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The Anderson metal-insulator transport transition

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Abstract. We discuss a new approach to the metal-insulator transition for random operators, based on transport instead of spectral properties. It applies to random Schrödinger operators, acoustic operators in random media, and Maxwell operators in random media. We define a local transport exponent $\beta(E)$, and set the *metallic transport region* to be the part of the spectrum with nontrivial transport (i.e., $\beta(E) > 0$). The *strong insulator region* is taken to be the part of the spectrum where the random operator exhibits strong dynamical localization in the Hilbert-Schmidt norm, and hence no transport. For the standard random operators satisfying a Wegner estimate, these metallic and insulator regions are shown to be complementary sets in the spectrum of the random operator, and the local transport exponent $\beta(E)$ provides a characterization of the *metal-insulator transport transition.* If such a transition occurs, then $\beta(E)$ has to be discontinuous at a *transport mobility edge*: if the transport is nontrivial then $\beta(E) \geq \frac{1}{2bd}$, where d is the space dimension and $b \geq 1$ is the power of the volume in Wegner's estimate. We also examine the transport transition for random polymer models, where the random dimer models provide explicit examples of the transport transition and of a transport mobility edge.

1. Introduction

In this survey we discuss a new approach to the metal-insulator transition for random operators based on transport instead of spectral properties, introduced in [**GK3**]. This new point of view, in addition to being closer to the physical meaning of a "metal-insulator" transition, is shown to give a better understanding of the transition.

By a random operator we always mean a $\mathbb{Z}^d$-ergodic random self-adjoint operator H_ω on either $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$, where ω belongs to a probability set Ω with a probability measure $\mathbb{P}$ and expectation $\mathbb{E}$. Note that ergodicity implies the existence of a nonrandom set Σ, such that $\sigma(H_\omega) = \Sigma$ with probability one, where $\sigma(A)$ denotes the spectrum of the operator A. In addition, there are nonrandom sets Σ_{pp}, Σ_{ac}, and Σ_{sc}, which are the pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum of H_ω, respectively, with probability one (see [**KM**]).

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For a Schrödinger operator with a random potential and spectrum of the form $[E_0, \infty)$, the following picture of a metal-insulator transition has emerged (e.g., [**LGP**, Section 4.2]): Near the bottom of the spectrum E_0 the spectrum is controlled by large fluctuations of the potential, with the corresponding states localized primarily in the regions of such fluctuations. But in three or more dimensions, at very large energies the kinetic term should dominate the fluctuations of the potential to produce extended states. Thus a transition must occur from an *insulator regime*, characterized by localized states, to a very different *metallic regime* characterized by extended states. The energy E_{me} at which this *metal-insulator transition* occurs is called the *mobility edge*. The medium should have zero conductivity in the *insulator region* $[E_0, E_{\mathrm{me}}]$ and nonzero conductivity in the *metallic region* $[E_{\mathrm{me}}, \infty)$.

The standard mathematical interpretation of the metal-insulator transition is as a spectral transition: the random Schrödinger operator should have pure point spectrum with exponentially decaying eigenstates in the interval $[E_0, E_{\mathrm{me}}]$ and absolutely continuous spectrum on the interval $[E_{\mathrm{me}}, \infty)$.

The existence of exponential localization (i.e., pure point spectrum with exponentially decaying eigenstates) is by now well established (e.g., [**GMP, KS, FS, HM, FMSS, CKM, vDK, AM, Ai, CH1, Klo2, KSS, FLM, ASFH, Wa2, GK1, Klo3, Klo4, U, GK4**]). But there are no mathematical results on the existence of continuous spectrum and a metal-insulator transition. (Except for the special case of the Anderson model on the Bethe lattice, where one of us has proved that for small disorder the random operator has purely absolutely continuous spectrum in a nontrivial interval [**Kle1**] and nontrivial transport [**Kle2**].)

In [**GK3**] we proposed a new approach to the metal-insulator transition based on transport instead of spectral properties, which we describe in this survey. It relies on a local transport exponent $\beta(E)$; the *metallic transport region* is set to be the part of the spectrum with nontrivial transport (i.e., $\beta(E) > 0$). The *strong insulator region* is defined as the part of the spectrum where the random operator exhibits strong dynamical localization in the Hilbert-Schmidt norm, and hence no transport. There is a natural definition of a *transport mobility edge* between the strong insulator and the metallic transport regions. (See Section 2.)

For the standard random operators satisfying a Wegner estimate, these metallic and insulator regions are shown to be complementary sets in the spectrum of the random operator. (This rules out the possibility of trivial transport, i.e., transport with $\beta(E) = 0$.) Thus the local transport exponent $\beta(E)$ provides a characterization of the *metal-insulator transport transition*. Moreover, $\beta(E)$ is discontinuous at a transport mobility edge: if the transport is nontrivial then $\beta(E) \geq \frac{1}{2bd}$, where d is the space dimension and $b \geq 1$ is the power of the volume in Wegner's estimate. (See Section 3.)

We illustrate these concepts in the case of random polymer models. They may have critical energies, and Jitomirskaya, Schulz-Baldes and Stolz [**JSBS**] showed that critical energies creates transport. We extend their results to local tranport exponents, proving that critical energies must be in the metallic transport region. For the special case of the random dimer model, we combine this fact with the results of De Bièvre and Germinet [**DBG**] to give a complete description of the transport transition and exhibit tranport mobility edges. (See Section 4.)

2. The transport transition

Let H_ω be a random operator on $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ (or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$). By χ_x we denote the characteristic function of the the cube of side 1 centered at $x \in \mathbb{R}^d$. If $x \in \mathbb{R}^d$ we write $\langle x \rangle = \sqrt{1 + |x|^2}$, and let $\langle X \rangle$ denote the operator given by multiplication by the function $\langle x \rangle$. Given an open interval $I \subset \mathbb{R}$, we denote by $C_c^\infty(I)$ the class of real valued infinitely differentiable functions on $\mathbb{R}$ with compact support contained in I, with $C_{c,+}^\infty(I)$ being the subclass of nonnegative functions. The Hilbert-Schmidt norm of an operator A is written as $\|A\|_2$, i.e., $\|A\|_2^2 = \operatorname{tr} A^* A$.

The (random) moment of order $n \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the cube of side one around the origin, and "localized" in energy by the function $\mathcal{X} \in C_{c,+}^\infty(\mathbb{R})$, is given by

$$M_\omega(n, \mathcal{X}, t) = \left\| \langle X \rangle^{\frac{n}{2}} \mathrm{e}^{-itH_\omega} \mathcal{X}(H_\omega) \chi_0 \right\|_2^2 , \tag{1}$$

its expectation by

$$\mathbf{M}(n, \mathcal{X}, t) = \mathbb{E}\{M_\omega(n, \mathcal{X}, t)\} , \tag{2}$$

and its time averaged expectation by

$$\mathcal{M}(n, \mathcal{X}, T) = \frac{2}{T} \int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \mathbf{M}(n, \mathcal{X}, t) \, \mathrm{d}t. \tag{3}$$

These quantities are finite for the usual random operators, including random Schrödinger operators, random Landau Hamiltonians, and acoustic and Maxwell operators in random media. (The required properties are a trace estimate and a kernel polynomial decay estimate as stated in [**GK3**, Lemmas A.4 and A.5]; they hold in great generality.) If $\mathcal{X}(H_\omega) \neq 0$ ($\mathcal{X}(H_\omega)$ is either $= 0$ or $\neq 0$ with probability one) we have [**GK3**, Proposition 3.1]:

$$0 \leq M_\omega(0, \mathcal{X}, 0) \leq M_\omega(n, \mathcal{X}, t) \leq C_{d,\mathcal{X},n} \langle t \rangle^{\left[n + \frac{3d}{2}\right] + 3} \quad \text{for } \mathbb{P}\text{-a.e. } \omega , \tag{4}$$

$$0 < \mathbf{M}(0, \mathcal{X}, 0) \leq \mathbf{M}(n, \mathcal{X}, t) \leq C_{d,\mathcal{X},n} \langle t \rangle^{\left[n + \frac{3d}{2}\right] + 3} , \tag{5}$$

$$0 < \mathbf{M}(0, \mathcal{X}, 0) \leq \mathcal{M}(n, \mathcal{X}, T) \leq C'_{d,\mathcal{X},n} \langle T \rangle^{\left[n + \frac{3d}{2}\right] + 3}, \tag{6}$$

where $[u]$ denotes the largest integer $\leq u$.

To measure the rate of growth of moments of initially spatially localized wave packets under the time evolution, "localized" in energy by $\mathcal{X} \in C_{c,+}^\infty(\mathbb{R})$ with $\mathcal{X}(H_\omega) \neq 0$, we compute the (lower) transport exponent

$$\beta(n, \mathcal{X}) = \liminf_{T \to \infty} \frac{\log \mathcal{M}(n, \mathcal{X}, T)}{n \log T} . \tag{7}$$

If $\mathcal{X}(H_\omega) = 0$ we set $\beta(n, \mathcal{X}) = 0$. We define the *n-th transport exponent* in an open interval I by

$$\beta(n, I) = \sup_{\mathcal{X} \in C_{c,+}^\infty(I)} \beta(n, \mathcal{X}) , \tag{8}$$

and the *n-th local transport exponent* at the energy E by

$$\beta(n, E) = \inf_{I \ni E} \beta(n, I) . \tag{9}$$

($\beta(n,E)$ provides a measure of the rate of transport for which E is responsible.) The exponent $\beta(n,E)$ is increasing in n and satisfy the ballistic bound

$$0 \le \beta(n,\mathcal{X}),\ \beta(n,I),\ \beta(n,E) \le 1\,, \tag{10}$$

as shown in [**GK3**, Proposition 3.2]. Note that $\beta(n,E)=0$ if $E \notin \Sigma$.

The *local (lower) transport exponent* may be now defined by

$$\beta(E) = \lim_{n\to\infty} \beta(n,E) = \sup_n \beta(n,E)\,, \tag{11}$$

and we have $0 \le \beta(E) \le 1$, with $\beta(E)=0$ if $E \notin \Sigma$. Note that $\beta(E)>0$ if and only if $\beta(n,E)>0$ for some $n>0$.

This motivates the following definition.

Definition 2.1. *The metallic transport region Σ_{MT} for H_ω is defined as the set of energies with nontrivial transport:*

$$\Sigma_{\mathrm{MT}} = \{E \in \mathbb{R},\ \beta(E)>0\} = \{E\in\Sigma,\ \beta(E)>0\}\,. \tag{12}$$

Its complementary set in the spectrum will be called the trivial transport region Σ_{TT} (note that logarithmic transport is not excluded a priori):

$$\Sigma_{\mathrm{TT}} = \Sigma\backslash\Sigma_{\mathrm{MT}} = \{E\in\Sigma,\ \beta(E)=0\}\,. \tag{13}$$

To make the connection with the absolutely contimuous spectrum, we recall that the Guarneri-Combes-Last bound [**Gu, Co, La**] says that $\beta(E)\ge\frac{1}{d}$ if $E\in\Sigma_{\mathrm{ac}}$. (While the Guarneri-Combes-Last bound is stated for a fixed self-adjoint operator, the same bound follows for random operators using Fatou's Lemma and Jensen's inequality.) Thus

$$\Sigma_{\mathrm{ac}} \subset \left\{E\in\Sigma,\ \beta(E)\ge\frac{1}{d}\right\} \subset \Sigma_{\mathrm{MT}}\,. \tag{14}$$

But the converse to the Guarneri-Combes-Last bound is not true, a lower bound on the local transport exponent does not specify the spectrum (e.g., [**DRJLS, La, DBF, BGT, CM, GKT**]).

The trivial transport region should be connected to localization, and indeed it is, but to the right kind of localization. Pure point spectrum does not imply trivial transport, there are counterexamples [**DRJLS, JSBS, GKT**]. The right notion of localization is dynamical localization (which implies pure point spectrum).

Definition 2.2. *The random operator H_ω exhibits strong HS-dynamical localization in the open interval I if for all $\mathcal{X}\in C^\infty_{c,+}(I)$ we have*

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} M_\omega(n,\mathcal{X},t)\right\} < \infty \quad \textit{for all } n\ge 0\,. \tag{15}$$

The random operator H_ω exhibits strong HS-dynamical localization at the energy $E\in\mathbb{R}$ if there exists an open interval I, with $E\in I$, such that there is strong HS-dynamical localization in the open interval I.

The intuitive idea behind the last definition is that the moments of an initially localized wave packet remain uniformly bounded under time evolution "localized" in an open interval around the energy E. The Hilbert-Schmidt norm takes into account all possible wave packets localized in a given bounded region.

Definition 2.3. *The strong insulator region Σ_{SI} for H_ω is defined as*

$$\Sigma_{\mathrm{SI}} = \{E\in\Sigma\,;\ H_\omega \textit{ exhibits strong HS-dynamical localization at } E\}\,.$$

We clearly have

$$\Sigma_{\mathrm{SI}} \subset \Sigma_{\mathrm{TT}}\,. \tag{16}$$

DEFINITION 2.4. *An energy* $E \in \Sigma_{\mathrm{MT}} \cap \overline{\Sigma}_{\mathrm{SI}}$ *will be called a transport mobility edge.*

Since the strong insulator region is a relatively open subset of the spectrum Σ, we have $\Sigma_{\mathrm{SI}} = \left\{\bigcup_{j=1}^{N} I_j\right\} \cap \Sigma$, where the I_j's are disjoint open intervals; N may be either finite or infinite. A transport mobility edge must be an edge of one of the intervals I_j.

3. A characterization of the transport transition

Instead of giving a technical definition of the relevant class of random operators, let us simply say that in this section by a standard random operator we mean one of the following:

The Anderson model:

$$H_\omega = -\Delta + V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z}^d)\,, \tag{17}$$

where Δ is the finite difference Laplacian and $\{V_\omega(x);\ x \in \mathbb{Z}^d\}$ are independent identically distributed bounded random variables. (E.g., [**FS, FMSS, CKM, vDK, AM, Ai, ASFH, Wa2**].)

Random Schrödinger operators (Anderson Hamiltonians):

$$H_\omega = -\Delta + V_{\mathrm{per}} + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{18}$$

where Δ is the Laplacian operator, V_{per} is a periodic potential (by rescaling we take the period to be one) of the form $V_{\mathrm{per}} = V_{\mathrm{per}}^{(1)} + V_{\mathrm{per}}^{(2)}$, with $V_{\mathrm{per}}^{(i)}$, $i = 1, 2$, periodic with period one, $0 \le V_{\mathrm{per}}^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$, $V_{\mathrm{per}}^{(2)}$ relatively form-bounded with respect to $-\Delta$ with relative bound < 1, and V_ω a random potential of the form

$$V_\omega(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} \omega_i\, u(x - i), \tag{19}$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables, u is a real valued measurable function with compact support, $u \in \mathrm{L}^p(\mathbb{R}^d, \mathrm{d}x)$ with $p > \frac{d}{2}$ if $d \ge 2$ and $p = 2$ if $d = 1$. (See [**CH1, Klo2, KSS, GK3, GK4**].)

Random Landau Hamiltonians:

$$H_\omega = H_0 + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{20}$$

where $H_0 = (-i\nabla - A)^2$, $A = \frac{B}{2}(x_2, -x_1)$, and the random potential V_ω is as in (19) with $q = 1$ and $u(x)$ bounded. (See [**CH2, Wa1, GK4**].)

Maxwell operators in random media:

$$H_\omega = \frac{1}{\sqrt{\mu_\omega(x)}}\nabla\times\frac{1}{\varepsilon_\omega(x)}\nabla\times\frac{1}{\sqrt{\mu_\omega(x)}} \quad \text{on} \quad \mathrm{L}^2\left(\mathbb{R}^3, dx; \mathbb{C}^3\right) \tag{21}$$

where $\nabla\times$ is the operator given by the curl, $\varepsilon_\omega(x)$ is the random dielectric constant and $\mu_\omega(x)$ is the random magnetic permeability. We take

$$\varepsilon_\omega(x) \;=\; \varepsilon_0(x)\gamma_\omega(x) \ \text{, with } \gamma_\omega(x) = 1 + \sum_{i\in\frac{1}{q}\mathbb{Z}^3} \omega_i u(x-i), \tag{22}$$

$$\mu_\omega(x) \;=\; \mu_0(x)\beta_\omega(x) \ \text{, with } \beta_\omega(x) = 1 + \sum_{i\in\frac{1}{q}\mathbb{Z}^3} \omega_i v(x-i), \tag{23}$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables taking values in the interval $[-1,1]$, $\varepsilon_0(x)$ and $\mu_0(x)$ are periodic measurable functions (by rescaling we take the period to be one), such that $0 < \varepsilon_- \le \varepsilon\,(x) \le \varepsilon_+ < \infty$ and $0 < \mu_- \le \mu\,(x) \le \mu_+ < \infty$ for some constants $\varepsilon_\pm$ and $\mu_\pm$, $u(x)$ and $v(x)$ are nonnegative measurable real valued functions with compact support, such that

$$0 \le U_- \le U(x) \equiv \sum_{i\in\frac{1}{q}\mathbb{Z}^3} u_i(x) \quad\le\quad U_+ < \infty, \tag{24}$$

$$0 \le V_- \le V(x) \equiv \sum_{i\in\frac{1}{q}\mathbb{Z}^3} v_i(x) \quad\le\quad V_+ < \infty, \tag{25}$$

for some constants $U_\pm$ and $V_\pm$, with $U_- + V_- > 0$ and $\max\{U_+, V_+\} < 1$. (See [**FK2, Kle3, KK2**].)

Acoustic operators in random media:

$$H_\omega = -\frac{1}{\sqrt{\kappa_\omega(x)}}\nabla\cdot\frac{1}{\rho_\omega(x)}\nabla\frac{1}{\sqrt{\kappa_\omega(x)}} \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{26}$$

where ∇ is the gradient operator, and, the random compressibility $\kappa_\omega(x)$ and the random mass density $\varrho_\omega(x)$ are of the same form as $\varepsilon_\omega(x)$ and $\mu_\omega(x)$ in (22) and (23). (See [**FK1, KK2**]).

These random operators satisfy all the requirements for the bootstrap multiscale analysis [**GK1**] in appropriate intervals, with the possible exception of a Wegner estimate. They also satisfy an appropriate interior estimate [**GK3**, Lemma A.2], and the kernel polynomial decay estimate of [**GK2**, Theorem 2] with nonrandom constants. (To be precise, a standard random operator is a random operator satisfying Assumptions SLI, EDI, IAD, NE, and SGEE in [**GK1**], the interior estimate of [**GK3**, Lemma A.2], and the Assumptions of [**GK4**] with nonrandom constants; these properties are routinely verified for the usual random operators.)

Although we frame our discussion for random operators on the continuum, our results apply also to random operators on the lattice.

In our results a Wegner estimate in an open interval (Assumption W in [**GK1**]) will be an explicit hypothesis. To state it we need to consider the restriction of a random operator H_ω to a finite box. By $\Lambda_L(x)$ we denote the open box (or cube) of side $L > 0$:

$$\Lambda_L(x) = \{y \in \mathbb{R}^d;\ \|y - x\| < L/2\}, \tag{27}$$

and by $\overline{\Lambda}_L(x)$ the closed box, where $\|x\| = \max\{|x_i|,\ i = 1,\dots,d\}$. (We will use $|x|$ to denote the usual Euclidean norm.) We will also use the notation

$$\chi_{x,L} = \chi_{\Lambda_L(x)}\,, \ \text{ in particular } \ \chi_x = \chi_{x,1} = \chi_{\Lambda_1(x)}\,. \tag{28}$$

The operator $H_{\omega,x,L}$ is defined as an appropriate restriction of H_ω to the box $\Lambda_L(x)$ (e.g., to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition). We write $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ for the finite volume resolvent.

We say that the random operator H_ω satisfies a Wegner estimate in an open interval $\mathcal{I}$ if there exist constants $b \geq 1$, $0 < \eta_{\mathcal{I}} \leq 1$, and $Q_{\mathcal{I}}$, such that

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_{\omega,x,L}), E) \leq \eta\} \leq Q_{\mathcal{I}} \eta L^{bd} \ , \tag{29}$$

for all $E \in \mathcal{I}$, $0 < \eta \leq \eta_{\mathcal{I}}$, $x \in \mathbb{Z}^d$, and $L \in 2\mathbb{N}$.

Wegner estimates have been proven for a large variety of random operators [**Weg, HM, CKM, CH1, Klo2, CH2, CHM, Ki, FK1, FK2, Wa1, KSS, St, CHN, HK, CHKN, KK2**], under certain assumptions. (E.g., for random Schrödinger operators the probability distribution of the random variable ω_i is assumed to have a bounded density, and $u(x) \geq 0$.) Usually $b = 1$ or 2.

If we have a Wegner estimate, we can prove a lower bound on the local transport exponent in the mettalic transport region [**GK3**, Theorem 2.10], which may be compared to the Guarneri-Combes-Last bound (see the paragraph containing (14)). We use the notation $B^{\mathcal{I}} = B \cap \mathcal{I}$ for a subset B of $\mathbb{R}$.

THEOREM 3.1. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. If $\beta(E) > 0$ for some $E \in \mathcal{I}$ then $\beta(E) \geq \frac{1}{2bd}$, i.e. the metallic transport region in $\mathcal{I}$ is given by*

$$\Sigma_{\mathrm{MT}}^{\mathcal{I}} = \left\{ E \in \mathcal{I},\ \beta(E) \geq \frac{1}{2bd} \right\} . \tag{30}$$

In fact, if $E \in \Sigma_{\mathrm{MT}}^{\mathcal{I}}$, then $\beta(n, E) \geq \frac{1}{2bd} - \frac{9b+2}{2bn}$ for all $n \geq 0$.

If the standard random operator satisfies a Wegner estimate we have equality in (16) [**GK3**, Theorem 2.8]:

THEOREM 3.2. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Then*

$$\Sigma_{\mathrm{SI}}^{\mathcal{I}} = \Sigma_{\mathrm{TT}}^{\mathcal{I}} . \tag{31}$$

In particular, the strong insulator region and the metallic transport region are complementary sets in the spectrum $\Sigma^{\mathcal{I}}$ of H_ω in $\mathcal{I}$, i.e.,

$$\Sigma_{\mathrm{SI}}^{\mathcal{I}} \cap \Sigma_{\mathrm{MT}}^{\mathcal{I}} = \emptyset \quad \textit{and} \quad \Sigma_{\mathrm{SI}}^{\mathcal{I}} \cup \Sigma_{\mathrm{MT}}^{\mathcal{I}} = \Sigma^{\mathcal{I}} . \tag{32}$$

Theorem 3.2 shows that the local transport exponent $\beta(E)$ provides a characterization of the metal-insulator transport transition. Theorem 3.1 says that if this transition occurs, $\beta(E)$ has to be discontinuous at a transport mobility edge.

The existence of a nontrivial strong insulator region is now proven for standard random operators. It is a consequence of well established results on Anderson localization and of the bootstrap multiscale analysis [**GK1**, Theorem 3.4] that yields strong HS-dynamical localization [**GK1**, Corollary 3.10]. (See also [**GDB, DS**]. On the lattice strong HS-dynamical localization turns out to be the same as strong dynamical localization (of wave packets) and was originally proven by Aizenman [**Ai, ASFH**].) The relevant results, adapted for this article, will now be stated.

Given $x \in \mathbb{Z}^d$, $L \in \mathbb{N}$, we set

$$\Gamma_{x,L} = \sum_{y \in \Upsilon_L(x)} \chi_y \ , \ \text{where} \ \ \Upsilon_L(x) = \left\{ y \in \mathbb{Z}^d;\ \|y - x\| = \tfrac{L}{2} - 1 \right\} . \tag{33}$$

Given $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-*suitable* for H_ω if $E \notin \sigma(H_{\omega,x,L})$ and

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| \leq \frac{1}{L^\theta}\,. \tag{34}$$

Theorem 3.3. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Given $\theta > d$, for each $E \in \mathcal{I}$ there exists a finite scale $\overline{\mathcal{L}}_\theta(E)$ (depending on θ, E, $Q_{\mathcal{I}}$, d), bounded in compact subintervals of $\mathcal{I}$, such that, if for some $E \in \Sigma \cap \mathcal{I}$ we can verify at some finite scale $\mathcal{L} > \overline{\mathcal{L}}_\theta(E)$ that*

$$\mathbb{P}\{\Lambda_{\mathcal{L}}(0) \ \textit{ is } (\theta, E)\textit{-suitable}\} > 1 - \frac{1}{841^d}\ , \tag{35}$$

then there exists an open interval $I \ni E$, such that H_ω has pure point spectrum in I, with exponentially decaying eigenfunctions, and exhibits strong HS-dynamical localization in I. In particular, $E \in \Sigma_{\mathrm{SI}}$.

Theorem 3.3 suggests the following definition.

Definition 3.4. *The multiscale analysis region Σ_{MSA} is defined as the set of energies where we can perform the bootstrap multiscale analysis:*

$$\Sigma_{\mathrm{MSA}} = \{E \in \Sigma\,;\ \ \textit{the hypotheses of Theorem 3.3 hold at } E\}\,.$$

(Note that if Σ_{AM} denotes the set of energies in the spectrum satisfying the starting hypothesis of the Aizenman-Molchanov method [**AM, Ai, ASFH**], we have $\Sigma_{\mathrm{AM}} = \Sigma_{\mathrm{MSA}}$.)

It follows that

$$\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}} \subset \Sigma_{\mathrm{TT}}\,. \tag{36}$$

With a Wegner estimate we have equality in (36) [**GK3**, Theorem 2.8]:

Theorem 3.5. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Then $\Sigma^{\mathcal{I}}_{\mathrm{TT}} \subset \Sigma^{\mathcal{I}}_{\mathrm{MSA}}$ and hence*

$$\Sigma^{\mathcal{I}}_{\mathrm{MSA}} = \Sigma^{\mathcal{I}}_{\mathrm{SI}} = \Sigma^{\mathcal{I}}_{\mathrm{TT}}\,. \tag{37}$$

The equality (37) shows that the strong insulator region is canonical in the sense that it may be defined by three equivalent conditions or properties, all very natural. In fact the number of such conditions/properties is actually much larger [**GK3**, Theorem 4.2]. In the analogy with classical statistical mechanics the strong insulator region corresponds to the region of complete analyticity [**DoSh1, DoSh2**].

Theorem 3.2 is a corollary of Theorem 3.5. Theorems 3.1 and 3.5 are consequences of the fact that slow transport cannot take place for random operators satisfying our assumptions [**GK3**, Theorem 2.11]. (A weaker form of this result has been discussed by Martinelli and Scoppola [**MS**, Section 8] for the discrete Anderson model.)

Theorem 3.6. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{I}$, $\alpha \geq 0$, and $n > 2bd\alpha + (9b+2)d$. If*

$$\liminf_{T \to \infty} \frac{1}{T^\alpha} \mathcal{M}(n, \mathcal{X}, T) < \infty\,, \tag{38}$$

then $J \cap \Sigma \subset \Sigma_{\mathrm{MSA}}$, and hence $J \cap \Sigma \subset \Sigma_{\mathrm{SI}}$.

Theorem 3.6 has the following immediate corollary, which can be read as follows: if the transport at an energy E is too slow (i.e., $\beta(n,E) < \frac{1}{2bd} - \frac{9b+2}{2bn}$ for some $n > (9b+2)d$), then strong HS-dynamical localization has to hold at E.

Corollary 3.7. *Let H_ω be a standard random operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. If $E \in \mathcal{I} \cap \Sigma$ and $\beta(n,E) < \frac{1}{2bd} - \frac{9b+2}{2bn}$ for some $n > (9b+2)d$, then $E \in \Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$.*

Theorem 3.5 follows immediately from Corollary 3.7, since $\beta(E) = 0 \Rightarrow \beta(n,E) = 0$ for all $n \geq 0$. The same is true for Theorem 3.1, since if $\beta(n,E) < \frac{1}{2bd} - \frac{9b+2}{2bn}$ for some $n > (9b+2)d$, it follows from Corollary 3.7 that $E \in \Sigma_{\mathrm{SI}}$ and hence $\beta(E) = 0$.

Some comments on the proof of Theorem 3.6 are in order. Its main hypothesis, condition (38), is formulated in terms of the dynamics, but the starting hypothesis of the bootstrap multiscale analysis, condition (35), is stated in terms of resolvents. The first step is to reformulate condition (38) in terms of resolvents. To do so, we used the Hilbert-Schmidt norm when we defined the moments in (1), so we can use Plancherel Theorem to get

$$\mathcal{M}(n,\mathcal{X},T) = \frac{1}{\pi T}\int_{\mathbb{R}} \mathbb{E}\left\| \langle X\rangle^{\frac{n}{2}} R_\omega(E + i\tfrac{1}{T})\mathcal{X}(H_\omega)\chi_0 \right\|_2^2 \mathrm{d}E\,, \tag{39}$$

for any $n > 0$, $T > 0$, and $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, where $R_\omega(z) = (H_\omega - z)^{-1}$. Thus, if we set

$$\Omega_\varepsilon(n,\mathcal{X},E) = \mathbb{E}\left(\left\| \langle X\rangle^{\frac{n}{2}} R_\omega(E + i\varepsilon)\mathcal{X}(H_\omega)\chi_0 \right\|_2^2 \right)\,, \tag{40}$$

condition (38) in Theorem 3.6 is the same as

$$\Omega \equiv \liminf_{\varepsilon \to 0^+} \varepsilon^{1+\alpha} \int_{\mathbb{R}} \Omega_\varepsilon(n,\mathcal{X},E)\,\mathrm{d}E < \infty\,. \tag{41}$$

The proof of Theorem 3.6 requires that we obtain the finite volume condition (35) for the bootstrap multiscale analysis out of the infinite volume condition (41). The kernel polynomial decay estimate from [**GK2**] plays an important role in the proof, as well as the Wegner estimate. We refer to [**GK3**] for the details.

4. Transport transition in random polymer models

Following Jitomirskaya, Schulz-Baldes and Stolz [**JSBS**], we define a random polymer model by

$$H_\omega = -\Delta + V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z})\,, \tag{42}$$

where Δ is the centered discrete Laplacian and the random potential V_ω is defined as follows: Let $\hat{v}_\pm = (\hat{v}_\pm(1), \cdots, \hat{v}_\pm(L_\pm))$ be two finite sequences of real numbers, where $L_\pm \in \mathbb{N}$. A discrete one dimensional random polymer is constructed by randomly juxtaposing blocks of sign $+$ and sign $-$: let $\omega = \{\omega_i\,;\ i \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables taking the value $+$ with probability $p \in (0,1)$ and $-$ with probability $1-p$; $\{V_\omega(x);\ x \in \mathbb{Z}\}$ is the sequence of blocks $(\ldots, \hat{v}_{\omega_0}, \hat{v}_{\omega_1}, \ldots)$. Note that H_ω is a $\mathbb{Z}$-ergodic self-adjoint random operator [**JSBS**].

The spectrum of a random polymer model is always pure point with exponentially decaying eigenfunctions, so there is no spectral transition. Moreover, they exhibit strong HS-dynamical localization on energy intervals not containing a discrete set of special energies. (See [**DBG**] for the random dimer model, for random polymer models these results can be proven by the methods of [**DSS**].) Nevertheless, random polymer models may exhibit a transport transition, as we shall see.

Random polymer models are similar to Bernoulli models (i.e., $L_\pm = 1$); they do not satisfy a Wegner estimate as stated in Section 2, only a subexponential Wegner estimate (i.e., with $\eta \le \mathrm{e}^{-L^\beta}$ with $0 < \beta < 1$ in (29), see [**CKM**, Theorem 4.1]) on appropriate intervals. The bootstrap multiscale analysis can still be performed with this subexponential Wegner estimate [**GK3**, Remark 3.13], and it gives pure point spectrum with exponentially decaying eigenfunctions, and strong HS-dynamical localization. But Theorem 3.6 (and hence Theorems 3.1 and 3.2) is not established for these models, since the proof relies on the polynomial decay estimates of [**GK2**], which can only beat polynomial bounds obtained by using the Wegner estimate (i.e., $\eta = L^{-s}$, $s > 0$ in (29)). Nevertheless, we can say a lot about the metal-insulator tranport transition for random polymer models.

Random polymer models may have *critical energies*, which we now define following [**JSBS**]. The transfer matrices associated to the blocks $\pm$ at the energy $E \in \mathbb{R}$ ae given by:

$$T_\pm^E = \prod_{L_\pm}^{i=1} T(\hat{v}_\pm(i) - E)\ , \quad \text{where} \quad T(v) = \begin{pmatrix} v & -1 \\ 1 & 0 \end{pmatrix} . \tag{43}$$

An energy $E_c \in \mathbb{R}$ is said to be critical for the random operator H_ω if the matrices $T_\pm^{E_c}$ are either elliptic (i.e., $\left|\mathrm{Tr}T_\pm^{E_c}\right| < 2$) or equal to $\pm\mathrm{I}$, and they commute (i.e., $\left[T_+^{E_c}, T_-^{E_c}\right] = 0$).

Critical energies may or may not exist. An example is given by the random dimer model: $L_\pm = 2$, $\hat{v}_\pm(i) = \pm\lambda$ for $i = 1, 2$, with $0 < \lambda$. De Bièvre and Germinet [**DBG**] proved:

(i): If $0 < \lambda \le 1$ there are exactly two critical energies: $\pm\lambda$.
(ii): If $1 < \lambda$ there are no critical energies.

Dunlap, Philipps and Wu [**DWP**] found numerically an interesting phenomenon of delocalization for the random dimer model at the critical energies, due to the absence of reflection at these energies (and thus perfect transmission). Such a phenomenon has been contested by physicists (e.g., see [**LGP**]). De Bièvre and Germinet [**DBG**] showed that energies that are not critical cannot contribute to this delocalization, and strong dynamical localization holds outside the critical energies (if $\lambda \ne \frac{1}{\sqrt{2}}, \sqrt{2}$). Jitomirskaya, Schulz-Baldes and Stolz [**JSBS**] confirmed by a rigorous proof the numerical computations of [**DWP**].

The transport exponents for random polymer models are defined as in Section 2. Note that in the lattice χ_0 is simply the orthogonal projection on the vector δ_0, the element of $\ell^2(\mathbb{Z})$ taking the value 1 at site 0 and the value 0 everywhere else. Thus (1) may be rewritten as

$$M_\omega(n, \mathcal{X}, t) = \left\| \langle X \rangle^{\frac{n}{2}} \mathrm{e}^{-itH_\omega} \mathcal{X}(H_\omega)\delta_0 \right\|^2 . \tag{44}$$

Since the ramdom operator H_ω is bounded, uniformly in ω, we may remove the restriction of $\mathcal{X}$ having compact support; in particular we may calculate moments and exponents with $\mathcal{X} \equiv 1$. It follows from [**JSBS**, Main Theorem], using Fatou's

Lemma and Jensen's inequality, that if H_ω has a critical energy E_c we have

$$\beta(n, \mathcal{X} \equiv 1) \geq 1 - \frac{1}{n} \quad \text{for all} \quad n \geq 0\,. \tag{45}$$

We can extend this result to local transport exponents, proving that critical energies lie in the metallic transport region, and hence the existence of the transport transition.

Theorem 4.1. *Let H_ω be a random polymer Hamiltonian as in* (42)*. If E_c is a critical energy, then*

$$\beta(n, E_c) \geq 1 - \frac{1}{n} \quad \textit{for all} \quad n \geq 0\,. \tag{46}$$

In particular, $\beta(E_c) = 1$ and $E_c \in \Sigma_{\mathrm{MT}}$.

The proof of Theorem 4.1 is given at the end of this section.

Under an extra hypothesis on the transfer matrices $T_\pm^{E_c}$, the bound (45) can be improved to $\beta(n, \mathcal{X} \equiv 1) \geq 1 - \frac{1}{2n}$ [**JSBS**, Theorem 4]; such a derivation requires a large deviation estimate for products of transfer matrices. In this case the proof of Theorem 4.1 yields $\beta(n, E_c) \geq 1 - \frac{1}{2n}$.

We may summarize the results on a transport transition for random polymer models in the following theorem

Theorem 4.2. *Let H_ω be a random polymer Hamiltonian as in* (42)*. Then $\Sigma\backslash\Sigma_{\mathrm{SI}}$ is a discrete set containing the critical energies (and hence $\Sigma_{\mathrm{SI}} \neq \emptyset$). Moreover, the set of critical energies is contained in Σ_{MT}, so if there are critical energies we have $\Sigma_{\mathrm{MT}} \neq \emptyset$ and there is a transport transition.*

For the special case of the random dimer model, we may use the results of [**DBG**] to get more detailed information.

Theorem 4.3. *Let Let H_ω be a random dimer Hamiltonian i.e., H_ω is as in* (42) *with $L_\pm = 2$, $\hat{v}_\pm(i) = \pm\lambda$ for $i = 1, 2$, with $0 < \lambda$. Then*

(i): *If $0 < \lambda \leq 1$, we have a transport transition and $\pm\lambda$ are transport mobility edges. Moreover,*

$$\Sigma_{\mathrm{MT}} = \{-\lambda, \lambda\}\,, \quad \Sigma_{\mathrm{SI}} = \Sigma\backslash\Sigma_{\mathrm{MT}} \quad \textit{if} \quad \lambda \neq \frac{1}{\sqrt{2}}\,, \tag{47}$$

$$\Sigma_{\mathrm{MT}} \supset \left\{-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\}, \quad \Sigma_{\mathrm{SI}} \supset \Sigma\backslash\left\{-\tfrac{3}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\right\} \textit{ if } \lambda = \frac{1}{\sqrt{2}}\,. \tag{48}$$

(ii): *If $1 < \lambda$, $\lambda \neq \sqrt{2}$, we do not have a transport transition:*

$$\Sigma_{\mathrm{MT}} = \emptyset \quad \textit{and} \quad \Sigma_{\mathrm{SI}} = \Sigma\,. \tag{49}$$

(iii): *If $\lambda = \sqrt{2}$ we have*

$$\Sigma_{\mathrm{SI}} \supset \Sigma\backslash\{0\}\,. \tag{50}$$

Proof of Theorem 4.1. Let E_c be a critical energy, and consider open bounded intervals I, I', with $E_c \in I \subset I'$, $\delta = \mathrm{dist}(I, \mathbb{R}\backslash I') > 0$, and $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on I'.

As in [**JSBS**, Proposition 1], it is an immediate consequence of the definition of a critical energy that $T_\pm^{E_c}$ can be represented as two rotations in some appropriate basis, and thus, there exists a constant $C > 0$ such that, for $z \in \mathbb{C}$ with $|z - E_c|$

small enough, we have $\| \prod_{i=k}^{l} T_{\omega_i}^{z} \| \le C \exp(C|z - E_c| \, |k - l|)$. Following [**JSBS**, Section 6, Proof of Theorem 1], or [**DT**, Section 2, Proof of Theorem 1.2], it implies that there exists a constant $C > 0$ such that, for all $\omega \in \Omega$,

$$\frac{2\pi}{T} \int_I \left\| \langle X \rangle^{n/2} R_\omega \left(E + iT^{-1} \right) \delta_0 \right\|^2 \mathrm{d}E \ge C T^{n-1} . \tag{51}$$

We exploit this lower bound together with the kernel decay estimate of [**GK2**, Theorem 2] to get a lower bound on $\mathcal{M}_\omega(n, \mathcal{X}, t)$, which is defined as in (3) but without taking the expectation. Using Plancherel's Theorem as in [**GK3**, Lemma 6.3], writing $\mathcal{X} = 1 - (1 - \mathcal{X})$, and using the inequality $(a-b)^2 \ge \frac{1}{2} a^2 - b^2$, for a, b real, we get

$$\frac{1}{T} \int_0^\infty e^{-2t/T} M_\omega(n, \mathcal{X}, t) \; \mathrm{d}t \tag{52}$$

$$= \frac{2\pi}{T} \int_{\mathbb{R}} \left\| \langle X \rangle^{n/2} R_\omega \left(E + iT^{-1} \right) \mathcal{X}(H_\omega) \delta_0 \right\|^2 \mathrm{d}E, \tag{53}$$

$$\ge \frac{1}{2T} \int_I \left\| \langle X \rangle^{n/2} R_\omega \left(E + iT^{-1} \right) \delta_0 \right\|^2 \mathrm{d}E \tag{54}$$

$$- \frac{1}{T} \int_I \left\| \langle X \rangle^{n/2} R_\omega \left(E + iT^{-1} \right) \left(1 - \mathcal{X}(H_\omega) \right) \delta_0 \right\|^2 \mathrm{d}E . \tag{55}$$

Proceeding as in [**GK3**, Eq. (6.34)], using [**GK2**, Theorem 2], we get

$$\sup_{E \in I} \sup_{T > 0} \left\| \langle X \rangle^{n/2} R_\omega \left(E + iT^{-1} \right) \left(1 - \mathcal{X}(H_\omega) \right) \delta_0 \right\|^2 \le C(n, \delta). \tag{56}$$

where $\delta = \operatorname{dist}(I, \mathbb{R} \backslash I') > 0$, and the finite constant $C(n, \delta)$ does not depend on ω. Combining (51) and (56), we get that there exists a constant $C' > 0$, depending only on the constants of the polymer, I, I', and $\mathcal{X}$, such that, for all $\omega \in \Omega$,

$$\frac{1}{T} \int_0^\infty e^{-2t/T} M_\omega(n, \mathcal{X}, t) \; \mathrm{d}t \ge C' T^{n-1} \tag{57}$$

for all $T > 0$. Taking the expectation, we obtain

$$\mathcal{M}(n, \mathcal{X}, T) \ge C' T^{n-1} \tag{58}$$

for all $T > 0$. It follows that $\beta(n, I) \ge 1 - \frac{1}{n}$, and since $I \ni E_c$ is arbitrary, $\beta(n, E_c) \ge 1 - \frac{1}{n}$. The theorem follows. □

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