Elliptic two-dimensional invariant tori for the planetary three-body problem*

Luca Biasco

Settore di Analisi Funzionale e Applicazioni

Scuola Internazionale Superiore di Studi Avanzati

Via Beirut 2-4, 34014 Trieste (Italy)

(biasco@sissa.it)

Luigi Chierchia

Dipartimento di Matematica

Università "Roma Tre"

Largo S. L. Murialdo 1, 00146 Roma (Italy)

(luigi@mat.uniroma3.it)

Enrico Valdinoci

Istituto Matematico

Scuola Normale Superiore

Piazza dei Cavalieri 7, 56126 Pisa (Italy)

(enrico@math.sns.it)

July 29, 2002

Abstract

The spatial planetary three-body problem (i.e., one "star" and two "planets", modelled by three massive points, interacting through gravity in a three dimensional space) is considered. It is proven that, near the limiting stable solutions given by the two planets revolving around the star on Keplerian ellipses with small eccentricity and small non-zero mutual inclination, the system affords two-dimensional, elliptic, quasi-periodic solutions, provided the masses of the planets are small enough compared to the mass of the star and provided the osculating Keplerian major semi-axes belong to a two-dimensional set of density close to one.

Keywords: three-body problem, nearly-integrable Hamiltonian systems, averaging theory, KAM theory, lower dimensional elliptic tori, small divisors.

MSC2000 numbers: 34C27, 37J40, 70K43, 70F07, 70F15, 70K65, 70H08.

*Supported by M.U.R.S.T. *Variational Methods and Nonlinear Differential Equations*.

Contents

1 Introduction and Results

1.1 In this paper we consider *the (non-planar) planetary three-body problem*, namely, the mechanical system made up of three massive points, one of which (the "star") has a significantly larger mass than the other two points (the "planets"), and which interact through a Newtonian gravitational field; the masses of the planets are regarded as small parameters. In particular, motivated by astronomical data, we are interested in *nearly circular* planetary motions ("small eccentricities") taking place along *nearly co-planar* orbits ("small mutual inclinations").

As well known, such system has been a source of extremely deep (and difficult) studies, among which stem out the contributions of Charles Eugene Delaunay and, especially, of Henri Poincaré. According to Delaunay and Poincaré, the three-body problem is described by a *nearly-integrable Hamiltonian system* on an eight-dimensional phase space, equipped with real-analytic action-angle variables. Such system turns out to be *properly degenerate*, i.e., the integrable limit (in which the three-body problem is described by two decoupled and integrable two-body systems) depends only on *two* (action) variables: in the integrable limit, *all motions lie on two-dimensional invariant tori* run by quasi-periodic motions with two frequencies (related, by Kepler's law, to the major semi-axis of the two limiting Keplerian ellipses). Furthermore, in the small-eccentricity-inclination regime, these two-tori are *linearly stable* (i.e., linearizing the full system along the unperturbed tori one finds a linear system with *purely imaginary eigenvalues*).

A basic natural question is: *what is the destiny of these two-dimensional tori when the full system is considered?*

Surprisingly enough, such question is, up to now, unsolved. In a 1966 paper [JM66], W.H. Jefferys and Jürgen Moser established the persistence of two-dimensional invariant tori for the planetary three-body problem in the case of large mutual inclinations; in such a case the unperturbed tori (as well as the perturbed ones) are *unstable* (or partially hyperbolic). Actually, Jefferys and Moser considered explicitly the above question (relative to the small-eccentricity-inclination regime), but were unable to settle it, leaving the reader with the feeling that they did not believe in the persistence of two-dimensional, elliptic tori[1].

In this paper, we answer the above question, showing that, for values of the "initial" semi-axis of the osculating ellipses in a set of nearly-full (two-dimensional) measure, *the above described unperturbed tori do persist in the full system*, provided the masses of the planets are small enough.

1.2 Let us, now, give an *analytical formulation of our result*. To do this we recall the classical Hamiltonian (action-angle) formulation of the planetary (non planar) three-body problem (for small eccentricity and small mutual inclination) according to Delaunay and Poincaré. Denote the three massive points ("bodies") by P_0, P_1, P_2 and let m_0, m_1, m_2 be their masses interacting through gravity (with constant of gravitation 1). Assume that, for some $0 < \bar\kappa \le 1$,

$$\bar\kappa\varepsilon \le \frac{m_1}{m_0}\,,\ \frac{m_2}{m_0} \le \varepsilon \le 1\,.$$

The number ε is regarded as a small parameter: the point P_0 represents "the star" and the points P_1 and P_2 "the planets". For $j = 1, 2$, consider the "osculating ellipses" of the two-body problems associated to the planets P_j and the star[2] P_0 and *assume that the eccentricities of such ellipses are small and that the intersection angle, $\hat\imath$, between the two planes containing the two osculating ellipses* ("mutual inclination") *is also small.* As customary in celestial mechanics, we denote the major semi-axes of such ellipses by a_j and their eccentricity by e_j. Let us, also, denote the mean anomaly by ℓ_j and the longitude of the perihelion by g_j (see figures below).

[1] "The details of the existence proof completed, it should be observed that the solutions found are of the elliptic-hyperbolic type, and hence are unstable. It would be desirable to establish similarly the existence of such solutions in the stable case. In this case, however, there is an essential difficulty,..." ([JM66], section (7), page 575); and also: "[$\cdots$] however, there are good reasons to conjecture that in general the stable solutions need not persist...." ([JM66], section (1), page 568).

[2] The "osculating ellipses (at time t_0)" of the two-body problems associated to (P_0, P_j), ($j = 1$ or $j = 2$), is defined as follows. Let $u^{(0)}$ and $u^{(j)}$ denote the coordinates (in some reference frame) of the points P_0 and P_j at time t_0 and let $\dot u^{(0)}$ and $\dot u^{(j)}$ denote the respective velocities. The "osculating plane" is defined as the plane spanned by $(u^{(j)} - u^{(0)})$ and $(\dot u^{(j)} - \dot u^{(0)})$; the "osculating ellipse is defined as the Keplerian ellipse (lying on the osculating plane) defined by the Kepler solution, with initial data $(u^{(0)}, u^{(j)})$ and $(\dot u^{(j)} - \dot u^{(0)})$, of the two-body problem (P_0, P_j) obtained disregarding (for $t \ge t_0$) the third body P_i ($i \ne j$); see Appendix C for details.

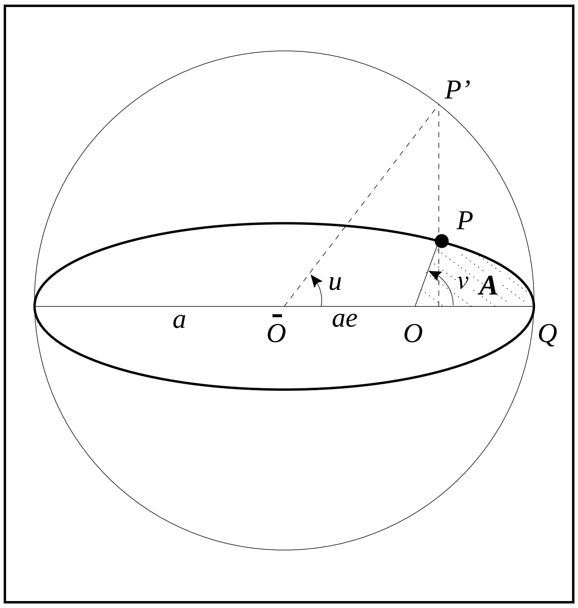

Figure 1 (Keplerian ellipse)
$\bar{O}$=center, O=focus, Q=perihelion, a=major semi-axis= $\mathrm{dist}\,(\bar{O},Q)$, e=eccentricity= $\mathrm{dist}\,(\bar{O},O)/a$
u = eccentric anomaly, v=true anomaly, A=area shaded region=$\frac{a^2\sqrt{1-e}}{2}(u - e\sin u)$.

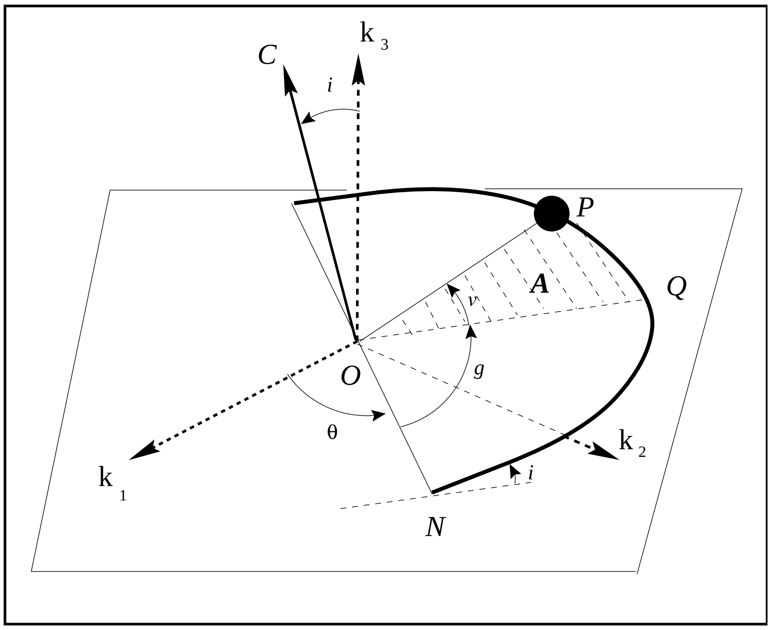

Figure 2 (orbital elements)
$\{k_1, k_2, k_3\}$=inertial frame, C=angular momentum of the 2-body system, N=node $\in$ ellipse plane $\cap$ span$\{k_1, k_2\}$
i=inclination, θ=longitude of the node, g=argument of the perihelion, ℓ=mean anomaly=$2\pi\frac{A}{A_{\rm tot}} = u - e\sin u$
w=true longitude=$\theta + g + v$, w^*=$g + v$

Let

$$\Lambda_j^* = \kappa_j^* \sqrt{a_j}\ , \qquad \kappa_j^* := \frac{m_j}{\varepsilon}\, \frac{1}{\sqrt{m_0(m_0+m_j)}}\ ,$$

(κ_j^* is a dimensionless constant satisfying $\frac{\bar\kappa}{\sqrt{2}} < \kappa_j^* < 1$) and, following notations common in celestial mechanics, define

$$\begin{aligned}
\lambda_j^* &= \ell_j + g_j\ , \\
H_j &= \Lambda_j^*\,(1-\sqrt{1-e_j^2})\ , \\
\xi_j^* &= \sqrt{2H_j}\cos g_j\ , \\
\eta_j^* &= -\sqrt{2H_j}\sin g_j\ .
\end{aligned}$$

Since we are interested in small eccentricities, collisions are avoided by requiring that the major semi-axes $a_j = a_j(\Lambda^*) := (\Lambda_j^*/\kappa_j^*)^2$ are different (and different from zero). We, therefore, *fix, once and for all,*

$$0 < a_{\rm min} < a_{\rm max} \qquad \text{and} \qquad 0 < \alpha_{\rm max} < 1 \tag{1.1}$$

and, from now on, we shall consider (attaching the index 1 to the "inner planet") values of Λ^* in the compact set

$$\{\Lambda^* \in \mathbb{R}^2 : a_{\rm min} \le a_1 < a_2 \le a_{\rm max}\ \text{ and }\ \frac{a_1}{a_2} \le \alpha_{\rm max}\}\ . \tag{1.2}$$

For a given set $A \subset \mathbb{R}^n$ and a given number $r > 0$, we shall denote by A_r the set in $\mathbb{C}^n$ at distance less than r from A, i.e., the set

$$A_r := \bigcup_{I\in A} D_r^n(I) \subset \mathbb{C}^n\ , \tag{1.3}$$

where $D_r^n(I)$ denotes the (open) complex n-ball of radius r centered at I while the *real* n-ball of radius r centered at I will be denoted $B_r^n(I)$; complex or real n-balls centered at the origin will be simply denoted by D_r^n or, respectively, B_r^n. The following classical result holds.

Theorem 1.1 (Delaunay, Poincaré) *There exist positive constants $\bar\varepsilon$, $\bar\delta$, $\imath_{\rm max}$ and $e_{\rm max}$ such that the above introduced variables (Λ^*, λ^*) are, for $0 < \varepsilon < \bar\varepsilon$, standard, real-analytic symplectic variables describing all motions of the spatial three-body problem in a $O(\bar\delta)$-neighborhood of any point Λ_0^* in the compact set* (1.2)*, for $e_j \le e_{\rm max}$ and for non-vanishing mutual inclinations $\hat\imath$ not bigger than $\imath_{\rm max}$; the Hamiltonian governing such motions (with respect to the standard symplectic form $\sum_i d\Lambda_i^* \wedge d\lambda_i^* + \sum d\eta_i^* \wedge d\xi_i^*$) is the real-analytic function*

$$-\frac{1}{2}\sum_{j=1}^{2} \frac{\kappa_j}{\Lambda_j^{*2}} + \varepsilon F(\Lambda^*, \lambda^*, \eta^*, \xi^*)\ , \tag{1.4}$$

where $\kappa_j := \left(\frac{m_j}{\varepsilon}\right)^3 \frac{1}{m_0^2(m_0+m_j)}$ *are positive constant of order 1* ($\frac{\bar\kappa^3}{2} < \kappa_j < 1$) *and* F *is a suitable function real-analytic in a neighborhood of* $B^2_{\bar\delta}(\Lambda_0^*) \times \mathbb{T}^2 \times \{(0,0)\}$. *More precisely, fix* Λ_0^* *in the compact set* (1.2) *and fix* $0 < \imath_{\min} < \imath_{\max}$, *then, for any* $0 < \varepsilon < \bar\varepsilon$, $\imath_{\min} \le |\hat\imath| \le \imath_{\max}$, *the function* F *can be taken to be real-analytic on the complex set*

$$\mathcal{I}_{\sigma_0} \times \mathbb{T}^2_{2s_0} \times D^4_{2\rho_0} \subset \mathbb{C}^8 \ ,$$

where σ_0, s_0, ρ_0 *are suitable positive numbers and where*

$$\mathcal{I} := [\Lambda_{01}^* - \delta, \Lambda_{01}^* + \delta] \times [\Lambda_{02}^* - \delta, \Lambda_{02}^* + \delta] \ ,$$

for some $\delta \ge \bar\delta$.

Furthermore, there exists a real-analytic, symplectic change of variables $\Psi_{\mathrm{W}} : (I, \varphi, p, q) \mapsto (\Lambda^*, \lambda^*, \eta^*, \xi^*)$ *of the form*

$$\Lambda^* = I \ , \qquad \lambda^* = \varphi + \hat\ell(I,p,q) \ , \qquad \binom{\eta^*}{\xi^*} = A(I)\binom{p}{q} \ , \tag{1.5}$$

where $\hat\ell$ *and* A *are real-analytic and* A *is a* (4×4) *symplectic matrix, such that, in the variables* (I,φ,p,q), *the Hamiltonian* (1.4) *takes the form*

$$\mathcal{H}(I,\varphi,p,q) = h(I) + f(I,\varphi,p,q) \,, \tag{1.6}$$

with

$$\begin{aligned} & h := -\frac{1}{2}\sum_{i=1}^2 \frac{\kappa_i}{I_i^2} \,, \quad f := \varepsilon f_1(I,p,q) + \varepsilon f_2(I,\varphi,p,q) \,, \\ & f_1 := f_{1,0}(I) + \sum_{i=1}^2 \bar\Omega_i(I)(p_i^2+q_i^2) + \tilde f_1(I,p,q) \ , \\ & \int_{\mathbb{T}^2} f_2 \, d\varphi = 0 \ , \qquad \sup_{\mathcal{I}_{\sigma_0}} |\tilde f_1| \le \mathrm{const}|(p,q)|^4 \ ; \end{aligned} \tag{1.7}$$

$\hat\ell$, A *and* f_i *are real-analytic and uniformly bounded on*

$$\mathcal{D}_0 := \mathcal{I}_{\sigma_0} \times \mathbb{T}^2_{s_0} \times D^4_{\rho_0} \subset \mathbb{C}^8 \ ,$$

and $\tilde f$ *is even in* (p,q). *Finally,*

$$\begin{aligned} & \inf_{I\in\mathcal{I}} \bar\Omega_i > \inf_{I\in\mathcal{I}_{\sigma_0}} |\bar\Omega_i| > \ \mathrm{const}\ > 0 \ , \\ & \inf_{I\in\mathcal{I}} \left(\bar\Omega_2 - \bar\Omega_1\right) > \inf_{I\in\mathcal{I}_{\sigma_0}} |\bar\Omega_2 - \bar\Omega_1| > \ \mathrm{const}\ > 0 \,. \end{aligned} \tag{1.8}$$

Remark 1.1 (i) Even though the results listed in this theorem are classical, the analytical formulation presented here (together with a detailed proof of it) seems not easy to be found in the literature. An effort to remedy to this breach has been done at the Bureau des longitudes in Paris in the late 80's and we refer the interested reader to the *Notes scientifiques et techniques du B.D.L.* by A. Chenciner and J. Laskar ([Ch88], [L88]). For completeness, we present the proof of Theorem 1.1 in[3] Appendix C.
(ii) We sketch here, very briefly, the ideas beyond Theorem 1.1, referring for details to Appendix C. First of all, by elementary mechanics, one can cast the three-body problem into a nine-degree-of-freedom Hamiltonian formalism. Then, reduction of the center of mass makes it possible to lower the number of degrees of freedom to six. In the planetary case considered here (one "star" and two comparatively small "planets"), such Hamiltonian system may be seen as a perturbation of two decoupled Kepler problems. Hence, classical Delaunay variables may be exploited to integrate the decoupled Kepler problems. Such variables present however certain singularities, which H. Poincaré showed how to avoid, by introducing an analytic set of variables, now, called "Poincaré variables" (or, more precisely, "osculating Poincaré variables[4]"). Finally, the reduction of the angular momentum (also known in the literature as Jacobi's "reduction of the nodes") introduces two extra integrals of motion - called *Poincaré integrals* in Appendix C - which allow to lower the degrees of freedom bringing the system into its final form of a four-degree-of-freedom Hamiltonian system.
The non-planarity condition $\hat{\imath} \neq 0$ is only needed in order to *define* the osculating Poincaré variables, while the smallness assumption on the inclination $\hat{\imath}$ (i.e. $|\hat{\imath}| < \imath_{\rm max}$) is related to the linear stability of the limiting motions, which, mathematically, reflects in $\bar\Omega_j$ being real; compare also with next item (iii). We point out, however, that, in considering *very* small mutual inclinations, one has to take, also, *very* small eccentricities and, in addition, information about the size of the complex analyticity domain get lost: in fact, as $\imath_{\rm min}$ goes to zero, $\sigma_0 \sim \rho_0^2 \sim \imath_{\rm min}^2$; compare (ii) of Remark C.2 in Appendix C.
Finally, we stress that the function F *does depend upon the chosen $O(1)-$neighborhood of Λ_0^* but obeys uniform bounds in the whole domain* (1.2). Such dependence is hidden in the dependence of the perturbation function upon the above mentioned Poincaré integrals, which, for small eccentricities, are related to the mutual inclination of the osculating ellipses.
(iii) The change of variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*) \mapsto (I, \varphi, p, q)$ is, simply, the (straightforward) completion of the linear symplectic diagonalization[5] of the (η^*, ξ^*)-quadratic part of the

[3]Beware that the variables (p, q) appearing in Theorem 1.1 correspond to the variables (η', ξ') of Appendix C (and are not to be confused with the variables (p, q) used in Appendix C).

[4]In connection with three-body problems, there are two sets of Poincaré variables, differentiated in Appendix C by means of a "$*$": the "Poincaré variables" $(\Lambda, \lambda, \eta, \xi) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}^4$ are particularly suited to treat the *planar* three-body problem, while the "osculating Poincaré variables" (defined only for non-vanishing mutual inclinations) $(\Lambda^*, \lambda^*, \eta^*, \xi^*) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}^4$ are more convenient in dealing with the *spatial* three-body problem; the word "osculating" refers to the fact that these variables "live" on the two planes associated to the two osculating ellipses.

[5]"Symplectic diagonalization" of a quadratic Hamiltonian $\mathcal{Q}(z)$, $z \in \mathbb{R}^{2n}$, means diagonalization, by a linear symplectic map, of the constant matrix $S_{2n}\mathcal{Q}''$, S_{2n} being the standard $(2n \times 2n)$ symplectic

"secular term" of the Hamiltonian (1.4), i.e., of the average over the angles λ^* of εF, namely, εf_1. Under the assumption that $\hat{\imath}$ is small enough, such quadratic part turns out to be *positive definite* and, hence, $\pm\sqrt{-1}(2\varepsilon\bar{\Omega}_j)$ are, simply, the purely imaginary eigenvalues of the (4×4)-matrix $S_4\mathcal{Q}''$, where S_4 denotes, here, the standard (4×4) symplectic unity matrix and $\mathcal{Q}=\mathcal{Q}(\eta^*,\xi^*;\Lambda^*)$ denotes the (positive-definite) quadratic part of $\varepsilon\fint_{\mathbb{T}^2}Fd\lambda^*$ (thought of as a function of (η^*,ξ^*) and parameterized by Λ^*). This diagonalization procedure, already known to Weierstrass, requires, in the case of the three-body problem, certain calculations, involving Laplace coefficients (which, in turn, are simply related to the Gauss hyper-geometric function): such calculations are sketched in Appendix C (see, also, [LR95] and [R95]).
(iv) The Hamiltonian $\mathcal{H}$ describes a nearly-integrable, properly degenerate systems: the integrable limit ($\varepsilon=0$) depends only on the two action variables I_1, I_2. The frequency vector associated to the integrable limit is $\big(\kappa_1/I_1^3,\kappa_2/I_2^3\big)$, which is a vector of order one. This means that the conjugated angles φ may be regarded as *fast angles* and, in "first approximation", the $\mathcal{H}$-motions are governed by the averaged Hamiltonian $h+\varepsilon f_1$: such Hamiltonian, which in case of the spatial three-body problem is *non integrable*, is sometimes referred to as "secular Hamiltonian[6]". By the discussion in the preceding item (iii) and from (1.7), it follows that $p=0=q$ is an *elliptic equilibrium* for the Hamiltonian f_1 and hence, for any I,

$$\{\varphi\in\mathbb{T}^2\}\times\{p=0=q\} \tag{1.9}$$

is an *elliptic two-torus* for the averaged Hamiltonian $h+\varepsilon f_1$ run by the linear flow $\varphi\to\varphi+\nabla(h+\varepsilon f_{1,0})t$. *These are the secular motions that we shall prove to persist for* $0<\varepsilon\ll 1$ *and for special, but nearly-full measure, values of* I.

We can, now, give a precise formulation of our main result. Let $\bar{\varepsilon}$, $\bar{\delta}$, $\imath_{\max}$, $e_{\max}$, Λ_0^*, $\mathcal{I}$, $\imath_{\min}$, σ_0, s_0, ρ_0 be as in Theorem 1.1 above and let meas_n denote the n-dimensional Lebesgue measure.

Theorem 1.2 *Fix* $\tau>1$ *and pick two numbers* b_i *such that*

$$0<b_1<\frac{1}{2}\ ,\qquad 0<b_2<\Big(\frac{1}{2}-b_1\Big)\frac{1}{\tau+1}\ .$$

Then, there exist $0<\varepsilon_0<\bar{\varepsilon}$ *and* $C>1$, *such that, for any* $0<\varepsilon<\varepsilon_0$, *one can find a Cantor set* $\mathcal{I}_*\subset\mathcal{I}$, *with*

$$\mathrm{meas}_2(\mathcal{I}\setminus\mathcal{I}_*)\le C\varepsilon^{b_1}\ , \tag{1.10}$$

such that the following holds. There exist a Lipschitz homeomorphism $\omega_*:\mathcal{I}_*\to\mathbb{R}^2$ *and a Lipschitz continuous family of tori embedding*

$$\phi:\ (\theta,J)\in\mathbb{T}^2\times\mathcal{I}_*\mapsto\Big(I_\phi(\theta;J),\ \varphi_\phi(\theta;J),\ p_\phi(\theta;J),\ q_\phi(\theta;J)\Big)\in\mathcal{I}\times\mathbb{T}^2\times B^4_{\rho_*}\ ,$$

unit matrix; $\mathcal{Q}''$ denotes the Hessian matrix of the second derivatives of $\mathcal{Q}$.

[6] A computer-assisted KAM theory for the secular Hamiltonian of the spatial three-body problem is studied in [LG00].

with $\rho_* := C\varepsilon^{b_2}$*, such that, for any* $J \in \mathcal{I}_*$*,* $\phi(\mathbb{T}^2, J)$ *is a real-analytic (elliptic)* $\mathcal{H}$*-invariant torus, on which the* $\mathcal{H}-$*flow is analytically conjugated to the linear flow* $\theta \mapsto \theta + \omega_* t$*. Furthermore,* $\phi(\cdot, J)$ *is real-analytic on* $\mathbb{T}^2_{s_0/8}$ *and the following bounds hold, uniformly on* $\mathbb{T}_{s_0/8} \times \mathcal{I}_*$*:*

$$\begin{aligned} |I_\phi(\theta, J) - J| &\leq C\varepsilon^{\frac{1}{2}+b_2} , \\ |p_\phi(\theta, J)| + |q_\phi(\theta, J)| &\leq C\varepsilon^{b_2} , \\ |\omega_*(J) - \nabla h(J)| &\leq C\varepsilon . \end{aligned} \tag{1.11}$$

Also, there exists a Lipschitz continuous function $\Omega_* : \mathcal{I}_* \longrightarrow \mathbb{R}^2$ *such that*

$$|\Omega_*(J) - \varepsilon\bar{\Omega}(J)| \leq C\varepsilon^{1+b_2} , \tag{1.12}$$

and[7]

$$|\omega_* \cdot k + \Omega_* \cdot \ell| \geq \frac{\varepsilon}{C(1 + |k|^\tau)} \tag{1.13}$$

for any $(k, \ell) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \setminus \{(0, 0)\}$ *with* $|\ell| \leq 2$*.*

1.3 We make, now, a few **comments and remarks**.

1.3.1 (*Amplification*) The proof presented below allows to catch also the *limiting case* "$b_1 \to 1/2$": compare also (ii), Remark 2.1 below. In this case, b_1 and b_2 are not defined, nevertheless, the conclusion of Theorem 1.2, hold with the following estimates replacing, respectively, (1.10), (1.11) and (1.12) ((1.13) remains unchanged):

$$\mathrm{meas}_2(\mathcal{I} \setminus \mathcal{I}_*) \leq C\varepsilon^{\frac{1}{2}}\Big(\log\frac{1}{\varepsilon}\Big)^{\tau+1} ; \tag{1.14}$$

$$\begin{aligned} |I_\phi(\theta, J) - J| &\leq C\frac{\varepsilon^{\frac{1}{2}}}{\log\frac{1}{\varepsilon}} ; \\ |p_\phi(\theta, J)| + |q_\phi(\theta, J)| &\leq C\frac{1}{\log\frac{1}{\varepsilon}} , \\ |\omega_*(J) - \nabla h(J)| &\leq C\varepsilon ; \end{aligned} \tag{1.15}$$

$$|\Omega_*(J) - \varepsilon\bar{\Omega}(J)| \leq C\frac{\varepsilon}{\log\frac{1}{\varepsilon}} . \tag{1.16}$$

1.3.2 (*On the proof of Theorem 1.2*) The proof of Theorem 1.2 presented below, rests upon a *KAM theorem for lower dimensional elliptic tori*, stated by V.K. Melnikov in a 1965 paper [M65] and proved only in 1988 by H. Eliasson [E88] and, independently, by S.B. Kuksin [K88]. In fact, we shall use a *quantitative version of Melnikov's theorem*

[7]Dot "$\cdot$" denotes the standard inner product: $x \cdot y := \sum_j x_j y_j$.

adapted from a paper by J. Pöschel [P96]. In order to apply Melnikov's theorem to the three-body problem, the difficulties introduced by the proper degeneracy of the model have to be overcome. At this purpose, we have to use a quantitatively refined version of *averaging theory*: the angles associated to the unperturbed two-dimensional motions are *fast* and the system "feels", at "low orders", only averaged effects. Averaging theory allows to exploit this phenomenon and to "reduce" significantly the effect of the perturbation. The averaging theory we use is not standard and we present it (in a self-contained way) in Appendix A. After averaging, two more symplectic change of variables are needed in order to cast the planetary three-body (small-eccentricity-inclinations) system in a form suitable for KAM theory.

We stress that a non-trivial part of the proof consists in setting up, in a precise analytical way, the Hamiltonian formalism. As mentioned above, this formalism goes back to the classical theory developed, mainly, by Delaunay and Poincaré. However, since it is not easy to find, in the published literature, an exhaustive, analytical presentation of such theory, we devote to it a detailed appendix (Appendix C).

1.3.3 (*A physical comment*) The linearly stable quasi-periodic orbits obtained by Theorem 1.2 are seen to be the continuation of the secular orbits $X^t_{h+\varepsilon f_1}(I,\varphi,0,0)$, for suitable initial values of the major semi-axis of the two osculating ellipses; compare, also, point (iv) of Remark 1.1. Eccentricities and inclinations of the persistent orbits may be described as follows:

- the eccentricities are small with ε: in fact, by the second estimate in (1.11), we will have $e_1 + e_2 \le \text{ const}\, \varepsilon^{b_2}$;
- admissible "initial" inclinations range between inclinations of order one in ε (and close to $\imath_{\max}$) and small-with-ε inclinations: choosing b_2 small (i.e., $b_2 < \frac{1}{2(5+\tau)}$) the inclinations can be of order ε^{b_2} (compare (1.11), (2.14), points (i) and (ii) of Remark C.2 and (C.61) below);
- during the true motions, however, the inclinations vary little with ε: because of (1.11) and the relation between the Poincaré integrals and the inclinations, one finds that $|i^2(t) - i^2(0)| = O(\varepsilon^{2b_2})$ (compare (2.14), Remark C.2 and (C.61)).

1.3.4 (*KAM and the n-body problem*) As well known, KAM theory has been mainly motivated (by the founding authors) by celestial mechanics, and, in particular, by the n-body problem, to which Arnold devoted one of the fundamental paper of this theory ([A63]). In [A63] the problem of the existence of *maximal* quasi-periodic solutions for the n-body problem is considered and the author proves existence of such quasi-periodic solutions for the planar, three-body problem and gives some indications about how to extend his theorem to more general situations. In 1995, Laskar and Robutel ([LR95] and [R95], to which we refer for further references) extended Arnold's result to the spatial three-body problem. Notice, however, that such results do not give any indication about the question posed in **1.1** (as they deal with the existence of *maximal* invariant tori).

Michael Herman ([H95]) announced a complete (and lengthy) proof of Arnold's theorem for the n-body problem. Unfortunately his untimely decease deprived us of a certainly beautiful (and, probably, quite technical, as it was in Herman's style) piece of mathematics.

In connection with quasi-periodic solutions for the *planar* three-body problem, we mention, also, the recent preprint[8] [F02].

2 Proof of Theorem 1.2

The proof of Theorem 1.2 is based on three well separated steps, which we, now, proceed to explain.

First of all, pick a number b so that

$$0 < b_2 < b < \Big(\frac{1}{2} - b_1\Big)\frac{1}{\tau + 1} \ . \tag{2.1}$$

Remark 2.1 (i) The estimate that we shall get in this proof (and which are expressed in terms of the constant b) are slightly better than those stated in Theorem 1.2; in the comparison keep in mind (2.1) and that $\frac{1}{2} - b(\tau+1) > b_1$.
(ii) To get the limiting case "$b_1 \to 1/2$", disregard (2.1) and let, in what follows, $b := 0$ (and keep in mind that, in such a case, b_1 and b_2 are not defined).

Step 1 ("fast averaging") The starting point is the Hamiltonian formulation given in Theorem 1.1 and the first step will consist in "removing" the angle-dependence of the perturbation function f in (1.6)-(1.7) to higher order in ε. To do this, we shall make use of "averaging theory" (or "normal form theory"), and, in particular of the proposition, which we shall shortly state, after having introduced some notations.

Let (I, φ, p, q) be standard symplectic variables in $U \times \mathbb{T}^2 \times V$ (with respect to the symplectic form $\sum_i dI_i \wedge d\varphi_i + \sum_i dp_i \wedge dq_i$), where $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^4$; let[9]

$$W_{r,\rho,s} := U_r \times \mathbb{T}^2_s \times V_\rho := \bigcup_{I\in U} D^2_r(I) \times \mathbb{T}^2_s \times \bigcup_{(p,q)\in V} D^4_\rho((p,q)) \subset \mathbb{C}^8 \ ; \tag{2.2}$$

and for a function real-analytic on $W_{r,\rho,s}$ denote by $\|f\|_{r,\rho,s}$ its "sup-Fourier" norm given by[10]

$$\|f\|_{r,\rho,s} := \sum_{k\in\mathbb{Z}^2} \left(\sup_{(I,p,q)\in U_r\times V_\rho} |f_k(I,p,q)| \right) \ e^{|k|s} \ , \tag{2.3}$$

[8] After finishing our manuscript we received the preprint [F02], where an unpublished C^∞ KAM result by M. Herman (together with the inverse Nash-Moser implicit function theorem in the context of "tame Fréchet spaces") is used to derive the existence of quasi-periodic motions in the *planar thee-body problem*. We remark that in the planar three-body problem, the averaged system ("secular Hamiltonian") is *integrable*, a fact that in [F02] is heavily used (on the other hand, in the non-planar case, such integrability is no more available; compare, also, footnote 6).

[9] Recall the notations in (1.3).

[10] If $k \in \mathbb{Z}^n$ $|k| := \sum_{i=1}^n |k_i|$.

where $f_k(I,p,q)$ denotes the Fourier coefficient of index $k \in \mathbb{Z}^2$ of the periodic function $\varphi \to f_k(I,\varphi,p,q)$.

Proposition 2.1 (Averaging Theorem) *Let $H := h(I) + f(I,\varphi,p,q)$ be a real-analytic Hamiltonian on $W_{r,\rho,s}$ and denote by $\omega := h' := \nabla h$ the gradient of h. Assume that there exist $\alpha, K > 0$, satisfying $Ks \geq 6$, such that*

$$|\omega(I) \cdot k| \geq \alpha \ , \qquad \forall\, k \in \{k \in \mathbb{Z}^2 : \ 0 < |k| \leq K\} \ , \quad \forall\, I \in U_r \ . \tag{2.4}$$

Assume, also, that, if $d := \min\{rs, \rho^2\}$, then

$$\|f\|_{r,\rho,s} < \frac{\alpha d}{c\ Ks} \ , \tag{2.5}$$

where $c > 1$ is a suitable (universal) constant. Then, there exists a real-analytic symplectic transformation

$$\Psi : (I',\varphi',p',q') \in W_{r/2,\rho/2,s/6} \mapsto (I,\varphi,p,q) = \Psi(I',\varphi',p',q') \in W_{r,\rho,s} \tag{2.6}$$

and a real-analytic function $g = g(I',p',q')$ such that

$$H_* := H \circ \Psi = h + g + f_* \ , \tag{2.7}$$

and the following bounds hold[11]*:*

$$\sup_{(I',p',q') \in U_{r/2} \times V_{\rho/2}} |g(I',p',q') - f_0(I',p',q')| \leq \frac{c}{\alpha d}\ \|f\|^2_{r,\rho,s} \ , \tag{2.8}$$

$$\|f_*\|_{r/2,\rho/2,s/6} \leq e^{-Ks/6}\ \|f\|_{r,\rho,s} \ . \tag{2.9}$$

Furthermore, for each $(I',\varphi',p',q') \in W_{r/2,\rho/2,s/6}$, $(I,\varphi,p,q) = \Psi(I',\varphi',p',q')$ satisfies

$$s\ |I - I'|, \quad r\ |\varphi - \varphi'|, \quad \rho\ |p - p'|, \quad \rho\ |q - q'| \leq \frac{c\|f\|_{r,\rho,s}}{\alpha} \ . \tag{2.10}$$

Remark 2.2 (i) In order not to interrupt the proof of Theorem 1.2, we present the proof of (a more general form of) the Averaging Theorem in Appendix A.
(ii) Notice that, at difference with standard normal form theory, in the above proposition, one controls also the "dummy" symplectic variables (p,q).
(iii) A qualitatively similar statement can be found in [N77] (Lemma 10.3, p. 45); however, the quantitative bounds proven in [N77] are not enough for our purposes (compare, in particular, the estimates reported in Remark 10.4, p. 46 of [N77] with the stronger estimate (2.8) above).
(iv) (*Notational conventions*) In the rest of the proof, we shall denote by "const" (or c_i, c_*, C, etc.) positive constants of order one in ε, which may depend upon τ, b, b_1, b_2, $\bar\varepsilon$, $\bar\delta$, $\imath_{\max}$, $e_{\max}$ $\sup_{\mathcal{I}_{\sigma_0}} |h''|$, and the sup-Fourier norm of f_i in their analyticity domains. Also, the expression "f real-analytic (or, simply, analytic) on $A \subset \mathbb{C}^n$" will be short for: *f is real-analytic on A with uniformly bounded sup-Fourier norm* (2.3).

[11] f_0 denotes the 0-Fourier coefficient of f, i.e., its φ-average.

Let, now, $\mathcal{H} = h + f$ be as in Theorem 1.1 and let, as above, $\omega(I) := h'(I)$. Define

$$\bar\gamma = c_* \, \varepsilon^{\frac12 - b(\tau+1)} \Big(\log \frac{1}{\varepsilon}\Big)^{\tau+1} , \tag{2.11}$$

with $c_* > 0$ to be chosen later. Consider the set of points I in $\mathcal{I}$ for which $\omega(I)$ is $(\bar\gamma, \tau)$-Diophantine:

$$\mathcal{I}_{\bar\gamma,\tau} := \Big\{ I \in \mathcal{I} \ : \ |\omega(I)\cdot k| \geq \frac{\bar\gamma}{|k|^\tau} , \ \forall k \in \mathbb{Z}^2 \setminus \{0\} \Big\} . \tag{2.12}$$

Notice that (as it is standard to prove)

$$\mathrm{meas}_2 \Big(\mathcal{I} \setminus \mathcal{I}_{\bar\gamma,\tau} \Big) \leq \mathrm{const}\, \bar\gamma = \mathrm{const}\, \varepsilon^{\frac12 - b(\tau+1)} \Big(\log \frac{1}{\varepsilon}\Big)^{\tau+1} . \tag{2.13}$$

Next, let us choose the sets and the parameters involved in Proposition 2.1 as follows:

$$\begin{aligned}
&K = \frac{12}{s_0\ \varepsilon^b} \log\frac{1}{\varepsilon} , \qquad \alpha = \frac{\bar\gamma}{2\, K^\tau} , \qquad s = s_0 , \\
&r = \frac{\bar\gamma}{2K^{\tau+1}\, \sup_{\mathcal{I}_{\sigma_0}} |h''|} , \qquad \rho := C^* \frac{\varepsilon^b}{\log \frac{1}{\varepsilon}} < \rho^* \leq \rho_0 , \\
&U = \mathcal{I}_{\bar\gamma,\tau} , \qquad U_r = \bigcup_{I' \in \mathcal{I}_{\bar\gamma,\tau}} D_r(I') , \qquad V_\rho = D^4_\rho \subset D^4_{\rho_0} ,
\end{aligned} \tag{2.14}$$

where C^* a suitable large constant to be fixed later. Notice that, from these definitions, there follows (for ε small enough) that

$$\begin{aligned}
&\alpha = \ \mathrm{const}\, \varepsilon^{\frac12 - b} \log\frac{1}{\varepsilon} , \qquad r = \ \mathrm{const}\, \frac{\alpha}{K} = \ \mathrm{const}\, \sqrt{\varepsilon} , \qquad d = \ \mathrm{const}\, r , \\
&\alpha d = \ \mathrm{const}\, \varepsilon^{1-b} \log\frac{1}{\varepsilon} , \qquad \frac{\alpha d}{cKs} = \ \mathrm{const}\, \varepsilon c_*^2 ,
\end{aligned} \tag{2.15}$$

(clearly, in the last evaluation, "const" does not involve c_*).

Now, it is not difficult to check that, choosing c_* big enough and letting ε be small enough, assumptions (2.4) and (2.5) are met. In fact, observing that f in (1.6)-(1.7) is such that

$$\|f\|_{r,\rho,s} \leq \ \mathrm{const}\, \varepsilon ,$$

(2.5) follows from last equality in (2.15), by choosing c_* large enough. As for (2.4), observe that for any point in $I \in U_r$ there is a point $I_0 \in \mathcal{I}_{\bar\gamma,\tau}$ at distance less than r from it. Hence, by (2.12), by the definitions of α and r and by Cauchy estimates[12], for

[12]As well known, "Cauchy estimates" allow to bound n-derivatives of analytic functions on a set A in terms of their sup-norm on larger domains $A' \supset A$ divided by $\mathrm{dist}\,(\partial A, \partial A')^n$; compare, also, Lemma A.1 of Appendix A.

any $I \in U_r$ and any $0 < |k| \le K$,

$$\begin{aligned} |h'(I) \cdot k| &\ge |h'(I_0) \cdot k| - |h'(I_0) - h'(I)|\ |k| \\ &\ge \frac{\bar\gamma}{K^\tau} - \sup_{\mathcal{I}_{\sigma_0}} |h''|\ r\ K \\ &= \frac{\bar\gamma}{2K^\tau}\ , \end{aligned}$$

which proves also (2.4). Thus, by Proposition 2.1, there exists a symplectic transformation

$$\Psi : (I', \varphi', p', q') \in \mathcal{D}_1 := U_{\frac{r}{2}} \times \mathbb{T}^2_{\frac{s}{6}} \times D^4_{\frac{\rho}{2}} \to (I, \varphi, p, q) \in U_r \times \mathbb{T}^2_{s_0} \times D^4_{\rho_0} \subset \mathcal{D}_0,$$

such that

$$\begin{aligned} &|I' - I| \le \text{ const } \frac{\varepsilon}{\alpha} = \text{ const } \frac{\varepsilon^{\frac12 + b}}{\log \frac{1}{\varepsilon}} \\ &|p' - p|\ ,\ |q' - q| \le \text{ const } \frac{\varepsilon}{\alpha\rho} = \text{ const } \sqrt{\varepsilon}\ , \end{aligned} \tag{2.16}$$

and which casts the Hamiltonian $\mathcal{H}$ into $\mathcal{H}' := \mathcal{H} \circ \Psi$ with

$$\mathcal{H}'(I', \varphi', p', q') := h(I') + g(I', p', q') + f_*(I', \varphi', p', q')\ , \tag{2.17}$$

where (since, by (1.7), f_0 coincides with $\varepsilon f_1(I, p, q)$)

$$\begin{aligned} \sup_{(I',p',q') \in U_{r/2} \times V_{\rho/2}} |g - \varepsilon f_1| &\le \text{ const } \frac{\varepsilon^2}{\alpha r} = \text{ const } \frac{\varepsilon^{1+b}}{\log(1/\varepsilon)}\ , \\ \|f_*\|_{r/2, \rho/2, s/6} &\le \text{ const } \varepsilon e^{-Ks/6} \le \varepsilon^3\ . \end{aligned} \tag{2.18}$$

Notice that if $b > 0$ then $\|f_*\|$ is *exponentially* small with $1/\varepsilon$, while if $b = 0$ then the above estimates yield *exactly* ε^3. Thus, setting $g =: \varepsilon \bar g$, $f_* =: \varepsilon^3 \bar f$, we see that $\mathcal{H}'$ can be rewritten as

$$\begin{aligned} \mathcal{H}' &:= h(I') + \varepsilon \bar g(I', p', q') + \varepsilon^3 \bar f(I', \varphi', p', q')\ , \\ \bar g &= f_1(I', p', q') + \frac{\varepsilon^b}{\log(1/\varepsilon)} \bar f_1(I', p', q') \end{aligned} \tag{2.19}$$

with $\bar f$ and $\bar f_1$ real-analytic on $\mathcal{D}_1$ (compare (2.14) and recall the convention in (iv) of Remark 2.2).

Step 2 We now look for *elliptic equilibria of the Hamiltonian* $\bar g$ in (2.19). Set

$$G(I', p', q') := \Big(\partial_{p'} \bar g(I', p', q'),\ \partial_{q'} \bar g(I', p', q') \Big)\ .$$

Recalling (2.19) and the definition of f_1 in (1.7), we see that

$$G(I', 0, 0)\Big|_{\varepsilon=0} = 0 \quad \text{and} \quad \det \partial_{(p',q')} G(I', 0, 0)\Big|_{\varepsilon=0} = 16(\bar\Omega_1 \bar\Omega_2)^2 > 0\ ,\ \forall\ I' \in U_{r/2}.$$

Therefore, by the Implicit Function Theorem, we infer that, for any $I' \in U_{r/2}$ and for ε small enough, there exist a suitable constant C and real-analytic functions, so that, choosing $C^* > C$,

$$I' \in U_{r/2} \to \Big(p'(I',\varepsilon),\, q'(I',\varepsilon)\Big) \in B_{C\varepsilon^b/\log\frac{1}{\varepsilon}} \subset B_\rho \subset B_{\rho^*}\,,$$

and

$$\partial_{p'}\bar g\Big(I', p'(I',\varepsilon), q'(I',\varepsilon)\Big) = 0 = \partial_{q'}\bar g\Big(I', p'(I',\varepsilon), q'(I',\varepsilon)\Big)\,. \tag{2.20}$$

For ε small enough, we can consider the following analytic symplectic transformation, which leaves fixed the I'-variable and is $O(\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}})-$close to the identity[13],

$$\Phi' : (J', \psi', v', u') \in U_{r/2} \times \mathbb{T}^2_{s/7} \times D_{\rho/3} \;\mapsto\; (I', \varphi', p', q') \in U_{r/2} \times \mathbb{T}^2_{s/6} \times D^4_{\rho/2}\,,$$

given by

$$\begin{array}{rcl} I' &=& J'\,,\\ \varphi' &=& \psi' + p'(J',\varepsilon)\,\partial_{I'}q'(J',\varepsilon) + \partial_{I'}q'(J',\varepsilon)\,v' - \partial_{I'}p'(J',\varepsilon)\,u'\,,\\ p' &=& v' + p'(J',\varepsilon)\,,\\ q' &=& u' + q'(J',\varepsilon)\,. \end{array}$$

In view of (2.20), the new Hamiltonian $\hat{\mathcal{H}} := \mathcal{H}' \circ \Phi'$ has the form

$$\hat{\mathcal{H}}(J', \psi', v', u') = h(J') + \varepsilon\tilde g(J', v', u') + \varepsilon^3 \tilde f(J', \psi', v', u')\,,$$

with $\tilde f$ and $\tilde g$ analytic in $U_{r/2} \times \mathbb{T}^2_{s/7} \times D_{\rho/3}$ and

$$\partial_{v',u'}\tilde g(J', 0, 0) = \partial_{p',q'}\bar g(I', p'(I',\varepsilon), q'(I',\varepsilon)) = 0\,, \qquad \forall\; I' \in U_{r/2}\;.$$

Also, the eigenvalues of the symplectic quadratic part of $\tilde g$ are given by $\pm i\tilde\Omega_j(J')$, for $j = 1, 2$, where

$$\tilde\Omega_j \in \mathbb{R} \qquad \text{and} \qquad |\tilde\Omega_j - \bar\Omega_j| \le \text{const}\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}}\,. \tag{2.21}$$

Thus, by a well known result by Weierstrass on the symplectic diagonalization of quadratic Hamiltonians, we can find an analytic transformation $O(\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}})-$close to the identity

$$\tilde\Phi : (\tilde J, \tilde\psi, \tilde v, \tilde u) \in U_{r/2} \times \mathbb{T}^2_{s/8} \times D^4_{\rho/4} \;\mapsto\; (J', \psi', v', u') \in U_{r/2} \times \mathbb{T}^2_{s/7} \times D^4_{\rho/3}\,,$$

so that $J' = \tilde J$ and the quadratic part of $\tilde g$ becomes, simply, $\sum_{i=1}^2 \tilde\Omega_i(\tilde J)\,(\tilde u_j^2 + \tilde v_j^2)$. Whence, the Hamiltonian $\hat{\mathcal{H}}$ takes the form $\tilde{\mathcal{H}} := \hat{\mathcal{H}} \circ \tilde\Phi$, with

$$\tilde{\mathcal{H}}(\tilde J, \tilde\psi, \tilde v, \tilde u) = h_0(\tilde J) + \varepsilon \sum_{i=1}^2 \tilde\Omega_i(\tilde J)\,(\tilde u_i^2 + \tilde v_i^2) + \varepsilon\tilde g_0(\tilde J, \tilde v, \tilde u) + \varepsilon^3 \tilde f_0(\tilde J, \tilde\psi, \tilde v, \tilde u)\,, \tag{2.22}$$

[13] Φ' has generating function $J' \cdot \varphi' + \Big(v' + p'(J',\varepsilon)\Big) \cdot \Big(q' - q'(J',\varepsilon)\Big)$.

where

$$h_0(\tilde{J}) := h(\tilde{J}) + \varepsilon \tilde{g}(\tilde{J}, 0, 0) , \tag{2.23}$$

$\tilde{g}_0$, $\tilde{f}_0$, $\tilde{\Omega}_j$ are real-analytic for $(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u})$ in

$$\mathcal{D}_2 := U_{r/2} \times \mathbb{T}^2_{s/8} \times D^4_{\rho/4} \tag{2.24}$$

and

$$\sup_{\tilde{J} \in U_{r/2}} |\tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u})| \leq \text{ const } |(\tilde{v}, \tilde{u})|^3 . \tag{2.25}$$

Finally, because of (2.21), the non-degeneracy condition (1.8) implies (for ε small enough)

$$\begin{aligned} &\inf_{\tilde{J} \in U} \tilde{\Omega}_i > \inf_{\tilde{J} \in U_{r/2}} |\tilde{\Omega}_i| > \text{ const } > 0 , \\ &\inf_{\tilde{J} \in U} \left(\tilde{\Omega}_2 - \tilde{\Omega}_1\right) > \inf_{\tilde{J} \in U_{r/2}} |\tilde{\Omega}_2 - \tilde{\Omega}_1| > \text{ const } > 0 . \end{aligned} \tag{2.26}$$

Step 3 (KAM) We are, now, in position to apply a KAM result in order to find *two-dimensional elliptic tori*. The KAM Theorem we shall use is, basically, the version in [P96] of a result first proved by H. Eliasson ([E88]) and S.B. Kuksin ([K88]) about the conservation of lower dimensional invariant elliptic tori[14]. To state the KAM theorem, we need a bit of preparation.

Consider a Hamiltonian system with symplectic variables $(y, x, v, u) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^{2m}$ (endowed with the standard symplectic form $\sum dy_j \wedge dx_j + \sum dv_i \wedge du_i$) and consider a real-analytic Hamiltonian function of the form

$$H(y, x, v, u; \xi) = N(y, v, u; \xi) + P(y, x, v, u; \xi) , \tag{2.27}$$

where, ξ is a parameter running over a *compact set* $\Pi \subset \mathbb{R}^n$ *of positive Lebesgue measure*, N *is in normal form*:

$$N(y, v, u; \xi) = e(\xi) + \omega(\xi) \cdot y + \sum_{j=1}^{m} \Omega_j(\xi)(u_j^2 + v_j^2) , \tag{2.28}$$

and P is a small *perturbation*.

Notice that the Hamiltonian $N(\cdot; \xi)$ affords, for any given value of the parameter $\xi \in \Pi$, the $n-$*dimensional elliptic torus*

$$\{y = 0\} \times \mathbb{T}^n \times \{v = u = 0\} ,$$

which is invariant for the Hamiltonian vector field X_N, the flow being, simply, $x \mapsto x + \omega(\xi)t$.

[14]Actually, [K88] and [P96] cover also infinite-dimensional (in (v, u)) cases.

Assume that P is real-analytic on

$$D(\bar r,\bar s):=\{(y,x,v,u)\in\mathbb{C}^{2(n+m)}\ :\ |y|<\bar r^2\,,x\in\mathbb{T}^n_{\bar s},\ \ |v|+|u|<\bar r\}\,,\tag{2.29}$$

and that the dependence of ω, Ω_j and P is Lipschitz in $\xi\in\Pi$.

Let $\tau>n-1$ and, as in [P96], let us introduce the following weighted norms:

$$\begin{aligned}
|\omega|^{\rm Lip}_{\Pi} &:= \sup_{\xi\neq\xi'\in\Pi}\frac{|\omega(\xi)-\omega(\xi')|}{|\xi-\xi'|}\,,\\
\|X_H\|_{\bar r,D(\bar r,\bar s)} &:= \sup_{D(\bar r,\bar s)\times\Pi}\Big(|\partial_y H|+\frac{1}{\bar r^2}|\partial_x H|+\frac{1}{\bar r}(|\partial_v H|+|\partial_u H|)\Big)\,,\\
\|X_H\|^{\rm Lip}_{\bar r,D(\bar r,\bar s)} &:= \sup_{D(\bar r,\bar s)}\Big(|\partial_y H|^{\rm Lip}_{\Pi}+\frac{1}{\bar r^2}|\partial_x H|^{\rm Lip}_{\Pi}+\frac{1}{\bar r}(|\partial_v H|^{\rm Lip}_{\Pi}+|\partial_u H|^{\rm Lip}_{\Pi})\Big)\,.
\end{aligned}$$

Proposition 2.2 (KAM Theorem for elliptic tori) *Assume that ω is a Lipschitz homeomorphism onto its image. Let L and M be so that*[15]

$$|\omega|^{\rm Lip}_{\Pi}+|\Omega|^{\rm Lip}_{\Pi}\le M\,,\qquad |\omega^{-1}|^{\rm Lip}_{\Pi}\le L\,.$$

Assume that there exists $\gamma_0>0$ such that

$$\min_{\xi\in\Pi,\,i\neq j}\{|\Omega_i(\xi)|\,,\ |\Omega_i(\xi)-\Omega_j(\xi)|\}\ge\gamma_0\ ,\tag{2.30}$$

$$|\omega(\xi)\cdot k+\Omega(\xi)\cdot\ell|\ge\gamma_0\,,\quad\forall\ 0<|k|\le K_0\,,\ |\ell|\le 2\,,\tag{2.31}$$

where K_0 is a suitable constant (depending, also, on n and τ), which is assumed to be bigger than $16LM$. Let $\gamma\in(0,\gamma_0/2]$ and denote

$$\|X_H\|_{\bar r,\bar s,\gamma}:=\|X_H\|_{\bar r,D(\bar r,\bar s)}+\frac{\gamma}{M}\,\|X_H\|^{\rm Lip}_{\bar r,D(\bar r,\bar s)}\,.$$

Then, there exist suitable constants $c=c(\bar s,\tau,n)$ and $a=a(\tau,n)>1$ such that, if

$$\frac{c}{\gamma}\,(LM)^a\,\|X_P\|_{\bar r,\bar s,\gamma}\le 1\,,\tag{2.32}$$

then the following holds. There exists a Cantor set of parameters $\Pi_*\subset\Pi$ and a Lipschitz continuous family of torus embedding

$$\begin{aligned}
\Phi\ \ :\ \ &(\theta,\xi)\in\mathbb{T}^n\times\Pi_*\longrightarrow\Big(y_\Phi(\theta;\xi),\,x_\Phi(\theta;\xi),\,v_\Phi(\theta;\xi)\,,u_\Phi(\theta;\xi)\Big)\,,\\
&\Phi(\mathbb{T}^n\times\Pi_*)\subset\{|y|<\bar r^2\}\times\mathbb{T}^n\times\{|v|+|u|<\bar r\}\,,
\end{aligned}$$

a Lipschitz homeomorphism ω_* on Π_* and a Lipschitz continuous function Ω_* on Π_*, such that, for any $\xi\in\Pi_*$, $\Phi(\mathbb{T}^n,\xi)$ is a real-analytic (elliptic) H-invariant $n-$dimensional

[15] Denote $\Omega:=(\Omega_1,\dots,\Omega_m)$

torus, on which the flow X_H is analytically conjugated to the linear flow $\theta \mapsto \theta + \omega_* t$. Furthermore, $\Phi(\cdot, \xi)$ is real-analytic on $\mathbb{T}^n_{\bar s/2}$ and the following bounds hold:

$$\begin{aligned}
& \frac{1}{\bar r^2}|y_\Phi| + |x_\Phi - \mathrm{Id}| + \frac{1}{\bar r}(|v_\Phi| + |u_\Phi|) + \\
+ \quad & \frac{\gamma}{M}\Big(\frac{1}{\bar r^2}|y_\Phi|^{\mathrm{Lip}}_{\Pi_*} + |x_\Phi|^{\mathrm{Lip}}_{\Pi_*} + \frac{1}{\bar r}(|v_\Phi|^{\mathrm{Lip}}_{\Pi_*} + |u_\Phi|^{\mathrm{Lip}}_{\Pi_*})\Big) \le c\,\frac{\|\!| X_P \|\!|_{\bar r, \bar s, \gamma}}{\gamma}\,, \qquad (2.33)\\
& |\omega_* - \omega| + \frac{\gamma}{M}|\omega_* - \omega|^{\mathrm{Lip}}_{\Pi_*} \le c\,\|\!| X_P \|\!|_{\bar r, \bar s, \gamma}\,, \qquad (2.34)\\
& |\Omega_* - \Omega| + \frac{\gamma}{M}|\Omega_* - \Omega|^{\mathrm{Lip}}_{\Pi_*} \le c\,\|\!| X_P \|\!|_{\bar r, \bar s, \gamma}\,, \qquad (2.35)\\
& |\omega_* \cdot k + \Omega_* \cdot \ell| \ge \frac{\gamma}{1 + |k|^\tau}\,, \quad \forall (k, \ell) \ne (0,0)\,,\ |\ell| \le 2\,, \qquad (2.36)\\
& \mathrm{meas}_n(\Pi \setminus \Pi_*) \le c\,\frac{\gamma}{M}\,(LM\ \mathrm{diam}\,\Pi)^{n-1}\,. \qquad (2.37)
\end{aligned}$$

Remark 2.3 This Theorem is a summary (in the finite dimensional case) of Theorems A and B and Corollary C of [P96], to which we refer for the proof[16]. Notice that (2.30) and (2.31) here play the role of the "Non-degeneracy Assumption A" of [P96], while Assumptions B and C of [P96] are trivially satisfied in the finite dimensional case. Assumptions (2.30) and (2.31) imply the measure estimate (2.37), as briefly shown in Appendix B.

To apply Proposition 2.2 to our case, we let

$$\begin{aligned}
& n = 2\,, \quad \omega_0 = h_0'\,, \quad \Omega_i = \varepsilon \tilde\Omega_i\,, \quad L = \text{const}\,, \quad M = \text{const}\,, \quad K_0 := 16LM\,, \\
& J = \xi \in \Pi = U = \mathcal{I}_{\bar\gamma, \tau}\,, \quad e(\xi) = h_0(\xi)\,, \quad \bar r = \varepsilon^{3/4}\,, \quad \bar s = \frac{s}{8}\,, \\
& N(y, v, u; \xi) = e(\xi) + \omega_0(\xi)\cdot y + \sum_{i=1}^{2} \tilde\Omega_i(\xi)(u_i^2 + v_i^2)\,, \quad (x, v, u) = (\tilde\psi, \tilde v, \tilde u)\,, \\
& H(y, x, v, u; \xi) = \tilde{\mathcal{H}}(J + y, \tilde\psi, \tilde v, \tilde u)\,. \qquad (2.38)
\end{aligned}$$

Recall that $r \sim \sqrt{\varepsilon}$, which is much larger than $\bar r = \varepsilon^{3/4}$, so that

$$\Pi_{\bar r^2} := \bigcup_{\xi \in \Pi} D_{\bar r^2}(\xi) \subset U_{r/2}\,,$$

and also $\bar r \ll \rho/4$ so that the Hamiltonian H in (2.38) is real-analytic for $|y| < \bar r^2$, $|u| + |v| < \bar r$ and $|\mathrm{Im}\, x| < \bar s$, for any $\xi \in \Pi$.

Next, we observe that the perturbation P may be written as $P = \sum_{1 \le k \le 4} P_k$, with

$$P_1 \;=\; h_0(\xi + y) - h_0(\xi) - \omega_0(\xi)\cdot y\,,$$

[16]For comparison purposes, we have kept the notations as close as possible to the notations in [P96]; notice, however, that (in order to avoid confusion with other parameters introduced in our paper) r and s in [P96] are denoted here $\bar r$ and $\bar s$.

$$
\begin{aligned}
P_2 &= \sum_{i=1}^{2}\Big(\Omega_i(\xi+y)-\Omega_i(\xi)\Big)\,(u_i^2+v_i^2)\,,\\
P_3 &= \varepsilon\tilde g_0(\xi+y,v,u)\,,\\
P_4 &= \varepsilon^3\tilde f_0(\xi+y,x,v,u)\,.
\end{aligned}
$$

By (2.26) and (2.38), we see that (2.30) holds true, provided $\gamma_0 = \text{const}\,\varepsilon$.

To check (2.31), take $0<|k|\le K_0$ and $|\ell|\le 2$. Then, observing that $|\Omega_j|\le \text{const}\,\varepsilon$, recalling the definitions of $U=\Pi=\mathcal{I}_{\bar\gamma,\tau}$ and $\bar\gamma$ in (2.14), (2.12), (2.11) and (2.38), for any $0<|k|\le K_0$ and $|\ell|\le 2$, one has that

$$
\begin{aligned}
|\omega_0(\xi)\cdot k+\Omega(\xi)\cdot\ell| &\ge |\omega_0(\xi)\cdot k| - \text{const}\,\varepsilon \ge\\
&\ge \frac{\bar\gamma}{K_0^\tau} - \text{const}\,\varepsilon > \varepsilon\,,
\end{aligned}
$$

proving (2.31). Finally, recalling the definition of the weighted norms introduced before Proposition 2.2, (2.25) and (2.38), we infer that

$$
\begin{aligned}
\|X_P\|_{\bar r,\bar s,\gamma} &\le \text{const}\left(\bar r^2+\varepsilon\bar r+\frac{\varepsilon^3}{\bar r^2}\right) = \text{const}\left(\varepsilon^{3/2}+\varepsilon^{7/4}+\varepsilon^{3/2}\right)\le\\
&\le \text{const}\,\varepsilon^{3/2}\,,
\end{aligned}
$$

so that, letting (say)

$$\gamma := \gamma_0/2 = \text{const}\,\varepsilon\ , \tag{2.39}$$

one finds

$$\frac{\|X_P\|_{\bar r,\bar s,\gamma}}{\gamma}\le \text{const}\sqrt{\varepsilon}\,. \tag{2.40}$$

Therefore, the assumptions of Proposition 2.2 are fulfilled and the existence of the elliptic quasi-periodic orbits follows from Proposition 2.2: the parameterization ϕ in Theorem 1.2 of the elliptic tori is given by

$$
\begin{aligned}
&\Big(\,I_\phi(\theta;J),\ \varphi_\phi(\theta;J),\ p_\phi(\theta;J),\ q_\phi(\theta;J)\,\Big)\\
&:= \Psi\circ\Phi'\circ\widetilde\Phi\Big(J+y_\Phi(\theta;J),\ x_\Phi(\theta;J),\ v_\Phi(\theta;J)\,,u_\Phi(\theta;J)\,\Big)\ ,
\end{aligned}\tag{2.41}
$$

where the parameter $J=\xi$ varies in

$$J\in\mathcal{I}_* := \Pi_*\ . \tag{2.42}$$

The estimates (1.11), (1.12) (or (1.15), (1.16)) and (1.13) follows, now, easily, from (2.16), the fact that Φ' and $\widetilde\Phi$ leave fixed the variables J' and $\tilde J$ and are $\varepsilon^b/\log\frac{1}{\varepsilon}$-close to the identity in the other variables, (2.33)÷(2.36), (2.16), (2.21) and (2.23).

Finally, we turn to the measure estimates (1.10). It follows from (2.37) and (2.39) that the 2−dimensional elliptic tori are described by a set of parameters Π_*, with

$$\text{meas}_2(\Pi\setminus\Pi_*)\le\text{const}\,\varepsilon\,. \tag{2.43}$$

Thus, from (2.13), (2.42) and (2.43), there follows

$$\begin{aligned}
\mathrm{meas}_2(\mathcal{I}\setminus\mathcal{I}_*) &\le \mathrm{meas}_2(\mathcal{I}\setminus\mathcal{I}_{\bar\gamma,\tau}) + \mathrm{meas}_2(\mathcal{I}_{\bar\gamma,\tau}\setminus\mathcal{I}_*)\\
&:= \mathrm{meas}_2(\mathcal{I}\setminus\mathcal{I}_{\bar\gamma,\tau}) + \mathrm{meas}_2(\Pi\setminus\Pi_*)\\
&\le \mathrm{const}\,\Big(\varepsilon^{\frac12-b(\tau+1)}\Big(\log\frac{1}{\varepsilon}\Big)^{\tau+1}+\varepsilon\Big)\\
&\le \mathrm{const}\,\varepsilon^{\frac12-b(\tau+1)}\Big(\log\frac{1}{\varepsilon}\Big)^{\tau+1},
\end{aligned}$$

completing the proof of Theorem 1.2. ∎

A Averaging Theory

In this appendix, we prove a general result (Proposition A.1 below) in averaging theory, which shall immediately imply Proposition 2.1. The techniques used here are similar to techniques used in [P93].

Let us first fix some notation. As above (I,φ) and (p,q) denotes sets of standard symplectic conjugate variables. We will use for $I\in\mathbb{R}^n$ the usual Euclidean norm $|I|:=|I|_2:=\Big(\sum_{i=1}^n|I_i|^2\Big)^{1/2}$, instead for $p,q\in\mathbb{R}^m$ or $\varphi\in\mathbb{T}^n$ we will use the norm $|p|:=|p|_\infty:=\max_{1\le i\le m}|p_i|$, $|q|:=|q|_\infty:=\max_{1\le i\le m}|q_i|$, $|\varphi|:=|\varphi|_\infty:=\max_{1\le i\le n}|\varphi_i|$, (mod 2π). If such variables are considered in complex domains we shall use the corresponding conventions. If $d>0$ and A is a subset of $\mathbb{R}^l$ with $l=n$ or $l=m$ we define

$$A_d:=\{z\in\mathbb{C}^l\ ,\quad \text{s.t.}\quad \exists\, x\in A\quad\text{with}\quad |z-x|_j<d\}$$

where $j=2$ or $j=\infty$ (according to whether the set is considered in the space of the actions I's or in the space of the other symplectic variables p,q or φ). Let

$$D\subset\mathbb{R}^n\ ,\quad E,F\subset\mathbb{R}^m\ ,\quad U:=D\times E\times F\ ,\quad W:=U\times\mathbb{T}^n\ ,$$

and let $r,r_p,r_q,s>0$ and $v:=(r,r_p,r_q)$. For a function

$$f(u,\varphi)=\sum_{k\in\mathbb{Z}^n}f_k(u)e^{ik\cdot\varphi}\ ,\qquad u:=(I,p,q)\ ,$$

real-analytic for $(u,\varphi)\in W_{v,s}:=U_v\times\mathbb{T}^n_s$, with $U_v:=D_r\times E_{r_p}\times F_{r_q}$, we shall use the norm[17]

$$\|f\|_{v,s}:=\sum_{k\in\mathbb{Z}^n}\sup_{u\in U_v}|f_k(u)|e^{|k|s}\ .$$

Finally, let Λ a sub-lattice of $\mathbb{Z}^n$ and, if $f=\sum f_ke^{ik\cdot\varphi}$, we set

$$T_Kf:=\sum_{|k|\le K}f_ke^{ik\cdot\varphi}\ ,\qquad P_\Lambda f:=\sum_{k\in\Lambda}f_ke^{ik\cdot\varphi}\ .$$

[17] If $k\in\mathbb{Z}^n$ $|k|:=\sum_{i=1}^n|k_i|$.

Proposition A.1 *Let $H := h(I) + f(u,\varphi)$ be a real-analytic Hamiltonian on $W_{v,s}$. Denoting $\omega := h'$ and $c_m := e(1+em)/2$, we suppose that*

$$|\omega(I)\cdot k| \geq \alpha > 0 \ , \qquad \forall\, |k| \leq K \ , \quad k \notin \Lambda \ , \quad \forall\, I \in D_r \ , \tag{A.1}$$

$Ks \geq 6$ and

$$\|f\|_{v,s} =: \varepsilon < \frac{\alpha d}{2^7 c_m K s} \qquad \text{where} \qquad d := \min\{rs, r_p r_q\} \ . \tag{A.2}$$

Then, there exists a real-analytic symplectic transformation

$$\Psi : (u', f') \in W_{v_*,s_*} \longrightarrow (u,\varphi) \in W_{v,s}$$

with $v_* := v/2$, $s_* := s/6$, such that

$$H_* := H \circ \Psi = h + g + f_* \ , \tag{A.3}$$

with g in normal form:

$$g = \sum_{k\in\Lambda} g_k(u') e^{ik\cdot\varphi'} \ . \tag{A.4}$$

Moreover, we have that (denoting the projection of $\Psi(I',p',q',\varphi')$ onto the $I-$variables by $I(I',p',q',\varphi')$, etc.),

$$\|g - P_\Lambda T_K f\|_{v_*,s_*} \leq \frac{12}{11}\,\frac{2^7 c_m \varepsilon}{\alpha d}\,\varepsilon \leq \frac{1}{4}\varepsilon \ , \tag{A.5}$$

$$\|f_*\|_{v_*,s_*} \leq \frac{2^9 c_m \varepsilon}{\alpha d} e^{-Ks/6}\varepsilon \leq e^{-Ks/6}\varepsilon \ , \tag{A.6}$$

$$s\ |I(u',f') - I'|,\ r_q\ |p(u',f') - p'|,\ r_p\ |q(u',f') - q'|,\ r\ |\varphi(u',f') - f'| \leq 9\varepsilon/\alpha \ . \tag{A.7}$$

The proof of this result rests upon a series of technical elementary lemmata, which we now proceed to state:

Lemma A.1 *Let*[18] *$0 < \nu < v$ and $0 < \sigma < s$, where $\nu := (\rho, \rho_p, \rho_q)$. Then*

$$\sum_{1\leq i\leq n} \|\frac{\partial f}{\partial \varphi_i}\|_{v,s-\sigma} \leq \frac{1}{e\sigma}\|f\|_{v,s} \ , \qquad \max_i \|\frac{\partial f}{\partial I_i}\|_{r-\rho,r_p,r_q,s} \leq \frac{1}{\rho}\|f\|_{r,r_p,r_q,s}$$

$$\max_l \|\frac{\partial f}{\partial p_l}\|_{r,r_p-\rho_p,r_q,s} \leq \frac{1}{\rho_p}\|f\|_{v,s} \ , \qquad \max_l \|\frac{\partial f}{\partial q_l}\|_{r,r_p,r_q-\rho_q,s} \leq \frac{1}{\rho_q}\|f\|_{v,s} \ .$$

This Lemma is a precise version of classical Cauchy estimates. We omit the well known proof; just notice that the estimates relative to φ-derivatives is a consequence of the choice of the (Fourier) norm. ∎

An immediate corollary of Lemma A.1 is the following Lemma on estimates for Poisson's brackets[19].

[18] If $a := (a_1,\ldots,a_j)$, $b := (b_1,\ldots,b_j) \in \mathbb{R}^j$ then $a < b$ means $a_i < b_i$ for all $1 \leq i \leq j$.

[19] $\{f,g\} := \sum_j \partial_{\varphi_j} f\ \partial_{I_j} g - \partial_{I_j} f\ \partial_{\varphi_j} g + \sum_i \partial_{q_i} f\ \partial_{p_i} g - \partial_{p_i} f\ \partial_{q_i} g$.

Lemma A.2 *Let $0 < v - \nu < \tilde v$, where $\tilde v := (r_0, \tilde r_p, \tilde r_q)$. Then,*

$$\begin{aligned}
\|\{f,g\}\|_{v-\nu,s-\sigma} &\leq \left[\frac{1}{e(r_0 - r + \rho)\sigma} + \frac{1}{e(\tilde s - s + \sigma)\rho} + \right. \\
&+ \left. \frac{m}{(\tilde r_p - r_p + \rho_p)\rho_q} + \frac{m}{(\tilde r_q - r_q + \rho_q)\rho_p}\right] \|f\|_{\tilde v, \tilde s}\ \|g\|_{v,s}.
\end{aligned}$$

If ϕ is a Hamiltonian function, we denote by X^t_ϕ the Hamiltonian flow of ϕ at the time t. If f and ϕ are analytic, expanding in Taylor series in time the function $f \circ X^t_\phi$, one gets

$$f \circ X^t_\phi = \sum_j \frac{t^j}{j!} L^j_\phi f \ ,$$

where $L_\phi f := \{f, \phi\}$, $L^0_\phi := Id$ and

$$L^j_\phi := \overbrace{L_\phi \circ \ldots \circ L_\phi}^{j \text{ times}} \ .$$

Lemma A.3 *Let $0 < \nu < v \leq \tilde v - \nu$, $0 < \sigma < s \leq \tilde s - \sigma$ and*

$$\|\phi\|_{\tilde v, \tilde s} < G := \frac{2}{e}\left(\frac{\rho \rho_p \rho_q \sigma}{\rho_p \rho_q + em\rho\sigma}\right) \ . \tag{A.8}$$

Then

$$\|f \circ X^1_\phi\|_{v-\nu,s-\sigma} \leq \left(1 - \frac{\|\phi\|_{\tilde v, \tilde s}}{G}\right)^{-1} \|f\|_{v,s} \ .$$

Proof Fix $h \geq 1$ and let $\bar\nu := \nu/h$, $\bar\sigma := \sigma/h$ $\|\cdot\|_i := \|\cdot\|_{v-i\bar\nu, s-i\bar\sigma}$, for all i with $1 \leq i \leq h$. We will use Lemma A.2 with $\nu = \bar\nu$ and $\sigma = \bar\sigma$. Then, $v - i\bar\nu = [v - (i-1)\bar\nu]$ and $s - i\bar\sigma = [s - (i-1)\bar\sigma]$. Hence,

$$\begin{aligned}
\|L^i_\phi f\|_i = \|\{L^{i-1}_\phi, \phi\}\|_i &\leq \left[\frac{1}{e(r_0 - r + i\bar\rho)\bar\sigma} + \frac{1}{e(\tilde s - s + i\bar\sigma)\bar\rho}\right. \\
&\qquad \left. + \frac{m}{(\tilde r_p - r_p + i\bar\gamma)\bar\kappa} + \frac{m}{(\tilde r_q - r_q + i\bar\kappa)\bar\gamma}\right] \|\phi\|_{\tilde v, \tilde s}\ \|L^{i-1}_\phi\|_{i-1} \\
&\leq \left[\frac{2}{e\bar\rho\bar\sigma} + \frac{2m}{\bar\gamma\bar\kappa}\right] \frac{1}{h+i} \|\phi\|_{\tilde v, \tilde s} \|L^{i-1}_\phi\|_{i-1} \ .
\end{aligned}$$

Iterating h times the previous estimate we obtain

$$\begin{aligned}
\|L^h_\phi\|_h &= \|L^h_\phi\|_{v-\nu,s-\sigma} \leq \left[\frac{2}{e\bar\rho\bar\sigma} + \frac{2m}{\bar\gamma\bar\kappa}\right]^h \frac{h!}{(2h)!} \|\phi\|^h_{\tilde v, \tilde s} \|f\|_{v,s} \\
&= \left[\frac{2}{e\rho\sigma} + \frac{2m}{\rho_p\rho_q}\right]^h \frac{h^{2h} h!}{(2h)!} \|\phi\|^h_{\tilde v, \tilde s} \|f\|_{v,s} \\
&\leq \left[\frac{e}{2\rho\sigma} + \frac{e^2 m}{2\rho_p\rho_q}\right]^h h! \|\phi\|^h_{\tilde v, \tilde s} \|f\|_{v,s} = \frac{1}{G} h! \|\phi\|^h_{\tilde v, \tilde s} \|f\|_{v,s} \ .
\end{aligned}$$

Finally, summing over h, we get

$$\begin{aligned}\|f\circ X^1_\phi\|_{v-\nu,s-\sigma} &= \|\sum_h \frac{1}{h!}L^h_\phi f\|_{v-\nu,s-\sigma} \le \sum_h \frac{1}{h!}\|L^h_\phi f\|_{v-\nu,s-\sigma} \\ &\le \|f\|_{v,s}\sum_h \frac{1}{G^h}\|\phi\|^h_{\tilde v,\tilde s} = \left(1-\frac{\|\phi\|_{\tilde v,\tilde s}}{G}\right)^{-1}\|f\|_{v,s}. \quad \blacksquare\end{aligned}$$

Next lemma is an immediate consequence of Hamilton equations (and the trivial proof is omitted).

Lemma A.4 *Let $0 < v-\nu < v$, $0 < s-\sigma < s$ and suppose that*

$$\max_{1\le i\le n}\|\frac{\partial\phi}{\partial I_i}\|_{v,s} \le \bar\sigma \le \sigma\ , \qquad \sum_{i=1}^n \|\frac{\partial\phi}{\partial\varphi_i}\|_{v,s} \le \bar\rho \le \rho\ ,$$
$$\max_{1\le l\le m}\|\frac{\partial\phi}{\partial p_l}\|_{v,s} \le \bar\rho_q \le \rho_q\ , \qquad \max_{1\le l\le m}\|\frac{\partial\phi}{\partial q_l}\|_{v,s} \le \bar\rho_p \le \rho_p\ .$$

Let $(u_0,\varphi_0) := (u(0),\varphi(0)) \in W_{v-\nu,s-\sigma}$. Then $X^t_\phi(u(0),\varphi(0)) = (u(t),\varphi(t)) \in W_{v,s}$ for all t with $0\le t\le 1$. More precisely,

$$|I(t)-I(0)| \le \sum_{i=1}^n |I_i(t)-I_i(0)| \le \bar\rho\ , \qquad \max_i |\varphi_i(t)-\varphi_i(0)| \le \bar\sigma\ ,$$
$$\max_{1\le l\le m}|p_l(t)-p_l(0)| \le \bar\rho_p\ , \qquad \max_{1\le l\le m}|q_l(t)-q_l(0)| \le \bar\rho_q\ .$$

Lemma A.5 *Let $H(u,\varphi) := h(I)+g(u,\varphi)+f(u,\varphi)$ real-analytic on $W_{v,s}$ with $g = \sum_{k\in\Lambda} g_k(u)e^{ik\cdot\varphi}$. Let $\nu < v/2$ and $\sigma < s/2$. Suppose that*

$$|\omega(I)\cdot k| \ge \alpha > 0\ , \qquad \forall\,|k|\le K\ , \quad k\notin\Lambda\ , \quad \forall\, I\in D_r\ , \tag{A.9}$$

and

$$\|f\|_{v,s} < \alpha\delta/c_m \qquad \text{where} \qquad \delta := \min\{\rho\sigma, \rho_p\rho_q\}\ . \tag{A.10}$$

Then, there exists a real-analytic symplectic transformation

$$\Phi : (\tilde u,\tilde\varphi) \in W_{v-2\nu,s-2\sigma} \longrightarrow (u,\varphi)\in W_{v,s}$$

such that

$$H\circ\Phi = h + g_+ + f_+\ , \qquad g_+ - g = P_\Lambda T_K f\ . \tag{A.11}$$

Here, $\Phi := X^1_\phi$ for a suitable $\phi : W_{v,s}\to\mathbb{C}$ such that

$$\|f_+\|_{v-2\nu,s-2\sigma} \le \left(1-\frac{c_m}{\alpha\delta}\|f\|_{v,s}\right)^{-1}\left[\frac{c_m}{\alpha\delta}\|f\|^2_{v,s} + \|\{g,\phi\}\|_{v-\nu,s-\sigma} + e^{-K\sigma}\|f\|_{v,s}\right]\ . \tag{A.12}$$

Furthermore,

$$\sigma\ |I(\tilde u,\tilde\varphi)-\tilde I|,\ \rho_q\ |p(\tilde u,\tilde\varphi)-\tilde p|,\ \rho_p\ |q(\tilde u,\tilde\varphi)-\tilde q|,\ \rho\ |\varphi(\tilde u,\tilde\varphi)-\tilde\varphi| \le \|f\|_{v,s}/\alpha\ . \tag{A.13}$$

Proof We have $H \circ \Phi = h \circ \Phi + (g + T_K f) \circ \Phi + (f - T_K f) \circ \Phi$ and we can write

$$\begin{aligned} h \circ \Phi &= h + \{h, \phi\} + \int_0^1 (1-t)\{\{h,\phi\},\phi\} \circ X_\phi^t \, dt \ , \\ (g + T_K f) \circ \Phi &= (g + T_K f) + \int_0^1 \{(g + T_K f), \phi\} \circ X_\phi^t \, dt \ . \end{aligned}$$

Since we want $g_+ = [\{h,\phi\} + g + T_K f]$ and $g_+ - g = P_\Lambda T_K f$ we have to solve $\{h,\phi\} + T_K f = P_\Lambda T_K f$, that is

$$\{h, \phi\} = \sum_{|k| \le K, k \notin \Lambda} f_k(u) e^{ik\cdot\varphi} \ .$$

The solution of such equation is explicitly given by

$$\phi(u,\varphi) = \sum_{|k| \le K, k \notin \Lambda} \frac{f_k(u)}{ik \cdot \omega(I)} e^{ik\cdot\varphi} \qquad \text{with} \qquad \|\phi\|_{v,s} \le \frac{1}{\alpha} \|f\|_{v,s} \ . \tag{A.14}$$

¿From (A.14), Lemma A.1 and Lemma A.4, we obtain (A.13) and the inclusion

$$\Phi : W_{v-2\nu, s-2\sigma} \longrightarrow W_{v,s} \ .$$

If $f_t := (1-t)(g_+ - g) + tT_K f$, we have

$$f_+ = \int_0^1 \{(g + f_t), \phi\} \circ X_\phi^t \, dt + (f - T_K f) \circ \Phi \ .$$

We can estimate G in (A.8) with $G \ge \delta / c_m$. Then, $\forall\, F$ and $\forall\, t$ with $0 \le t \le 1$, substituting $\tilde{v} \to v$, $v \to v - \nu$, $\nu \to \nu$, we have (by Lemma A.3) that

$$\|F \circ X_\phi^t\|_{v-2\nu, s-2\sigma} \le C \|F\|_{v-\nu, s-\sigma} \ ,$$

with $C := (1 - c_m \|\phi\|_{v,s}/\delta)^{-1}$. Then, choosing $F := \{f_t, \phi\}$, and using Lemma A.2 we have

$$\begin{aligned} \|\int_0^1 \{f_t, \phi\} \circ X_\phi^t \, dt\|_{v-2\nu, s-2\sigma} &\le \int_0^1 \|\{f_t,\phi\} \circ X_\phi^t\|_{v-2\nu,s-2\sigma} \, dt \\ &\le C \int_0^1 \|\{f_t, \phi\}\|_{v-\nu, s-\sigma} \, dt \\ &\le C \int_0^1 \left(\frac{2}{e\rho\sigma} + \frac{2m}{\rho_p \rho_q}\right) \|f_t\|_{v,s} \|\phi\|_{v,s} \, dt \\ &\le C \frac{c_m}{\delta} \int_0^1 \|f_t\|_{v,s} \|\phi\|_{v,s} \, dt \le C \frac{c_m}{\alpha\delta} \|f\|_{v,s}^2 \ , \end{aligned} \tag{A.15}$$

where in the last inequality we have used the simple fact that $\|f_t\|_{v,s} \le \|f\|_{v,s}$. Similarly we obtain

$$\|\int_0^1 \{g, \phi\} \circ X_\phi^t \, dt\|_{v-2\nu, s-2\sigma} \le C \|\{g,\phi\}\|_{v-\nu, s-\sigma} \tag{A.16}$$

and

$$\|(f - T_K f) \circ X^1_\phi\|_{v-2\nu,s-2\sigma} \le C\|f - T_K f\|_{v-\nu,s-\sigma} \le C\, e^{-K\sigma}\|f\|_{v,s}\ . \tag{A.17}$$

Collecting (A.15), (A.16), (A.17) we have (A.12). ∎

We are now ready for the

Proof of Proposition A.1 Let $\varepsilon_0 := \varepsilon$, $\nu_0 := v/8$, $\sigma_0 := s/6$, $\delta_0 := \min\{\rho_0\sigma_0, \rho_{p0}\rho_{q0}\}$. Suppose that[20]

$$e^{-Ks/6} \le 32c_m\varepsilon/\alpha d \tag{A.18}$$

Substituting $\nu \to \nu_0$, $\sigma \to \sigma_0$, $\delta \to \delta_0$, we can use Lemma A.5 since $\delta_0 \ge d/64$ and (A.2) implies (A.10). Defining $W_1 := W_{v_1,s_1}$ with $v_1 := v - 2\nu_0 = 3/4v$ and $s_1 := s - 2\sigma_0 = 2/3s$, we obtain an analytic symplectic transformation $\Phi_0 : W_1 \to W_{v,s}$ with $H \circ \Phi_0 = h + g_0 + f_1$, where $g_0 = P_\Lambda T_K f$. Moreover from (A.2), (A.18), (A.12) we obtain

$$\|f_1\|_1 := \|f_1\|_{v_1,s_1} =: \varepsilon_1 \le \left(1 - \frac{64c_m\varepsilon}{\alpha d}\right)^{-1}\left[\frac{64c_m\varepsilon}{\alpha d} + e^{-Ks}\right]\varepsilon \le \frac{9}{11}\,\frac{2^7c_m\varepsilon}{\alpha d}\varepsilon \le \frac{1}{6}\varepsilon\ . \tag{A.19}$$

Letting $(u,\varphi) = \Phi_0(u^{(1)}, \varphi^{(1)})$, from (A.13) we have, $\forall\,(u,\varphi) \in W_1$,

$$s\ |I^{(1)} - I|,\ r_p\ |p^{(1)} - p|,\ r_q\ |q^{(1)} - q|,\ r\ |\varphi^{(1)} - \varphi|,\ \le 8\varepsilon/\alpha\ . \tag{A.20}$$

Let $L \in \mathbb{N}$ be such that

$$L \le \frac{Ks}{12\ln 2} < L + 1\ , \qquad \Big(\implies Ks \ge 8L\Big)\ . \tag{A.21}$$

Let $\nu := v/8L$, $\sigma := s/4L$, and define for $1 \le i \le L$, $v_i := v_{i-1} - 2\nu = v_1 - 2(i-1)\nu$, $s_i := s_{i-1} - 2\sigma = s_1 - 2(i-1)\sigma$, $\Phi_i : W_{i+1} \to W_i := W_{v_i,s_i}$ with $H_i := H \circ \Phi_{i-1} =: h + g_{i-1} + f_i$ on W_i and $\varepsilon_i := \|f_i\|_{W_i} =: \|f_i\|_i$. Observing that $W_{i+1} \subset W_i$, we can iterate Lemma A.5 with $v \to v_i$, $s \to s_i$, after verifying by induction (see below) that

$$\varepsilon_i \le \varepsilon_1\ , \qquad \forall\,1 \le i \le L\ . \tag{A.22}$$

In fact, in order to apply Lemma A.5 we have to verify (A.10), which is implied, for any $1 \le i \le L$, by (A.22) and the estimate[21]

$$\varepsilon_i \le \varepsilon_1 \le 2^{-6}(\alpha d/64c_mL^2)\ , \tag{A.23}$$

which follows directly from (A.19), (A.2) and (A.21). We observe that $\forall\,1 \le i \le L$

$$\|g_i - g_{i-1}\|_i = \|P_\Lambda T_K f_i\|_i \le \|f_i\|_i = \varepsilon_i\ . \tag{A.24}$$

[20] The case in which in (A.18) holds “>” is even simpler. In fact it is sufficient to apply Lemma A.5 with $\nu := \frac{v}{4}$, $\sigma := \frac{s}{3}$, $v - 2\nu = v_*$, $s - 2\sigma = s_*$, $g = 0$, having $\Psi := \Phi$, $g_+ = P_\Lambda T_K f$, $f_* := f_+$. (A.6) and (A.7) follow from (A.2), (A.12) and from $Ks \ge 6$, as it is easy to verify.

[21] We observe that $\delta \ge d/64$.

We now prove (A.22). In order to estimate f_{i+1}, we evaluate $g_{i-1} = \sum_{j=0}^{i-1} \tilde g_j$ where $\tilde g_0 := g_0$ and $\tilde g_j := (g_j - g_{j-1})$ are defined on W_j. Since from (A.14) we have $\|\phi_i\|_i \le \|f_i\|_i/\alpha = \varepsilon_i/\alpha$, and

$$
\begin{aligned}
\|\{g_{i-1},\phi_i\}\|_{v_i-\nu,s_i-\sigma} &\le \sum_{j=0}^{i-1} \|\{\tilde g_j,\phi_i\}\|_{v_i-\nu,s_i-\sigma} \\
&\le \sum_{j=0}^{i-1} \Bigg[\frac{1}{e(r_j-r_i+\rho)\sigma} + \frac{1}{e(s_j-s_i+\sigma)\rho} \\
&\qquad + \frac{m}{(r_{pj}-r_{pi}+\rho_p)\rho_q} + \frac{m}{(r_{qj}-r_{qi}+\rho_q)\rho_p}\Bigg] \|\tilde g_j\|_j \|\phi_i\|_i \\
&\le \frac{2}{3}\left[\frac{1}{e\rho\sigma} + \frac{m}{\rho_p\rho_q}\right] \sum_{j=1}^{i-1} \|P_\Lambda T_K f_j\|_j \frac{1}{\alpha}\varepsilon_i + \frac{\varepsilon}{\alpha L}\varepsilon_i \left[\frac{1}{e\rho\sigma} + \frac{m}{\rho_q\rho_p}\right] \\
&\le \frac{64 c_m L^2 \varepsilon_i}{\alpha d}\left[\frac{2}{3}\sum_{j=1}^{i-1}\varepsilon_j + \frac{\varepsilon}{L}\right] \le \frac{64 c_m L^2\varepsilon_i}{\alpha d}\left[\varepsilon_1 + \frac{\varepsilon}{L}\right], \qquad \text{(A.25)}
\end{aligned}
$$

where we have used Lemma A.2 and the fact that $\tilde g_j = P_\Lambda T_K f_j$, considered separately the case $j = 0$ from $j > 0$, and observed that, if $j > 0$, then

$$
\begin{aligned}
v - v_j + \nu &\ge v - v_1 + \nu \ge \nu + v/4 = (2L+1)\nu \ , \\
s - s_j + \sigma &\ge s - s_1 + \sigma \ge \sigma + s/3 = (4L/3+1)\sigma \ .
\end{aligned}
$$

Using (A.21), (A.19), (A.23) and $\varepsilon_i \le \varepsilon_1$ we obtain from Lemma A.5

$$
\varepsilon_{i+1} = \|f_{i+1}\|_{i+1} \le \left(1 - \frac{64 c_m L^2\varepsilon_i}{\alpha d}\right)^{-1} \left[\frac{64 c_m L^2 \varepsilon_i}{\alpha d}(\varepsilon_i + \varepsilon_1 + \varepsilon/L) + e^{-Ks}\right]\varepsilon_i \le \frac{\varepsilon_i}{4} \ . \quad \text{(A.26)}
$$

Moreover, from (A.19), there follows

$$
\|f_*\|_{v_*,s_*} = \varepsilon_{L+1} \le 4^{-L}\varepsilon_1 \le \frac{2^9 c_m \varepsilon}{\alpha d} 4^{-(L+1)}\varepsilon \le \frac{2^9 c_m\varepsilon}{\alpha d} 4^{-\frac{Ks}{12\ln 2}}\varepsilon = \frac{2^9 c_m \varepsilon}{\alpha d} e^{-Ks/6}\varepsilon \ ,
$$

and

$$
\|g - g_0\|_{v_*,s_*} \le \sum_{i=1}^{L} \|g_i - g_{i-1}\|_{v_i,s_i} \le \sum_{i=1}^{L}\varepsilon_i \le 4\sum_{i=1}^{L}(\frac{1}{4})^i \varepsilon_1 = \frac{4}{3}\varepsilon_1 \ ,
$$

from which (A.5) and (A.6) follow.

Let, now, $\Psi := \phi_0 \circ \cdots \circ \phi_L$ and $(u^{(i)}, \varphi^{(i)}) \in W_i$. Using (A.21), (A.20) and Lemma A.5 we have

$$
|I - I'| = |I^{(L+1)} - I^{(0)}| \le |I^{(1)} - I^{(0)}| + \sum_{i=1}^{L} |I^{(i+1)} - I^{(i)}| \le \frac{8\varepsilon}{\alpha s} + \frac{4L}{\alpha s}\sum_{i=1}^{L}\varepsilon_i \le \frac{9}{\alpha s}\varepsilon \ .
$$

The estimates for $|p - p'|$, $|q - q'|$ and $|\varphi - \varphi'|$ are analogous. ▮

B KAM measure estimates

Here, we show how assumptions (2.30) and (2.31) imply the measure estimate (2.37).

Indeed, the set Π_* is obtained as $\Pi_* = \bigcap_{\nu\in\mathbb{N}} \Pi_\nu$, where $\Pi_0 := \Pi$ and, recursively,

$$\Pi_{\nu+1} := \Pi_\nu \setminus \Big(\bigcup_{\substack{(k,\ell)\in\mathbb{Z}^{n+m}\setminus\{0\} \\ |\ell|\le 2,\ |k|>K_\nu}} \mathcal{R}_{k\ell}^{\nu+1} \Big) ,$$

with $K_\nu := K_0\, 2^\nu$, and $\mathcal{R}_{k\ell}^{\nu+1}$ is a suitable "resonant set" to be discarded at the $\nu^{\rm th}$ step of the KAM iteration (compare with the Iterative Lemma in Section 4 of [P96]). The sets $\mathcal{R}_{k\ell}^{\nu+1}$ satisfy the measure estimate

$$\mathrm{meas}_n(\mathcal{R}_{kl}^{\nu+1}) \le \frac{\lambda}{|k|^{\tau+1}} , \qquad \lambda := \ \mathrm{const}\,(LM)^n \frac{\gamma}{M}(\,\mathrm{diam}\,\Pi)^{n-1} , \tag{B.1}$$

for any $|k| \ge K_0$, $\nu \ge 0$, $|\ell| \le 2$ (see Lemma 5 in [P96]). Therefore,

$$\begin{aligned} \mathrm{meas}_n(\Pi_{\nu+1}) \ &\ge \ \mathrm{meas}_n(\Pi_\nu) - \ \mathrm{const}\,\lambda \sum_{|k|>K_\nu} |k|^{-(\tau+1)} \ge \\ &\ge \ \mathrm{meas}_n(\Pi_\nu) - \ \mathrm{const}\,\lambda \frac{1}{K_\nu^{\tau-n+1}} \ . \end{aligned}$$

Iterating this relation and using the definition of K_ν, we get

$$\mathrm{meas}_n(\Pi_{\nu+1}) \ \ge \ \mathrm{meas}_n(\Pi) - \ \mathrm{const}\,\frac{\gamma}{M}(LM\ \ \mathrm{diam}\,\Pi)^{n-1} ,$$

which proves (2.37). ∎

C The Delaunay-Poincaré theory of the planetary three-body problem

In this appendix we discuss, in a self-contained way, the Hamiltonian formulation of the planetary (non-planar) three-body problem, discussing, in particular, the classical Delaunay-Poincarè Theorem 1.1 and its proof. The appendix is divided in two parts dealing, respectively, with the canonical treatment of the two-body problem and with the (partial) extension of such theory to the three-body problem.

PART I - Canonical variables for the two-body problem

I.1 Integration of the Kepler problem

Consider two bodies P_0, P_1 of masses m_0, m_1 and spatial position $u^{(0)}$, $u^{(1)} \in \mathbb{R}^3$, interacting through gravity, with gravitational constant 1; the (inertial) frame $\mathbb{R}^3$ is chosen so that its origin coincides with the center of mass. Let

$$\mathrm{M} := \mathrm{m}_0 + \mathrm{m}_1 \ , \quad \mathrm{m} := \frac{\mathrm{m}_0\mathrm{m}_1}{\mathrm{M}} \ , \quad x := u^{(1)} - u^{(0)} \ , \quad X := \mathrm{m}\dot{x} \ . \tag{C.1}$$

Then, the motion of the two bodies is governed by the Hamiltonian

$$\mathcal{K}(X,x) = \frac{1}{2\mathrm{m}}|X|^2 - \frac{\mathrm{mM}}{|x|}\,, \tag{C.2}$$

with $(X,x) \in \mathbb{R}^3 \times \mathbb{R}^3$ conjugate variables[22], i.e., the equations of motion are $\dot{x} = \partial_X \mathcal{K}$, $\dot{X} = -\partial_x \mathcal{K}$.

As well known such system is integrable and for $\mathcal{K} < 0$ the (x-projection of the) orbits are ellipses. More precisely, one has

Proposition C.1 *Fix $\Lambda_- > 0 > \mathcal{K}_0$ and let $\Lambda_+ := \left(\frac{\mathrm{m}^3\mathrm{M}^2}{-2\mathcal{K}_0}\right)^{\frac{1}{2}}$. Then, there exist $\hat{\rho} > 0$ and a real-analytic symplectic transformation*[23]

$$\Psi_{\mathrm{DP}} : \big((\Lambda,\eta,\mathrm{p}),(\lambda,\xi,\mathrm{q})\big) \in \big([\Lambda_-,\Lambda_+]\times B^2_{\hat{\rho}}\big) \times \big(\mathbb{T}\times B^2_{\hat{\rho}}\big) \mapsto (X,x) \in \{|x| \geq \frac{\hat{\rho}^2}{\mathrm{m}^2\mathrm{M}}\}\ ,$$

casting (C.2) *into the integrable Hamiltonian* $(-\mathrm{m}^3\mathrm{M}^2)/(2\Lambda^2)$.

This classical proposition is due to Poincaré ([Poi1905], chapter III) and the variables $(\Lambda,\eta,\mathrm{p},\lambda,\xi,\mathrm{q})$ are, usually, called **Poincaré variables**. The proof of Proposition C.1 is particularly interesting from the physical point of view and rests upon the introduction of three different (famous) change of variables, which we, now, proceed to describe briefly (for more details, see [Ch88]).

Recall that ℓ, θ and g denote, respectively, the mean anomaly, the longitude of the (ascending) node and the argument of the perihelion (see Figure 2 above).

Step 1. The system is set in *"symplectic" spherical polar variables*: namely, we consider the symplectic map $\Psi_{\mathrm{spc}} : \big((R,\Omega,\Phi),(r,\omega,\varphi)\big) \mapsto (X,x)$ (where $r > 0$, $0 < \omega < \pi$ and $0 \leq \varphi < 2\pi$) given by[24]

$$\Psi_{\mathrm{spc}} : \begin{cases} x_1 = r\sin\omega\ \cos\varphi \\ x_2 = r\sin\omega\ \sin\varphi \\ x_3 = r\cos\omega \end{cases} ,\quad X = \begin{pmatrix} \sin\omega\ \cos\varphi & \frac{\cos\omega\cos\varphi}{r} & -\frac{\sin\varphi}{r\sin\omega} \\ \sin\omega\ \sin\varphi & \frac{\cos\omega\sin\varphi}{r} & \frac{\cos\varphi}{r\sin\omega} \\ \cos\omega & -\frac{\sin\omega}{r} & 0 \end{pmatrix} \begin{pmatrix} R \\ \Omega \\ \Phi \end{pmatrix} \tag{C.3}$$

and consider the new Hamiltonian $\mathcal{K}_{\mathrm{spc}} := \mathcal{K} \circ \Psi_{\mathrm{spc}}$.

[22]Often, in this appendix, upper-lower case letters indicate couples of standard symplectic conjugate momentum-coordinate variables.

[23]Recall that B^n_r, D^n_r, $B^n_r(x_0)$ and $D^n_r(x_0)$ denote, respectively, the real n-ball of radius r centered at the origin, the complex n-ball of radius r centered at the origin, the real n-ball of radius r centered at x_0 and the complex n-ball of radius r centered at x_0.

[24]The matrix in (C.3) is the transpose of the inverse of the Jacobian $\frac{\partial x}{\partial(r,\omega,\varphi)}$.

Step 2. Using the method of Hamilton-Jacobi, one can find a symplectic map $\Psi_{\rm D}$: $\big((L,G,\Theta),(\ell,g,\theta)\big) \mapsto \big((R,\Omega,\Phi),(r,\omega,\varphi)\big)$ that integrates the system: $\Psi_{\rm D}$ is the symplectic transformation with generating function

$$S(L,G,\Theta,r,\omega,\varphi) = \int \sqrt{-\frac{\mathrm{m}^4\mathrm{M}^2}{L^2} + \frac{2\mathrm{m}^2\mathrm{M}}{r} - \frac{G^2}{r^2}}\, dr + \int \sqrt{G^2 - \frac{\Theta^2}{\sin^2\omega}}\, d\omega + \Theta\varphi\,. \quad \text{(C.4)}$$

The variables $\big((L,G,\Theta),(\ell,g,\theta)\big)$ are known as *Delaunay variables*. In such variables, the new Hamiltonian becomes

$$\mathcal{K}_{\rm D} := \mathcal{K}_{\rm spc} \circ \Psi_{\rm D} = -\frac{\mathrm{m}^3\mathrm{M}^2}{2L^2}\,.$$

Let C be the angular momentum of the planet, let a be the major semi-axis and let i be its inclination, i.e., the angle between a fixed reference plane and the Keplerian ellipse plane; compare Figure 2 (later, such reference plane will be taken to be the "total angular momentum plane"). By construction, the following relations hold

$$G = |C|\,, \quad \Theta = G\,\cos i \quad \text{and} \quad L = \mathrm{m}\sqrt{\mathrm{M}a}\,.$$

Step 3. To remove singularities, following Poincaré, we proceed as follows. First, we introduce *Poincaré action-angle variables* by means of the linear symplectic transformation $\Psi_{\rm P_{aa}} : \big((\Lambda,H,Z),(\lambda,h,\zeta)\big) \mapsto \big((L,G,\Theta),(\ell,g,\theta)\big)$ given by

$$\Psi_{\rm P_{aa}} : \quad \begin{cases} \Lambda = L\,, \quad H = L - G\,, \quad Z = G - \Theta\,, \\ \lambda = \ell + g + \theta\,, \quad h = -g - \theta\,, \quad \zeta = -\theta\,. \end{cases} \qquad \text{(C.5)}$$

Then, we let[25] $\Psi_{\rm P} : \big((\Lambda,\eta,\mathrm{p}),(\lambda,\xi,\mathrm{q})\big) \mapsto \big((\Lambda,H,Z),(\lambda,h,\zeta)\big)$ be the symplectic map defined by the relations

$$\begin{array}{lll} H = \frac{\eta^2+\xi^2}{2}\,, & \sqrt{2H}\,\cos h = \eta\,, & \sqrt{2H}\,\sin h = \xi\,, \\ Z = \frac{p^2+q^2}{2}\,, & \sqrt{2Z}\,\cos\zeta = \mathrm{p}\,, & \sqrt{2Z}\,\sin\zeta = \mathrm{q}\,. \end{array} \qquad \text{(C.6)}$$

As Poincaré showed (see below), *the symplectic map* $\Psi_{\rm DP} : \big((\Lambda,\eta,\mathrm{p}),(\lambda,\xi,\mathrm{q})\big) \mapsto (X,x)$ *with*

$$\Psi_{\rm DP} := \Psi_{\rm spc} \circ \Psi_{\rm D} \circ \Psi_{\rm P_{aa}} \circ \Psi_{\rm P} \qquad \text{(C.7)}$$

is real-analytic in a neighborhood of $\big([\Lambda_-,\Lambda_+] \times \{(0,0)\}\big) \times \big(\mathbb{T} \times \{(0,0)\}\big)$, (and the two-body Hamiltonian, in Poincaré variables, is $\mathcal{K} \circ \Psi = -\frac{\mathrm{m}^3\mathrm{M}^2}{2\Lambda^2}$).

[25] Do not confuse the variables (p,q) here with the variables (p,q) used in the text (and, in particular, in Theorem 1.2, where the variables (p,q) correspond to the variables (η',ξ') introduced below).

Remark C.1 (i) If we denote $(X,x)=\Phi_{\rm DP}\Big((\Lambda,\eta,{\rm p}),(\lambda,\xi,{\rm q})\Big)$ then[26] ,

$$X=\frac{{\rm m}^4{\rm M}^2}{\Lambda^3}\frac{\partial x}{\partial\lambda}\,.$$

(ii) Let us collect, here, some important relations among the above introduced quantities. Let, as usual, e denote the eccentricity of the Keplerian ellipse and let a and i denote the major semi-axis and the inclination. Then, by construction, one sees that

$$\begin{aligned}&\Lambda={\rm m}\sqrt{{\rm M}a}\,,\\&\sqrt{\xi^2+\eta^2}=\sqrt{\Lambda}\,e\,(1+O(e^2))\ ,\\&\sqrt{{\rm p}^2+{\rm q}^2}=\sqrt{\Lambda}\,i\,(1+O(e^2)+O(i^2))\,.\end{aligned}\tag{C.8}$$

A more explicit link between H, the eccentricity and the major semi-axis is given by

$$H=\Lambda\,(1-\sqrt{1-e^2})=\Lambda\ \frac{e^2}{2}(1+O(e^2))\ ,\tag{C.9}$$

$$e(H,\Lambda)=\sqrt{\frac{H}{\Lambda}\Big(2-\frac{H}{\Lambda}\Big)}\,.\tag{C.10}$$

Also, if C is the angular momentum of the system, one infers that

$$|C|=\Lambda\sqrt{1-e^2}=\Lambda(1+O(e^2))\ ,\tag{C.11}$$

$$Z=|C|\,(1-\cos i)=\Lambda\sqrt{1-e^2}\,(1-\cos i)=|C|\ \frac{i^2}{2}(1+O(i^2))\,.\tag{C.12}$$

Poincaré's argument ([Poi1905]) for proving the analyticity of $\Psi_{\rm DP}$ goes as follows.
Let us define $\alpha:=g+v$, $\psi:=\varphi-\theta$, $\mathcal{X}:=r\cos(v-\ell)$, $\mathcal{Y}:=r\sin(v-\ell)$. Let us also denote by "trig" sin *or* cos and by $\mathcal{Z}$ the vector $(\mathcal{X},\mathcal{Y})$. By (C.3), one gets an analytic expression of x in terms of

$$\Big(\mathcal{Z},\ \cos^2\frac{i}{2}\ {\rm trig}\,\lambda,\ \sin^2\frac{i}{2}\ {\rm trig}\,(\lambda+2\zeta),\ \sin i\ {\rm trig}\,(\lambda+\zeta)\Big)\,.\tag{C.13}$$

By geometric considerations, $r\cos v=a(\cos u-e)$ and $r\sin v=a\sqrt{1-e^2}\sin u$. Thus, one gets an analytic expression of $\mathcal{Z}$ of the form

$$\mathcal{Z}=\mathcal{Z}(a,\,e^2,\ {\rm trig}\,(u-\ell),\,e^2\,{\rm trig}\,(u+\ell),\,e\,{\rm trig}\,\ell)\,.\tag{C.14}$$

By geometric considerations, $u-\ell=e\sin u$, from which $u-\ell$ results to be an analytic function of $e\,{\rm trig}\,\ell$. Hence, by standard trigonometric computations, $e^2\,{\rm trig}\,(u+\ell)$ is proved to be an analytic function of $e\,{\rm trig}\,\ell$.
Thus, from (C.14), we get an analytic representation $\mathcal{Z}=\mathcal{Z}(a,\,e^2,\,e\,{\rm trig}\,\ell)$. By $e^2=(e\sin\ell)^2+(e\cos\ell)^2$ and the first of (C.8), it follows that $\mathcal{Z}=\mathcal{Z}(\Lambda,\,e\,{\rm trig}\,\ell)$.
From (C.5), (C.6) and (C.9), $\mathcal{Z}=\mathcal{Z}(\Lambda,\,\lambda,\,\eta,\,\xi)$.

[26] By Hamilton equations one sees that : $\dot\lambda=\partial_\Lambda\Big(-\frac{{\rm m}^3{\rm M}^2}{2\Lambda^2}\Big)=\frac{{\rm m}^3{\rm M}^2}{\Lambda^3}$, and $\dot\Lambda=\dot\xi=\dot\eta=\dot{\rm p}=\dot{\rm q}=0$. Thus, by the chain rule, $X={\rm m}\dot x={\rm m}(\partial_\lambda x)\,\dot\lambda=\frac{{\rm m}^4{\rm M}^2}{\Lambda^3}\frac{\partial x}{\partial\lambda}$.

Hence, we come back to the expression in (C.13). To complete the proof of the analytic dependence of x w.r.t. the Poincaré variables, we need to find an expression of $\sin\frac{i}{2}\,\mathrm{trig}\,\zeta$ and $\mathrm{trig}\,\frac{i}{2}$. From (C.6),

$$\sin\frac{i}{2}\cos\zeta = \frac{p}{2\sqrt{\Lambda - H}}\,, \quad \sin\frac{i}{2}\sin\zeta = \frac{q}{2\sqrt{\Lambda - H}}\,, \quad H = \frac{\xi^2+\eta^2}{2}\,.$$

Using again trigonometric relations, one sees that $\sin i\,\mathrm{trig}\,\zeta$ is an analytic function of $(\Lambda, \lambda, \eta, \xi, p, q)$. Moreover, by (C.6), (C.11) and (C.12), one has

$$1 - \cos i = \frac{p^2+q^2}{2\Lambda - (\eta^2+\xi^2)}\,,$$

and an analytic expression of $\mathrm{trig}\,\frac{i}{2}$ in terms of $(\Lambda, \eta, \xi, p, q)$ easily follows. Hence, one obtains an analytic expression of x in terms of $(\Lambda, \lambda, \eta, \xi, p, q)$.
From point (i) of Remark C.1, one finally gets that X is analytic in $(\Lambda, \lambda, \eta, \xi, p, q)$. ∎

I.2 "Osculating" Poincaré variables

Following Poincaré, we introduce a new set of action-angle variables (linearly related to the Delaunay variables), $\big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\big) \in (\mathbb{R}^3 \times \mathbb{T}^3)$, by letting

$$\Psi_{\mathrm{P}^*_{\mathrm{aa}}} : \quad \begin{cases} \Lambda^* = \Lambda\,, & H^* = H\,, \quad Z^* = Z - \Lambda + H\,, \\ \lambda^* = \lambda + \zeta\,, & h^* = h - \zeta\,, \quad \zeta^* = \zeta\,. \end{cases} \tag{C.15}$$

The physical interpretation of these variables follows from the above construction. In particular,

$$\lambda^* = \ell + g\,, \quad h^* = -g\,, \quad \zeta^* = -\theta\,. \tag{C.16}$$

Observe that *the angles* (λ^*, h^*) *are defined in the orbital plane*: for this reason, we shall call the set of variables $\big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\big)$ **osculating action-angle variables**[27]. Notice that, by (C.11),

$$Z^* = Z - \Lambda\sqrt{1-e^2} = -\Lambda + O(e^2) + O(i^2)$$

is negative for small eccentricity and inclination.

In a way analogous to (C.6), we can introduce symplectic **osculating Poincaré variables**, $\Psi_{\mathrm{P}^*} : \big((\Lambda^*, \eta^*, \mathrm{p}^*), (\lambda^*, \xi^*, \mathrm{q}^*)\big) \mapsto \big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\big)$, through the relations

$$H^* = \frac{\eta^{*2}+\xi^{*2}}{2}\,, \qquad \sqrt{2H^*}\cos h^* = \eta^*\,, \qquad \sqrt{2H^*}\sin h^* = \xi^*\,, \tag{C.17}$$

[27]Obviously, in the two-body problem the "osculating plane" coincides with the orbital plane, but we shall use these symplectic variables also for the spatial three-body problem, where the two two-body systems considered (star+planet P_j) will not move on fixed planes and, in such a case, it makes only sense to speak about "osculating planes"; we also anticipate that, choosing as reference plane the "total angular momentum plane", ζ^* will turn out to be a *cyclic* variable and, hence, Z_j^* will be integrals of the motions (*Poincaré integrals*).

$$-Z^* = \frac{\mathrm{p}^{*2} + \mathrm{q}^{*2}}{2} \ , \qquad \sqrt{-2Z^*}\, \cos\zeta^* = \mathrm{p}^* \ , \qquad \sqrt{-2Z^*}\, \sin\zeta^* = \mathrm{q}^* \ . \tag{C.18}$$

Notice that (C.17) and (C.18) are singular for $H^* = 0$ and $Z^* = 0$, respectively[28]. However, the Hamiltonian formalism in osculating Poincaré variables is analytic for $\mathrm{p}^2 + \mathrm{q}^2 > 0$ (which, for small eccentricities and in view of (C.8), means *for non-zero inclinations*):

Proposition C.2 *There exists $0 < \rho_0 < \sqrt{2\min\{\Lambda_-, \Lambda_+ - \Lambda_-\}}$, such that the symplectic transformation $\Psi_{\mathrm{DP}^*} : \big((\Lambda^*, \eta^*, \mathrm{p}^*), (\lambda^*, \xi^*, \mathrm{q}^*)\big) \mapsto (X, x) \in \{|x| \geq \frac{\hat{\rho}^2}{\mathrm{m}^2\mathrm{M}}\}$ given by*

$$\Psi_{\mathrm{DP}^*} = \Psi_{\mathrm{spc}} \circ \Psi_{\mathrm{D}} \circ \Psi_{\mathrm{P}^*_{\mathrm{aa}}} \circ \Psi_{\mathrm{P}^*}$$

is real-analytic for $\big((\Lambda^*, \lambda^*), (\eta^*, \xi^*), (\mathrm{p}^*, \mathrm{q}^*)\big) \in \mathcal{C}$ where $\mathcal{C} = \mathcal{C}(\Lambda_-, \Lambda_+, \rho_0)$ is the "conical" region defined as

$$\mathcal{C} := \bigcup_{\substack{\Lambda_- \leq \tilde{\Lambda} \leq \Lambda_+ \\ 0 \leq \tilde{\rho} < \rho_0}} \Big(\{\tilde{\Lambda}\} \times \mathbb{T}\Big) \times \{|(\eta^*, \xi^*)| = \tilde{\rho}\} \times \Big\{(\mathrm{p}^*, \mathrm{q}^*) : \ \tilde{\Lambda} - \frac{\tilde{\rho}^2 + \rho_0^2}{2} < -Z^* < \tilde{\Lambda} - \frac{\tilde{\rho}^2}{2}\Big\} \, .$$

Furthermore, there exist positive numbers $\beta, \bar{\beta}, \rho^*$ satisfying

$$2\beta + \bar{\beta} < \frac{\rho_0^2}{2} \ , \qquad \rho^{*2} < \rho_0^2 - 2(\bar{\beta} + 2\beta) \ , \tag{C.19}$$

such that, for any $\Lambda_0^* \in [\Lambda_- + \beta, \Lambda_+ - \beta]$,

$$([\Lambda_0^* - \beta, \Lambda_0^* + \beta] \times \mathbb{T}) \times B^2_{\rho^*} \times \Big\{(\mathrm{p}^*, \mathrm{q}^*) : \ \Lambda_0^* - \beta - \frac{\rho^{*2}}{2} - \bar{\beta} < -Z^* < \Lambda_0^* - \beta - \frac{\rho^{*2}}{2}\Big\} \subset \mathcal{C} \ . \tag{C.20}$$

For a suitable $s > 0$ and for any $(\mathrm{p}^*, \mathrm{q}^*)$ s.t. $\Lambda_0^* - \beta - \frac{\rho^{*2}}{2} - \bar{\beta} < -Z^* < \Lambda_0^* - \beta - \frac{\rho^{*2}}{2}$, the map $((\Lambda^*, \lambda^*), (\eta^*, \xi^*)) \to \Psi_{\mathrm{DP}^*}\big((\Lambda^*, \eta^*, \mathrm{p}^*), (\lambda^*, \xi^*, \mathrm{q}^*)\big)$ is analytic on the complex domain

$$((\Lambda^*, \lambda^*), (\eta^*, \xi^*)) \in (\mathcal{E}_{\rho^{*2}} \times \mathbb{T}_s) \times D^2_{\rho^*} \ , \tag{C.21}$$

where

$$\mathcal{E} := \Big[\Lambda_0^* - \beta + \rho^{*2}, \Lambda_0^* + \beta - \rho^{*2}\Big] \ , \qquad \mathcal{E}_{\rho^{*2}} := \bigcup_{\tilde{\Lambda} \in \mathcal{E}} D^1_{\rho^{*2}}(\tilde{\Lambda}) \ . \tag{C.22}$$

In the osculating variables the two-body Hamiltonian is given by $\mathcal{K} \circ \Psi_{\mathrm{DP}^*} = -\frac{\mathrm{m}^3\mathrm{M}^2}{2(\Lambda^*)^2}$.

Proof By (C.17), (C.15) and (C.6),

$$\eta^* = \frac{\eta\mathrm{p} + \xi\mathrm{q}}{\sqrt{\mathrm{p}^2 + \mathrm{q}^2}} \qquad \text{and} \qquad \xi^* = \frac{\xi\mathrm{p} - \eta\mathrm{q}}{\sqrt{\mathrm{p}^2 + \mathrm{q}^2}} \, .$$

[28] The singularity $Z^* = 0$ would not be actually a problem since in our case $Z^* \sim -\Lambda$ which is bounded away from 0.

Also, (C.6) is regular for $(\mathrm{p},\mathrm{q}) \neq 0$, so that, for non-zero inclination, we can express $Z = Z(\mathrm{p},\mathrm{q}) = (\mathrm{p}^2+\mathrm{q}^2)/2$ and $\zeta = \zeta(\mathrm{p},\mathrm{q})$ as analytic functions (more precisely, $\zeta(\mathrm{p},\mathrm{q})$ is analytic on the pinched complex torus $\{\mathbb{C}/(2\pi\mathbb{Z})\} \setminus 2\pi\mathbb{Z}$). Thus, in light of (C.15) and (C.18),

$$\begin{aligned}
\lambda^* &= \lambda + \zeta(\mathrm{p},\mathrm{q})\,, \\
\mathrm{p}^* &= \sqrt{2\Lambda - \mathrm{p}^2 - \mathrm{q}^2 - \eta^2 - \xi^2}\, \sin\zeta(\mathrm{p},\mathrm{q})\,, \\
\mathrm{q}^* &= \sqrt{2\Lambda - \mathrm{p}^2 - \mathrm{q}^2 - \eta^2 - \xi^2}\, \cos\zeta(\mathrm{p},\mathrm{q})\,.
\end{aligned}$$

Therefore, *the diffeomorphism*

$$\Psi := \Psi_{\mathrm{P}_*}^{-1} \circ \Psi_{\mathrm{P}^*_{\mathrm{aa}}}^{-1} \circ \Psi_{\mathrm{P}_{\mathrm{aa}}} \circ \Psi_{\mathrm{P}} : \Big((\Lambda,\eta,\mathrm{p}),(\lambda,\xi,\mathrm{q})\Big) \mapsto \Big((\Lambda^*,\eta^*,\mathrm{p}^*),(\lambda^*,\xi^*,\mathrm{q}^*)\Big)\,,$$

which maps the Poincaré variables into the osculating Poincaré variables is analytic for $\Lambda_- < \Lambda < \Lambda_+$*,* $\lambda \in \mathbb{T}$*,* $(\eta,\xi) \in B^2_{\rho_0}$*,* $(\mathrm{p},\mathrm{q}) \in B^2_{\rho_0} \setminus \{0\}$*, for a suitably small* ρ_0.
By construction, $\Psi_{\mathrm{DP}^*} = \Psi_{\mathrm{DP}} \circ \Psi^{-1}$, where Ψ_{DP} has been defined in (C.7), and, as shown in Figure 3 below,

$$\begin{aligned}
&\Psi\Big(\{\Lambda = \tilde\Lambda,\ \lambda \in \mathbb{T},\ |(\eta,\xi)| = \tilde\rho,\ |(\mathrm{p},\mathrm{q})| = \tilde r\}\Big) \\
&= \{\Lambda^* = \tilde\Lambda,\ \lambda^* \in \mathbb{T},\ |(\eta^*,\xi^*)| = \tilde\rho,\ |(\mathrm{p}^*,\mathrm{q}^*)| = \sqrt{2\tilde\Lambda - \tilde\rho^2 - \tilde r^2}\}\,.
\end{aligned}$$

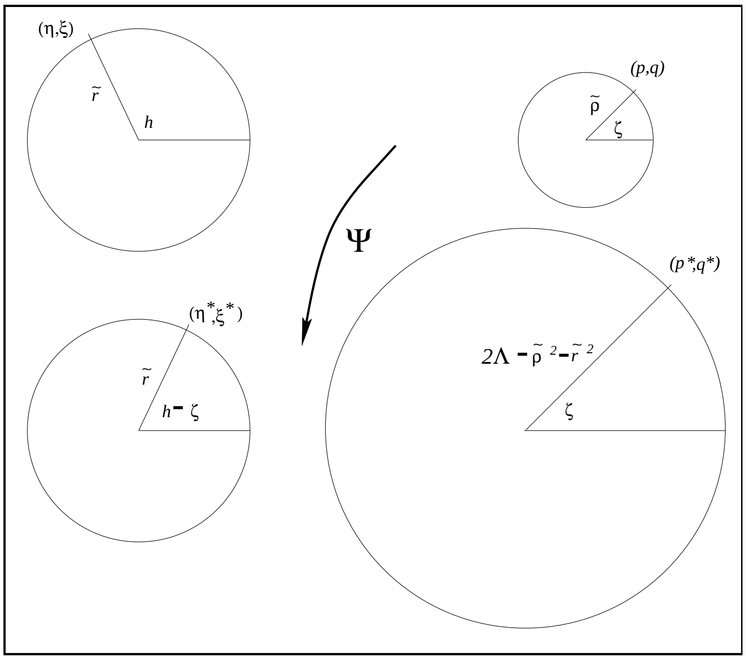

Figure 3

Now, observe that, in view of the above relation,

$$
\begin{aligned}
&\Psi\Big(\Big\{\Lambda\in[\Lambda_-,\Lambda_+],\ \lambda\in\mathbb{T},\ (\eta,\xi)\in B^2_{\rho_0},\ (\mathrm{p},\mathrm{q})\in B^2_{\rho_0}\setminus\{0\}\Big\}\Big)\\
=&\bigcup_{\substack{\tilde\Lambda\in[\Lambda_-,\Lambda_+]\\ \tilde\rho\in[0,\rho_0)\\ \tilde r\in(0,\rho_0)}}\Big\{\Lambda^*=\tilde\Lambda,\ \lambda^*\in\mathbb{T},\ |(\eta^*,\xi^*)|=\tilde\rho,\ |(\mathrm{p}^*,\mathrm{q}^*)|=\sqrt{2\tilde\Lambda-\tilde\rho^2-\tilde r^2}\Big\}\\
=&\bigcup_{\substack{\tilde\Lambda\in[\Lambda_-,\Lambda_+]\\ \tilde\rho\in[0,\rho_0)}}\Big\{\Lambda^*=\tilde\Lambda,\lambda^*\in\mathbb{T}\ ,\ |(\eta^*,\xi^*)|=\tilde\rho,\ \sqrt{2\tilde\Lambda-\tilde\rho^2-\rho_0^2}<|(\mathrm{p}^*,\mathrm{q}^*)|<\sqrt{2\tilde\Lambda-\tilde\rho^2}\Big\},
\end{aligned}
$$

which proves the real-analyticity of Ψ_{DP^*} on $\mathcal{C}$. *Any* choice of $\beta,\bar\beta,\rho^*$ satisfying (C.19) implies easily the inclusion (C.20) and (C.21). ∎

Remark C.2 (i) The Poincaré integral Z^* is (in the small inclination-eccentricity regime considered here), for fixed values of H^* and Λ^*, in one-to-one correspondence with the squared inclination i^2. In fact, by (C.9)÷(C.12) and (C.15),

$$
\frac{i^2}{2}\Big(1+O(i^2)\Big)=1-\cos i=\frac{Z^*+\Lambda^*\sqrt{1-e(H^*,\Lambda^*)^2}}{\Lambda^*\sqrt{1-e(H^*,\Lambda^*)^2}}=1+\frac{Z^*}{\Lambda^*}+O(e^2)\ . \qquad \text{(C.23)}
$$

(ii) In view of the preceding remark, the set in (C.20) corresponds to absolute values of the inclination between $O(\rho^*)$ and $O\Big((\bar\beta)^{\frac12}\Big)$: indeed, choosing, for $\beta'\in(0,\bar\beta)$,

$$
-Z^*=\Lambda^*_0-\beta-\frac{\rho^{*2}}{2}-\beta'\ ,\quad \Lambda^*=\Lambda^*_0-\beta+\rho^{*2}\ ,\quad H^*=0\ ,
$$

one has (by (C.23)) that

$$
i^2=O\Big(\frac{Z^*+\Lambda^*}{\Lambda^*}\Big)=O\Big(\rho^{*2}+\beta'\Big)\ .
$$

(iii) Finally, we point out that, as above (compare Remark C.1 and the relative proof in footnote), if $(X,x)=\Phi_{\mathrm{DP}^*}\Big((\Lambda^*,\eta^*,\mathrm{p}^*),(\lambda^*,\xi^*,\mathrm{q}^*)\Big)$ is as in Proposition C.2, then,

$$
X=\frac{\mathrm{m}^4\mathrm{M}^2}{\Lambda^{*3}}\frac{\partial x}{\partial\lambda^*}\,.
$$

(iv) The use of osculating variables will turn out to be particularly useful in the reduction of the angular momentum (see PART II, below).

I.3 Some orbital elements in terms of osculating Poincaré variables

In this paragraph we show a way to express some of the classical orbital elements as functions of the osculating Poincaré variables $(\Lambda^*,\lambda^*,\eta^*,\xi^*)$. Denote the eccentric anomaly by u and the true anomaly by v (compare Figure 1 above); denote, also, by d the distance between the planet and its star, by w the *true longitude* $w := v + g + \theta$ and let $w^* := w - \theta = v + g$. The target of this paragraph is, also, to find (for later use) analytic expressions d and w^* as functions of $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$. From the definition of ℓ and (C.16) one has that

$$\lambda^* + h^* = \ell = u - e \sin u\,. \tag{C.24}$$

Inverting (C.24), one finds $u = u(\Lambda^*, \lambda^*, h^*, e)$. Then, from (C.10), one obtains u in terms of $(\Lambda^*, \lambda^*, H^*, h^*)$. Furthermore, one has:

$$d = a\,(1 - e\,\cos u) \tag{C.25}$$

$$= \frac{a\,(1 - e^2)}{1 + e\,\cos v}\,. \tag{C.26}$$

In light of (C.25), (C.10), the first of (C.8) and the above expression of u, one gets a representation d in terms of $(\Lambda^*, \lambda^*, H^*, h^*)$. Therefore, via (C.17), we obtain a representation of d in terms of osculating Poincaré variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$.

Let us now express w^* in osculating Poincaré variables. From (C.25) and (C.26),

$$\cos v = \frac{\cos u - e}{1 - e \cos u}\,. \tag{C.27}$$

Hence, recalling (C.10) and the expressions of u and v above, one readily expresses v as a function of $(\Lambda^*, \lambda^*, H^*, h^*)$. Making use of (C.16), one infers that $w^* = v - h^*$. Hence, from the above construction and (C.17), we obtain a representation of w^* in terms of osculating Poincaré variables $(\lambda^*, \eta^*, \xi^*)$.

PART II - Canonical variables for the three-body problem

II.1 The Poincaré Hamiltonian of the planetary three-body problem

The discussion of this paragraph follows [Ch88] and [L88]. We consider three bodies P_0, P_1, P_2 of mass m_0, m_1, m_2 interacting through gravity (with constant of gravitation 1). Assume that

$$\bar\kappa\varepsilon \le \frac{m_1}{m_0}\,,\ \frac{m_2}{m_0} \le \varepsilon < 1\ , \tag{C.28}$$

for a fixed constant $0 < \bar\kappa \le 1$ and a small parameter $\varepsilon > 0$. We consider a (inertial) frame $\{k_1, k_2, k_3\}$ in $\mathbb{R}^3$ *with origin in the center of mass of the system and with vertical axis parallel to the (conserved) total angular momentum*. Thus, if $u^{(i)}$ denotes the position of P_i, $U^{(i)} := m_i \dot u_i$ denotes the momentum of P_i and C is the total angular momentum,

$$C = \sum_{i=0}^{2} u^{(i)} \times U^{(i)}\,, \tag{C.29}$$

our choices imply

$$\sum_{i=0}^{2} m_i u^{(i)} = 0\ , \qquad \frac{C}{|C|} = k_3\ . \tag{C.30}$$

Newton's laws imply that the three-body problem is governed by the nine-degree-of-freedom Hamiltonian

$$\bar{\mathcal{H}}^{\rm tb}(U^{(1)},U^{(2)},U^{(3)},u^{(1)},u^{(2)},u^{(3)}) := \sum_{i=0}^{2}\frac{1}{2m_i}|U^{(i)}|^2 - \sum_{0\le i<j\le 2}\frac{m_im_j}{|u^{(i)}-u^{(j)}|}\,,$$

where $U=(U^{(0)},U^{(1)},U^{(2)})\in\mathbb{R}^9$ and $u=(u^{(0)},u^{(1)},u^{(2)})\in\mathbb{R}^9$ are conjugate symplectic variables.

We, now, introduce **canonical heliocentric variables** $(R^{(0)},R^{(1)},R^{(2)},r^{(0)},r^{(1)},r^{(2)})$ by means of the linear symplectic transformation

$$\begin{aligned} &u^{(0)}=r^{(0)}\ , \quad u^{(1)}=r^{(0)}+r^{(1)}\ , \quad u^{(2)}=r^{(0)}+r^{(2)}\ ,\\ &U^{(0)}=R^{(0)}-R^{(1)}-R^{(2)}\ , \quad U^{(1)}=R^{(1)}\ , \quad U^{(2)}=R^{(2)}\ . \end{aligned} \tag{C.31}$$

The angular momentum is preserved by this transformation:

$$\sum_{i=0}^{2} r^{(i)}\times R^{(i)} = C\,. \tag{C.32}$$

By means of (C.30), $R^{(0)}=\sum_{i=0}^{2}U^{(i)}=0$. Hence, the three-body Hamiltonian $\bar{\mathcal{H}}^{\rm tb}$ in $(R^{(i)},r^{(i)})$-variables takes the form

$$\tilde{\mathcal{H}}^{\rm tb}(R^{(1)},R^{(2)},r^{(1)},r^{(2)}) := \sum_{i=1}^{2}\Big(\frac{m_0+m_i}{2m_0m_i}|R^{(i)}|^2-\frac{m_0m_i}{|r^{(i)}|}\Big)+\frac{R^{(1)}\cdot R^{(2)}}{m_0}-\frac{m_1m_2}{|r^{(1)}-r^{(2)}|}\,.$$

We have, therefore, obtained a six-degree of freedom Hamiltonian of conjugated variables $(R^{(1)},R^{(2)})\in\mathbb{R}^6$ and $(r^{(1)},r^{(2)})\in\mathbb{R}^6$: the number of degrees of freedom decreased by three units because of the above **reduction of the center of mass**.

The masses m_j appear in the definition of the momenta $R^{(1)}$ and $R^{(2)}$, which are both of order ε. In order to remove this singularity (as $\varepsilon\to 0$), we introduce new symplectic variables[29]

$$X^{(i)}=\frac{R^{(i)}}{\varepsilon m_0^{5/3}}\ , \qquad x^{(i)}=\frac{r^{(i)}}{m_0^{2/3}}\ . \tag{C.33}$$

In such variables, the $\tilde{\mathcal{H}}^{\rm tb}$-flow is equivalent to the $\mathcal{H}^{\rm tb}$-flow with

$$\mathcal{H}^{\rm tb}(X^{(1)},X^{(2)},x^{(1)},x^{(2)}) := \mathcal{H}_0^{\rm tb}(X^{(1)},X^{(2)},x^{(1)},x^{(2)})+\mathcal{H}_1^{\rm tb}(X^{(1)},X^{(2)},x^{(1)},x^{(2)}) \tag{C.34}$$

where

$$\begin{aligned} &\mathcal{H}_0^{\rm tb} := \sum_{i=1}^{2}\Big(\frac{1}{2\mathrm{m_i}}|X^{(i)}|^2-\frac{\mathrm{m}_i\mathrm{M}_i}{|x^{(i)}|}\Big)\,, \quad \mathcal{H}_1^{\rm tb} := \varepsilon\, X^{(1)}\cdot X^{(2)}-\frac{m_1m_2}{\varepsilon\, m_0^2}\,\frac{1}{|x^{(1)}-x^{(2)}|}\,,\\ &\mathrm{m}_i := \frac{m_i}{\varepsilon}\,\frac{1}{m_0+m_i}\ , \qquad \mathrm{M}_i := 1+\frac{m_i}{m_0}\,. \end{aligned} \tag{C.35}$$

[29]Recall that, if $\alpha>0$ and $\beta>0$ are two "rescaling factors", the Hamiltonian flow governed by a Hamiltonian function $h(X,x)$ (w.r.t. the symplectic form $dX\wedge dx$) coincides with the Hamiltonian flow governed by the Hamiltonian $\frac{1}{\alpha\beta}h(\alpha\tilde{X},\beta\tilde{x})$ (w.r.t. the symplectic form $d\tilde{X}\wedge d\tilde{x}$) with initial data $\tilde{X}(0)=\frac{1}{\alpha}X(0)$ and $\tilde{x}(0)=\frac{1}{\beta}x(0)$.

Recalling (C.28), it follows that m_i and M_i are bounded and bounded away from zero (uniformly in ε):

$$\frac{\bar\kappa}{1+\bar\kappa} \le \mathrm{m}_i \le 1\ , \qquad 1 \le \mathrm{M}_i \le 2\ . \tag{C.36}$$

Notice that *the Hamiltonian $\mathcal{H}_0^{\rm tb}$, which is of order one, is simply the sum of two uncoupled Kepler problems, while $\mathcal{H}_1^{\rm tb}$ is of order ε and will be considered as a perturbation.*

We can, therefore, *introduce osculating Poincaré variables associated to the osculating orbital elements relative to* (P_0, P_i) *with* $i = 1, 2$ *and masses* m_0, m_i. More precisely, let

$$(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*) := \Psi_{\rm DP^*}^{-1}(X^{(i)}, x^{(i)})\ ,$$

where $\Psi_{\rm DP^*}$ is defined in Proposition C.2 (and the reference plane for computing the orbital elements relative to (P_0, P_i) is the plane spanned by $\{k_1, k_2\}$, i.e., the total angular momentum plane). For example, the expression of Λ_i^* with respect to the major semi-axis of the planet P_i is

$$\Lambda_i^* \;=\; \kappa_i^* \sqrt{a_i}\ , \qquad \kappa_i^* := \frac{m_i}{\varepsilon}\, \frac{1}{\sqrt{m_0(m_0+m_i)}}\ . \tag{C.37}$$

In osculating Poincaré variables

$$(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*) := \Big((\Lambda_1^*, \Lambda_2^*), (\eta_1^*, \eta_2^*), (\mathrm{p}_1^*, \mathrm{p}_2^*), (\lambda_1^*, \lambda_2^*), (\xi_1^*, \xi_2^*), (\mathrm{q}_1^*, \mathrm{q}_2^*) \Big)\ , \tag{C.38}$$

the unperturbed Hamiltonian $\mathcal{H}_0^{\rm tb}$ becomes, simply,

$$\mathcal{H}_0^*(\Lambda^*) := -\sum_{i=1}^{2} \frac{\kappa_i}{2(\Lambda_i^*)^2}\,, \qquad \kappa_i := \Big(\frac{m_i}{\varepsilon}\Big)^3 \frac{1}{m_0^2(m_0+m_i)}\ , \tag{C.39}$$

(with κ_i of order one), and the full three-body Hamiltonian becomes

$$\mathcal{H}^*(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*) = \mathcal{H}_0^*(\Lambda^*) + \mathcal{H}_1^*(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*)\ , \tag{C.40}$$

with $\mathcal{H}_1^*$ real-analytic on $\mathcal{C}\times\mathcal{C}$ (compare Proposition C.2). It is customary to split the $O(\varepsilon)$-perturbation $\mathcal{H}_1^*$ as the sum of two functions:

$$\mathcal{H}_1^* := \mathcal{H}_1^{*,\rm compl} + \mathcal{H}_1^{*,\rm princ}\ , \tag{C.41}$$

where $\mathcal{H}_1^{*,\rm compl}$, called the **complementary part** of the perturbation, is the function $\varepsilon\, X^{(1)}\cdot\ X^{(2)}$ expressed in Poincaré osculating variables (C.38), while $\mathcal{H}_1^{*,\rm princ}$, called the **principal part** of the perturbation, is the function $\frac{m_1 m_2}{\varepsilon m_0^2}\,\frac{1}{|x^{(1)}-x^{(2)}|}$ expressed in Poincaré osculating variables (C.38).

Notice that, because of point (iii) of Remark C.2, the λ^*-average of $\mathcal{H}_1^{*,\rm compl}$ vanishes: let $x^{(i)} = x^{(i)}(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*)$ and $X^{(i)} = X^{(i)}(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*)$. Then

$$\begin{aligned}\int_0^{2\pi}\!\!\int_0^{2\pi} \mathcal{H}_1^{*,\rm compl}(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*)\, d\lambda_1^*\, d\lambda_2^* &= \varepsilon \int_0^{2\pi}\!\!\int_0^{2\pi} X^{(1)}\cdot\ X^{(2)}\, d\lambda_1^*\, d\lambda_2^* \\ = \varepsilon\,{\rm const} \int_0^{2\pi}\!\!\int_0^{2\pi} \partial_{\lambda_1^*} x^{(1)}\cdot\ \partial_{\lambda_2^*} x^{(2)}\, d\lambda_1^*\, d\lambda_2^* = 0\ . \end{aligned} \tag{C.42}$$

II.2 Reduction of the angular momentum

The conservation of the total angular momentum allows to lower by two more units the number of degrees of freedom ("*reduction of the angular momentum*"). Recall that we are *excluding planar motions*, i.e., motions with vanishing mutual inclinations.

Proposition C.3 *The function $\mathcal{H}_1^*$ in* (C.40) *is independent of ζ^*. More precisely, let $r_j^* := \sqrt{\mathrm{p}^{*2}_j + \mathrm{q}^{*2}_j}$ and $r^* := (r_1^*, r_2^*)$. Then, for any $\zeta_j^* \in \mathbb{T}$ and for each $r_j^* > 0$ (for which $(\mathrm{p}^*, \mathrm{q}^*)$ belong to the real part of their domain of definition; see Proposition C.2), one has*

$$\mathcal{H}_1^*(\Lambda^*, \eta^*, (r_1^* \cos\zeta_1^*, r_2^* \cos\zeta_2^*), \lambda^*, \xi^*, (r_1^* \sin\zeta_1^*, r_2^* \sin\zeta_2^*)) = \mathcal{H}_1^*(\Lambda^*, \eta^*, r^*, \lambda^*, \xi^*, (0,0)). \tag{C.43}$$

In particular, $Z_j^* := -\frac{r_j^{*\,2}}{2}$ are analytic integrals: $\{Z_j^*\ ,\ \mathcal{H}_1^*\} = 0$.

We shall call the Z_j^*'s the **Poincaré integrals** of the non-planar three-body-problem.

Proof Looking at the force field, one sees that, if at some time t_0 the mutual inclination of the planets vanishes, it vanishes at every instant of time: therefore the motions of the three-body problem are either *planar*, so that the mutual inclination is identically zero, or *non-planar*, so that the mutual inclination is always non-zero.

Now, recall our choice of reference plane as the total angular momentum plane, fix a time t_0 and consider the inclination $i_j = i_j(t_0)$ of the instant orbital plane ("osculating plane") associated to the planet P_j (i.e., the plane spanned by the position and velocity of P_j at time t_0): $i_j(t_0)$ is the angle between the $\{k_1, k_2\}$-plane and the plane spanned by the position and velocity of P_j at time t_0. If $i_1 \neq i_2$, we can define *the line of the nodes* as the intersection of the two osculating planes; let $N \neq P_0$ be a node, i.e., a point in the line of the nodes. Let, also, the vector V be the difference between N and the position of the star P_0. By construction, V lies in the intersection of the osculating planes spanned by $(r_j(t_0), R_j(t_0))$, $j = 1, 2$. Therefore, recalling (C.32), $V \cdot C = 0$. Whence, the difference between the longitudes of the ascending nodes of the planets is constant: indeed,

$$\theta_1 - \theta_2 = \pi\,. \tag{C.44}$$

Since $i_1 \neq i_2$ in a neighborhood of t_0, we can perform the transformation (C.18): we obtain a Hamiltonian $\mathcal{H}^{**}(\Lambda^*, \lambda^*, \eta^*, \xi^*, Z^*, \zeta^*)$, with

$$\zeta_2^* - \zeta_1^* = \theta_1 - \theta_2 = \pi\,. \tag{C.45}$$

Let us consider a rotation $\mathcal{R}_\vartheta$ of an angle ϑ around C. By construction (see the relations (C.8), (C.9), (C.11) and (C.16) above), the variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*, Z^*)$ are *preserved* by $\mathcal{R}_\vartheta$, while (ζ_1^*, ζ_2^*) is sent into $(\zeta_1^* + \vartheta, \zeta_2^* + \vartheta)$. Since the energy of the system is also *preserved* by $\mathcal{R}_\vartheta$, we obtain that $\mathcal{H}^{**}(\ldots, \zeta_1^*, \zeta_2^*) = \mathcal{H}^{**}(\ldots, \zeta_1^* + \vartheta, \zeta_2^* + \vartheta)$. Thus, the Hamiltonian $\mathcal{H}^{**}$ (and, hence, the function $\mathcal{H}_1^*$) does not depend on ζ_1^* and ζ_2^* separately, but *only on their difference*. The thesis follows, now, from (C.45). ∎

Hence, *we can consider the Hamiltonian* $\mathcal{H}^*$ *as depending only on* $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$ *and on the initial value of Poincaré integral* Z^*. We, therefore, let[30]

$$\varepsilon F := \varepsilon F(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) := \mathcal{H}_1^* \Big(\Lambda^*, \eta^*, \big(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\big), \lambda^*, \xi^*, (0,0)\Big) \ ,$$
$$Z^* = (Z_1^*, Z_2^*) \ , \quad Z_j^* := -\frac{\mathrm{p}_\mathrm{j}^{*2} + \mathrm{q}_\mathrm{j}^{*2}}{2} \ . \tag{C.46}$$

Analogously, we set

$$\mathcal{H}_1^{\mathrm{compl}}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) := \mathcal{H}_1^{*,\mathrm{compl}} \Big(\Lambda^*, \eta^*, \big(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\big), \lambda^*, \xi^*, (0,0)\Big) ,$$
$$\mathcal{H}_1^{\mathrm{princ}}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) := \mathcal{H}_1^{*,\mathrm{princ}} \Big(\Lambda^*, \eta^*, \big(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\big), \lambda^*, \xi^*, (0,0)\Big) . \tag{C.47}$$

From now on the values of the Poincaré integrals Z_j^* will be taken to be real in the domains described in Proposition C.2. Notice that, physically, *changing values of the Poincaré integrals correspond to consider different ranges of mutual inclinations*; compare (C.9), (C.12), (C.15), In fact, choosing Z_j^* as in (C.20), (C.21) implies that $\imath_{\min} \sim \rho^*$.

Often, however, *the value of the Poincaré integral will be omitted from the notation.*

II.3 The principal part of the perturbation

In this paragraph, we will obtain an "explicit" representation in terms of osculating Poincaré variables of the principal part of the perturbation $\mathcal{H}_1^{\mathrm{princ}}$.

Let, as above, $w_i := \theta_i + g_i + v_i$ be the *true longitude* of the planet P_i and $\hat{S}$ the angle between the planets P_1 and P_2 (see Figure 4 below).

Let $\hat{\imath}$ be the mutual inclination of the planets and $w_i^* = w_i - \theta_i$. Notice that, by (C.44), $w_2^* - w_1^* = w_2 - w_1 + \pi$. By elementary trigonometry

$$\cos \hat{S} = -\cos w_1^* \cos w_2^* - \sin w_1^* \sin w_2^* \cos \hat{\imath} \, . \tag{C.48}$$

[30]Compare (1.4) in Theorem 1.1.

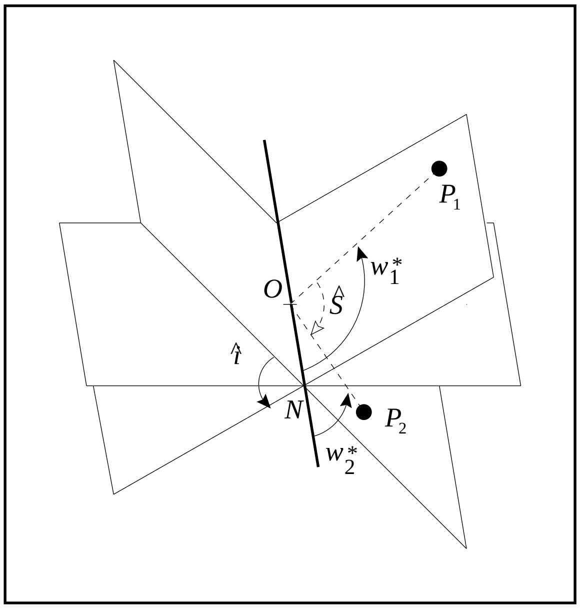

Figure 4

Denote the angular momentum of the planet P_i by $C^{(i)}$, so that $C = C^{(1)} + C^{(2)}$. Then, using (C.11), we infer that

$$\begin{aligned} |C|^2 &= |C^{(1)}|^2 + |C^{(2)}|^2 + 2\,|C^{(1)}|\,|C^{(2)}|\,\cos\hat{\imath} = \\ &= \Lambda_1^2\,(1-e_1^2) + \Lambda_2^2\,(1-e_2^2) + 2\Lambda_1\Lambda_2\sqrt{(1-e_1^2)(1-e_2^2)}\,\cos\hat{\imath}\,. \end{aligned} \tag{C.49}$$

Then, using (C.49) and (C.10), we can express $\cos\hat{\imath}$ in terms of the osculating Poincaré variables (and the initial value parameter C). Explicitly:

$$\cos\hat{\imath} = \frac{|C|^2 - \sum_{i=1}^{2}\left(\Lambda_i^* - H_i^*\right)^2}{2\,\prod_{i=1}^{2}(\Lambda_i^* - H_i^*)} = \frac{|C|^2 - \Lambda_1^{*2} + \Lambda_2^{*2}}{2\Lambda_1^*\Lambda_2^*} + O(e_1^2) + O(e_2^2)\ . \tag{C.50}$$

Also, as proven in §**I.3**, $w_i^* = w_i^*(\Lambda_i^*, \lambda_i^*, \eta_i^*, \xi_i^*)$. Therefore, by (C.48), the expression of $\cos\hat{S}$ in terms of osculating Poincaré variables follows.

Also,

$$|r^{(1)} - r^{(2)}|^2 = |r^{(1)}|^2 + |r^{(2)}|^2 - 2\,|r^{(1)}|\,|r^{(2)}|\,\cos\hat{S}\,.$$

Thus, from the definition of $x^{(i)}$, we have

$$\frac{m_0^2}{|x^{(1)} - x^{(2)}|} = \frac{1}{|r^{(1)} - r^{(2)}|} = \frac{1}{\sqrt{|r^{(1)}|^2 + |r^{(2)}|^2 - 2\,|r^{(1)}|\,|r^{(2)}|\,\cos\hat{S}}} \tag{C.51}$$

Since we expressed $\cos\hat{S}$ and $|r^{(i)}|$ in osculating Poincaré variables, the expression of the principal part of the perturbation in osculating Poincaré variables $(\Lambda^*, \eta^*, \lambda^*, \xi^*)$ readily follows.

Also, one can see, by (C.18) and a 180 degree rotation of the perihelia, that the λ^*-average of $\mathcal{H}_1^{\rm princ}$ in (C.47) is even in (η^*, ξ^*):

Proposition C.4 *Let*

$$f_1^*(\Lambda^*, \eta^*, \xi^*) := \frac{1}{\varepsilon} \int_{\mathbb{T}^2} \mathcal{H}_1^{\rm princ}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*)\ d\lambda^* .$$

Then, $f_1^*(\Lambda^*, -\eta^*, -\xi^*) = f_1^*(\Lambda^*, \eta^*, \xi^*)$.

The rescaling by $\frac{1}{\varepsilon}$ is made so that f_1^* is a (real-analytic) uniformly bounded (by an order-one constant) function.

Proof The system is invariant under the map $\mathcal{R}(\Lambda^*, \eta^*, \lambda^*, \xi^*) = (\Lambda^*, -\eta^*, \lambda^* + \pi, -\xi^*)$ (usually referred to as "space inversion"). Thus, the thesis follows by observing that $\int_{\mathbb{T}^2} \mathcal{H}^* \circ \mathcal{R}\ d\lambda^* = \int_{\mathbb{T}^2} \mathcal{H}^*\ d\lambda^*$, and making use of (C.42). ∎

Thus, dropping the explicit dependence upon the Poincaré integrals, the perturbation function εF (see (1.4) and (C.46)), has the form

$$F(\Lambda^*, \eta^*, \lambda^*, \xi^*) =: f_1^*(\Lambda^*, \eta^*, \xi^*) + f_2^*(\Lambda^*, \eta^*, \lambda^*, \xi^*) , \tag{C.52}$$

where

$$\frac{1}{4\pi^2} \int_{\mathbb{T}^2} f_2^*\, d\lambda^* = 0 \tag{C.53}$$

and f_1^* even in (η^*, ξ^*) (Proposition C.4). Thus, we can split f_1^* as

$$f_1^* = f_{1,0}^*(\Lambda^*) + f_{1,2}^*(\Lambda^*, \eta^*, \xi^*) + \tilde{f}_1^*(\Lambda^*, \eta^*, \xi^*) ,$$

with $f_{1,2}^*$ quadratic in (η^*, ξ^*) and, uniformly in Λ^* (complex),

$$|\tilde{f}_1^*(\Lambda^*, \eta^*, \xi^*)| \le \text{const}\, |(\eta^*, \xi^*)|^4 .$$

II.4 Symplectic diagonalization and conclusion

The final step consists in showing that $f_{1,2}^*$ is *a positive definite* quadratic form and in finding "explicitly" the (purely imaginary) eigenvalues of

$$Q := S_4 \partial^2_{\eta^*, \xi^*} f_{1,2}^*(\Lambda^*, 0, 0) \tag{C.54}$$

(S_4 being the standard (4×4)-symplectic matrix). This calculation has been performed in detail in [R95] (§ 3.4, § 3.5), which, here, we follow[31].

[31] Notational remark: the Hamiltonian $\mathcal{H}^*$ here differs from the one in [R95] for a scaling factor of size $\varepsilon^3 m_0^5$.

Let us introduce some notations (which we shall keep similar to those used in [R95]): let α denote the ratio of the planetary semi-axis, $\alpha := a_1/a_2$, (recall that by our assumptions $\alpha \le \alpha_{\max} < 1$); let

$$h := \frac{m_1}{m_2}\sqrt{\frac{m_0+m_2}{m_0+m_1}}$$

(which is close to the planetary mass ratio for small ε and is a quantity of order one); let

$$D := \frac{(\Lambda_1^*+\Lambda_2^*)^2 - |C|^2}{\Lambda_1^*\Lambda_2^*} = \hat{\imath}^2 + O(\hat{\imath}^4) + O(e_1^2) + O(e_2^2)\ ,$$

(where the asymptotic evaluation is consequence of (C.50)); let $\pm\sqrt{-1}(2\varepsilon\bar{\Omega}_j)$ denote the eigenvalues of the matrix Q in (C.54) and let

$$\mathcal{L}_i := \frac{\Lambda_1}{c}\bar{\Omega}_i\ , \qquad c := -\frac{2m_1m_2^3}{\varepsilon^4 m_0^3(m_0+m_2)\,\Lambda_2^2}\ , \tag{C.55}$$

(notice that c is a quantity of order one). Finally, recall the well known definition of Laplace coefficients for $0 \le \alpha < 1$:

$$b_s^{(k)}(\alpha) = 2\,\frac{s(s+1)\ldots(s+k-1)}{k!}\,\alpha^k\,\Big(1+\sum_{\ell\ge 0}\big(\frac{s\ldots(s+\ell)}{\ell!}\,\frac{(s+k)\ldots(s+k+\ell)}{(k+1)\ldots(k+1+\ell)}\,\alpha^{2(\ell+1)}\big)\Big)\ .$$

After the above preparation, (and after quite a bit of algebra), one finds that (compare with [R95]):

$$\mathcal{L}_i = \mathcal{L}_i^{(0)}(\alpha) + O(D)\,,$$

($O(D)$ being a *real* quantity of order D) with

$$\begin{aligned}
\mathcal{L}_1^{(0)}(\alpha) &= \frac{\alpha}{32}\left[3(1+h\sqrt{\alpha})b_{3/2}^{(1)}(\alpha) - \sqrt{\mathcal{D}(\alpha)}\right]\ ,\\
\mathcal{L}_2^{(0)}(\alpha) &= \frac{\alpha}{32}\left[3(1+h\sqrt{\alpha})b_{3/2}^{(1)}(\alpha) + \sqrt{\mathcal{D}(\alpha)}\right]\ ,\\
\mathcal{D}(\alpha) &:= (1-h\sqrt{\alpha})^2\left(b_{3/2}^{(1)}(\alpha)\right)^2 + 4h\sqrt{\alpha}\left(b_{3/2}^{(2)}(\alpha)\right)^2\ ,
\end{aligned} \tag{C.56}$$

showing, in particular, that $\mathcal{L}_i$ and hence $\bar{\Omega}_i$ *are real*. Furthermore, from the definition of Laplace coefficients, there follows that $b_{3/2}^{(2)}(\alpha) < b_{3/2}^{(1)}(\alpha)$ for $\alpha \in (0,1)$. Thus, because of (C.56), there exist suitable (order-one) constants $\bar{D} > 0$ and $\bar{c} > 0$ such that, if $D \le \bar{D}$ (i.e., if the mutual inclination is sufficiently small), then, uniformly in Λ^*,

$$\inf \mathcal{L}_i > \bar{c} > 0\ , \qquad \inf\Big(\mathcal{L}_2 - \mathcal{L}_1\Big) > \bar{c} > 0\,. \tag{C.57}$$

Finally, by a standard argument going back to Weierstrass, one can find, for any *fixed* Λ^*, a linear symplectic transformation

$$\binom{\eta'}{\xi'} \mapsto \binom{\eta^*}{\xi^*} = A(\Lambda^*)\binom{\eta'}{\xi'}\,, \tag{C.58}$$

which sends $f^*_{1,2}$ into

$$f_{1,2} := \frac{c}{\Lambda_1}\Big(\mathcal{L}_1(\Lambda^*)\cdot((\eta'_1)^2+(\xi'_1)^2)+\mathcal{L}_2(\Lambda^*)\cdot((\eta'_2)^2+(\xi'_2)^2)\Big)\,. \tag{C.59}$$

By classical generating function theory, one sees easily that the transformation (C.58) can be extended to a symplectic transformation on the whole phase space $\Psi_{\rm W} : (\Lambda', \eta', \lambda', \xi') \mapsto (\Lambda^*, \eta^*, \lambda^*, \xi^*)$ with

$$\Lambda^* = \Lambda' \qquad \text{and} \qquad \lambda^* = \lambda' + \hat\ell(\Lambda', \eta', \xi')\,, \tag{C.60}$$

for a suitable function $\hat\ell$.

Letting $(I, \varphi) := (\Lambda', \lambda')$, $(p, q) := (\eta', \xi')$, $f_1 := f_1^* \circ \Psi_{\rm W}$, $f_2 := f_2^* \circ \Psi_{\rm W}$,

$$\sigma_0 := {\rho^*}^2\,, \qquad \delta := \beta - {\rho^*}^2\,, \qquad \rho_0 := \rho^*\,, \tag{C.61}$$

one recognizes that the proof of the Delaunay-Poincaré Theorem 1.1 is completed.

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