REGULARITY PROPERTIES OF POTENTIALS FOR JOINT MEASURES OF RANDOM SPIN SYSTEMS

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ABSTRACT. We consider general quenched disordered lattice spin models on compact local spin spaces with possibly dependent disorder. We discuss their corresponding joint measures on the product space of disorder variables and spin variables in the infinite volume. These measures often possess pathologies in a low temperature region reminiscent of renormalization group pathologies in the sense that they are not Gibbs measures on the product space. Often the joint measures are not even almost Gibbs, but it is known that there is always a potential for their conditional expectations that may however only be summable on a full measures set, and not everywhere. In this note we complement the picture from the non-pathological side. We show regularity properties for the potential in the region of interactions where the joint potential is absolutely summable everywhere. We prove unicity and Lipschitz-continuity, much in analogy to the two fundamental regularity theorems proved by van Enter, Fernandez, Sokal for renormalization group transformations.

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1. Introduction

The study of joint measures of quenched random systems on the product space of spin-variables and disorder-variables is interesting for two reasons. First, there is interest in their behavior coming from theoretical physics, which was put forward in the so-called grand ensemble approach to disordered systems going back to [Mo64], and pursued e.g. in [Ku96],[KM94]. Second, as it was found much later starting from the example in [EMSS00] and investigated in generality in [K99], [K01], [KLR02], joint measures provide a whole class of measures with very interesting 'pathological' behavior w.r.t their Gibbsian properties. Therefore they are useful examples from a merely theoretical point of view to sharpen our notions of generalisations of Gibbsian theory and show "what can go wrong and what cannot go wrong". See also the discussion in [EKM00], [KM00] on the relevance for physics.

So, the study of these joint measures parallels and complements the study of so-called renormalized measures that were the first source of natural measures outside the Gibbsian class. For the latter, see e.g. [EFS93], [BKL98], [MRSV00] and references therein, for interesting examples of non-Gibbsian measures arising in different contexts, see e.g. [EFHR02],[MMRTV00]. In particular the joint measures of the random field Ising model with i.i.d. random fields serve as a complete example that illustrates the distinctions between the classes of weakly Gibbsian and almost Gibbsian measures, and the failure of the variational principle for a weakly Gibbsian measure [KLR02]. For recent results restoring the variational principle for generalized Gibbs measures under continuity conditions on the conditional expectations, see [KLR02].

In previous research about joint measures we mainly focused on the "pathological" or negative side of the behavior of the joint measures. In contrast to that we focus in this note on general results in the non-pathological regime that show "what cannot go wrong" if the joint measures stay in the class of ordinary Gibbs measures. We are led here by the two "fundamental theorems" stated as the central "positive" results in the huge basic paper on renormalization group pathologies [EFS93]. In brief, these theorems say: 1) When an absolutely convergent interaction potential exists for a measure that is the image of a Gibbs measure under a renormalization group transform, it must be unique (up to physical equivalence). This means that it is independent of the choice of the initial measure within the Gibbs measures corresponding to a fixed potential. 2) The map from potentials to such renormalized measures is Lipschitz-continuous in a suitable norm, wherever it is defined.

It it natural that we ask the same questions, when we replace a "renormalization group transform of a translation invariant measure" by the "joint measure of a quenched random system". The good news of this paper is that we get analogous answers in the setting of joint measures as for renormalized measures, in an appropriate setup. Without much additional effort we can allow in our setup the case of dependent disordered variables distributed according to another Gibbs measure, instead of just i.i.d ones. (Think of the case where the random fields of a random field Ising model are given by an Ising model.)

Our regularity results are in brief as follows. Theorem 1: When an absolutely convergent interaction potential exists for a joint measure coming from the same disordered specification, it must be unique. Moreover, it does not depend on the choice of the phase of the distribution of the disorder variables.

Theorem 2: The map from the pair (potential defining the quenched system)× (potential for the disorder distribution) ↦ (potential for the joint measure) is Lipschitz-continuous in the appropriate norm, wherever it is defined.

To appreciate the uniqueness-result for the joint potential we stress the "pathological" fact that the unicity statement of Theorem 1) is proved to go wrong when the resulting translation-invariant interaction potential is only a.s. absolutely convergent (weakly Gibbs), for the random field Ising model at low temperatures, small disorder, in at least 3 dimensions [KLR02].

The paper is organized as follows. In Section 2 we introduce our formal setup. In Section 3 we prove Theorem 1 which is rather simple. In Section 4 we prove Theorem 2, and also give an alternative norm-estimate on the difference of joint potentials, stated as Theorem 3. In Section 5 we illustrate the results with the random field Ising model.

2. Joint measures of quenched random systems - Setup

We consider disordered models of the following general type. We assume that the configuration space of the quenched model is $\Omega = E^{\mathbb{Z}^d}$ where E is a compact separable metric space, and Ω is equipped with the product topology. The spin variables are denoted by $\sigma \in \Omega$. We assume that Ω is equipped with some fixed product Borel probability measure $\nu(d\sigma)$. Additionally we assume that there are also disorder variables $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ entering the game, and we assume that they take values in an infinite product space $\bar{\Omega} = (\bar{E})^{\mathbb{Z}^d}$, where again $\bar{E}$ is a compact metric space, equipped with some product Borel probability measure $\bar{\nu}(d\eta)$.

We denote the *joint variables* by $\xi = (\xi_x)_{x\in\mathbb{Z}^d} = (\sigma\eta) = (\sigma_x\eta_x)_{x\in\mathbb{Z}^d}$. Here $\sigma_x\eta_x$ is a pair, not a product (that would make no sense in general). The first essential ingredient of the quenched model is given by the *defining potential* $\Phi = (\Phi_A)_{A\subset\mathbb{Z}^d}$ depending on the joint variables $\xi = (\sigma\eta)$. $\Phi_A(\xi)$ is a continuous function and depends on ξ only through ξ_A.

We need to specify the Banach spaces of potentials Φ and the other potentials we will be dealing with. We follow here [EFS93] in spirit and in notation, but additionally we need to make explicit the underlying spin spaces. We write $\mathcal{B}^0(\Omega\times\bar{\Omega})$ for the space of translation-invariant continuous interactions Φ on the joint space $\Omega\times\bar{\Omega}$ with finite norm $\|\Phi\|_{\mathcal{B}^0} := \sum_{A\ni 0}\frac{1}{|A|}\sup_{\sigma\eta}|\Phi(\sigma\eta)|$. We write $\mathcal{B}^1(\Omega\times\bar{\Omega})$ for the smaller space of translation-invariant continuous interactions Φ with finite norm $\|\Phi\|_{\mathcal{B}^1(\Omega\times\bar{\Omega})} := \sum_{A\ni 0}\sup_{\sigma\eta}|\Phi(\sigma\eta)|$. We also use the analogous notations like $\mathcal{B}^0(\Omega), \mathcal{B}^0(\bar{\Omega}), \dots$, for potentials on the marginal spaces Ω and $\bar{\Omega}$. We assume that $\Phi \in \mathcal{B}^1(\Omega\times\bar{\Omega})$. When we fix a realization of the disorder η we have a potential for the spin-variables σ that is typically non-translation invariant. We then define the corresponding *quenched Gibbs specification* by the definition

$$\mu_\Lambda^{\Phi;\bar\sigma}[\eta](B) := \frac{1}{Z_\Lambda^{\Phi;\bar\sigma}[\eta]}\int \nu(d\sigma_\Lambda)1_B(\sigma_\Lambda\bar\sigma_{\mathbb{Z}^d\setminus\Lambda})e^{-\sum_{A:A\cap\Lambda\neq\emptyset}\Phi_A(\sigma_\Lambda\bar\sigma_{\mathbb{Z}^d\setminus\Lambda},\eta)} \tag{2.1}$$

where $Z_\Lambda^{\Phi;\bar\sigma}[\eta]$ is the η-dependent normalization factor called the quenched partition function. The symbol $\sigma_\Lambda\bar\sigma_{\mathbb{Z}^d\setminus\Lambda}$ denotes the configuration in Ω that is given by σ_x for $x\in\Lambda$ and by $\bar\sigma_x$ for $x\in\mathbb{Z}^d\setminus\Lambda$.

The second ingredient of the quenched model is the distribution of the disorder variables $\mathbb{P}(d\eta)$. Most of the times in the theory of disordered systems one considers the case of i.i.d. variables only, but we assume more generally that this distribution is a translation-invariant Gibbs measure for a translation invariant potential $\varphi\in\mathcal{B}^1(\bar\Omega)$ for an a priori product measure $\bar\nu(d\eta)$. That is, a version of the conditional expectation $\mathbb{P}(\,\cdot\,|\bar\eta_{\Lambda^c})$ is given by the specification

$$\bar\gamma_\Lambda^{\varphi;\bar\eta}(\bar B) := \frac{1}{\bar Z_\Lambda^{\varphi;\bar\eta}}\int \bar\nu(d\eta_\Lambda)1_{\bar B}(\eta_\Lambda\bar\eta_{\mathbb{Z}^d\setminus\Lambda})e^{-\sum_{A:A\cap\Lambda\neq\emptyset}\varphi_A(\eta_\Lambda\bar\eta_{\mathbb{Z}^d\setminus\Lambda})} \tag{2.2}$$

The objects of interest will then be the infinite volume *joint measures* $K^{\bar\sigma}(d\xi)$, by which we understand any limiting measure of $\lim_{\Lambda\uparrow\infty}\mathbb{P}(d\eta)\mu_\Lambda^{\Phi;\bar\sigma}[\eta](d\sigma)$ in the product topology on the space of joint variables. Of course, there are examples for different joint measures of the same quenched Gibbs specification for different spin boundary conditions $\bar\sigma$. Also there can even be different ones for the same spin-boundary condition $\bar\sigma$, depending on the sub-sequence.

The crucial question is: Given a joint measure K, is there a potential $\Psi\in\mathcal{B}^1(\Omega\times\bar\Omega)$ such that K can be written as a Gibbs measure on the space of joint variables, such that a version of the conditional expectation $K(\,\cdot\,|\bar\xi_{\Lambda^c})$ is given by the specification

$$\gamma_\Lambda^{\Psi;\bar\xi}(\hat B) := \frac{1}{\hat Z_\Lambda^{\bar\xi}}\int \hat\nu(d\xi_\Lambda)1_{\bar B}(\xi_\Lambda\bar\xi_{\mathbb{Z}^d\setminus\Lambda})e^{-\sum_{A:A\cap\Lambda\neq\emptyset}\Psi_A(\xi_\Lambda\bar\xi_{\mathbb{Z}^d\setminus\Lambda})} \tag{2.3}$$

If this is true we write $K\in\mathcal{G}(\Psi)$. Very often such a regular potential in $\mathcal{B}^1(\Omega\times\bar\Omega)$ does not exist in a low temperature situation, but a potential with only a.s. convergence properties always exists (see [K01]). The theorems we are about to prove apply to the nice situation where a regular potential in $\mathcal{B}^1(\Omega\times\bar\Omega)$ does exist. We refer the reader to [EFS93] for an excellent pedagogical discussion of the relevance and the differences of the spaces $\mathcal{B}^0$ and $\mathcal{B}^1$, and why the space $\mathcal{B}^1$ can be considered as the natural setup for ordinary Gibbsian theory.

For all of this the reader might think of the concrete example of the *Random Field Ising model.* Here the spin variables σ_x take values in $\{-1,1\}$. The defining potential of the quenched specification $\Phi_{\beta,h}(\sigma,\eta)$ is given by $\Phi_{\beta,h;\{x,y\}}(\sigma,\eta) = -\beta\sigma_x\sigma_y$ for nearest neighbors $x,y\in\mathbb{Z}^d$, $\Phi_{\beta,h;\{x\}}(\sigma,\eta) = -h\eta_x\sigma_x$, and $\Phi_{\beta,h;A} = 0$ else. Let us choose the disorder variables $\eta_x\in\{-1,1\}$ distributed according to a Gibbs measure $\mathbb{P}_{\bar\beta}$ for the standard nearest neighbor Ising potential $\varphi_{\bar\beta}$ given by $\varphi_{\bar\beta,\{x,y\}}(\eta) = -\bar\beta\eta_x\eta_y$, and $\varphi_{\bar\beta;A} = 0$ else. For $\bar\beta = 0$ the usual random field Ising model with symmetric i.i.d. fields is recovered. Denote by $K^{\bar\sigma}_{\bar\beta;\beta,h}(d\sigma d\eta) = \lim_{\Lambda\uparrow\infty}\mathbb{P}_{\bar\beta}(d\eta)\mu_\Lambda^{\Phi_{\beta,h};\bar\sigma}[\eta](d\sigma)$ any translation-invariant joint measure obtained with the parameters $\bar\beta;\beta,h$. By monotonicity arguments one has in particular that the limit with plus boundary conditions $\mu_\Lambda^{\Phi_{\beta,h};+}[\eta](d\sigma) =: \mu^{\Phi_{\beta,h};+}[\eta](d\sigma)$ exists for any fixed η, and $K^{+}_{\bar\beta;\beta,h}(d\sigma d\eta) = \mu^{\Phi_{\beta,h};+}[\eta](d\sigma)$ is translation-invariant. We come back to this model to exemplify our results in Section 5.

3. First fundamental theorem - Single-valuedness of joint potential

In this setup we have the following unicity result.

Theorem 1. *Denote by*

$$\begin{aligned} K_1(d\eta d\sigma) &= \lim_{N\uparrow\infty}\mathbb{P}_1(d\eta)\mu_{\Lambda_N}^{\Phi;\bar\sigma_1}[\eta](d\sigma),\\ K_2(d\eta d\sigma) &= \lim_{N\uparrow\infty}\mathbb{P}_2(d\eta)\mu_{\Lambda'_N}^{\Phi;\bar\sigma_2}[\eta](d\sigma) \end{aligned} \tag{3.1}$$

two (possibly different) translation-invariant joint measures for the same defining potential $\Phi \in \mathcal{B}^1(\Omega \times \bar\Omega)$, obtained with any two spin boundary conditions $\bar\sigma_1$ (and $\bar\sigma_2$ respectively), along any subsequences Λ_N (and Λ'_N respectively), for translation invariant (possibly different) disorder distributions $\mathbb{P}_1(d\eta), \mathbb{P}_2(d\eta)$ that are Gibbs measures for the same potential $\varphi \in \mathcal{B}^1(\bar\Omega)$. Then either of the following is true:

(i) There exists a translation-invariant joint interaction potential $\Psi \in \mathcal{B}^1(\Omega \times \bar\Omega)$ for which both K_1 and K_2 are Gibbs measures.

(ii) Both K_1 and K_2 are not Gibbs measures for any potential $\Psi \in \mathcal{B}^1(\Omega \times \bar\Omega)$

Proof. The proof of the theorem is simple. We will prove below that the relative entropy density vanishes, i.e.

$$h(K_1|K_2) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\int K_1(d\sigma_\Lambda d\eta_\Lambda)\log\Bigl(\frac{dK_1}{dK_2}\Big|_\Lambda(\sigma_\Lambda\eta_\Lambda)\Bigr) = 0 \tag{3.2}$$

for an increasing sequence of cubes $\Lambda \uparrow \mathbb{Z}^d$ (or more generally any van Hove sequence). This is a slight generalization of the statement in [KLR02] to the present setup of Gibbsian disorder variables and more general spin spaces. But from here the proof follows from the same argument as the proof of Theorem 3.4 of [EFS93]: Assume that K_2 has a translation-invariant potential $\Psi \in \mathcal{B}^1(\Omega \times \bar\Omega)$. Then the classical variational principle for Gibbs measures ([Geo88] Theorem 15.37) implies that also K_1 is a Gibbs measure for the same interaction. Since the roles of K_1 and K_2 are symmetric, the statement of the theorem follows.

For convenience of the reader let us give the argument that leads to the vanishing of the relative entropy density in this context which is slightly more general than that of

[KLR02]. Let us define an auxiliary finite volume joint measure $K^0_\Lambda(d\sigma_\Lambda d\eta_\Lambda)$ by putting free boundary conditions on both disorder variables and spin variables. That is, we put

$$K^0_\Lambda(d\sigma_\Lambda d\eta_\Lambda) := \int \mathbb{P}^{\varphi;0}_\Lambda(d\eta_\Lambda) \int \mu^{\Phi;0}_\Lambda[\eta_\Lambda](d\sigma_\Lambda) \tag{3.3}$$

where

$$\frac{d\mathbb{P}^{\varphi;0}_\Lambda}{d\bar{\nu}_\Lambda}(\eta_\Lambda) := \frac{e^{-\sum_{A\subset\Lambda}\varphi_A(\eta_\Lambda)}}{\bar{Z}^{\varphi;0}_\Lambda} \tag{3.4}$$

and

$$\frac{d\mu^{\Phi;0}_\Lambda[\eta]}{d\nu_\Lambda}(\sigma_\Lambda) := \frac{e^{-\sum_{A\subset\Lambda}\Phi_A(\sigma_\Lambda\eta_\Lambda)}}{Z^{\Phi;0}_\Lambda[\eta]} \tag{3.5}$$

Note that for any joint local event A_Λ depending only on $\sigma_\Lambda\eta_\Lambda$ we have

$$e^{-r(\Lambda)}K^0_\Lambda(A_\Lambda) \le \lim_{N\uparrow\infty}\int \mathbb{P}_1(d\eta)\int \mu^{\Phi;\bar{\sigma}_1}_{\Lambda_N}[\eta](d\sigma)(A_\Lambda) \le e^{r(\Lambda)}K^0_\Lambda(A_\Lambda) \tag{3.6}$$

with $\lim_\Lambda r(\Lambda)/|\Lambda| = 0$ when Λ runs along a sequence of increasing cubes (or more generally a van Hove sequence.) The requirement on the summability properties on both potentials, i.e. $\Phi \in \mathcal{B}^1(\Omega\times\bar{\Omega})$ and $\varphi \in \mathcal{B}^1(\bar{\Omega})$, is needed at this point to control the contribution from the boundary and ensure that $\sum_{A\cap\Lambda\neq\emptyset, A\cap\Lambda^c\neq\emptyset}\sup_{\sigma,\eta}|\Phi_A(\sigma\eta)| = o(|\Lambda|)$ and the similar statement for φ. (See [EFS93] Proposition 2.45 (d). To make this statement hold one needs $\mathcal{B}^1$ instead of just $\mathcal{B}^0$, and this is the reason for dealing with the smaller space $\mathcal{B}^1$.) From (3.6) follows that

$$\frac{dK_1}{dK_2}\Big|_\Lambda = \frac{dK_1|_\Lambda}{dK^0_\Lambda}\Big(\frac{dK_2|_\Lambda}{dK^0_\Lambda}\Big)^{-1} \le e^{2r(\Lambda)} \tag{3.7}$$

and this implies (3.2). ♢

4. Second fundamental theorem - Norm estimates on the joint potential

To formulate the continuity results, we need to introduce the appropriate norms on the various spaces of potentials we are considering taking account physical equivalence. We follow closely the notations of [EFS93] and adopt them to the present situation of the product space $\Omega\times\bar{\Omega}$. The bounded measurable functions on this space are denoted by $B(\Omega\times\bar{\Omega})$, equipped with the corresponding (essential) sup-norm $\|\cdot\|_{B(\Omega\times\bar{\Omega})}$. We also use the obvious notations $\|\cdot\|_{B(\Omega)}$ (and $\|\cdot\|_{B(\bar{\Omega})}$) for functions on Ω (and $\bar{\Omega}$).

Let us fix one of the spaces Ω, $\bar{\Omega}$, or $\Omega\times\bar{\Omega}$. Denote by $\mathcal{J}$ the space of continuous functions on this space having zero expectation for any translation-invariant measure. Given a potential Φ on this space define the energy density function $f_\Phi := \sum_{A\ni 0}\frac{\Phi_A}{|A|}$.

Using the same symbol define also the sub-spaces of potentials $\mathcal{J} := \{\Phi : f_\Phi \in \mathcal{J}\}$ and $\mathcal{J} + \mathit{Const} := \{\Phi : f_\Phi \in \mathcal{J} + \mathit{Const}\}$. To stress the importance of this space we recall the basic fact about physical equivalence of translation-invariant potentials which states (see [EFS93] Theorem 2.42.): Two potentials in $\mathcal{B}^1$ on (say) $\bar{\Omega}$ give rise to the same specifications if and only if their difference is in $\mathcal{J} + \mathit{Const}$ (if every open set in $\bar{\Omega}$ gets positive $\bar{\nu}$-a priori measure.) Correspondingly, denote $\|\Phi\|_{\mathcal{B}^0/(\mathcal{J}+\mathit{Const})} := \inf_{\Phi' \in \mathcal{J}+\mathit{Const}} \|\Phi - \Phi'\|_{\mathcal{B}^0}$. The necessity of this quotient norm for Gibbs potentials is clear because physical equivalence must be divided out. On the level of functions one needs the norm $\|f\|_{B(\Omega\times\bar{\Omega})/\mathit{Const}} = \inf_{c\in\mathbb{R}} \|f - c\|_{B(\Omega\times\bar{\Omega})}$.

Next we will have to make explicit again the underlying spaces Ω and $\bar{\Omega}$. For our set-up of quenched random systems we will then also use the obvious notation $\|\Phi\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{B}^1(\bar{\Omega})+\mathcal{J}+\mathit{Const})} = \inf_{\varphi'\in\mathcal{B}^0(\bar{\Omega})} \|\Phi - \varphi'\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{J}+\mathit{Const})}$. The occurrence of this norm for the defining potentials Φ of the quenched specification is natural because it factors out purely η-dependent parts φ' that give rise to the same quenched specification (2.1). Now we are able to formulate the continuity result.

Theorem 2 (Lipschitz continuity). *The map $(\varphi, \Phi) \to \Psi$ is Lipschitz-continuous in the $\mathcal{B}^0/(\mathcal{J} + \mathit{Const})$-norm wherever it is defined.*

More precisely, assume that

$$\begin{aligned} K_1(d\eta d\sigma) &= \lim_{N\uparrow\infty} \mathbb{P}_1(d\eta)\mu_{\Lambda_N}^{\Phi_1;\bar{\sigma}}[\eta](d\sigma), \\ K_2(d\eta d\sigma) &= \lim_{N\uparrow\infty} \mathbb{P}_2(d\eta)\mu_{\Lambda'_N}^{\Phi_2;\bar{\sigma}'}[\eta](d\sigma) \end{aligned} \tag{4.1}$$

are two translation-invariant joint measures, for possibly different defining potentials $\Phi_1, \Phi_2 \in \mathcal{B}^1(\Omega\times\bar{\Omega})$, and possibly different disorder-distributions $\mathbb{P}_1 \in \mathcal{G}(\varphi_1), \mathbb{P}_2 \in \mathcal{G}(\varphi_2)$, for potentials $\varphi_1, \varphi_2 \in \mathcal{B}^1(\bar{\Omega})$, for the same a priori product measure $\bar{\nu}$.

Assume that there are joint potentials $\Psi_1, \Psi_2 \in \mathcal{B}^1(\Omega\times\bar{\Omega})$ such that $K_1 \in \mathcal{G}(\Psi_1), K_2 \in \mathcal{G}(\Psi_2)$, for the same a priori product measure $\hat{\nu}$. Then

$$\begin{aligned} &\|\Psi_1 - \Psi_2\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{J}+\mathit{Const})} \\ &\leq \|\varphi_1 - \varphi_2\|_{\mathcal{B}^0(\bar{\Omega})/(\mathcal{J}+\mathit{Const})} + 2\|\Phi_1 - \Phi_2\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{B}^1(\bar{\Omega})+\mathcal{J}+\mathit{Const})} \end{aligned} \tag{4.2}$$

Proof. The proof follows from the general fact that one can reconstruct a translation-invariant Hamiltonian (modulo physical equivalence) from the densities in finite volume, up to corrections of the order of the boundary. More precisely, one has

$$\|\Psi_1 - \Psi_2\|_{\mathcal{B}^0/(\mathcal{J}+\mathit{Const})} = \lim_{\Lambda} \frac{1}{|\Lambda|} \left\| \log \frac{dK_1}{dK_2}\Big|_{\Lambda} \right\|_{B(\Omega\times\bar{\Omega})/\mathit{Const}} \tag{4.3}$$

for any sequence of increasing cubes Λ (or more generally van Hove sequence). This follows from Proposition 2.46 formula (2.65) from [EFS93] under the hypothesis that K_1 and K_2 are both Gibbs measures for interactions $\Psi_1, \Psi_2 \in \mathcal{B}^1$.

As in the proof of Theorem 1 we approximate both joint measures by finite volume approximations with free boundary conditions of the form (3.3), introducing

$$\begin{aligned} K^{0,1}_\Lambda(d\sigma_\Lambda d\eta_\Lambda) &:= \int \mathbb{P}^{\varphi_1;0}_\Lambda(d\eta_\Lambda)\int \mu^{\Phi_1;0}_\Lambda[\eta_\Lambda](d\sigma_\Lambda)\\ K^{0,2}_\Lambda(d\sigma_\Lambda d\eta_\Lambda) &:= \int \mathbb{P}^{\varphi_2;0}_\Lambda(d\eta_\Lambda)\int \mu^{\Phi_2;0}_\Lambda[\eta_\Lambda](d\sigma_\Lambda) \end{aligned} \tag{4.4}$$

We write

$$\frac{dK_1}{dK_2}\Big|_\Lambda = \frac{dK_1|_\Lambda}{dK^{0,1}_\Lambda}\frac{dK^{0,1}_\Lambda}{dK^{0,2}_\Lambda}\Big(\frac{dK_2|_\Lambda}{dK^{0,2}_\Lambda}\Big)^{-1} \tag{4.5}$$

As in the proof of the first theorem we have that

$$e^{-2r(\Lambda)} \le \frac{dK_1|_\Lambda}{dK^{0,1}_\Lambda}\Big(\frac{dK_2|_\Lambda}{dK^{0,2}_\Lambda}\Big)^{-1} \le e^{2r(\Lambda)} \tag{4.6}$$

with a function $r(\Lambda) = o(|\Lambda|)$ that depends on $\varphi_1, \varphi_2, \Phi_1, \Phi_2$. This shows that we only need to look at the middle term of the r.h.s. of (4.5) to control the r.h.s of (4.3).

We may then rewrite the term of interest in the form

$$\begin{aligned} &\frac{dK^{0,1}_\Lambda}{dK^{0,2}_\Lambda}(\sigma_\Lambda\eta_\Lambda) = \frac{dK^{0,1}_\Lambda}{d(\nu\times\bar\nu)_\Lambda}(\sigma_\Lambda\eta_\Lambda)\Big(\frac{dK^{0,2}_\Lambda}{d(\nu\times\bar\nu)_\Lambda}(\sigma_\Lambda\eta_\Lambda)\Big)^{-1}\\ &= \exp\Big(-\sum_{A\subset\Lambda}[\varphi_{1-2,A}(\eta_\Lambda) + \Phi_{1-2,A}(\sigma_\Lambda,\eta_\Lambda)]\Big)\frac{\bar Z^{\varphi_2;0}_\Lambda}{\bar Z^{\varphi_1;0}_\Lambda}\frac{Z^{\Phi_2;0}_\Lambda[\eta_\Lambda]}{Z^{\Phi_1;0}_\Lambda[\eta_\Lambda]} \end{aligned} \tag{4.7}$$

where we have used the short notations $\varphi_{1-2} := \varphi_1 - \varphi_2$ and $\Phi_{1-2} := \Phi_1 - \Phi_2$.

It is obvious that this expression does not depend on the choice of the representatives of the defining potential of the quenched specification, $\Phi^0_1 \in \Phi_1 + \mathcal{B}^1(\bar\Omega)$, $\Phi^0_2 \in \Phi_2 + \mathcal{B}^1(\bar\Omega)$. This factors out the purely disorder-dependent terms. We also use that the first (but not the second!) quotient of partition functions in the first line is just a constant and so we have that

$$\begin{aligned} &\Big\|\log\frac{dK^{0,1}_\Lambda}{dK^{0,2}_\Lambda}\Big\|_{B(\Omega\times\bar\Omega)/Const}\\ &= \Big\|\sum_{A\subset\Lambda}[\varphi_{1-2,A}(\eta_\Lambda = \cdot) + \Phi^0_{1-2,A}(\sigma_\Lambda, \eta_\Lambda = \cdot)] - \log\frac{Z^{\Phi^0_2;0}_\Lambda[\eta_\Lambda = \cdot]}{Z^{\Phi^0_1;0}_\Lambda[\eta_\Lambda = \cdot]}\Big\|_{B(\Omega\times\bar\Omega)/Const}\\ &\le \Big\|\sum_{A\subset\Lambda}\varphi_{1-2,A}\Big\|_{B(\bar\Omega)/Const} + \Big\|\sum_{A\subset\Lambda}\Phi^0_{1-2,A}\Big\|_{B(\Omega\times\bar\Omega)/Const} + \Big\|\log\frac{Z^{\Phi^0_2;0}_\Lambda[\eta_\Lambda = \cdot]}{Z^{\Phi^0_1;0}_\Lambda[\eta_\Lambda = \cdot]}\Big\|_{B(\bar\Omega)/Const} \end{aligned} \tag{4.8}$$

where of course $\Phi^0_{1-2} := \Phi^0_1 - \Phi^0_2$.

The control of the first two terms is well-known. From [EFS93] Proposition 2.44 we have for the first term that

$$\begin{aligned}&\Big\|\sum_{A\subset\Lambda}\varphi_{1-2,A}\Big\|_{B^0(\bar\Omega)/Const} = \inf_c\sup_\eta\Big|\sum_{A\subset\Lambda}\varphi_{1-2,A}(\eta_\Lambda)-c\Big|\\ &= |\Lambda|\cdot\|\varphi_{1-2}\|_{\mathcal{B}^0(\bar\Omega)/(\mathcal{J}+Const)}+o(|\Lambda|)\end{aligned}\tag{4.9}$$

whenever $\varphi_{1-2}\in B^1(\bar\Omega)$. Similarly the second is bounded by $|\Lambda|\cdot\|\Phi_{1-2}\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{J}+Const)}+o(|\Lambda|)$. Now, to control the quotient of the two quenched partition functions we use the following lemma. It will be used again in the proof of Theorem 3.

Lemma 1 (quenched partition functions). *Suppose that* $\Phi,\Phi'\in\mathcal{B}^1(\Omega\times\Omega')$. *Then we have that*

$$\left\|\log\frac{Z_\Lambda^{\Phi;0}[\eta_\Lambda=\cdot]}{Z_\Lambda^{\Phi';0}[\eta_\Lambda=\cdot]}\right\|_{B(\bar\Omega)/Const}\le|\Lambda|\cdot\|\Phi-\Phi'\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/\mathcal{J}+Const}+o(|\Lambda|)\tag{4.10}$$

Assuming the lemma, the proof of Theorem 2 follows.

Proof of Lemma 1. Indeed, we have that

$$\begin{aligned}&\inf_c\sup_\eta\left|\log\frac{Z_\Lambda^{\Phi;0}[\eta_\Lambda]}{Z_\Lambda^{\Phi';0}[\eta_\Lambda]}-c\right|\\ &=\inf_c\sup_\eta\left|\log\int\mu_\Lambda^{\Phi';0}[\eta_\Lambda](d\tilde\sigma)\exp\Big(-\sum_{A\subset\Lambda}[\Phi_A(\tilde\sigma_\Lambda,\eta_\Lambda)-\Phi'_A(\tilde\sigma_\Lambda,\eta_\Lambda)-c]\Big)\right|\\ &\le\inf_c\sup_{\sigma\eta}\left|\sum_{A\subset\Lambda}[\Phi_A(\sigma_\Lambda,\eta_\Lambda)-\Phi'_A(\sigma_\Lambda,\eta_\Lambda)-c]\right|\\ &=|\Lambda|\cdot\|\Phi-\Phi'\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/\mathcal{J}+Const}+o(|\Lambda|)\end{aligned}\tag{4.11}$$

Here the last line follows from [EFS93] Proposition 2.44.◇◇

In the case where the defining potentials of the quenched specification do not depend on the disorder variable, i.e. when $\Phi_1\in\mathcal{B}^1(\Omega)$ and $\Phi_2\in\mathcal{B}^1(\Omega)$, K_1 and K_2 are trivially Gibbs with the joint potentials $\varphi_1+\Phi_1$ and $\varphi_2+\Phi_2$. In this case the l.h.s. of (4.2) becomes equal to the sum of norms $\|\varphi_1-\varphi_2\|_{\mathcal{B}^0(\bar\Omega)/(\mathcal{J}+Const)}+\|\Phi_1-\Phi_2\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{B}^1(\bar\Omega)+\mathcal{J}+Const)}$, whereas the r.h.s. carries a factor 2. Responsible for this fact was the estimation of the quotient of quenched partition functions. So, let us finally give an alternative estimate that is better for ‘small randomness’ and becomes sharp in the situation where the quenched specification does not depend on the disorder variable.

Theorem 3. *Assume in the situation of Theorem 2 that* $\Phi_1^0 \in \Phi_1 + \mathcal{B}^1(\bar\Omega)$ *is a fixed representative of the defining potential, and similarly* $\Phi_2^0 \in \Phi_2 + \mathcal{B}^1(\bar\Omega)$*. Then we have the alternative estimate*

$$\begin{aligned} &\|\Psi_1 - \Psi_2\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{J}+Const)} \\ &\leq \|\varphi_1 - \varphi_2\|_{\mathcal{B}^0(\bar\Omega)/(\mathcal{J}+Const)} + \|\Phi_1^0 - \Phi_2^0\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{B}^1(\bar\Omega)+\mathcal{J}+Const)} \\ &+ \|\Phi_1^0\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{B}^1(\Omega)+\mathcal{J}+Const)} + \|\Phi_2^0\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/(\mathcal{B}^1(\Omega)+\mathcal{J}+Const)} \end{aligned} \tag{4.12}$$

Remark. In the last two terms we have factored out the disorder-independent contributions of the defining potential of the quenched specification. Trivially, these terms vanish when the quenched specification is disorder-independent, and so the bound becomes sharp in that case.

Proof. The proof follows from (4.8) and an alternative bound on the quotient of quenched partition functions that is obtained by comparison with non-random partition functions for potentials $\chi_1, \chi_2 \in \mathcal{B}^1(\Omega)$. Writing

$$\frac{Z_\Lambda^{\Phi_2^0;0}[\eta_\Lambda = \cdot]}{Z_\Lambda^{\Phi_1^0;0}[\eta_\Lambda = \cdot]} = \frac{Z_\Lambda^{\Phi_2^0;0}[\eta_\Lambda = \cdot]}{Z_\Lambda^{\chi_2;0}} \frac{Z_\Lambda^{\chi_2;0}}{Z_\Lambda^{\chi_1;0}} \frac{Z_\Lambda^{\chi_1;0}}{Z_\Lambda^{\Phi_1^0;0}[\eta_\Lambda = \cdot]} \tag{4.13}$$

we get from Lemma 1 that

$$\begin{aligned} &\left\| \log \frac{Z_\Lambda^{\Phi_2^0;0}[\eta_\Lambda = \cdot]}{Z_\Lambda^{\Phi_1^0;0}[\eta_\Lambda = \cdot]} \right\|_{B(\bar\Omega)/Const} \\ &\leq \left\| \log \frac{Z_\Lambda^{\Phi_2^0;0}[\eta_\Lambda = \cdot]}{Z_\Lambda^{\chi_2;0}} \right\|_{B(\bar\Omega)/Const} + \left\| \log \frac{Z_\Lambda^{\Phi_1^0;0}[\eta_\Lambda = \cdot]}{Z_\Lambda^{\chi_1;0}} \right\|_{B(\bar\Omega)/Const} \\ &\leq |\Lambda| \cdot \Bigl(\|\Phi_1^0 - \chi_1\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/\mathcal{J}+Const} + \|\Phi_2^0 - \chi_2\|_{\mathcal{B}^0(\Omega\times\bar\Omega)/\mathcal{J}+Const} \Bigr) + o(|\Lambda|) \end{aligned} \tag{4.14}$$

Optimizing over χ_1, χ_2 this proves Theorem 3.$\diamondsuit$

5. Example: Random Field Ising model with dependent disorder fields

We use the notations that were introduced at the end of Section 2. According to Theorem 1 we know that, *if* there is a joint potential $\Psi_{\bar\beta;\beta,h} \in \mathcal{B}^1(\Omega \times \bar\Omega)$ for a particular joint measure $K_{\bar\beta;\beta,h}^{\bar\sigma}(d\sigma d\eta)$, it is necessarily a joint potential for any, possibly different translation-invariant measure for the same $\bar\beta, \beta, h$. In particular K^+ and K^- (obtained with minus-boundary conditions) must have the same potential. We remark that there is a pretty complete analysis of the Gibbsian properties for $\bar\beta = 0$. Here we know that exponential decay of $\sup_\eta |\mu_{\beta,h}^+[\eta](\sigma_x\sigma_y) - \mu_{\beta,h}^+[\eta](\sigma_x)\mu_{\beta,h}^+[\eta](\sigma_y)|$ in the distance $|x-y|$

implies that there really is a joint potential for K^+ in $\mathcal{B}^1$. (See the remark after (2.11) in [K01].) Exponential decay uniform in the realization η can be shown e.g. by cluster expansion methods for β sufficiently small, in any dimension, for any h. So, our Theorems 1-3 apply in this situation. In contrast to that, there is a regime where the joint measure does not possess a potential in $\mathcal{B}^1$, but a translation-invariant potential that converges only a.s. and this happens in the phase transition regime of the quenched model, i.e for β large, h small and $d \geq 3$. Indeed, the conditional expectations of K^+ and K^- were shown to be different in this case, even though there is a joint potential for K^+ and a different one for K^- that are translation-invariant and decay like a stretched exponential on a full measure set. This is shown in [KLR02], based on the general representation of the joint potential given in [K01] and the renormalization group analysis of the quenched specification of the random field Ising model of [BK88].

Let us come to the case of dependent disorder, that is the inverse temperature of the random fields $\bar{\beta}$ is positive. We remark first that the joint potential constructed in [K01] for $\bar{\beta} = 0$ is of the form $\Phi_{\beta,h}(\sigma,\eta) + U_{\beta,h}(\eta)$ where the non-trivial potential $U_{\beta,h}(\eta)$ was shown to be uniformly summable if the quenched correlations are uniformly exponentially decaying. More generally, the following is true: Whenever $U_{\beta,h} \in \mathcal{B}^1(\bar{\Omega})$ we have that a joint potential $\Phi_{\beta,h}(\sigma,\eta) + \varphi_{\bar{\beta}}(\eta) + U_{\beta,h}(\eta)$ is a Gibbs potential for any corresponding joint measure, for any $\bar{\beta}$, with *the same* $U_{\beta,h}(\eta)$. In particular the plus-state $\mathbb{P}^+$ and minus-state $\mathbb{P}^-$ for the random fields give rise to the same joint potential for the corresponding joint measures $\mathbb{P}^+(d\eta)\mu^+[\eta](d\sigma)$ (resp. $\mathbb{P}^-(d\eta)\mu^+[\eta](d\sigma)$), in the regime of small inverse spin-temperature β. This can be seen by going through the arguments of Section 3 in [K01] (including Lemma 4.1 therein). These arguments consist of two parts: The first is a formal computation in finite volume, and the second deals with the infinite volume limit. Both parts stay correct when the i.i.d. distribution for the disorder variables is replaced by a Gibbs measure with finite range potential. The focus of [K01] however was to show almost sure summability of the potential $U(\eta)$ by abstract arguments for general models, even when it fails to be summable everywhere. The generalization of *this* part of [K01] to dependent disorder variables is less obvious, so we leave a discussion of the non-Gibbsian regime to a future publication.

Let us now illustrate the application of the Theorems 2 and 3 to the present situation.

Corollary from Theorem 2. *Assume that there are joint potentials* $\Psi_{\bar{\beta}_1;\beta_1,h_1}, \Psi_{\bar{\beta}_2;\beta_2,h_2} \in \mathcal{B}^1(\Omega \times \bar{\Omega})$ *for the translation-invariant joint measures* $K_{\bar{\beta}_1;\beta_1,h_1}, K_{\bar{\beta}_2;\beta_2,h_2}$*, obtained as weak limits. Then*

$$\begin{aligned}&\|\Psi_{\bar{\beta}_1;\beta_1,h_1} - \Psi_{\bar{\beta}_2;\beta_2,h_2}\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{J}+Const)}\\ &\leq d|\bar{\beta}_1 - \bar{\beta}_2| + 2d|\beta_1 - \beta_2| + 2|h_1 - h_2|\end{aligned} \tag{5.1}$$

Corollary from Theorem 3. *In the same situation we have the alternative estimate*

$$\begin{aligned}&\|\Psi_{\bar{\beta}_1;\beta_1,h_1} - \Psi_{\bar{\beta}_2;\beta_2,h_2}\|_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{J}+Const)}\\ &\leq d|\bar{\beta}_1 - \bar{\beta}_2| + d|\beta_1 - \beta_2| + |h_1 - h_2| + |h_1| + |h_2|\end{aligned} \tag{5.2}$$

Remark. While the first bound shows the continuity w.r.t. the parameters of the defining potential and potential of a priori distribution, the second bound is better for small magnetic fields. In particular for $h_1 = h_2 = 0$ the disorder variables and the spin variables decouple, the joint potential is just the sum $\varphi + \Phi$, and we have that

$$||\Psi_{\bar{\beta}_1;\beta_1,h_1=0} - \Psi_{\bar{\beta}_2;\beta_2,h_2=0}||_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{J}+Const)} = d|\bar{\beta}_1 - \bar{\beta}_2| + d|\beta_1 - \beta_2| \tag{5.3}$$

(see below) and so the bound is sharp.

Proof of Corollaries. To compute the norm of the potentials occurring, define some corresponding energy density function and compute its sup-norm modulo constants.

For the defining potential of the quenched specification $\Phi_{\beta,h}(\sigma\eta)$ we put e.g. $f'_{\beta,h}(\sigma\eta) := -\sigma_0[\beta\sum_{i=1}^{d}\sigma_{e_i} - h\eta_0]$ where the sum is over all unity vectors e_i spanning $\mathbb{Z}^d$ and pointing in positive directions. Then we have that

$$||\Phi_{\beta,h}||_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{B}^1(\bar{\Omega})+\mathcal{J}+Const)} = \inf_{c}\sup_{\sigma\eta}\Big|f'_{\beta,h}(\sigma\eta) - c\Big| = d|\beta| + |h| \tag{5.4}$$

For the first equality, see [EFS93] Proposition A.11. Similarly we have $||\varphi_{\bar{\beta}}||_{\mathcal{B}^0(\bar{\Omega})/(\mathcal{J}+Const)} = d|\bar{\beta}|$. Finally we have

$$||\Phi_{\beta,h}||_{\mathcal{B}^0(\Omega\times\bar{\Omega})/(\mathcal{B}^1(\Omega)+\mathcal{J}+Const)} = \sup_{\sigma\eta}\Big|-h\sigma_0\eta_0\Big| = |h| \tag{5.5}$$

by subtraction of the purely σ-dependent part.◇

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