Homotopy arguments for quantized Hall conductivity

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Abstract

Using the strong localization bounds obtained by the Aizenman-Molcanov method for a particle in a magnetic field and a disordered potential, we show that the zero-temperature Hall conductivity of a gas of such particles is quantized and constant as long as both Fermi energy and disorder coupling parameter vary in a region of strong localization of the corresponding two-dimensional phase diagram.

1 Introduction

In quantum Hall effect experiments [1], one measures the Hall resistance of an effectively two-dimensional electron gas. For a wide range of the experimental parameters (such as the magnetic field and the particle density), the Hall resistance is an integer multiple of the fundamental constant e^2/h (here e is the charge of an electron and h is Planck's constant). The two most remarkable facts of this so-called integer quantum Hall effect, namely

1) the appearence of an integer number and
2) its stability with respect to changes of the parameters

can both be explained within a one-particle framework [1, 2]. The integers are known to be of topological origin, whereas their stability is due to Anderson localization in the sample. Here we focus on this second issue and study a one-particle Hamiltonian $H^\lambda = H_0 + \lambda V$, $\lambda \in \mathbf{R}$, given by the sum of a free magnetic operator H_0 of the Harper type and a disordered potential V. We prove that in a region of the (λ, μ)-plane of coupling constant λ and Fermi level μ in which localization estimates hold, the Hall conductivity is constant and equal to an integer multiple of e^2/h.

2 Models and results

2.1 Covariant operators

Let $(\Omega, T, \mathbf{Z}^2, \mathbf{P})$ be a dynamical system given by a compact probability space Ω endowed with a probability measure $\mathbf{P}$ which is invariant and ergodic with respect to the action T of the group $\mathbf{Z}^2$. A family $A = (A_\omega)_{\omega\in\Omega}$ of bounded operators on $\ell^2(\mathbf{Z}^2)$ is called covariant with respect to the action if it is weakly continuous (in ω) and it satisfies the covariance relation

$$U^n A_\omega U^{*n} \;=\; A_{T^n\omega}\;, \qquad n \in \mathbf{Z}^2\;.$$

Here the magnetic translation operators U^n, $n = (n_x, n_y)$, acting on $\ell^2(\mathbf{Z}^2)$ are defined by

$$U^n\psi(m) \;=\; \exp\left(\frac{\imath}{2}\,\theta\, n\wedge m\right)\psi(m-n)\;, \qquad m = (m_x, m_y) \in \mathbf{Z}^2\;, \qquad \psi \in \ell^2(\mathbf{Z}^2)\;,$$

where $a \wedge n = a_x n_y - a_y n_x$ and θ is the magnetic flux per unit cell. Given a covariant operator family $A = (A_\omega)_{\omega\in\Omega}$, Birkhoff's ergodic theorem implies that its trace per unit volume

$$\mathcal{T}(A) \;=\; \lim_{\Lambda\to\mathbf{Z}^2}\frac{1}{|\Lambda|}\,\mathrm{Tr}_\Lambda(A_\omega) \;=\; \lim_{\Lambda\to\mathbf{Z}^2}\frac{1}{|\Lambda|}\sum_{n\in\Lambda}\langle 0|A_{T^n\omega}|0\rangle\;,$$

is $\mathbf{P}$-almost surely independent of $\omega \in \Omega$ and is equal to $\int d\mathbf{P}(\omega)\,\langle 0|A_\omega|0\rangle$. If X and Y denote the components of the position operator on $\ell^2(\mathbf{Z}^2)$, new covariant operator families $\nabla_x A$ and $\nabla_y A$ are defined by

$$(\nabla_x A)_\omega \;=\; \imath[X, A_\omega]\;, \qquad (\nabla_y A)_\omega \;=\; \imath[Y, A_\omega]\;,$$

whenever the right-hand-sides are bounded operators. Let now $P = (P_\omega)_{\omega\in\Omega}$ be a covariant family of projection operators (*i.e.* $P_\omega^2 = P_\omega^* = P_\omega$). Its Chern number is defined by

$$\mathrm{Ch}(P) \;=\; 2\pi\imath\,\mathcal{T}(P[\nabla_x P, \nabla_y P])\;. \tag{1}$$

The Chern number is a topological invariant of the equivalence class of covariant projections defined by P. For further details and a more algebraic formulation of these notions, please consult [2].

2.2 Hamiltonian and physical quantities

We consider disordered two-dimensional magnetic operators in the tight-binding appoximation on the rectangular lattice Hilbert space $\ell^2(\mathbf{Z}^2)$ which are of the form:

$$H_\omega^\lambda \;=\; H_0 + \lambda V_\omega\;, \qquad V_\omega \;=\; \sum_{n\in\mathbf{Z}^2}\omega_n\,|n\rangle\,\langle n|\;, \tag{2}$$

Here $H_0 = H_0^*$ is a polynomial in the magnetic translations. The most simple non-trivial example is given by the Harper Hamiltonian $H_{\mathrm{Har}} = U^{(1,0)} + U^{(-1,0)} + U^{(0,1)} + U^{(0,-1)}$. Furthermore, $\omega = (\omega_n)_{n\in\mathbf{Z}^2}$ is a point in the space of disorder configurations given by the topological product

$\Omega = [-1,1]^{\otimes \mathbf{Z}^2}$. We suppose that the ω_n's are independent and identically distributed random variables with a probability distribution ρ. The product measure $\mathbf{P} = \rho^{\otimes \mathbf{Z}^2}$ then defines a probability on Ω which is invariant and ergodic with respect to the shift action T of $\mathbf{Z}^2$ on Ω. Hence we dispose of a dynamical system $(\Omega, T, \mathbf{Z}^2, \mathbf{P})$ as in Section 2.1.

The family $H = (H^\lambda_\omega)_{\omega\in\Omega}$ and any function $f(H)$ of it (f measurable) are covariant. Given a Fermi level $\mu \in \mathbf{R}$, we consider the associated Fermi projection $P^{\lambda,\mu} = \chi_{(-\infty,\mu]}(H^\lambda)$ where $\chi_{(-\infty,\mu]}$ is the characteristic function of the interval $(-\infty,\mu]$. Whenever well-defined, its Chern number is linked to the bulk Hall conductivity $\sigma_\perp(\lambda,\mu)$ via the Kubo-Chern formula [2]:

$$\sigma_\perp(\lambda,\mu) \;=\; \frac{e^2}{h}\,\mathrm{Ch}(P^{\lambda,\mu}) \;.$$

Furthermore we consider the density of states of H^λ, notably the probability measure $\mathcal{N}^\lambda$ defined by

$$\int d\mathcal{N}^\lambda(E)\, f(E) \;=\; \mathcal{T}(f(H^\lambda))\;, \qquad f \in C_0(\mathbf{R})\;.$$

2.3 Localization regime

Here we shall study the quantities $\sigma_\perp(\lambda,\mu)$ and $\mathcal{N}^\lambda$ in the

Aizenman-Molcanov Localization Regime (AMLR): *A spectral interval* $\Delta \subset \mathbf{R}$ *is in the* AMLR *for the covariant Hamiltonian* $H = (H_\omega)_{\omega\in\Omega}$ *if for any* $s \in (0,1)$ *there exist positive constants* Λ *and* $c_{s,\epsilon}$ *such that*

$$\int d\mathbf{P}(\omega)\;\left|\langle n|(E+\imath\epsilon - H_\omega)^{-1}|m\rangle\right|^s \;\leq\; c_{s,\epsilon}\, e^{-\Lambda|n-m|} \qquad (3)$$

for all $E \in \Delta$ *and* $\epsilon \neq 0$.

Under appropriate regularity conditions on the probability measure ρ (absolute continuity is sufficient), the AMLR is known to hold in the strong disorder regime (large λ) [3, 4] as well as at the band edges in the weak disorder regime [5, 6]. Furthermore note that the AMLR holds for a ($\mathbf{P}$-almost sure) gap of the spectrum of H^λ by a standard Combes-Thomas argument. However, it is not known rigorously (as suggest, for example, the numerical calculations of Tan [7] discussed below) that the AMLR holds for all but a set of critical energies for the 2D magnetic models defined by equation (2).

2.4 Main result and phase diagram

Theorem 1 *Let* $\mathcal{R} \subset \{(\mu,\lambda) \in \mathbf{R}^2\}$ *be a connected set such that for every* λ *the set* $\Delta(\lambda) = \{\mu|(\mu,\lambda) \in \mathcal{R}\}$ *is an interval for which the* AMLR *holds for* H^λ*. Then:*
(i) *The map* $(\mu,\lambda) \in \mathcal{R} \mapsto \mathcal{N}^\lambda((-\infty,\mu])$ *is Hölder continuous for any exponent less than* $1/2$.
(ii) $(\mu,\lambda) \in \mathcal{R} \mapsto \sigma_\perp(\lambda,\mu)$ *is constant and equal to an integer multiple of* q^2/h*.*

The fact that $\mu \mapsto \sigma_\perp(\lambda,\mu)$ is constant as long as μ varies in an interval for which the AMLR holds was already proven in [2] (see also [4]). Hence the new result concerns constancy in λ. Concerning (i), the continuity of the DOS for fixed λ is well known (by Wegner estimate even absolute continuity [8]), so that again the main issue is continuity in λ. Even though $\lambda \mapsto H^\lambda$ is norm continuous, this is non-trivial because $\mathcal{N}^\lambda((-\infty,\mu])$ is expressed in terms of a discontinuous function of H^λ.

Theorem 1 (ii) combined with a recent work by Avron and Sadun [10] allows to clarify the (μ,λ)-phase diagramm (Fig. 1) which was obtained numerically by Y. Tan [7] for the family of disordered Harper Hamiltonians $H_{\rm Har}+\lambda V_\omega$ where ρ is the uniform distribution on $[-1,1]$. There are regions where the Hall conductivity is constant. These regions correspond to a localization regime, whereas the lines indicate critical lines where the Hall conductivity jumps. Generically, such jumps are by unity [10], which is actually satisfied except on the line $\lambda = 0$ where the integrable character of Harper's equation lead to singularities in this respect.

Let us further note that at large disorder the whole spectrum belongs to the phase with vanishing Chern number and hence zero Hall conductance. This is compatible with the proof of AMLR for large disorder [3]. There is a corresponding result of Nakamura and Bellissard [11] for the continuous case showing that for fixed Fermi energy, there is a value of the disorder coupling constant λ_c such that for $\lambda > \lambda_c$ the Hall conductivity vanishes. In this situation, the current carrying lines do not annihiliate as in Fig. 1, but bend towards higher energies [12].

An interesting open problem is to show that the edge Hall conductivity as defined in [13] is well-defined and also remains constant in the regions of the above phase-diagram.

3 Proofs

Let us denote the disorder average w.r.t. $\mathbf{P}$ by $\mathbf{E}$. For technical reasons, we consider a covariant family $H^{(\nu)} = (H^{(\nu)}_\omega)_{\omega\in\Omega}$ given by

$$H^{(\nu)}_\omega \;=\; H^\lambda_\omega + \nu\, W_\omega \;,$$

where $W = (W_\omega)_{\omega\in\Omega}$ is a family of uniformely bounded covariant operators which is diagonal in the basis $(|n\rangle)_{n\in\mathbf{Z}^2}$. If $W = V$, then $H^{(\nu)} = H^{\lambda+\nu}$, but we will also be interested in the case $W = 1$ to study continuity in the energy. The following result also holds for a (magnetic) Anderson model in arbitrary dimension.

Lemma 1 *Let f be a real, piece-wise continuous and bounded function such that the discontinuities of f are within* AMLR *of $H^{(\nu)}$ for all $\nu \in [\nu_1,\nu_2]$. Then for any $s < 1/2$,*

$$\mathbf{E}\left(\left|\langle n|f(H^{(\nu_1)}_\omega)|m\rangle) \;-\; \langle n|f(H^{(\nu_2)}_\omega)|m\rangle\right|\right) \;\leq\; c\,|\nu_1-\nu_2|^s\, e^{-\Lambda|n-m|} \;. \qquad (4)$$

Proof. First let us recall the resolvent identity

$$(z-H^{(\nu_1)}_\omega)^{-1} - (z-H^{(\nu_2)}_\omega)^{-1} \;=\; (\nu_1-\nu_2)\,(z-H^{(\nu_1)}_\omega)^{-1}W_\omega(z-H^{(\nu_2)}_\omega)^{-1} \;.$$

Now using the convexity inequality $|\sum_j x_j|^s \leq \sum_j |x_j|^s$ and the fact that W is diagonal and $|W_\omega(k)| \leq c$, one obtains therefore:

$$\Big|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle - \langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\Big|^s \\ \leq\; c|\nu_1-\nu_2|^s \sum_{k\in\mathbf{Z}^2} |\langle n|(z-H_\omega^{(\nu_1)})^{-1}|k\rangle|^s\;|\langle k|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle|^s\;.$$

This gives

$$\Big|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle - \langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\Big| \\ \leq\; c|\nu_1-\nu_2|^s \sum_{k\in\mathbf{Z}^2} |\langle n|(z-H_\omega^{(\nu_1)})^{-1}|k\rangle|^s\;|\langle k|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle|^s \\ \cdot\left(\Big|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle\Big|^{1-s}+\Big|\langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\Big|^{1-s}\right)\;.$$

Now we use the bound $|\langle n|(z-H_\omega^{(\nu_{1,2})})^{-1}|m\rangle|\leq|\Im m(z)|^{-1}$, then take the disorder average and apply the Cauchy-Schwarz inequality in order to get:

$$\mathbf{E}\left(\Big|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle - \langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\Big|\right)\;\leq\; c|\nu_1-\nu_2|^s\;|\Im m(z)|^{s-1} \\ \cdot\sum_{k\in\mathbf{Z}^2}\Big(\mathbf{E}(|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|k\rangle|^{2s})\Big)^{1/2}\Big(\mathbf{E}(|\langle k|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle|^{2s})\Big)^{1/2}\;,$$

so that replacing the AMLR bound we finally obtain by using for example the Young inequality:

$$\mathbf{E}\left(\Big|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle - \langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\Big|\right)\;\leq\; c|\nu_1-\nu_2|^s\;|\Im m(z)|^{s-1}\;e^{-\Lambda|m-n|} \qquad (5)$$

for $s<1/2$.

Now let f be an analytic function on an interval $\Delta=[E_l,E_r]$ the end points of which fall into the AMLR. Then we use holomorphic functional calculus in the following way:

$$f(H_\omega^{(\nu)})\;=\;Q_f(H_\omega^{(\nu)}-E_r)-R_f(H_\omega^{(\nu)})-Q_f(H_\omega^{(\nu)}-E_l)+R_f(H_\omega^{(\nu)})^*\;, \qquad (6)$$

where

$$Q_f(z)\;=\;\int_{-1}^{1}\frac{dt}{2\pi}\,\frac{f(\imath t)}{\imath t-z}\;,\qquad R_f(z)\;=\;\int_{E_l}^{E_r}\frac{dt}{2\pi}\,\frac{f(t)}{t-\imath-z}\;.$$

Because we are in the AMLR, the spectrum is pure-point and it is then well-known that E_l and E_r are almost surely not in the spectrum of $H_\omega^{(\nu)}$. Hence the intergrals in (6) are almost surely well-defined (a similar argument was already used in [4]). Now we can estimate the differences for different coupling parameters ν:

$$\begin{aligned}
&\mathbf{E}\left(\left|\langle n|Q_f(H_\omega^{(\nu_1)}-E_r)|m\rangle-\langle n|Q_f(H_\omega^{(\nu_2)}-E_r)|m\rangle\right|\right)\\
&\quad\leq\;\left(\sup_{t'\in[-1,1]}|f(\imath t')|\right)\int_{-1}^{1}\frac{dt}{2\pi}\,\mathbf{E}\left(\left|\langle n|(z-H_\omega^{(\nu_1)})^{-1}|m\rangle-\langle n|(z-H_\omega^{(\nu_2)})^{-1}|m\rangle\right|\right) \qquad (7)\\
&\quad\leq\;c\,|\nu_1-\nu_2|^s\int_{-1}^{1}\frac{dt}{|t|^{1-s}}\;e^{-\Lambda|m-n|}\\
&\quad\leq\;c'\,|\nu_1-\nu_2|^s\;e^{-\Lambda|m-n|}\;.
\end{aligned}$$

Of course, the terms with E_l instead of E_r can be treated in the same way. Concerning the R_f-contributions in (6), let us first remark that a bound like (3) but with $s=1$ actually holds for all complex energies off the real axis (by a Combes-Thomas argument, *cf.* [4]). Using this fact, one can either proceed as above or more directly use the resovent identity. We skip the details.

Combining these estimates, we see that (4) holds for an analytic function f on Δ. Because the above estimate is continuous w.r.t. the maximum norm on the space of continuous functions (*cf.* the bound (7), for example), a standard Stone-Weierstrass approximation argument allows to obtain the same bound also for every continuous function f. Finally, for a piecewise continuous function we can treat each continuous part seperately provided the points of discontinuity of f are within the AMLR. $\square$

Proof of Theorem 1 (i): Because the indicator function χ_μ on the interval $(-\infty,\mu]$ is piecewise continuous and the discontinuity varies within the AMLR according to hypothesis, we can apply the above lemma with $W=V$ and $W=1$ respectively. With the notation $\mathcal{N}^\lambda(\mu)=\mathcal{N}^\lambda((-\infty,\mu])$, we therefore obtain:

$$\begin{aligned}
|\mathcal{N}^{\lambda_1}(\mu_1)-\mathcal{N}^{\lambda_2}(\mu_2)|\;&\leq\;|\mathcal{N}^{\lambda_1}(\mu_1)-\mathcal{N}^{\lambda_2}(\mu_1)|\;+\;|\mathcal{N}^{\lambda_2}(\mu_1)-\mathcal{N}^{\lambda_2}(\mu_2)|\\
&\leq\;\mathbf{E}\left(\left|\langle 0|\chi_{\mu_1}(H^{\lambda_1})|0\rangle-\langle 0|\chi_{\mu_1}(H^{\lambda_2})|0\rangle\right|\right)\\
&\qquad+\mathbf{E}\left(\left|\langle 0|\chi_{\mu_1}(H^{\lambda_2})|0\rangle-\langle 0|\chi_{\mu_1}(H^{\lambda_2}+\mu_2-\mu_1)|0\rangle\right|\right)\\
&\leq\;c\,|\lambda_1-\lambda_2|^s+c\,|\nu_1-\nu_2|^s\qquad(s<1/2)\;.
\end{aligned}$$

This gives the desired Hölder continuity. $\square$

Proof of Theorem 1 (ii): As in [2], let us introduce a non-commutative Sobolov space as the set of all weakly continuous covariant operator families $A=(A_\omega)_{\omega\in\Omega}$ for which the norm

$$\|A\|_{\mathcal{S}}^2\;=\;\mathcal{T}(|A|^2)+\mathcal{T}(|\vec{\nabla}A|^2) \tag{8}$$

is finite (here $|\vec{\nabla}A|^2=|\nabla_x A|^2+|\nabla_y A|^2$). Note that the Chern number (1) is continuous w.r.t. this norm. An important result of [2] is the following: if $t\mapsto P(t)\in\mathcal{S}$ is a one-parameter family of covariant projection operators which is continuous w.r.t. to the Sobolev norm $\|.\|_{\mathcal{S}}$, then $t\mapsto \mathrm{Ch}(P(t))$ is constant and equal to an integer. Hence it is sufficient to show that the map $(\mu,\lambda)\in\mathcal{R}\mapsto\|P^{\lambda,\mu}\|_{\mathcal{S}}$ is Hölder continuous. According to part (i) proved above, this actually holds for the $\mathcal{T}(|\,.\,|^2)$-part of the norm. For the second part, we begin with

$$\left|\mathcal{T}(|\vec{\nabla}P^{\mu_1,\lambda_1}|^2) - \mathcal{T}(|\vec{\nabla}P^{\mu_2,\lambda_2}|^2)\right| \;\leq\; \sum_{m\in\mathbf{Z}^2} |m|^2\,\mathbf{E}\left(\left||\langle 0|\chi_{\mu_1}(H^{\lambda_1})|m\rangle|^2 - |\langle 0|\chi_{\mu_2}(H^{\lambda_2})|m\rangle|^2\right|\right)\;. \tag{9}$$

Now the bound $|\langle 0|\chi_\mu(H^\lambda)|m\rangle| \leq 1$ implies that

$$\left||\langle 0|\chi_{\mu_1}(H^{\lambda_1})|m\rangle|^2 - |\langle 0|\chi_{\mu_2}(H^{\lambda_2})|m\rangle|^2\right| \;\leq\; 2\left|\langle 0|\chi_{\mu_1}(H^{\lambda_1})|m\rangle - \langle 0|\chi_{\mu_2}(H^{\lambda_2})|m\rangle\right|\;,$$

so that an argument similar to that in the proof of (i) shows that

$$\mathbf{E}\left(\left||\langle 0|\chi_{\mu_1}(H^{\lambda_1})|m\rangle|^2 - |\langle 0|\chi_{\mu_2}(H^{\lambda_2})|m\rangle|^2\right|\right) \;\leq\; c(|\lambda_1-\lambda_2|^s + |\mu_1-\mu_2|^s)\,e^{-\Lambda|m|}\;.$$

Inserting this into (9) allows to conclude. □

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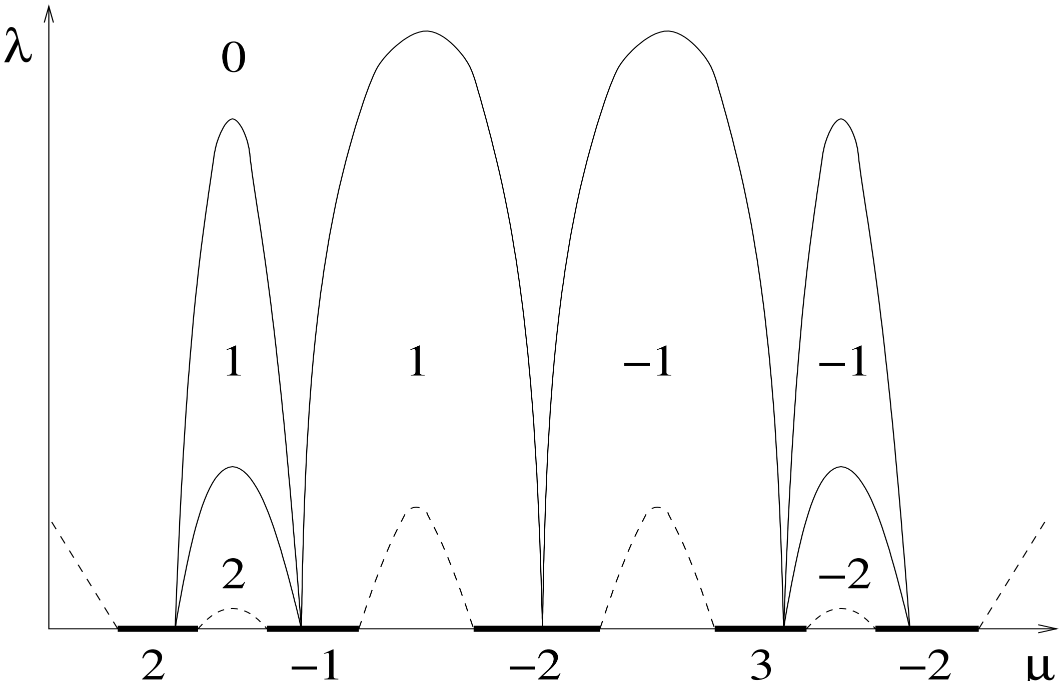
Figure Caption:

Phase diagram for the Hall conductivity in a disordered Harper model with rational flux $\theta = 2\pi \cdot 3/5$ (following [7]; The fat lines are the Bloch bands of the free Harper operator, each with its Chern number given below. These numbers can easily be calculated using the diophantine equation in [9]). The solid lines show the critical energies separating the regions of the phase diagram with constant Chern number; this number is written inside each region. The dashed lines show the boundaries of the spectrum.


λ
0
1
1
−1
−1
2
−2
2
−1
−2
3
−2
μ