Confinement to lowest Landau band and application to quantum current

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Abstract

In this paper we study neutral atoms of nuclear charge Z in a strong constant magnetic field of strength B. We improve the bound on confinement to lowest Landau band given by Lieb, Solovej and Yngvason in [LSY94a]. This permits to calculate the asymptotic form of the quantum current of atoms in the parameter region $Z^{4/3} \ll B \ll Z^{46/27}$.

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1 Introduction

1.1 Discussion of results

The behavior of atoms in strong magnetic fields has been the object of study of a number of papers in recent years. The mathematical analysis most notably started with the works [LSY94a] and [LSY94b] (see these for a discussion of earlier work).

In this paper we will study the local current (local magnetic moment) of the atoms (see precise definition below). The current (magnetic moment) is the variational derivative of the ground state energy of the atom with respect to the external magnetic vector potential (external magnetic field). In the previous work [Fou01b] the current of neutral atoms was calculated in the asymptotic regime $B, Z \to \infty$ with $B/Z^{4/3}$ bounded. This includes the regime where the magnetic field starts to play a role (see [LSY94a] and [LSY94b]), but fails to include the regime ($B \gg Z^{4/3}$) where magnetic effects become very strong (confinement to lowest Landau band). In the recent paper [Fou] the easier mean-field case was considered in a semiclassical limit corresponding to $Z^{4/3} \ll B \ll Z^3$. The essential new work there was to obtain (what can be understood as) an improved estimate on confinement to the lowest Landau band. Unfortunately, this confinement estimate is not readily generalizable to the N-body case. In this paper we use another approach to obtain a confinement estimate (Theorem 1.1 below) and collect the consequences for the current. Our result on the current in the present paper does not include the entire region $Z^{4/3} \ll B$ since the confinement estimate is weaker than the corresponding result in [Fou]. However, we do go (far) beyond the $B \approx Z^{4/3}$ range studied in [Fou01b].

Before further discussion, let us set the mathematical stage. The magnetic hamiltonian of an N-electron atom of nuclear charge Z in a magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$ is

$$\begin{aligned} H(N, Z, \mathbf{A}) &= \sum_{j=1}^{N} \left((\mathbf{p}^{(j)} + \mathbf{A}(x^{(j)}))^2 + \sigma^{(j)} \cdot \mathbf{B}(x^{(j)}) - \frac{Z}{|x^{(j)}|} \right) \\ &\quad + \sum_{1 \leq j < k \leq N} \frac{1}{|x^{(j)} - x^{(k)}|}. \end{aligned} \tag{1.1}$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator $H(N, Z, \mathbf{A})$ acts on the electronic Hilbert space (including spin) $\mathcal{H} = \wedge_{j=1}^{N} L^2(\mathbb{R}^3; \mathbb{C}^2)$. As usual $\mathbf{p} = (-i\nabla)$ and a superscript (j) denotes that the corresponding operator acts on the j-th factor in the antisymmetric tensor product defining $\mathcal{H}$. In particular $x^{(j)}$ is the coordinate of the j-th electron. In the following we will use the notation $\mathbf{p}_{\mathbf{A}} = (\mathbf{p} + \mathbf{A})$.

Let $\psi \in \mathcal{H}$ be a ground state of $H(N, Z, \mathbf{A})$ with energy $E(N, Z, \mathbf{A})$. It is easy to see that (if the derivative exists)

$$\frac{d}{dt} E(N, Z, \mathbf{A} + t\mathbf{a}) \Big|_{t=0} = \langle \psi; J(\mathbf{a}) \psi \rangle. \tag{1.2}$$

Here we have introduced the notation $\langle \cdot; \cdot \rangle$ for the scalar product in $\mathcal{H}$ and we have for $\mathbf{b} = \operatorname{curl} \mathbf{a} = (b_1, b_2, b_3)$,

$$J(\mathbf{a}) = \sum_{j=1}^{N} \mathbf{a}(x^{(j)}) \cdot \mathbf{p}_{\mathbf{A}}^{(j)} + \mathbf{p}_{\mathbf{A}}^{(j)} \cdot \mathbf{a}(x^{(j)}) + \sigma^{(j)} \cdot \mathbf{b}(x^{(j)}). \tag{1.3}$$

In what follows we will fix $\mathbf{B} = (0, 0, B)$ and $\mathbf{A}(x) = \frac{1}{2} \mathbf{B} \times x$. The parameter B, which measures the strength of the magnetic field, will tend to infinity. With this choice of $\mathbf{B}, \mathbf{A}$ we will write $H(N, Z, B)$ instead of $H(N, Z, \mathbf{A})$. Notice that since $\mathbf{B} \cdot \sigma = B\sigma_3$, we can replace the term $\sigma^{(j)} \cdot \mathbf{b}(x^{(j)})$ in (1.3) by $b_3(x^{(j)})\sigma_3^{(j)}$.

Define the ground state energy of the atom by

$$E(N,Z,B) = \inf \operatorname{Spec} H(N,Z,B). \tag{1.4}$$

We will sometimes denote this energy by $E^{\mathrm{Q}}(N,Z,B)$ to distinguish it from other energies present in the paper. Furthermore, if ψ is a normalized ground state of $H(N,Z,B)$, i.e. a normalized function with $\langle \psi; H(N,Z,B)\psi\rangle = E(N,Z,B)$, we define the ground state density as the following function in $L^1(\mathbb{R}^3)$:

$$\rho^{\mathrm{Q}}(x) = N \int_{\mathbb{R}^{3N-3}} |\psi(x, x^{(2)}, \ldots, x^{(N)})|^2 \, dx^{(2)} \cdots dx^{(N)}.$$

(Here $|\cdot|$ is the euclidean norm in $\mathbb{C}^N$.) This definition of the density depends on the existence of a ground state. Notice that if the ground state is degenerate, then ρ^{Q} is not uniquely defined. This non-uniqueness is not important. The convergence results below will be true for any choice of the ground state in case of degeneracy.

The energy was the main object of study in [LSY94b, LSY94a]. There it was proved that if N/Z is held fixed as $N, Z, B \to \infty$ we have the following asymptotic behaviour (with $\beta = B/Z^{4/3}$) of the atomic ground state energy.

$$E(N,Z,B) \approx \begin{cases} Z^{7/3} & B/Z^{4/3} \text{ bounded} \\ Z^{7/3}\beta^{2/5} & B/Z^{4/3} \to \infty, B/Z^3 \text{ bounded} \\ Z^3(\log(B/Z^3))^2 & B/Z^3 \to \infty. \end{cases} \tag{1.5}$$

In this paper we will be interested in the last two regions above - i.e. $B/Z^{4/3} \to \infty$.

The first result of the present paper concerns confinement to the lowest Landau band and is a sharpening of [LSY94a, Theorem 1.3]. Knowing the asymptotic form of the energy (c.f. (1.5)), one can motivate this confinement as follows: It costs an energy of the order of B to place an electron in a Landau band which is not the lowest. When $B \gg Z^{4/3}$ and we have $N = Z$ electrons, the ground state energy would therefore be of order BZ if a nonvanishing fraction of the electrons were out of the lowest Landau band. But the energy given in (1.5) is clearly much smaller (for $B \gg Z^{4/3}$) than BZ.

To give the statement of the mathematical result we need a bit of notation. The projection Π_0 on the lowest Landau band for one electron has the integral kernel (see for instance [FGPY92])

$$\Pi_0(x,y) = \frac{B}{2\pi} e^{iB(x_1y_2 - x_2y_1)/2} e^{-B(x_\perp - y_\perp)^2/4} \delta(x_3 - y_3) P^\downarrow, \tag{1.6}$$

where $x_\perp = (x_1, x_2)$, and where

$$P^\downarrow = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

is the projection to the spin-down subspace.

We define Π_0^N as the projection in $\wedge_{j=1}^N L^2(\mathbb{R}^3, \mathbb{C}^2)$ to the space where all electrons are in the lowest Landau band, i.e. $\Pi_0^N = \otimes_{j=1}^N \Pi_0^{(j)}$. Now we can define the ground state energy for electrons in the lowest Landau band $E^{\mathrm{Q}}_{conf}(N,Z,B)$ as

$$\begin{aligned} E^{\mathrm{Q}}_{conf}(N,Z,B) &= \inf \operatorname{Spec} \Pi_0^N H(N,Z,B) \Pi_0^N \\ &= \inf_{\psi \in \operatorname{Ran} \Pi_0^N \setminus \{0\}} \frac{\langle \psi; H(N,Z,B)\psi \rangle}{\|\psi\|^2}. \end{aligned}$$

The improved estimate on localization to the lowest Landau band is the following (compare [LSY94a, Theorem 1.3]), proved in Section 3.

Theorem 1.1. *Let $\lambda, \Lambda, c > 0$ be given. Then there exists $C_1, C_2 > 0$ such that if $\lambda < N/Z < \Lambda$, $Z \geq C_2$, $B \leq cZ^3$, and $\beta = BZ^{-4/3} \geq C_2$. Then*

$$E^{\mathrm{Q}}_{conf}(N,Z,B) \geq E^{\mathrm{Q}}(N,Z,B) \geq E^{\mathrm{Q}}_{conf}(N,Z,B)(1 - C_1 R),$$

where $R = \begin{cases} \beta^{-9/10} & B \leq Z^{56/37} \\ \beta^{-6/19} Z^{-2/9} & B \geq Z^{56/37} \end{cases}$

Theorem 1.1 will permit us to calculate the current of neutral atoms up to $B \ll Z^{46/27}$. In subsection 1.3 we will state the results of this calculation in more detail.

We easily get the following corollary to Theorem 1.1, which will be proved in Section 3 after the proof of the theorem. Recall that $\mathbf{p_A} = (p_{\mathbf{A},1}, p_{\mathbf{A},2}, p_{\mathbf{A},3}) = \mathbf{p} + \mathbf{A}$. We now define the kinetic energy operator in the coordinates perpendicular to the magnetic field, $\hat{K}$, as $\hat{K} = p_{\mathbf{A},1}^2 + p_{\mathbf{A},2}^2 + \sigma \cdot \mathbf{B}$. The corresponding total perpendicular kinetic energy operator is

$$\hat{K}^N = \sum_{j=1}^N \hat{K}^{(j)}.$$

Corollary 1.2. *Let* $\lambda, \Lambda, c > 0$ *be given, let* $\psi = \psi_{N,Z,B}$ *be a ground state for* $H(N, Z, B)$ *and let* R *denote the error term from Theorem 1.1. Then there exist* $C_1, C_2 > 0$ *such that if* $\lambda \le N/Z \le \Lambda$, $Z \ge C_2$, $BZ^{-4/3} \ge C_2$, $B/(cZ^3) \le 1$, *then the perpendicular kinetic energy satisfies the estimate*

$$0 \le \langle \psi | \hat{K}^N | \psi \rangle \le C_1 R |E^{\mathrm{Q}}(N, Z, B)|.$$

In particular, for all $\epsilon > 0$ *exists* $C > 0$ *such that if* $\lambda \le N/Z \le \Lambda$, $Z \ge C$, $BZ^{-4/3} \ge C$, $BZ^{-46/27} \le C^{-1}$, *then*

$$\langle \psi | \hat{K}^N | \psi \rangle \le \epsilon \beta^{-3/5} |E^{\mathrm{Q}}(N, Z, B)|.$$

1.2 Short summary of relevant known results on the energy and density

To describe the results on the quantum current, we need to discuss the approximating theories in more detail. In the region $B \ll Z^3$, a Thomas-Fermi type theory – MTF theory – describes the ground state of the atom correctly to highest order. In this theory one assigns an energy to an electronic density $\rho \in L^1(\mathbb{R}^3)$ using the following functional

$$\mathcal{E}^{\mathrm{MTF}}_{Z,B}[\rho] = \int_{\mathbb{R}^3} \left(\tau^{\mathrm{MTF}}_B(\rho(x)) - Z\frac{\rho}{|x|} \right) dx + D(\rho, \rho),$$

where

$$D(f, g) = \frac{1}{2} \iint \frac{\overline{f(x)} g(y)}{|x - y|} \, dx dy. \tag{1.7}$$

Here the kinetic energy term, $\tau^{\mathrm{MTF}}_B(\rho)$, which in usual (non-magnetic) Thomas-Fermi theory is just $c_{TF}\rho^{5/3}$, is given as follows

$$\tau^{\mathrm{MTF}}_B(\rho) \;=\; \sup_{\nu \ge 0} [\rho\nu - P_B(\nu)],$$

with

$$P_B(\nu) \;=\; \frac{B}{3\pi^2} \left[\nu^{3/2} + 2 \sum_{j=1}^{\infty} [\nu - 2B\nu]_+^{3/2} \right].$$

We will also need the limit of $\mathcal{E}^{\mathrm{MTF}}$ for $B \gg Z^{4/3}$, which is the STF-theory. The functional $\mathcal{E}^{\mathrm{STF}}_{Z,B}[\rho]$ is given as $\mathcal{E}^{\mathrm{MTF}}_{Z,B}[\rho]$, but with $\tau^{\mathrm{MTF}}_{B}(\rho)$ replaced by

$$\tau^{\mathrm{STF}}_{B}(\rho) = \frac{4\pi^4}{3B^2}\rho^3.$$

For each of the above theories we define

$$E^{\#}(N,Z,B) = \inf\{\mathcal{E}^{\#}_{Z,B}[\rho]; 0 \le \rho(x), \int \rho(x)\,dx \le N\},$$

where # is either MTF or STF. In both cases it is proved in [LSY94b] that a unique minimizer $\rho^{\#}_{N,Z,B}$ exists.

When $Z^{4/3} \ll B$, we can also approximate by the density matrix (DM) theory. Here one uses a functional on the set $\mathcal{G}_B$ of admissible *density matrices* Γ. In this context a density matrix Γ is a map

$$\Gamma : \mathbb{R}^2 \ni x_\perp \mapsto \Gamma_{x_\perp} \in \mathcal{B}(L^2(\mathbb{R})),$$

where $\mathcal{B}$ denotes the set of bounded operators. The following condition (1.8), defining $\mathcal{G}_B$, is imposed on Γ:

— $x_\perp \mapsto \langle f; \Gamma_{x_\perp} f\rangle$ is measurable $\forall f \in L^2(\mathbb{R})$.

— $\Gamma_{x_\perp}$ is trace class on $L^2(\mathbb{R})$ for almost all $x_\perp \in \mathbb{R}^2$.

— $0 \le \Gamma_{x_\perp} \le \dfrac{B}{2\pi}$ (as operators on $\mathcal{B}(L^2(\mathbb{R}))$) for almost all $x_\perp \in \mathbb{R}^2$.

— $$\int_{\mathbb{R}^2} \mathrm{tr}_{L^2(\mathbb{R})}[(1-\partial^2_{x_3})\Gamma_{x_\perp}]\,dx_\perp < +\infty. \tag{1.8}$$

The functional is now defined as

$$\mathcal{E}^{\mathrm{DM}}_{Z,B}[\Gamma] = \int_{\mathbb{R}^2} \mathrm{tr}_{L^2(\mathbb{R})}[-\partial_3^2 \Gamma_{x_\perp}]\,dx_\perp - Z\int_{\mathbb{R}^3} \frac{\rho_\Gamma(x)}{|x|}\,dx + D(\rho_\Gamma,\rho_\Gamma),$$

where $\rho_\Gamma(x) = \rho_\Gamma(x_\perp, x_3) = \Gamma_{x_\perp}(x_3,x_3)$ and D was defined in (1.7). The corresponding energy is

$$E^{\mathrm{DM}}(N,Z,B) = \inf\{\mathcal{E}^{\mathrm{DM}}_{Z,B}[\Gamma]; \int \rho_\Gamma(x)\,dx \le N, \Gamma \text{ satisfies (1.8)}\}.$$

The result on convergence of the energies can be written as follows.

Theorem 1.3 (Energy asymptotics [LSY94a, LSY94b]). *Let* N/Z *be fixed and let* $\psi = \psi_{N,Z,B}$ *be a ground state of* $H(N,Z,B)$.

Suppose that $B/Z^3 \to 0$ *as* $Z \to \infty$, *then*

$$\frac{E^{\mathrm{Q}}(N,Z,B)}{E^{\mathrm{MTF}}(N,Z,B)} \to 1 \text{ as } Z \to \infty. \tag{1.9}$$

Furthermore, with $\ell = Z^{-1/3}(1 + B/Z^{4/3})^{-2/5}$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \rho^{\mathrm{Q}}(x) Z\ell^{-1} V(x/\ell)\, dx \\ = &\int_{\mathbb{R}^3} \rho^{\mathrm{MTF}}(x) Z\ell^{-1} V(x/\ell)\, dx + o(E^{\mathrm{Q}}(N,Z,B)), \end{aligned} \tag{1.10}$$

as $Z \to \infty$, *for all* $V \in L^{5/2}(\mathbb{R}^3)$, $\operatorname{supp} V$ *compact.*

If $B/Z^{4/3} \to \infty$ *and* $B/Z^3 \to 0$ *as* $Z \to \infty$, *then* (1.9) *and* (1.10) *hold with* MTF *replaced by* STF.

Finally, if $B/Z^{4/3} \to \infty$ *as* $Z \to \infty$, *then*

$$\frac{E^{\mathrm{Q}}(N,Z,B)}{E^{\mathrm{DM}}(N,Z,B)} \to 1 \text{ as } Z \to \infty. \tag{1.11}$$

Furthermore, if $B/Z^3 \geq c > 0$, *then for all* $W \in L^\infty(\mathbb{R}^3)$, $\operatorname{supp} W$ *compact, we have, with* $\eta = \max(2, \frac{B}{2\pi Z^3})$,

$$\begin{aligned} &Z^2(\log\eta)^2 \int_{\mathbb{R}^3} \rho^{\mathrm{Q}}(x) W(Z\sqrt{\eta}x_\perp, Z(\log\eta)x_3)\, dx \\ = \;&Z^2(\log\eta)^2 \int_{\mathbb{R}^3} \rho^{\mathrm{DM}}(x) W(Z\sqrt{\eta}x_\perp, Z(\log\eta)x_3)\, dx + o(E^{\mathrm{Q}}(N,Z,B)). \end{aligned} \tag{1.12}$$

Remark 1.4. Theorem 1.3 remains true, even if the ground state ψ does not exist. Then the ground state energy is just the bottom of the spectrum. The convergence of the densities still hold in the sense that the densities of approximate ground states will converge to the minimizers of the appropriate functionals (MTF, STF and DM).

For the current this is generally not true. One can construct (see the appendix to [Fou01a]) approximate ground states, the current of which do not converge to the current of the true ground state.

In particular, the result on the densities in Theorem 1.3 remains true if the ground state is not unique, i.e. if the ground state energy is a degenerate eigenvalue. This stability remains true for the current — i.e. the results below on the current do not assume the ground state energy to be a non-degenerate eigenvalue.

1.3 Results on the current

In order to calculate the current in strong magnetic fields a *virial type theorem* is very useful (see e.g. [Fou01b]). This virial theorem (Lemma 4.1) will be discussed further in Section 4. It is a necessary assumption in order to apply the virial theorem that there exists a ground state (i.e. that the bottom of the spectrum is an eigenvalue). This is the reason why, for the results on the current, we need to assume the existence of a ground state. Using the virial theorem we can write, for all eigenstates ψ of H,

$$B\langle \psi ; J(\mathbf{a})\psi \rangle = \langle \psi ; (J_{\text{KIN}} - J_{\text{INT}} + J_{\text{DENS}})\psi \rangle,$$

where the operators on the right, J_{KIN}, J_{INT} and J_{DENS} are given in (4.3). It turns out to be much easier to calculate the terms on the right hand side than the original expression $\langle \psi ; J(\mathbf{a})\psi \rangle$.

Most of our results will only be valid in (a part of) the region $Z^{4/3} \ll B \ll Z^3$. In this region both MTF (and the similar STF)-theory and DM-theory are asymptotically correct with respect to the energy. For MTF (and STF)-theory, one can easily define an MTF-current since MTF-theory readily generalizes to non-constant magnetic fields. This was carried out in [Fou01b]. The results below imply, using the calculations in [Fou01b], that MTF-theory not only gives correctly the energy and the density (as stated in Theorem 1.3) but also the current.

The other approximating theory, DM-theory, does not easily generalize to non-constant magnetic fields. It is therefore of interest that the mathematical analysis below indicates (even if not proved beyond $B \ll Z^{46/27}$) what the current in extreme magnetic fields should be.

Now we will discuss each of those three terms and how it is analyzed in the paper. The operator J_{DENS} is actually just a multiplication operator $J_{\text{DENS}} = \sum_{j=1}^{N} \tilde{V}(x^{(j)})$. Thus

$$\langle \psi ; J_{\text{DENS}}\psi \rangle = \int \rho_\psi \tilde{V}\, dx,$$

where ρ_ψ is the density of the atom. We therefore immediately get the asymptotics of $J_{\rm DENS}$ from Theorem 1.3. Let us first define the common notation for the results on the different parts of the current. We take an $\mathbf{a}_{sc} = (a_{sc,1}, a_{sc,2}, 0) \in C_0^4(\mathbb{R}^3;\mathbb{R}^3)$, i.e. a test function that lives on a scale of order 1. We then scale it to the size of an atom

$$\mathbf{a}(x) = \mathbf{a}_{Z,B}(x) = \begin{cases} \ell \mathbf{a}_{sc}(x/\ell) & \text{for } B \leq 2Z^3 \\ Z(\log\eta)^2 \mathbf{a}_{sc}(Z\eta^{1/2}x_\perp, Z(\log\eta)x_3) & \text{for } B \geq 2Z^3, \end{cases} \tag{1.13}$$

where $\eta = B/Z^3$ and $\ell = Z^{-1/3}\beta^{-2/5}$.

Remark 1.5. In order to calculate the current for general $\mathbf{a}$ (i.e. not assuming $a_3 \equiv 0$), one can follow the strategy of [Fou01b]. That argument uses the calculation of the current for $\mathbf{a}$'s with $a_3 \equiv 0$ and a symmetry property of the hamiltonian (1.1) together with gauge invariance. We will not repeat this argument here.

Theorem 1.6. *Let $\lambda, \Lambda > 0$ be given, let ψ be a ground state of $H(N,Z,B)$ and let $\mathbf{a}$ be as in (1.13). Then for all $\epsilon > 0$ exists $C > 0$ such that if*

$$\lambda \leq N/Z \leq \Lambda, \qquad Z \geq C, \qquad BZ^{-3} \leq C^{-1}, \tag{1.14}$$

then

$$\begin{aligned} &\left| \langle \psi; J_{\rm DENS}\psi\rangle - Z\int_{\mathbb{R}^3} \frac{x_2(a_1(x)-a_1(0)) - x_1(a_2(x)-a_2(0))}{|x|^3} \rho^{\rm MTF}(x)\,dx \right| \\ \leq\ & \epsilon \left| E^{\rm Q}(N,Z,B) \right|. \end{aligned}$$

The conclusion remains valid with MTF *replaced by* STF *or* DM *if we impose the further condition $BZ^{-4/3} \geq C$ in* (1.14).

The second operator, $J_{\rm INT}$, is very similar to the electron-electron interaction $\sum_{j<k} |x^{(j)} - x^{(k)}|^{-1}$. In order to analyze this term we need to slightly generalize the Lieb-Oxford correlation inequality. Furthermore, we need to study a density matrix theory which is perturbed by a resulting current term. This is done in Section 5. The result of all this is the following

Theorem 1.7. *Let the assumptions be as in Theorem 1.6, then*

$$\begin{aligned} &\left| \langle \psi; J_{\rm INT}\psi\rangle = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \rho^{\rm MTF}(x) \frac{(\tilde{a}(x)-\tilde{a}(y))\cdot(x-y)}{|x-y|^3} \rho^{\rm MTF}(y)\,dxdy \right| \\ \leq\ & \epsilon \left| E^{\rm Q}(N,Z,B) \right|. \end{aligned}$$

If we add the condition $BZ^{-4/3} \geq C$ *in* (1.14)*, then we may replace* MTF *by* STF *in the conclusion. If we replace the condition* $BZ^{-3} \leq C^{-1}$ *in* (1.14) *by the two conditions* $BZ^{-4/3} \geq C$, $BZ^{-14/3} \leq C^{-1}$*, then the conclusion holds with* MTF *replaced by* DM.

Finally, we have to deal with J_{KIN}. This is an operator which resembles the kinetic energy. However, it contains an off-diagonal part which couples the lowest Landau band with the second. This off-diagonal part causes some trouble. In general we would expect this term to be small:

Conjecture 1.8. *Let* $\lambda, \Lambda > 0$ *be given and let* $\psi = \psi_{N,Z,B}$ *be a ground state for* $H(N,Z,B)$*. Then for all* $\epsilon > 0$ *exists a constant* $C > 0$ *such that if*

$$\lambda \leq N/Z \leq \Lambda, \qquad Z \geq C, \qquad BZ^{-4/3} \geq C, \tag{1.15}$$

then

$$|\langle\psi; J_{\mathrm{KIN}}\psi\rangle| \leq \epsilon \left|E^{\mathrm{Q}}(N,Z,B)\right|.$$

Unfortunately, we are not able to prove this estimate in the entire parameter region $B \gg Z^{4/3}$. In fact, we get

Theorem 1.9. *Let the notation and conditions be as in Conjecture 1.8 but with the assumption* $BZ^{-46/27} \leq C^{-1}$ *added in* (1.15)*. Then the conclusion of Conjecture 1.8 holds.*

Remark 1.10. Notice that $46/27 > 45/27 = 5/3$.

The proof of this theorem is carried through in Sections 3 and 6. In Section 6 we reduce the proof to the estimate on confinement to the lowest Landau band; Theorem 1.1. That estimate is proved in Section 3. The estimate needed is a sharpening of the similar result in [LSY94a]. This improved confinement is interesting in its own right, independent of applications to the current.

2 Basic estimates

In this section we collect a few basic estimates that will be used in the paper. First we summarize some results on the different parts of the energy (kinetic, potential).

Lemma 2.1 ([LSY94a]). *Let* $\lambda, \Lambda > 0$ *be given and let* ψ *be a ground state for* $H(N, Z, B)$. *Then for all* $\epsilon > 0$ *exist constants* $c, C > 0$ *such that if*

$$\lambda \leq N/Z \leq \Lambda, \qquad \beta = BZ^{-4/3} \geq C, \tag{2.1}$$

then

$$\langle \psi; \hat{K}^N \psi \rangle \leq \epsilon \left| E^{\mathrm{Q}}(N, Z, B) \right| .$$

and

$$\left\langle \psi \middle| \sum_{j=1}^{N} (\mathbf{p}_{\mathbf{A},3}^{(j)})^2 \middle| \psi \right\rangle \leq c \left| E^{\mathrm{Q}}(N, Z, B) \right| .$$

Furthermore, if $V \in L^{5/2}(\mathbb{R}^3)$, $\operatorname{supp} V$ *compact, and the condition* $BZ^{-3} \leq C^{-1}$ *is added in* (2.1), *then*

$$\left| \left\langle \Pi_0^N \psi \middle| \sum_{j=1}^{N} Z\ell^{-1} V(x^{(j)}/\ell) \middle| \Pi_0^N \psi \right\rangle - \int \rho^{\mathrm{Q}}(x) Z\ell^{-1} V(x/\ell)\, dx \right| \\ \leq \epsilon \left| E^{\mathrm{Q}}(N, Z, B) \right| .$$

In particular,

$$\left| \left\langle \Pi_0^N \psi \middle| \sum_{j=1}^{N} Zl^{-1} V(x^{(j)}/l) \middle| \Pi_0^N \psi \right\rangle \right| \leq c \left| E^{\mathrm{Q}}(N, Z, B) \right| .$$

We will also need to know that the projection Π_0 commutes (to a very high accuracy) with regular functions. This is given more precisely by the following lemma.

Lemma 2.2. *Let* Π_0 *be the projection on the lowest Landau band given in* (1.6). *Suppose* f *is a function satisfying* $\nabla f \in L^\infty(\mathbb{R}^3)$ *(i.e.* $f \in C^{0,1}(\mathbb{R}^3)$ *the space of uniformly Lipschitz continuous functions). Then there exists a constant independent of* B *and* f *such that*

$$\|[\Pi_0, f]\| \leq cB^{-1/2} \|\nabla f\|_\infty .$$

Where $\|\nabla f\|_\infty$ *is the* L^∞*-norm of* ∇f.

Proof. The operator $[\Pi_0, f]$ has kernel $K(x, y) = \Pi_0(x, y)(f(y) - f(x))$. The lemma follows using $f(y) - f(x) = (y - x) \int_0^1 \nabla f(x + t(y - x))\, dt$, the explicit form of $\Pi_0(x, y)$ given in (1.6) and Schur's Lemma. ☐

3 Improved localization to lowest Landau band

The objective of this section is to prove the localization to the lowest Landau band — Theorem 1.1. The proof follows some of the ideas in [LSY94a, Section 6] but with more careful estimates. Furthermore, we are able to gain extra presicion by noticing that Π_0 essentially commutes with the potential away from the singularities of the potential.

Proof. Part 1. Splitting of H. We write

$$H = \sum_{j=1}^{N} \left(H_A^{(j)} - ZV(x^{(j)}) \right) + \sum_{j<k} V(x^{(j)} - x^{(k)}),$$

with $V(x) = |x|^{-1}$. We will split V as follows. Let f_1, f_2 be a smooth partition of unity on $\mathbb{R}$ satisfying $f_1(x) = 1$ for $|x| \leq 1$, $\operatorname{supp} f_1 \subset [-2, 2]$. Define

$$V_{j,\delta}(x) = \frac{f_j(\delta^{-1}|x|)}{|x|}.$$

Then $V = V_{1,\delta} + V_{2,\delta}$. Notice, using Lemma 2.2, that

$$\|\Pi_0 V_{2,\delta} \Pi_>\| = \|\Pi_0 [\Pi_0, V_{2,\delta}] \Pi_>\| \leq cB^{-1/2} \|\nabla V_{2,\delta}\|_\infty = cB^{-1/2}\delta^{-2}. \qquad (3.1)$$

We will now proceed as in [LSY94a, Section 6] but using (3.1). Define Π^α for $\alpha \subseteq \{1, \ldots, N\}$ by

$$\Pi^\alpha = \prod_{j\in\alpha} \Pi_0^{(j)} \prod_{j\notin\alpha} \Pi_>^{(j)}.$$

With this notation we will prove that for any $0 < \epsilon_1, \epsilon_2 < 1$,

$$H \geq \sum_\alpha \Pi^\alpha \left(\hat{H}^\alpha + \tilde{H}^\alpha \right) \Pi^\alpha, \qquad (3.2)$$

where

$$\begin{aligned} \hat{H}^\alpha =& \sum_{j\in\alpha} \left(H_A^{(j)} - ZV(x^{(j)}) - \epsilon_1 Z V_{1,\delta}(x^{(j)}) - c\epsilon_2 Z B^{-1/2}\delta^{-2} \right) \\ &+ \sum_{j<k, j,k\in\alpha} \left(V(x^{(j)} - x^{(k)}) - 3\epsilon_1 V_{1,\delta}(x^{(j)} - x^{(k)}) - c\epsilon_2 B^{-1/2}\delta^{-2} \right). \end{aligned}$$

and

$$\begin{aligned}\tilde{H}^\alpha =&\sum_{j\notin\alpha}\left(H_A^{(j)} - ZV(x^{(j)}) - \epsilon_1^{-1}ZV_{1,\delta}(x^{(j)}) - c\epsilon_2^{-1}ZB^{-1/2}\delta^{-2}\right)\\ &+\sum_{j<k,j,k\in\alpha}\left(V(x^{(j)}-x^{(k)}) - 3\epsilon_1^{-1}V_{1,\delta}(x^{(j)}-x^{(k)}) - c\epsilon_2^{-1}B^{-1/2}\delta^{-2}\right)\\ &+\sum_{j\in\alpha,k\notin\alpha}\left(V(x^{(j)}-x^{(k)}) - \tfrac{3}{2}(\epsilon_1+\epsilon_1^{-1})V_{1,\delta}(x^{(j)}-x^{(k)}) - c\epsilon_2^{-1}B^{-1/2}\delta^{-2}\right).\end{aligned}$$

In order to prove (3.2) notice first that it is an equality for the kinetic energy terms $H_A^{(j)}$, since these commute with the Π^α. For $V(x^{(j)})$ we proceed as follows:

$$\begin{aligned}V(x^{(j)}) =&\Pi_0^{(j)}V(x^{(j)})\Pi_0^{(j)} + \Pi_>^{(j)}V(x^{(j)})\Pi_>^{(j)}\\ &+\left(\Pi_>^{(j)}V_{1,\delta}(x^{(j)})\Pi_0^{(j)} + \Pi_0^{(j)}V_{1,\delta}(x^{(j)})\Pi_>^{(j)}\right)\\ &+\left(\Pi_>^{(j)}V_{2,\delta}(x^{(j)})\Pi_0^{(j)} + \Pi_0^{(j)}V_{2,\delta}(x^{(j)})\Pi_>^{(j)}\right). \qquad (3.3)\end{aligned}$$

For the cross terms with $V_{1,\delta}$ in (3.3), we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}&-\Pi_>^{(j)}V_{1,\delta}(x^{(j)})\Pi_0^{(j)} - \Pi_0^{(j)}V_{1,\delta}(x^{(j)})\Pi_>^{(j)}\\ &\qquad\geq -\epsilon_1\Pi_0^{(j)}V_{1,\delta}(x^{(j)})\Pi_0^{(j)} - \epsilon_1^{-1}\Pi_>^{(j)}V_{1,\delta}(x^{(j)})\Pi_>^{(j)}.\end{aligned}$$

To estimate the cross terms with $V_{2,\delta}$ in (3.3), we use (3.1) before applying Cauchy-Schwarz and get

$$-\Pi_>^{(j)}V_{2,\delta}(x^{(j)})\Pi_0^{(j)} - \Pi_0^{(j)}V_{2,\delta}(x^{(j)})\Pi_>^{(j)} \geq -\epsilon_2B^{-1/2}\delta^{-2}\Pi_0^{(j)} - \epsilon_2^{-1}B^{-1/2}\delta^{-2}\Pi_>^{(j)}.$$

This gives the terms in $\hat{H}^\alpha$ and $\tilde{H}^\alpha$ with one-particle potentials. For the two-particle terms $V(x^{(j)}-x^{(k)})$ we proceed similarly, but with projections Π_0 and $\Pi_>$ in both the j and k variables.

Part 2. Study of $\hat{H}^\alpha$. Notice, that we actually proved (3.2) as an operator inequality on all of $\otimes_{j=1}^N L^2(\mathbb{R}^3,\mathbb{C}^2)$. For the next steps we will need the antisymmetry of the electronic wavefunctions. Suppose that $\psi\in\wedge_{j=1}^N L^2(\mathbb{R}^3,\mathbb{C}^2)$, then $\Pi^\alpha\psi(x^{(1)},\ldots,x^{(N)})$ is antisymmetric separately in the variables in α and those in the complement, $\tilde{\alpha}$. We will bound $\Pi^\alpha\left(\hat{H}^\alpha+\tilde{H}^\alpha\right)\Pi^\alpha$ from below by

$\inf \operatorname{Spec} \Pi^\alpha \hat{H}^\alpha \Pi^\alpha + \inf \operatorname{Spec} \Pi^\alpha \tilde{H}^\alpha \Pi^\alpha$, where the (anti-)symmetry conditions above are imposed on each of the operators.

We clearly have

$$\begin{aligned}\hat{H}^\alpha \geq & \Big\{ \sum_{j\in\alpha} \Big(H_A^{(j)} - ZV(x^{(j)}) - \epsilon_1 Z V_{1,\delta}(x^{(j)}) \Big) \\ & + \sum_{j<k, j,k\in\alpha} \big(V(x^{(j)} - x^{(k)}) - 3\epsilon_1 V_{1,\delta}(x^{(j)} - x^{(k)}) \big) \Big\} - c\epsilon_2 Z^2 B^{-1/2} \delta^{-2}.\end{aligned} \tag{3.4}$$

As usual extra electrons can be sent to infinity, so we have

$$\inf \operatorname{Spec} \Pi^\alpha \hat{H}^\alpha \Pi^\alpha \geq \inf \operatorname{Spec} \Pi_0^N H^{\alpha_0} \Pi_0^N - c\epsilon_2 Z^2 B^{-1/2} \delta^{-2},$$

where H^{α_0} is the operator in $\{\cdot\}$ in (3.4) with $\alpha = \{1, \ldots, N\}$.

We will now estimate the difference between the ground state energy of the perturbed operator, $\Pi_0^N H^{\alpha_0} \Pi_0^N$, and and the confined energy E^{Q}_{conf}. The result will be

$$\inf \operatorname{Spec} \Pi_0^N H^{\alpha_0} \Pi_0^N \geq E^{\mathrm{Q}}_{conf}(N, Z, B) - c\epsilon_1 \delta \beta^{2/5} Z^{1/3} E^{\mathrm{Q}}(N, Z, B). \tag{3.5}$$

Notice that $\ell = Z^{-1/3}\beta^{-2/5}$ is the mean size (the size of an atom in magnetic Thomas Fermi theory) of a neutral atom with nuclear size Z in a magnetic field of magnitude B. Thus the error bound above, of order $\epsilon_1\delta/\ell E^{\mathrm{Q}}$, is very natural.

To prove (3.5) we decompose H^{α_0} as

$$H^{\alpha_0} = H_1^{\alpha_0} + \epsilon_1 \frac{\delta}{\ell} H_2^{\alpha_0} + \epsilon_1 \frac{\delta}{\ell} H_3^{\alpha_0},$$

where

$$\begin{aligned} H_1^{\alpha_0} &= (1 - 2\epsilon_1 \frac{\delta}{\ell}) \sum_{j=1}^N H_{\mathbf{A}}^{(j)} - \frac{Z}{|x^{(j)}|} + \sum_{j<k} \frac{1}{|x^{(j)} - x^{(k)}|}, \\ H_2^{\alpha_0} &= \sum_{j=1}^N H_{\mathbf{A}}^{(j)} - \frac{Z\ell}{\delta} V_{1,\delta}(x^{(j)}) - \sum_{j<k} \frac{\ell}{\delta} V_{1,\delta}(x^{(j)} - x^{(k)}), \\ H_3^{\alpha_0} &= \sum_{j=1}^N H_{\mathbf{A}}^{(j)} - \frac{2Z}{|x^{(j)}|} + \sum_{j<k} \frac{2}{|x^{(j)} - x^{(k)}|}. \end{aligned}$$

In order to estimate $H_2^{\alpha_0}$ from below we write

$$H_2^{\alpha_0} = \frac{1}{N-1}\sum_{k=1}^{N}\left\{ \sum_{j=1,j\neq k}^{N} H_{\mathbf{A}}^{(j)} - \frac{Z\ell}{\delta}V_{1,\delta}(x^{(j)}) - \frac{\ell(N-1)}{2\delta}V_{1,\delta}(x^{(j)}-x^{(k)})\right\}.$$

For fixed k we consider the operator in $\{\cdot\}$. Since there is no kinetic energy term $H_{\mathbf{A}}^{(k)}$ we may estimate

$$\begin{aligned}&\inf \operatorname{Spec} \Pi_0^N\{\cdot\}\Pi_0^N \\ &\geq \inf \operatorname{Spec} \Pi_0^N \left\{ \sum_{j=1,j\neq k}^{N} H_{\mathbf{A}}^{(j)} - \frac{Z\ell}{\delta}V_{1,\delta}(x^{(j)}) - \frac{\ell(N-1)}{2\delta}V_{1,\delta}(x^{(j)}-0)\right\}\Pi_0^N.\end{aligned}$$

Now we are in a position to use the Lieb-Thirring inequality for the lowest Landau band, Lemma 3.2, to get

$$\begin{aligned}\inf \operatorname{Spec} \Pi_0^N\{\cdot\}\Pi_0^N &\geq \frac{2}{3\pi}B\int_{\mathbb{R}^3}\left[\frac{\ell(Z+(N-1)/2)}{\delta}V_{1,\delta}(x)\right]^{3/2}dx \\ &= -c\beta^{2/5}Z^{7/3}.\end{aligned}$$

This proves the bound (3.5) for $\epsilon_1\frac{\delta}{\ell}H_2^{\alpha_0}$.

The same bound holds clearly for $\epsilon_1\frac{\delta}{\ell}H_3^{\alpha_0}$, using the magnetic Lieb-Thirring inequality. Thus we have proved (3.5).

We will choose ϵ_2, δ such that

$$\epsilon_2\delta^{-2}Z^2B^{-1/2} = \epsilon_1\delta\beta^{2/5}Z^{1/3}E^{\mathrm{Q}}(N,Z,B). \qquad (3.6)$$

Thus we get the following final bound, using (3.4), (3.5) and (3.6)

$$\inf \operatorname{Spec} \Pi^\alpha\hat{H}^\alpha\Pi^\alpha \geq E^{\mathrm{Q}}_{conf}(N,Z,B) - c\epsilon_1\delta\beta^{2/5}Z^{1/3}E^{\mathrm{Q}}(N,Z,B). \qquad (3.7)$$

Part 3. Study of $\tilde{H}^\alpha$. For $\tilde{H}^\alpha$ we get the following estimate for $j \notin \alpha$ $\Pi^\alpha H_A^{(j)}\Pi^\alpha \geq \frac{1}{2}\Pi^\alpha((p_A^{(j)})^2+2B)\Pi^\alpha$. This follows as in [LSY94a] since the jth electron is restricted away from the lowest Landau band (when $j \notin \alpha$). We want the error term from the commutator to be controled by the positive B above, so we get the condition

$$\epsilon_2^{-1}ZB^{-1/2}\delta^{-2} \leq B/2. \qquad (3.8)$$

Thus we get the following estimate on $\Pi^\alpha \tilde{H}^\alpha \Pi^\alpha$

$$\begin{aligned} & \Pi^\alpha \tilde{H}^\alpha \Pi^\alpha \\ \geq\ & \Pi^\alpha \Big\{ \sum_{j \notin \alpha} \Big((p_A^{(j)})^2 + B - 2\epsilon_1^{-1} Z V(x^{(j)}) \Big) - 3\epsilon_1^{-1} \sum_{j<k, j,k \in \alpha} V(x^{(j)} - x^{(k)}) \\ & -3\epsilon_1^{-1} \sum_{j \in \alpha, k \notin \alpha} V(x^{(j)} - x^{(k)}) \Big\} \Pi^\alpha. \end{aligned} \tag{3.9}$$

We will, without loss of generality, suppose that $\{1, \ldots, N\} \setminus \alpha = \{1, \ldots, n\}$. Then we can write the operator on the right hand side in (3.9), as

$$\begin{aligned} \frac{1}{n-1} \sum_{k=1}^{n} \Big\{ \sum_{j=1, j\neq k}^{n} \Big((p_A^{(j)})^2 + B - \frac{2\epsilon_1^{-1} Z}{|x^{(j)}|} - \frac{3\epsilon_1^{-1}(n-1)/2}{|x^{(j)} - x^{(k)}|} \\ - \sum_{s=n+1}^{N} \frac{3\epsilon_1^{-1}}{|x^{(j)} - x^{(s)}|} \Big) \Big\}. \end{aligned}$$

We will, for each fixed k, bound the operator in $\{\}$ independently of k. Notice that there is no kinetic energy term $(p_A^{(k)})^2$ nor terms with $(p_A^{(s)})^2$ for $s > n$. Therefore, (as described in [LSY94a, p. 571]) up to a translation $x^{(k)} \mapsto x^{(k)} + a^{(k)}$, $x^{(s)} \mapsto x^{(s)} + a^{(s)}$, for $s > n$, it is enough to bound

$$\begin{aligned} & \sum_{j=1, j\neq k}^{n} (p_A^{(j)})^2 + B - \frac{2\epsilon_1^{-1} Z}{|x^{(j)}|} - \frac{3\epsilon_1^{-1}(n-1)/2}{|x^{(j)}|} - \sum_{s=n+1}^{N} \frac{3\epsilon_1^{-1}}{|x^{(j)}|} \\ = & \sum_{j=1, j\neq k}^{n} (p_A^{(j)})^2 + B - W(x^{(j)}), \end{aligned}$$

where $W(x) = \epsilon_1^{-1} \frac{2Z - 3(n-1)/2 - 3(N-n)}{|x|}$. Using the *non-magnetic* Lieb-Thirring inequality ([LT93]) we can bound this operator from below by

$$-c_0 \int_{\mathbb{R}^3} [B - W(x)]_-^{5/2}\, dx \geq -c' B^{-1/2} (\epsilon_1^{-1} Z)^3.$$

Here we used that $\lambda \leq \frac{N}{Z} \leq \Lambda$. Thus we get our final lower bound on $\tilde{H}^\alpha$ uniformly in α as

$$\tilde{H}^\alpha \geq c_1 \epsilon_1^{-3} Z^{7/3} \beta^{-1/2} = c_2 \epsilon_1^{-3} \beta^{-9/10} E^{\mathrm{Q}}(N, Z, B). \tag{3.10}$$

Part 4. Choice of parameters. By combining (3.2), (3.7) and (3.10) (under the conditions (3.6) and (3.8)) we get

$$E^{\mathrm{Q}} \geq E^{\mathrm{Q}}_{conf} + c(\epsilon_1 \delta \beta^{2/5} Z^{1/3} + \epsilon_1^{-3} \beta^{-9/10}) E^{\mathrm{Q}}. \tag{3.11}$$

We choose

$$\delta = \epsilon_1^{-1/5} \beta^{-14/25} Z^{-7/15}, \qquad \epsilon_2 = \epsilon_1^{2/5} \beta^{-19/50} Z^{-1/15}, \tag{3.12}$$

in order to satisfy (3.6) and (3.8). That implies

$$\epsilon_1 \delta \beta^{2/5} Z^{1/3} = \epsilon_1^{4/5} \beta^{-4/25} Z^{-2/15},$$

and we optimize (3.11) in ϵ_1 under the condition $\epsilon_1 \leq 1$. This gives

$$\epsilon_1 = \begin{cases} 1 & B \leq Z^{56/37} \\ \beta^{-37/190} Z^{2/57} & B \geq Z^{56/37} \end{cases} \tag{3.13}$$

and the bound becomes

$$E^{\mathrm{Q}}(N, Z, B) \geq E^{\mathrm{Q}}_{conf}(N, Z, B) + cRE^{\mathrm{Q}}(N, Z, B),$$

where

$$R = \begin{cases} \beta^{-9/10} & B \leq Z^{56/37} \\ \beta^{-6/19} Z^{-2/19} & B \geq Z^{56/37} \end{cases}.$$

□

Remark 3.1. Notice that $\beta^{-6/19} Z^{-2/19} \ll \beta^{-3/5}$ iff $B \ll Z^{46/27}$. Notice furthermore, $46/27 - 56/37 = 190/999 > 0$.

In the proof we used the following Lieb-Thirring type inequality.

Lemma 3.2 (Lieb-Thirring inequality in the lowest Landau band). *Suppose* $[V(x)]_- \in L^{3/2}(\mathbb{R}^3)$. *Let* $H = H_{\mathbf{A}} + V(x)$ *on* $L^2(\mathbb{R}^3; \mathbb{C}^2)$. *Let* $\tilde{e}_j(B.V)$ *be the negative eigenvalues of* $\Pi_0 H \Pi_0$ *(counted with multiplicity). Then*

$$\sum_j |\tilde{e}_j(B.V)| \leq \frac{2}{3\pi} B \int_{\mathbb{R}^3} [V(x)]_-^{3/2}\, dx.$$

Proof. The proof is analogous, but a bit simpler than the proof of [LSY94a, Theorem 2.1]. We may clearly assume that $V(x) \leq 0$ for all $x \in \mathbb{R}^3$. Define the Birman-Schwinger kernel K_E for $E > 0$, by

$$K_E = \left|\Pi_0 V \Pi_0 + \frac{E}{2}\right|_-^{1/2} (\Pi_0 H_{\mathbf{A}} \Pi_0 + \frac{E}{2})^{-1} \left|\Pi_0 V \Pi_0 + \frac{E}{2}\right|_-^{1/2}.$$

The Birman-Schwinger principle says that the number N_E of eigenvalues of $\Pi_0 H \Pi_0$ below $-E$ is bounded above by the number of eigenvalues of K_E which are ≥ 1. This again is bounded above by $\mathrm{tr} K_E$, since K_E is a positive operator. Thus, we get

$$\sum_j |\tilde{e}_j(B.V)| = \int_0^\infty N_E \, dE \leq \int_0^\infty \mathrm{tr} K_E \, dE.$$

To calculate $\mathrm{tr} K_E$, we first notice that, since $E > 0$,

$$\left|\Pi_0 V \Pi_0 + \frac{E}{2}\right|_- = \Pi_0 \left|V + \frac{E}{2}\right|_- \Pi_0.$$

Therefore, $\left|\Pi_0 V \Pi_0 + \frac{E}{2}\right|_-^{1/2} = \Pi_0 \left|V + \frac{E}{2}\right|_-^{1/2} \Pi_0$. If we now use the definition of tr and the fact that is positive, we can estimate

$$\begin{aligned}
\mathrm{tr} K_E &= \mathrm{tr}\left(\Pi_0 \left|V + \frac{E}{2}\right|_-^{1/2} \Pi_0 (H_{\mathbf{A}} + \frac{E}{2})^{-1} \Pi_0 \left|V + \frac{E}{2}\right|_-^{1/2} \Pi_0\right) \\
&\leq \mathrm{tr}\left(\left|V + \frac{E}{2}\right|_-^{1/2} \Pi_0 (H_{\mathbf{A}} + \frac{E}{2})^{-1} \Pi_0 \left|V + \frac{E}{2}\right|_-^{1/2}\right) \\
&= \mathrm{tr}\left(\left|V + \frac{E}{2}\right|_- \Pi_0 (p_3^2 + \frac{E}{2})^{-1}\right).
\end{aligned}$$

From here the calculation is straight forward, (and identical to the one in [LSY94b]) using the explicit kernels of Π_0 and $(p_3^2 + \frac{E}{2})^{-1}$. $\square$

Since the Legendre transform, $F_B(t)$, of the function $w \mapsto \frac{2B}{3\pi} w^{3/2}$ is

$$F_B(t) = \frac{\pi^2}{3B^2} t^3,$$

we therefore get (c.f. [LSY94b, Cor. 2.2], [LT93]).

Corollary 3.3. *Let $\psi \in \wedge_{j=1}^{N} \left(\Pi_0^{(j)} L^2(\mathbb{R}^3;\mathbb{C}^2)\right)$ be normalized. Then*

$$\int \rho_\psi^3(x)\,dx \leq \frac{3B^2}{\pi^2}\langle\psi|\sum_{j=1}^{N} H_{\mathbf{A}}^{(j)}|\psi\rangle.$$

We finally give the short proof of Corollary 1.2.

Proof of Corollary 1.2. Let

$$\hat{H} = H - \tfrac{1}{2}\sum_{j=1}^{N}\hat{K}_j,$$

let $\hat{E}(N,Z,B)$ be the ground state energy of $\hat{H}$ and let ψ be a ground state of H. Then

$$0 \geq \frac{-1}{2}\langle\psi|\sum_{j=1}^{N}\hat{K}_j|\psi\rangle \geq \hat{E}(N,Z,B) - E(N,Z,B).$$

One now notices that (the proof of) Theorem 1.1 remains true with H replaced by $\hat{H}$. Thus

$$\hat{E}(N,Z,B) - E(N,Z,B) = O(RE^{\mathrm{Q}}(N,Z,B)).$$

This finishes the proof. □

4 Commutator formula

In calculations of the current the commutator formula (4.2) below is very useful. It makes it possible to replace the operator $J(\mathbf{a})$ by a sum of other operators that all are better behaved than $J(\mathbf{a})$.

Let $\mathbf{a}(x) = (a_1(x), a_2(x), 0) \in C_0^3(\mathbb{R}^3)$. Define

$$\tilde{a}(x) = (-a_2(x), a_1(x), 0) \tag{4.1}$$

In order to avoid worse singularities than $|x|^{-1}$, we also introduce $\tilde{a}_0(x) = \tilde{a}(x) - \tilde{a}(0)$. Then a calculation (see for instance [Fou01b]) gives

$$\frac{1}{2i}\Big[H(N,Z,B), \sum_{j=1}^{N}\tilde{a}(x^{(j)})\cdot p_{\mathbf{A}}^{(j)} + p_{\mathbf{A}}^{(j)}\cdot\tilde{a}(x^{(j)})\Big] = -BJ(\mathbf{a}) + J_{\mathrm{KIN}} - J_{\mathrm{INT}} + J_{\mathrm{DENS}}, \tag{4.2}$$

where

$$\begin{aligned}
J_{\mathrm{KIN}} &= \sum_{j=1}^{N}\left(p_{\mathbf{A}}^{(j)}\cdot M(x^{(j)})p_{\mathbf{A}}^{(j)} + Bb_3(x^{(j)})\sigma_3^{(j)} + \frac{1}{2}\Delta_\perp \operatorname{div}\tilde{a}(x^{(j)})\right),\\
J_{\mathrm{INT}} &= \sum_{1\le j<k\le N}\frac{(x^{(j)}-x^{(k)})\cdot(\tilde{a}(x^{(j)})-\tilde{a}(x^{(k)}))}{|x^{(j)}-x^{(k)}|^3},\\
J_{\mathrm{DENS}} &= \sum_{j=1}^{N}\left(Z\frac{\tilde{a}_0(x^{(j)})\cdot x^{(j)}}{|x^{(j)}|^3} + \frac{1}{2}\Delta_{\|}\operatorname{div}\tilde{a}(x^{(j)})\right), \qquad (4.3)
\end{aligned}$$

where $\Delta_\perp = \partial_{x_1}^2 + \partial_{x_2}^2$ and $\Delta_{\|} = \partial_{x_3}^2$. In the definition of J_{KIN} we introduced the symmetric matrix

$$M(x) = -\left(D\tilde{a} + (D\tilde{a})^t\right) = \begin{pmatrix} 2\partial_1 a_2 & \partial_2 a_2 - \partial_1 a_1 & \partial_3 a_2 \\ \partial_2 a_2 - \partial_1 a_1 & -2\partial_2 a_1 & -\partial_3 a_1 \\ \partial_3 a_2 & -\partial_3 a_1 & 0 \end{pmatrix}, \qquad (4.4)$$

where $D\tilde{a}$ is the Jacobian matrix of $\tilde{a}$ and $(\cdot)^t$ denotes the transpose. Notice that the term $\operatorname{div}\tilde{a}$ satisfies $\operatorname{div}\tilde{a} = \partial_2 a_1 - \partial_1 a_2 = -b_3$. Thus we can rewrite J_{KIN} and J_{DENS},

$$\begin{aligned}
J_{\mathrm{KIN}} &= \sum_{j=1}^{N}\left(p_{\mathbf{A}}^{(j)}\cdot M(x^{(j)})p_{\mathbf{A}}^{(j)} + Bb_3(x^{(j)})\sigma_3^{(j)} - \frac{1}{2}(\Delta_\perp b_3(x^{(j)}))\right)\\
J_{\mathrm{DENS}} &= \sum_{j=1}^{N}\left(Z\frac{\tilde{a}_0(x^{(j)})\cdot x^{(j)}}{|x^{(j)}|^3} - \frac{1}{2}(\Delta_{\|} b_3(x^{(j)}))\right).
\end{aligned}$$

Let now ψ be an eigenstate of $H = H(N,Z,B)$. Then formally (for instance it would be true for matrices) for all operators O we have

$$\langle\psi;[H,O]\psi\rangle = 0.$$

This result can easily be applied to (4.2) above and we have thereby proved:

Lemma 4.1. *Let ψ be an eigenstate of $H(N,Z,B)$. Then*

$$B\langle\psi; J(\mathbf{a})\psi\rangle = \langle\psi;(J_{\mathrm{KIN}} - J_{\mathrm{INT}} + J_{\mathrm{DENS}})\psi\rangle. \qquad (4.5)$$

The utility of this lemma comes from the fact that the terms on the right hand side are easier to analyze than $\langle\psi; J(\mathbf{a})\psi\rangle$. This is in particular due to the factor B which appears on the left hand side but not on the right hand side of (4.5).

5 Analysis of J_{INT}

5.1 A note on how and why

We wish to calculate $\langle\psi|J_{\mathrm{INT}}|\psi\rangle$ as $\frac{d}{d\alpha}\Big|_{\alpha=0} E_\alpha$, where E_α is the ground state energy of an operator H_α. This is essentially mimicking the standard proof of the convergence of the density (see for instance [LS77]). We will choose H_α to formally satisfy

$$H_\alpha = H(N,Z,B) + \alpha J_{\mathrm{INT}} + O(\alpha^2). \tag{5.1}$$

We then make a similar perturbation of the approximating theory (DM or MTF). If we have concavity of the energy as a function of α, then convergence of the energies implies convergence of their first derivatives in α - i.e. the desired formula for $\langle\psi|J_{\mathrm{INT}}|\psi\rangle$. In [Fou01b] this was carried out for MTF-theory by only including the α-term in (5.1) (i.e. defining the $O(\alpha^2)$-term to be zero). This choice gives concavity automatically, however, when proving convergence of E_α to the (similarly defined) perturbed MTF-energy $E_\alpha^{\mathrm{MTF}}(N,Z,B)$ one is lead to require that

$$\tilde{D}_\alpha(f,g) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \overline{f(x)}|x-y|^{-1}\left\{1+\alpha\frac{(x-y)\cdot(\tilde{a}(x)-\tilde{a}(y))}{|x-y|^2}\right\} g(y)\,dxdy$$

defines a nonnegative quadratic form for small α (remember that in the proof of correctness of TF-theory one needs the inequality $D(\rho_\psi-\rho^{TF},\rho_\psi-\rho^{TF})\geq 0$ in the lower bound). Though this statement is very likely true, it is not obvious, so in order to avoid this point we define slightly more complicated operators/functionals (still of the form (5.1)) in this paper.

We will only carry the argument through for DM-theory. For MTF-theory, one can either (roughly) replace DM by MTF in the proof below, or look at [Fou01b] — taking note of the remark above.

5.2 A concave function

Let $\alpha\in(-\epsilon_0,\epsilon_0)$, where ϵ_0 is so small that the function

$$x\mapsto \psi_\alpha(x)\equiv x+\alpha\tilde{a}(x),$$

is invertible on all of $\mathbb{R}^3$ for all $\alpha\in(-\epsilon_0,\epsilon_0)$, where $\tilde{a}$ was defined in (4.1). We denote the inverse of ψ_α by ϕ_α. This function satisfies clearly $\phi_\alpha(x) =$

$x-\alpha\tilde{a}(x)+O(\alpha^2)$. Notice also that since $\tilde{a}$ has compact support there exists an $R\in\mathbb{R}_+$ independent of α such that $\phi_\alpha(x)=x$ for all $|x|\geq R$, $\alpha\in(-\epsilon_0,\epsilon_0)$.

Consider, for $x,y\in\mathbb{R}^3, x\neq y$, the function

$$\alpha\mapsto F_{x,y}(\alpha)=\frac{|x-y|}{|\phi_\alpha(x)-\phi_\alpha(y)|}-k_0\alpha^2.$$

By explicit differentiation we see that for $k_0>0$ and sufficiently big, we have $\frac{d^2}{d\alpha^2}F_{x,y}\leq -1/2$ for $\alpha\in(-\epsilon_0,\epsilon_0)$, uniformly in x,y. Thus $\alpha\mapsto F_{x,y}(\alpha)$ is a concave function on $(-\epsilon_0,\epsilon_0)$. From now on we fix k_0 satisfying the conditions above. Notice that the concavity of $F_{x,y}$ implies that

$$\alpha\mapsto\left\langle u\left|\frac{F_{x,y}(\alpha)}{|x-y|}\right|u\right\rangle,$$

is concave (for all $u(x,y)$ with $\langle u\big||x-y|^{-1}\big|u\rangle<+\infty$).

5.3 DM-theory with a current term

We will need a perturbed density matrix theory. Let $D_\alpha(\rho,\rho)$ be defined by

$$D_\alpha(\rho,\rho)=\tfrac{1}{2}\iint\overline{\rho(x)}K_\alpha(x,y)\rho(y)\,dxdy,$$

where

$$K_\alpha(x,y)=\frac{1-k_0\alpha^2}{2}|x-y|^{-1}+\tfrac{1}{2}|\phi_\alpha(x)-\phi_\alpha(y)|^{-1}.$$

Notice that, for fixed ρ with $D(\rho,\rho)<+\infty$,

$$D_\alpha(\rho,\rho)=D(\rho,\rho)+\frac{\alpha}{4}\iint\overline{\rho(x)}\frac{(x-y)(\tilde{a}(x)-\tilde{a}(y))}{|x-y|^3}\rho(y)\,dxdy+O(\alpha^2).$$

We define the functional $\mathcal{E}^{\mathrm{DM}}_{Z,B,\alpha}$ for density matrices $\Gamma\in\mathcal{G}_B$ (see (1.8) for the definition of $\mathcal{G}_B$) by

$$\mathcal{E}^{\mathrm{DM}}_{Z,B,\alpha}[\Gamma]=\int_{\mathbb{R}^2}\mathrm{tr}_{L^2(\mathbb{R})}[-\partial_3^2\Gamma_{x_\perp}]\,dx_\perp-Z\int_{\mathbb{R}^3}\frac{\rho_\Gamma(x)}{|x|}\,dx+D_\alpha(\rho_\Gamma,\rho_\Gamma).$$

The corresponding energy is

$$E^{\mathrm{DM}}_\alpha(N,Z,B)=\inf\{\mathcal{E}^{\mathrm{DM}}_{Z,B,\alpha}[\Gamma];\int\rho_\Gamma(x)\,dx\leq N,\Gamma\in\mathcal{G}_B\}.$$

By a change of variables we see that, with $f_\alpha = |\det D\psi_\alpha(x)| f(\psi_\alpha(x))$, $g_\alpha = |\det D\psi_\alpha(x)| g(\psi_\alpha(x))$ (with $D\psi_\alpha$ denoting the Jacobian matrix of ψ_α),

$$\frac{1}{2}\iint \frac{\overline{f(x)}g(y)}{|\phi_\alpha(x)-\phi_\alpha(y)|}\,dxdy = D(f_\alpha, g_\alpha).$$

This implies, in particular, due to known properties of $D(\cdot,\cdot)$ that

$$D_\alpha(f,f) \geq 0 \quad \text{for all } f. \tag{5.2}$$

Furthermore, we have

$$\phi_\alpha(x)-\phi_\alpha(y) = x - y - \alpha(\tilde{a}(x)-\tilde{a}(y)) + \alpha^2\left(x^2\eta_\alpha(x) - y^2\eta_\alpha(y)\right),$$

where $\eta_\alpha \in C_0^\infty(\mathbb{R}^3)$. This implies that, for ϵ_0 sufficiently small, there exists a constant c_0 such that

$$(1-c_0\alpha)|x-y| \leq |\phi_\alpha(x)-\phi_\alpha(y)| \leq (1+c_0\alpha)|x-y|. \tag{5.3}$$

Therefore, for all $\rho \geq 0$, we have

$$(1-c_0'\alpha)D(\rho,\rho) \leq D_\alpha(\rho,\rho) \leq (1+c_0'\alpha)D(\rho,\rho). \tag{5.4}$$

These properties of D_α imply that the analysis of [LSY94a, Section 4] of the DM-functional goes through without changes for $\mathcal{E}_\alpha^{\mathrm{DM}}$. Before summarizing these results, we need to introduce a linearized functional $\mathcal{E}_{\alpha,\mathrm{lin}}^{\mathrm{DM}}$. This functional presupposes the existence of a unique minimizer $\Gamma_\alpha^{\mathrm{DM}}$ with associated density $\rho_\alpha^{\mathrm{DM}}$ of $\mathcal{E}_\alpha^{\mathrm{DM}}$. Using this density, we can define a mean-field potential $\phi_{x_\perp,\alpha}^{\mathrm{DM}}$ as

$$\phi_{x_\perp,\alpha}^{\mathrm{DM}}(x_3) = -Z|x|^{-1} + \int_{\mathbb{R}^3} K_\alpha(x,y)\rho_\alpha^{\mathrm{DM}}(y).$$

This gives us $h_{x_\perp,\alpha}^{\mathrm{DM}}$ and $\mathcal{E}_{\alpha,\mathrm{lin}}^{\mathrm{DM}}$ as

$$h_{x_\perp,\alpha}^{\mathrm{DM}} = -\frac{\partial^2}{\partial x_3^2} + \phi_{x_\perp,\alpha}^{\mathrm{DM}}(x_3)$$
$$\mathcal{E}_{\alpha,\mathrm{lin}}^{\mathrm{DM}}[\Gamma] = \int_{\mathbb{R}^2} \mathrm{tr}[h_{x_\perp,\alpha}^{\mathrm{DM}}\Gamma_{x_\perp}]\,dx_\perp.$$

We can now state the results on the perturbed DM-theory.

Theorem 5.1. *For* ϵ_0 *sufficiently small* $\mathcal{E}_\alpha^{\rm DM}$ *has the following properties:*

- *For each* $N \geq 0$ *there is a unique minimizing density matrix* $\Gamma_\alpha^{\rm DM} \in \mathcal{S}_B^{\rm DM}$ *with* $\int \rho_{\Gamma_\alpha^{\rm DM}}(x)\, dx \leq N$.
- *The minimizer* $\Gamma_\alpha^{\rm DM}$ *is also a minimizer for the linearized functional* $\mathcal{E}_{\alpha,\rm lin}^{\rm DM}$ *and satisfies*
$$\mathcal{E}_{\alpha,\rm lin}^{\rm DM}[\Gamma_\alpha^{\rm DM}] = \mathcal{E}_\alpha^{\rm DM}[\Gamma_\alpha^{\rm DM}] + D_\alpha(\rho_\alpha^{\rm DM}, \rho_\alpha^{\rm DM}).$$
- *The function* $\alpha \mapsto E_\alpha^{\rm DM}(N,Z,B)$ *is differentiable at* $\alpha = 0$ *with*
$$\frac{d}{d\alpha}\Big|_{\alpha=0} E_\alpha^{\rm DM}(N,Z,B) = \frac{1}{4}\iint \rho_0^{\rm DM}(x)\frac{(x-y)\cdot(\tilde{a}(x)-\tilde{a}(y))}{|x-y|^3}\, dxdy.$$
More precisely, there exists a function $f(x)$ *with* $\lim_{\alpha\to 0} f(\alpha) = 0$ *such that*
$$\begin{aligned}&\left|E_\alpha^{\rm DM}(N,Z,B) - E_0^{\rm DM}(N,Z,B) - \alpha\frac{d}{d\alpha}\Big|_{\alpha=0} E_\alpha^{\rm DM}(N,Z,B)\right| \\ &\leq \alpha f(\alpha)|E^{\rm Q}(N,Z,B)|. \end{aligned} \tag{5.5}$$

5.4 Calculating $J_{\rm INT}$.

In order to calculate $\langle\psi; J_{\rm INT}\psi\rangle$ we introduce a perturbed Hamiltonian for $\alpha \in (-\epsilon_0.\epsilon_0)$:

$$\begin{aligned} H_\alpha &= \sum_{j=1}^N \left(H_{\mathbf{A}}^{(j)} - \frac{Z}{|x^{(j)}|}\right) \\ &\quad + \sum_{j<k}\left\{\tfrac{1}{2}(1-k_0\alpha^2)|x^{(j)}-x^{(k)}|^{-1} + \tfrac{1}{2}|\phi_\alpha(x^{(j)}) - \phi_\alpha(x^{(k)})|^{-1}\right\}. \end{aligned}$$

Notice that $H_0(N,Z,B) = H(N,Z,B)$. We can and will assume, by choosing ϵ_0 sufficiently small, $1 - k_0\alpha^2 > 0$.

Let
$$E_\alpha(N,Z,B) = \inf \operatorname{Spec} H_\alpha.$$

The idea of the proof is to use the Feynman-Hellman theorem (this idea was first used in the context of Thomas-Fermi theory in [LS77]) to calculate $\langle\psi; J_{\rm INT}\psi\rangle$ by differentiating $E_\alpha(N,Z,B)$ at $\alpha = 0$.

Using (5.3), we see that for some $c > 0$,

$$(1-\alpha c)E(N,Z,B) \leq E_\alpha(N,Z,B) \leq (1+\alpha c)E(N,Z,B).$$

The proof of Theorem 1.7 starts by justifying the following calculation.

$$\begin{aligned} \frac{\alpha}{2}\langle \psi; J_{\mathrm{INT}}\psi\rangle &\geq \langle \psi; (H_\alpha - H)\psi\rangle \qquad (5.6)\\ &\geq E_\alpha(N,Z,B) - E(N,Z,B)\\ &= E_\alpha^{\mathrm{DM}}(N,Z,B) - E^{\mathrm{DM}}(N,Z,B) + o(E(N,Z,B)). \end{aligned}$$

The first inequality is just the concavity in α. The second inequality is a consequence of the variational principle for the energy, so we only need to justify the final equality (which is really a bound). Using the upper bound

$$E(N,Z,B) \leq E^{\mathrm{DM}}(N,Z,B) + o(E(N,Z,B)),$$

from [LSY94a], we only need to prove a lower bound

$$E_\alpha(N,Z,B) \geq E_\alpha^{\mathrm{DM}}(N,Z,B) + o(E(N,Z,B)). \qquad (5.7)$$

We state this as a lemma

Lemma 5.2. *Let $\lambda, \Lambda > 0$ be given. Let $\epsilon > 0$, then there exists $C > 0$ such that if $\lambda \leq N/Z \leq \Lambda$, $BZ^{-4/3} > C$, $BZ^{-14/3} < C^{-1}$, $Z > C$, then*

$$E_\alpha(N,Z,B) \geq E_\alpha^{\mathrm{DM}}(N,Z,B) - \epsilon |E^{\mathrm{Q}}(N,Z,B)|,$$

for all $\alpha \in (-\epsilon_0, \epsilon_0)$.

To prove Lemma 5.2 we state a version of the perturbed Lieb-Oxford inequality from [Fou01b].

Lemma 5.3. *There exists a constant C such that for $\alpha \in (-\epsilon_0, \epsilon_0)$ and all normalized $\psi \in L^2(\mathbb{R}^{3N})$ the following bound holds.*

$$\left\langle \psi \left| \sum_{j<k} \left\{ \tfrac{1}{2}(1-k_0\alpha^2)|x^{(j)} - x^{(k)}|^{-1} + \tfrac{1}{2}|\phi_\alpha(x^{(j)}) - \phi_\alpha(x^{(k)})|^{-1} \right\} \right| \psi \right\rangle$$
$$\geq D_\alpha(\rho_\psi, \rho_\psi) - C \int_{\mathbb{R}^3} \rho_\psi^{4/3}\, dx.$$

The proof of this lemma is simple, using the standard Lieb-Oxford inequality and the argument from [Fou01b, Appendix A], and will therefore be omitted.

Proof of Lemma 5.2. The argument is essentially as in [LSY94a, Section 8]. Let $\psi \in \Pi_0^N \left(\wedge_{j=1}^N L^2(\mathbb{R}^3;\mathbb{C}^2)\right)$ be normalized and satisfy $\langle \psi, H(N,Z,B)\psi\rangle \leq 0$. Then, using the magnetic Lieb-Thirring inequality we see that

$$\int\int_{\mathbb{R}^3} \rho_\psi^{4/3}\,dx \leq cZ^{17/15}B^{2/5} = o(E(N,Z,B)),$$

where the equality holds in the parameter region we consider in Lemma 5.2 ($Z^{4/3} \ll B \ll Z^{14/3}$). Thus the error term in the correlation inequality, Lemma 5.3, is of lower order.

Using (5.2) we see that $D_\alpha(\rho_\psi - \rho_\alpha^{\rm DM}, \rho_\psi - \rho_\alpha^{\rm DM}) \geq 0$. We combine this with the usual arguments (see [LSY94a, Section 7, 8] for more details) to get

$$\langle \psi; H_\alpha(N,Z,B)\psi\rangle \geq \mathcal{E}_{\alpha,\rm lin}^{\rm DM}[\Gamma^\psi] - D_\alpha(\rho_\alpha^{\rm DM},\rho_\alpha^{\rm DM}) + o(E(N,Z,B)).$$

We finally use Theorem 5.1 to conclude that

$$\mathcal{E}_{\alpha,\rm lin}^{\rm DM}[\Gamma^\psi] \geq \mathcal{E}_{\alpha,\rm lin}^{\rm DM}[\Gamma_\alpha^{\rm DM}] = \mathcal{E}_\alpha^{\rm DM}[\Gamma_\alpha^{\rm DM}] + D_\alpha(\rho_\alpha^{\rm DM},\rho_\alpha^{\rm DM}).$$

This finishes the proof of Lemma 5.2. □

At this point we have justified the calculation in (5.6). To finish the proof of Theorem 1.7 we only have to invoke (5.5).

6 Analysis of $J_{\rm KIN}$

Let M be the matrix from (4.4). It is clear from the definition of $\mathbf{a}, \tilde{a}$ that we can write $M = M_0(x/l)$, where $M_0 \in C^2(\mathbb{R}^3)$ is a matrix valued function independent of B, Z.

Define

$$\begin{aligned} M &= M_{\rm t} + N_1 + N_2 \qquad (6.1)\\ &= \begin{pmatrix} b_3 & 0 & 0\\ 0 & b_3 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix} N_{11} & N_{12} & 0\\ N_{12} & -N_{11} & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 & N_{13}\\ 0 & 0 & N_{23}\\ N_{13} & N_{23} & 0\end{pmatrix}.\end{aligned}$$

Notice that $\operatorname{tr}[N_1] = \operatorname{tr}[N_2] = 0$. With these definitions we write

$$\begin{aligned} J_{\mathrm{KIN,diag}} &= \sum_{j=1}^{N} \left(p_{\mathbf{A}}^{(j)} \cdot M_{\mathrm{t}}(x^{(j)}) p_{\mathbf{A}}^{(j)} + B b_3(x^{(j)}) \sigma_3^{(j)} - \frac{1}{2}(\Delta_\perp b_3(x^{(j)})) \right), \\ J_s &= \sum_{j=1}^{N} p_{\mathbf{A}}^{(j)} \cdot N_s(x^{(j)}) p_{\mathbf{A}}^{(j)} \qquad \text{for } s = 1, 2. \end{aligned} \tag{6.2}$$

The contribution to the current of each of these terms will be small. First the terms $J_{\mathrm{KIN,diag}}$ and J_2 are estimated in Lemma 6.1. The remaining operator, J_1, is further split in (6.6). The individual components of J_1 are then estimated in Lemma 6.2 and Lemma 6.3. The final lines of this section combine the estimates proved here with Corollary 1.2 to obtain a proof of Theorem 1.9.

Lemma 6.1. *Let $\lambda, \Lambda > 0$ be given and let ψ be a ground state for $H(N, Z, B)$. Let $\epsilon > 0$, then there exists $C > 0$ such that if $\lambda \leq N/Z \leq \Lambda$, $BZ^{-4/3} > C$, $BZ^{-3} < C^{-1}$, then*

$$|\langle \psi; J_{\mathrm{KIN,diag}} \psi \rangle| + |\langle \psi; J_2 \psi \rangle| \leq \epsilon |E^{\mathrm{Q}}(N, Z, B)|. \tag{6.3}$$

Proof. We will treat the terms seperately. Both for $J = J_{\mathrm{KIN,diag}}$ and $J = J_2$ we will decompose

$$J = \Pi_0 J \Pi_0 + \Pi_> J \Pi_> + (\Pi_0 J \Pi_> + \Pi_> J \Pi_0),$$

and estimate the resulting three terms seperately. Let us introduce the raising and lowering operators a, a^*:

$$a = p_{\mathbf{A},1} - i p_{\mathbf{A},2} \qquad a^* = p_{\mathbf{A},1} + i p_{\mathbf{A},2}. \tag{6.4}$$

We have $a^* a = \mathbf{p}_{\mathbf{A},1}^2 + \mathbf{p}_{\mathbf{A},1}^2 - B$ and $[a, a^*] = 2B$. It is now an easy calculation that

$$p_{\mathbf{A}} M_{\mathrm{t}} p_{\mathbf{A}} = (a^* b_3 a + a b_3 a^*)/2.$$

Therefore, we get by easy commutations

$$p_{\mathbf{A}} M_{\mathrm{t}} p_{\mathbf{A}} - B b_3 - \frac{1}{2} \Delta_\perp b_3 = \frac{1}{2} \left(b_3 a^* a + a^* a b_3 \right), \tag{6.5}$$

so, since $a \Pi_0 = 0$,

$$\Pi_0 \left(p_{\mathbf{A}} M_{\mathrm{t}} p_{\mathbf{A}} + B b_3 \sigma_3 - \frac{1}{2} \Delta_\perp b_3 \right) \Pi_0 = \frac{1}{2} \Pi_0 \left(b_3 a^* a + a^* a b_3 \right) \Pi_0 = 0.$$

We now use the original representation (6.2) for $J_{\mathrm{KIN,diag}}$ and the fact that $|\Delta_\perp b_3| \le c\ell^{-2} \le c'B$. Thereby we can estimate

$$\pm \Pi_> \left(p_{\mathbf{A}} M_{\mathrm{t}} p_{\mathbf{A}} + B b_3 \sigma_3 - \frac{1}{2} \Delta_\perp b_3 \right) \Pi_> \le c\hat{K},$$

and we know from Lemma 2.1 that

$$\langle \psi ; \hat{K}^N \psi \rangle = o(E^{\mathrm{Q}}).$$

We finally estimate the off-diagonal terms, using (6.5), $a\Pi_0 = 0$, commutation and Cauchy-Schwarz.

$$\begin{aligned} & \pm \left\{ \Pi_0 \left(p_{\mathbf{A}} M_{\mathrm{t}} p_{\mathbf{A}} + B b_3 \sigma_3 - \frac{1}{2} \Delta_\perp b_3 \right) \Pi_> + h.c. \right\} \\ = \; & \pm \frac{1}{2} \left\{ \Pi_0 [b_3, a^*] a \Pi_> + \Pi_> a^* [a, b_3] \Pi_0 \right\} \\ \le \; & \eta \ell^{-2} \Pi_0 |\phi(x/\ell)|^2 \Pi_0 + \eta^{-1} \hat{K}, \end{aligned}$$

for any $\eta > 0$. Here we wrote $[a, b_3] = -(i\partial_1 + \partial_2) b_3 \equiv \ell^{-1} \phi(x/\ell)$. Now,

$$\sum_{j=1}^{N} \langle \psi ; Z \ell^{-1} \Pi_0 |\phi(x/\ell)|^2 \Pi_0 \psi \rangle = O(E^{\mathrm{Q}}),$$

by Lemma 2.1, and since ℓ^{-1}/Z can be made arbitrarily small by taking Z big with B/Z^3 small, we get the bound

$$\begin{aligned} & \langle \psi ; \left(\Pi_0^{(j)} \left(p_{\mathbf{A}}^{(j)} M_{\mathrm{t}}(x^{(j)}) p_{\mathbf{A}}^{(j)} + B b_3(x^{(j)}) \sigma_3^{(j)} - \frac{1}{2} (\Delta_\perp b_3)(x^{(j)}) \right) \Pi_>^{(j)} + h.c. \right) \psi \rangle \\ & = o(E^{\mathrm{Q}}). \end{aligned}$$

This proves (6.3) for $J_{\mathrm{KIN,diag}}$.

The analysis of J_2 is similar. Notice that $p_{\mathbf{A}} N_2 p_{\mathbf{A}}$ consists of terms of the form $(p_{\mathbf{A},j} N_{j,3} p_{\mathbf{A},3} + h.c.)$. We use that $p_{\mathbf{A},j}$ and $p_{\mathbf{A},3}$ commute to calculate (for $j = 1, 2$)

$$\begin{aligned} \Pi_0 p_{\mathbf{A},j} N_{j,3} p_{\mathbf{A},3} \Pi_0 &= c \Pi_0 a N_{j,3} p_{\mathbf{A},3} \Pi_0 = c \Pi_0 [a, N_{j,3}] p_{\mathbf{A},3} \Pi_0 \\ &\le c' \left(\eta^{-1} \ell^{-2} \Pi_0 |\phi(x/\ell)|^2 \Pi_0 + \eta p_{\mathbf{A},3}^2 \right), \end{aligned}$$

for any $\eta > 0$. Once again we use Lemma 2.1 to bound the involved terms. For the other diagonal term, $\Pi_> p_{\mathbf{A},j} N_{j,3} p_{\mathbf{A},3} \Pi_>$, we estimate

$$\begin{aligned} \pm \left(\Pi_> p_{\mathbf{A},j} N_{j,3} p_{\mathbf{A},3} \Pi_> + h.c. \right) &\leq c\eta^{-1} \Pi_> p_{\mathbf{A},\perp}^2 \Pi_> + \eta p_{\mathbf{A},3}^2 \\ &\leq c'\eta^{-1} \Pi_> \hat{K} \Pi_> + \eta p_{\mathbf{A},3}^2, \end{aligned}$$

for any $\eta > 0$. This easily implies the desired bound.

Finally, the off-diagonal part. This is estimated combining the above ideas. □

The diagonal part of J_1 is also readily estimated. We write

$$\begin{aligned} J_{1,\mathrm{diag}} &= \sum_{j=1}^{N} \left(\Pi_0^{(j)} p_{\mathbf{A}}^{(j)} \cdot N_1(x^{(j)}) p_{\mathbf{A}}^{(j)} \Pi_0^{(j)} + \Pi_>^{(j)} p_{\mathbf{A}}^{(j)} \cdot N_1(x^{(j)}) p_{\mathbf{A}}^{(j)} \Pi_>^{(j)} \right) \\ J_{1,\mathrm{off}} &= \sum_{j=1}^{N} \left(\Pi_0^{(j)} p_{\mathbf{A}}^{(j)} \cdot N_1(x^{(j)}) p_{\mathbf{A}}^{(j)} \Pi_>^{(j)} + \Pi_>^{(j)} p_{\mathbf{A}}^{(j)} \cdot N_1(x^{(j)}) p_{\mathbf{A}}^{(j)} \Pi_0^{(j)} \right). \end{aligned} \tag{6.6}$$

Lemma 6.2. *Let $\lambda, \Lambda > 0$ be given and let $\psi = \psi_{N,Z,B}$ be a ground state for $H(N,Z,B)$. Let $\epsilon > 0$, then there exists $C > 0$ such that if $\lambda \leq N/Z \leq \Lambda$, $\beta > C$, $Z > C$, then*

$$|\langle \psi ; J_{1,\mathrm{diag}} \psi \rangle| \leq \epsilon |E^{\mathrm{Q}}(N,Z,B)|. \tag{6.7}$$

Proof. A simple calculation using the raising and lowering operators a, a^* from (6.4) gives

$$p_{\mathbf{A}} \cdot N_1 p_{\mathbf{A}} = \frac{1}{2} \left(a(N_{11} + iN_{12}) a + a^*(N_{11} - iN_{12}) a^* \right). \tag{6.8}$$

Using, $a\Pi_0 = 0$, we therefore get

$$\Pi_0 p_{\mathbf{A}} \cdot N_1 p_{\mathbf{A}} \Pi_0 = 0.$$

For the component with $\Pi_>$ we estimate N_1 by its norm to get

$$\pm \Pi_> p_{\mathbf{A}} \cdot N_1 p_{\mathbf{A}} \Pi_> \leq c \Pi_> p_{\mathbf{A},\perp}^2 \Pi_> \leq c' \hat{K}.$$

Therefore,

$$\pm J_{1,\mathrm{diag}} \leq c' \hat{K}^N,$$

and we finish the proof of Lemma 6.2 by invoking Lemma 2.1. □

We finally need to estimate the off-diagonal part of the operator J_1. This is the result of Lemma 6.3. Combined with Corollary 1.2 on the perpendicular kinetic energy Lemma 6.3 gives a useful bound on $J_{1,\mathrm{off}}$ for a range of magnetic field strengths.

Lemma 6.3. *Let ψ be a ground state of $H(N,Z,B)$. Then there exists a constant c (independent of N, Z, B) such that*

$$|\langle\psi; J_{1,\mathrm{off}}\psi\rangle| \leq cB^{1/2}N^{1/2}\sqrt{\langle\psi; \hat{K}^N\psi\rangle}. \tag{6.9}$$

Proof. Using (6.8) and $a\Pi_0 = 0$ we get, with $r(x) = N_{11}(x) - iN_{12}(x)$,

$$\Pi_0 p_{\mathbf{A}} \cdot N_1 p_{\mathbf{A}} \Pi_> = \Pi_0 a^* r a^* \Pi_>.$$

We now apply the Cauchy-Schwarz inequality and $\|a^*\Pi_0\| \leq cB^{1/2}$ to get

$$|\langle\psi; J_{1,\mathrm{off}}\psi\rangle| \leq c\sum_{j=1}^{N} B^{1/2}\left\langle \psi \left| \Pi_>^{(j)}(a^{(j)})^*|r(x^{(j)}|^2(a^{(j)})^*\Pi_>^{(j)}\right| \psi\right\rangle^{1/2}.$$

By using the antisymmetry of ψ this expression is estimated by

$$cB^{1/2}N^{1/2}\left\langle\psi\right|\sum_{j=1}^{N}\Pi_>^{(j)}(a^{(j)})^*|r(x^{(j)}|^2(a^{(j)})^*\Pi_>^{(j)}\left|\psi\right\rangle^{1/2}.$$

The proof of Lemma 6.3 finishes by estimating $|r(x^{(j)})| \leq c$, yielding

$$\Pi_>^{(j)}(a^{(j)})^*|r(x^{(j)}|^2(a^{(j)})^*\Pi_>^{(j)} \leq c'\hat{K}^{(j)}.$$

□

The proof of Theorem 1.9 is now straight forward.

Proof of Theorem 1.9. It is easy to see that Theorem 1.9 follows by combining Corollary 1.2 with Lemma 6.1, Lemma 6.2 and Lemma 6.3. □

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