_0)-(1-\chi)f(P_1)(1-\tilde{\chi}))-\mathrm{Tr}[\chi f(P_1)]\\
=&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}}g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\\
&+\sum_{z_j\in\mathrm{Res}(P_0)\cap\mathbb{R}}g(z_j)H_j(z_j-\lambda_0)^{-m_0}\\
&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}}g(z)\mathrm{Tr}[\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(z)]\,dz.
\end{aligned}$$

This, and (3.1) implies the following equality of measures

$$\begin{aligned}
&-\frac{1}{2i\pi}(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}\big(S'S^{-1}(\lambda)\big)\,d\lambda+\sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}}\delta_\lambda(\lambda-\lambda_0)^{-m_0}\\
=&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}(G_{\pm\theta}(\lambda))\,d\lambda\\
&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}\big(\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(\lambda)\big)\,d\lambda\\
&+\sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}}\delta_\lambda H_\lambda(\lambda-\lambda_0)^{-m_0}.
\end{aligned}$$

Taking the continuous part of the measures we get

$$\begin{aligned}
(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}(S'S^{-1}(\lambda))\,d\lambda&=\mathrm{Tr}(G_\theta(\lambda))d\lambda-\mathrm{Tr}(G_{-\theta}(\lambda))\,d\lambda\\
&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}[\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(\lambda)]\,d\lambda.
\end{aligned}\tag{3.3}$$

Due to the analyticity of the functions we get, for all $z \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$,

$$\mathrm{Tr}(S'S^{-1}(z)) = (z-\lambda_0)^{m_0}\mathrm{Tr}(G_\theta(z)) - (z-\lambda_0)^{m_0}\mathrm{Tr}(G_{-\theta}(z))$$
$$+\frac{1}{2i\pi}\mathrm{Tr}[\chi(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(z)] - \frac{1}{2i\pi}\mathrm{Tr}\big(\chi(P_{1,-\theta}-\lambda_0)^{-m_0}R_{1,-\theta}(z)\big).$$

A computation of $m_{R_1}(z)$ gives

$$\begin{aligned} 0 = m_{R_1}(z) &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_{1,\theta}(u)\,du \\ &= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P_{1,\theta}-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(P_{1,\theta}-\lambda_0)^{m_0}R_{1,\theta}(u)\,du\Big) \\ &= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P_{1,\theta}-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}R_{1,\theta}(u)\,du\Big) \\ &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(u)\,du. \end{aligned}$$

Similarly, for $m_{R_{0,\theta}}(z)$ we get

(3.4)

$$\begin{aligned} m_{R_{0,\theta}}(z) &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_{0,\theta}(u)\,du \\ &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P_{0,\theta}-\lambda_0)^{-m_0}R_{0,\theta}(u)\,du \\ &= -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(G_\theta(u)\big)\,du \\ &\quad +\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(\chi(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(u)\big)\,du. \end{aligned}$$

By integrating (3.3), using (3.4) we obtain

(3.5)

$$\begin{aligned} m_S(z) &= -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}\mathrm{Tr}S'S(u)^{-1}\,du \\ &= -\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}(G_{\pm\theta})(u)\,du \\ &\quad +\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(u)\big)\,du \\ &= m_{R_{0,\theta}}(z) - m_{R_{0,-\theta}}(z). \end{aligned}$$

Denote by $\mathcal{C}$ the operator $u \to \overline{u}$. For any operator A of trace class we obtain $\mathrm{Tr}(\mathcal{C}A\mathcal{C}) = \overline{\mathrm{Tr}A}$. We get $\mathcal{C}P_0\mathcal{C} = P_0$, so $\mathcal{C}(R_{0,-\theta}(z))\mathcal{C} = R_{0,\theta}(\overline{z})$, and therefore

$$m_{R_{0,-\theta}}(z) = \frac{1}{2i\pi}\mathrm{Tr}[\int_{\gamma_\varepsilon(z)} R_{0,-\theta}(u)\,du]$$
$$= \overline{\left(-\frac{1}{2i\pi}\mathrm{Tr}\big(\mathcal{C}(\int_{\gamma_\varepsilon(z)} R_{0,-\theta}(u)\,du)\mathcal{C}\big)\right)}$$
$$= \overline{\left(\frac{1}{2i\pi}\mathrm{Tr}\big(\int_{\gamma_\varepsilon(\overline{z})} R_{0,\theta}(u)\,du\big)\right)}$$
$$= \overline{m_{R_{0,\theta}}(\overline{z})} = m_{R_{0,\theta}}(\overline{z}).$$

The equation (3.5) implies

$$m_S(z) = m_{R_{0,\theta}}(z) - m_{R_{0,\theta}}(\overline{z}).$$

This proves Theorem 2.1. I like to thank M. Zworski for suggested me this problem.

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