Low temperature analysis of two dimensional Fermi systems with symmetric Fermi surface

G. Benfatto[*]
e-mail: benfatto@mat.uniroma2.it,

A. Giuliani[†]
e-mail: alessandro.giuliani@roma1.infn.it

V. Mastropietro[*]
e-mail: mastropi@mat.uniroma2.it

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Abstract

We prove the convergence of the perturbative expansion, based on Renormalization Group, of the two point Schwinger function of a system of weakly interacting fermions in $d = 2$, with symmetric Fermi surface and up to exponentially small temperatures, close to the expected onset of superconductivity.

1 Introduction and main results

1.1 Motivations

The unexpected properties of recently discovered materials, showing high-T_c superconductivity and significative deviations from Fermi liquid behavior in their normal phase (*i.e.* above T_c) [VLSAR], provides the main physical motivation for the search of well established results on models for interacting non relativistic fermions, describing the conduction electrons in metals.

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[*] Supported by M.I.U.R. - Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, I-00133, Roma

[†] Dipartimento di Fisica, Università di Roma “La Sapienza”, Piazzale Aldo Moro 5, I-00185, Roma

One can consider such models not only in $d = 3$, but also in $d = 1, 2$, to describe metals so anisotropic that the conduction electrons move essentially on a chain or on a plane. Renormalization Group (RG) methods provide a powerful technique for studying such models. While in $d = 1$ RG methods were applied since long time [So] and many rigorous results up to $T = 0$ were established (see for instance [BGPS], [BoM], [BM] and [GM] for an updated review), in $d > 1$ the application of RG methods is much more recent and started in [BG], [FT]. At the moment RG methods seem unable to get a rigorous control of such models in $d > 1$ up to $T = 0$, for the generic presence of phase transitions (for instance to a superconducting state) at low temperatures (unless such phase transitions are forbidden by a careful choice of the dispersion relation, see [FKT]). On the other hand, RG methods seem well suited to obtain rigorous information on the behavior of $d > 1$ models at temperatures above T_c, and to clarify the microscopic origin of Fermi or non Fermi liquid behavior in the normal phase. One can write, in the weakly interacting case, an expansion for the Schwinger functions based on RG ideas; the finite temperature acts as an infrared cut-off so that each perturbative order is trivially finite; the mathematical non trivial problem is to prove that the expansion is convergent, and it turns out that such problem is more and more difficult as the temperature of the system decreases. Indeed, if λ is the interaction strength, the cancellations due to the anticommutativity properties of fermions allow quite easily to prove convergence of naive perturbation theory for $T \geq |\lambda|^\alpha$, for some constant $\alpha > 0$. On the other hand, the critical temperature in the weak coupling case at which phase transitions are expected is $O(e^{-(a|\lambda|)^{-1}})$ where a is a constant essentially given by the second order contributions [AGD] of the expansion, *i.e.* it is exponentially small and so quite smaller than $|\lambda|^\alpha$, if λ is small enough. In [FMRT] and [DR] the perturbative expansion convergence was proved for the effective potential up to exponentially small temperatures, in the $d = 2$ *Jellium model*, describing fermions in the continuum, with dispersion relation $\varepsilon(\vec{k}) = |\vec{k}|^2/(2m)$ and a rotation invariant weak interaction. One of the main difficulties of the proof is that non perturbative bounds are naturally obtained in coordinate space, while one has to exploit the geometric properties of the Fermi surface (*i.e.* the set of momenta $\vec{k}$ such that $\varepsilon(\vec{k}) = \mu$), which are naturally investigated in momentum space. In [FMRT] and [DR] an expansion based on RG is considered, such that only the relevant (but not the marginal) terms are renormalized; this has the effect that one has convergence for $T \geq e^{-(c|\lambda|)^{-1}}$, with c related to an all order bound, hence much bigger than a. The proof uses in a crucial way the *rotation invariance* of the Jellium model, an hypothesis which is indeed quite unrealistic (it corresponds to completely neglecting the effect of the lattice).

The aim of this paper is to prove convergence of the perturbative expansion for the two point Schwinger function, in the case of an interacting system of fermions in a lattice or in the continuum. Since the interaction modifies the Fermi surface, we write the dispersion relation $\varepsilon_0(\vec{k})$ of the free model in the form $\varepsilon_0(\vec{k}) = \varepsilon(\vec{k}) + \delta\varepsilon(\vec{k})$ and try to choose the *counterterm* $\delta\varepsilon(\vec{k})$, which becomes part of the interaction, as a suitable function of the original interaction, so that the Fermi surface of the interacting system is the set $F = \{\vec{k} : \varepsilon(\vec{k}) = \mu\}$. We can face this problem if $\varepsilon(\vec{k})$ satisfies some conditions, implying mainly that F is a smooth, convex curve, symmetric with respect to the origin.

We prove convergence for weak coupling and up to temperatures $T \geq \exp\{-(c_0|\lambda|)^{-1}\}$, where c_0 depends on a few terms of the first and second order, implying a bound of the critical temperature very close to the expected value. Note that, in order to get this type of result, we consider an expansion in which both the relevant and marginal terms are renormalized.

1.2 The model

There are two main classes of models of interacting fermions, depending wether the Fermi operators space coordinates are continuous or discrete variables. Our analysis deals with both such possibilities, so we give the following definitions.

1) *Continuum models.* In such a case, given a square $[0, L]^2 \in \mathbb{R}^2$, the inverse temperature β and the (large) integer M, we introduce in $\Lambda = [0, L]^2 \times [0, \beta]$ a lattice Λ_M, whose sites are given by the *space-time points* $\mathbf{x} = (x_0, \vec{x}) = (n_0 a_0, n_1 a, n_2 a)$, $a = L/M$, $a_0 = \beta/M$, $n_1, n_2, n_0 = 0, 1, \ldots, M-1$. We also consider the set $\mathcal{D}$ of *space-time momenta* $\mathbf{k} = (k_0, \vec{k})$, with $\vec{k} = \frac{2\pi\vec{n}}{L}$, $\vec{n} \in \mathbb{Z}^2$, $\vec{n} = (n_1, n_2)$, $-M \leq n_i \leq M-1$ and $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$, $n_1, n_2, n_0 = 0, 1, \ldots, M-1$. With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmanian variables $\hat{\psi}^{\varepsilon}_{\mathbf{k},\sigma}$, $\varepsilon, \sigma \in \{+, -\}$. The lattice Λ_M is introduced only for technical reasons so that the number of Grassmanian variables is finite, and eventually the (essentially trivial) limit $M \to \infty$ is taken.

2) *Lattice models.* In such a case, given $[0, L]^2 \in \mathbb{Z}^2$, the inverse temperature β and the (large) integer M, we introduce in $\Lambda = [0, L]^2 \times [0, \beta]$ a lattice Λ_M, whose sites are given by the *space-time points* $\mathbf{x} = (x_0, \vec{x}) = (n_0 a_0, n_1, n_2), a_0 = \beta/M$, $n_1, n_2 = 0, .., L-1$ and $n_0 = 0, 1, \ldots, M-1$; this definition is obtained from the previous one by defining $a = 1$. In such a case $\mathcal{D}$ is a set of space-time momenta $\mathbf{k} = (k_0, \vec{k})$, with $k_0 = \frac{2\pi}{\beta}(n + \frac{1}{2})$, $n \in \mathbb{Z}$, $-M \leq n \leq M-1$; and $\vec{k} = \frac{2\pi\vec{n}}{L}$, $\vec{n} \in \mathbb{Z}^2$, $\vec{n} = (n_1, n_2)$, $-[\frac{L}{2}] \leq$

$n_i \leq [\frac{(L-1)}{2}]$. With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmanian variables $\hat{\psi}^{\varepsilon}_{\mathbf{k},\sigma}$, $\varepsilon, \sigma \in \{+,-\}$.

All the models are defined by introducing a linear functional $P(d\psi)$ on the Grassmanian algebra generated by the variables $\hat{\psi}^{\varepsilon}_{\mathbf{k},\sigma}$, such that

$$\int P(d\psi)\hat{\psi}^{-}_{\mathbf{k}_1,\sigma_1}\hat{\psi}^{+}_{\mathbf{k}_2,\sigma_2} = L^2\beta\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}(\mathbf{k}_1)\ , \tag{1.1}$$
$$\hat{g}(\mathbf{k}) = \frac{\bar{C}_0^{-1}(\vec{k})}{-ik_0 + \varepsilon(\vec{k}) - \mu}\ ,$$

where $\varepsilon(\vec{k})$, the *dispersion relation* of the model, is a function strictly positive for $\vec{k} \neq 0$ and equal to 0 for $\vec{k} = 0$, μ is the *chemical potential* and $\bar{C}_0^{-1}(\vec{k})$ is the *ultraviolet cut-off*. In the case of lattice models we choose $\bar{C}_0^{-1}(\vec{k}) = 1$, while for continuum models the function $\bar{C}_0^{-1}(\vec{k})$ is defined as $\bar{C}_0^{-1}(\vec{k}) = H\left(\varepsilon(\vec{k}) - \mu\right)$ where $H(t) \in C^\infty(\mathbb{R})$ is a smooth function of compact support such that, for example, $H(t) = 1$ for $t < 1$ and $H(t) = 0$ for $t > 2$.

We introduce the propagator in coordinate space:

$$g^{L,\beta}(\mathbf{x}-\mathbf{y}) \equiv \lim_{M\to\infty} \frac{1}{L^2\beta}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\hat{g}(\mathbf{k}) = \lim_{M\to\infty}\int P(d\psi)\psi^{-}_{\mathbf{x},\sigma}\psi^{+}_{\mathbf{y},\sigma}\ , \tag{1.2}$$

where the *Grassmanian field* $\psi^{\varepsilon}_{\mathbf{x}}$ is defined by

$$\psi^{\pm}_{\mathbf{x},\sigma} = \frac{1}{L^2\beta}\sum_{\mathbf{k}\in\mathcal{D}} \hat{\psi}^{\pm}_{\mathbf{k},\sigma} e^{\pm i\mathbf{k}\cdot\mathbf{x}}\ . \tag{1.3}$$

The “Gaussian measure” $P(d\psi)$ has a simple representation in terms of the “Lebesgue Grassmanian measure”

$$D\psi = \prod_{\substack{\mathbf{k}\in\mathcal{D},\sigma=\pm \\ C_0^{-1}(\vec{k})>0}} d\hat{\psi}^{+}_{\mathbf{k},\sigma}d\hat{\psi}^{-}_{\mathbf{k},\sigma}, \tag{1.4}$$

defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(\hat{\psi}^-, \hat{\psi}^+)$ in the variables $\hat{\psi}^{-}_{\mathbf{k},\sigma}, \hat{\psi}^{+}_{\mathbf{k},\sigma}$, its value is 0, except in the case $Q(\hat{\psi}^-, \hat{\psi}^+) = \prod_{\mathbf{k},\sigma}\hat{\psi}^{-}_{\mathbf{k},\sigma}\hat{\psi}^{+}_{\mathbf{k},\sigma}$, up to a permutation of the variable, in which case its value is 1. We define

$$P(d\psi) = N^{-1}D\psi\cdot\ \exp\left\{-\frac{1}{L^2\beta}\sum_{\substack{\mathbf{k}\in\mathcal{D},\sigma=\pm \\ \bar{C}_0^{-1}(\vec{k})>0}} \bar{C}_0(\vec{k})(-ik_0+\varepsilon(\vec{k})-\mu)\hat{\psi}^{+}_{\mathbf{k},\sigma}\hat{\psi}^{-}_{\mathbf{k},\sigma}\right\}\ , \tag{1.5}$$

with $N = \prod_{\mathbf{k}\in\mathcal{D},\sigma=\pm}[(L^2\beta)^{-1}(-ik_0+\varepsilon(\vec{k})-\mu)\bar{C}_0(\vec{k})]$.

The *Schwinger functions* are defined by the following *Grassmanian functional integral*

$$S(\mathbf{x}_1,\varepsilon_1,\sigma_1;..,\mathbf{x}_n,\varepsilon_n,\sigma_n) = \lim_{L\to\infty}\lim_{M\to\infty}\frac{\int P(d\psi)e^{-\mathcal{V}(\psi)-\mathcal{N}(\psi)}\psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1}...\psi^{\varepsilon_n,\sigma_n}_{\mathbf{x}_n}}{\int P(d\psi)e^{-\mathcal{V}(\psi)-\mathcal{N}(\psi)}}\ , \tag{1.6}$$

where

$$\mathcal{N}(\psi) = \frac{1}{L^2\beta}\sum_{\substack{\mathbf{k}\in\mathcal{D},\sigma=\pm\\ C_0^{-1}(\vec{k})>0}}\hat{\nu}(\vec{k},\lambda)\psi^+_{\mathbf{k},\sigma}\psi^-_{\mathbf{k},\sigma}, \tag{1.7}$$

and, if we use $\int d\mathbf{x}$ and $\delta(x_0-y_0)$ as short-hands for $\sum_{\mathbf{x}\in\Lambda_M}a_0a^2$ and $a_0^{-1}\delta_{x_0,y_0}$,

$$\mathcal{V}(\psi) = \lambda\sum_{\sigma,\sigma'}\int d\mathbf{x}d\mathbf{y}\delta(x_0-y_0)v_{\sigma,\sigma'}(\vec{x}-\vec{y})\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{y},\sigma'}\psi^-_{\mathbf{y},\sigma'}\ , \tag{1.8}$$

$v_{\sigma,\sigma'}(\vec{x})$ being smooth functions such that $\max_{\sigma,\sigma'}\int d\vec{x}(1+|\vec{x}|^2)|v_{\sigma,\sigma'}(\vec{x})|$ is bounded.

Note that $\hat{\nu}(\vec{k},\lambda)$ is related to the counterterm $\delta\varepsilon(\vec{k})$ introduced in §1.1 by the relation $\delta\varepsilon(\vec{k}) = \bar{C}_0^{-1}(\vec{k})\hat{\nu}(\vec{k},\lambda)$

In order to make more precise the model, we have to specify some properties of the dispersion relation. We will assume that $\varepsilon(\vec{k})$ verifies the following properties (whose consequences are discussed in §7. From now on $c, c_1, c_2, \ldots$, will denote suitable positive constants.

1. There exists e_0 such that, for $|e|\le e_0$, $\varepsilon(\vec{k})-\mu = e$ defines a regular C^∞ convex curve $\Sigma(e)$ encircling the origin, which can be represented in polar coordinates as $\vec{p} = u(\theta,e)\vec{e}_r(\theta)$ with $\vec{e}_r(\theta) = (\cos\theta,\sin\theta)$. Note that $e_0<\mu$, since $\varepsilon(\vec{k})>0$ for $\vec{k}\ne 0$ and $\varepsilon(\vec{0})=0$; moreover $u(\theta,e)\ge c>0$ and, if $r(\theta,e)$ is the curvature radius,
$$r(\theta,e)^{-1}\ge c>0\ . \tag{1.9}$$
2. e_0 is chosen so that, if $\vec{k}\in\Sigma(e)$ and $|e|\le e_0$, then $\bar{C}_0^{-1}(\vec{k}) = 1$.
3. If $|e|\le e_0$, then
$$0<c_1\le\vec{\nabla}\varepsilon(\vec{p})\cdot\vec{e}_r(\theta)\le c_2\ . \tag{1.10}$$
4. The following symmetry relation is satisfied
$$\varepsilon(\vec{p}) = \varepsilon(-\vec{p})\ , \tag{1.11}$$
implying that the curves $\Sigma(e)$ are symmetric by reflection with respect to the origin.

We will call $\Sigma_F \equiv \Sigma(0)$ the *Fermi surface* and we will put $u(\theta,0)\vec{e}_r(\theta) = \vec{p}_F(\theta)$ and $u(\theta) \equiv u(\theta,0) = |\vec{p}_F(\theta)|$.

Remarks - The Grassmanian functional integrals (1.6) are equal, in the limit $M \to \infty$, to the Schwinger functions of an *Hamiltonian* model of fermions in two dimensions, expressed in terms of fermionic creation or annihilation operators. Among the dispersion relations which are in the class we are considering is that of the *Hubbard model*, defined in a lattice with local interaction $v_{\sigma,\sigma'}(\vec{x}-\vec{y}) \equiv \delta_{\sigma,-\sigma'}\delta_{\vec{x},\vec{y}}$ (without the counterterm) and $\varepsilon(\vec{k}) = 2 - \cos k_1 - \cos k_2$, and that of the *jellium model*, defined in the continuum with $\varepsilon(\vec{k}) = |\vec{k}|^2/2m$. The index σ is the *spin index*; *in the following it will be not play any role and it will be omitted to shorten the notation.*

We are mainly interested in the two point Schwinger function $S(\mathbf{x}-\mathbf{y}) \equiv S(\mathbf{x},-;\mathbf{y},+)$, with $S(\mathbf{x},-;\mathbf{y},+)$ given by (1.6). For $\lambda = 0$ and $\hat{\nu}(\vec{k},\lambda) = 0$, $S(\mathbf{x}-\mathbf{y})$ is equal to the propagator (1.2), hence its Fourier transform is singular at $k_0 = 0$ (which is not an allowed value at finite temperature) and $\varepsilon(\vec{k}) = \mu$. As we said in §1.1, we want to fix $\hat{\nu}(\vec{k},\lambda)$ so that the location of this singularity does not change for $\lambda \neq 0$; this allows to study the model as a perturbation of the model with $\lambda = 0$.

Our main result is the following theorem

Theorem 1.1 *There exist two positive constants ε and c_0, the last one only depending on first and second order terms in the perturbative expansion, and a continuous function $\hat{\nu}(\vec{k},\lambda) = O(\lambda)$, such that, for all $|\lambda| \leq \varepsilon$ and $T \geq \exp\{-(c_0|\lambda|)^{-1}\}$,*

$$\hat{S}(\mathbf{k}) = \hat{g}(\mathbf{k})(1 + \lambda \hat{S}_1(\mathbf{k})) , \tag{1.12}$$

where $\hat{g}(\mathbf{k})$ is the free propagator at finite β (i.e. it is equal to the Fourier transform of $\lim_{L\to\infty} g^{L;\beta}(\mathbf{x}-\mathbf{y})$, see (1.2)) and $|\hat{S}_1(\mathbf{k})| \leq c$, for some constant c. In the continuum case with $\varepsilon(\vec{k}) = |\vec{k}|^2/2m$ and $v(\vec{r}) = \tilde{v}(|\vec{r}|)$, there exists another constant c_1 such that, if $|\lambda| \leq \varepsilon$ and $T \geq \exp\{-(c_1|\lambda|)^{-1}\}$, $\hat{\nu}(\vec{k},\lambda) = \nu(\lambda)$ is a constant.

This theorem says that the two point Schwinger function of the interacting system is close to the free one, for weak interactions and up to exponentially small temperatures; the condition on the temperature is not technical, as at temperatures low enough phase transitions are expected and a result like (1.12) cannot hold. The theorem is proved by an expansion similar to the one in [BG], in which the *relevant* and the *marginal* interactions are renormalized at any iteration of the Renormalization Group. One writes $\hat{S}(\mathbf{k})$ in terms of a set of *running coupling functions*, which obey recursive equations, the *beta function* of the model. We prove that the expansions of $\hat{S}(\mathbf{k})$ and of the beta function are convergent, if the running coupling functions are small in a suitable norm; the convergence proof is based on the *tree expansion* and the *determinant bounds* used for instance in [BM] and on a suitable generalization to the present problem of the *sector counting lemma* of [FMRT]. Finally we show, by choosing properly the counterterm $\hat{\nu}(\vec{k}, \lambda)$ and by solving iteratively the beta function, that the running coupling functions are small up to temperatures exponentially small $T \geq \exp -(c_0|\lambda|)^{-1}$; c_0 is expressed in terms of a few terms of first and second order, so much closer to the expected value for the onset of superconductivity. Our non perturbative definition of the beta function is interesting by itself, as it could be used to detect the main instabilities of the model at lower temperatures.

In order to complete our program, we should prove that $\hat{\nu}(\vec{k}, \lambda)$ and $\varepsilon(\vec{k})$ can be chosen in a space of functions with the same differentiability properties and that the relation $\varepsilon_0(\vec{k}) = \varepsilon(\vec{k}) + \hat{\nu}(\vec{k}, \lambda)$ can be solved with respect to $\varepsilon(\vec{k})$, given $\varepsilon_0(\vec{k})$ and λ. This would imply that the introduction of the counterterm is only a technical trick, but does not restrict the class of allowed dispersion relations; for example one could consider the Hubbard model away from half-filling.

We did not yet get this result, mainly because our bounds can only show that $\hat{\nu}(\vec{k}, \lambda)$ is a continuous function of compact support, whose Fourier transform is summable, while $\varepsilon(\vec{k})$ has to be a bit more regular than a twice differentiable function. A similar problem appears in [FKT] in which a result similar to Theorem 1.1 above is proved in a class of asymmetric Fermi surfaces (the asymmetry makes an equation like (1.12) valid up to $T = 0$). It is likely that an improvement in the differentiability properties of the counterterm could be obtained by applying the more detailed analysis on the derivatives of the self energy introduced in [DR].

This problem is not present in the Jellium model, where, by using rotational invariance (so that both the free and the interacting Fermi surfaces are circles), one can choose $\hat{\nu}(\vec{k}, \lambda)$ as a constant with respect to $\vec{k}$; this is the last statement in the Theorem, already proved in [DR]. In order to get this result in a simple way, we chose to give up the “close to optimal” upper bound on the critical temperature; in fact the constant c_1 depends on an all order bound, like in [DR]. However, even in this case, our result has some interest, since we get it without being involved in the delicate one particle irreducibility analysis of [DR].

2 Renormalization Group analysis

2.1 The scale decomposition

It is convenient, for clarity reasons, to start by studying the "free energy" of the model, defined as

$$\begin{aligned} & E_{L,\beta} = -\frac{1}{L^2\beta} \log \int P(d\psi^{(\leq 1)}) e^{-\mathcal{V}^{(1)}} , \\ & P(d\psi^{(\leq 1)}) \equiv P(d\psi), \qquad \mathcal{V}^{(1)} \equiv \mathcal{V} + \mathcal{N} . \end{aligned} \tag{2.1}$$

Note that our model has an ultraviolet cut-off in the $\vec{k}$ momentum, but the k_0 variable is unbounded in the limit $M \to \infty$. Hence, it is convenient to decompose the field as $\psi^{(\leq 1)} = \psi^{(+1)} + \psi^{(\leq 0)}$, where $\psi^{(+1)}$ and $\psi^{(\leq 0)}$ are independent fields whose covariances have Fourier transforms with support, respectively, in the *ultraviolet region* and the *infrared region*, defined in the following way.

Item 1) in the list of properties of the dispersion relation given in §1.2 implies that, if $H_0(t)$ is a smooth function of $t \in \mathbb{R}^1$, such that

$$H_0(t) = \begin{cases} 1 & \text{if } t < e_0/\gamma \ , \\ 0 & \text{if } t > e_0 \ , \end{cases} \tag{2.2}$$

$\gamma > 1$ being a parameter to be fixed below, then, since $\bar{C}_0^{-1}(\vec{k}) = 1$ if $|\varepsilon(\vec{k}) - \mu| \leq e_0$,

$$\begin{aligned} \bar{C}_0^{-1}(\vec{k}) &= C_0^{-1}(\mathbf{k}) + f_1(\mathbf{k}) \ , \\ C_0^{-1}(\mathbf{k}) &= H_0\left[\sqrt{k_0^2 + [\varepsilon(\vec{k}) - \mu]^2}\right] \ , \\ f_1(\mathbf{k}) &= \bar{C}_0^{-1}(\vec{k}) \left\{ 1 - H_0\left[\sqrt{k_0^2 + [\varepsilon(\vec{k}) - \mu]^2}\right]\right\} \ . \end{aligned} \tag{2.3}$$

The covariances $g^{(+1)}$ and $g^{(\leq 0)}$ of the fields $\psi^{(+1)}$ and $\psi^{(\leq 0)}$ are defined as in (1.8), with $f_1(\mathbf{k})$ and $C_0^{-1}(\mathbf{k})$ in place of $\bar{C}_0^{-1}(\vec{k})$.

If we perform the integration of the ultraviolet field variables $\psi^{(+1)}$, we get

$$e^{-L^2\beta E_{L,\beta}} = e^{-L^2\beta E_0} \int P(d\psi^{(\leq 0)}) e^{-\mathcal{V}^0(\psi^{\leq 0})} \ , \tag{2.4}$$

where $\mathcal{V}^{(0)}(\psi^{(\leq 0)})$, the *effective potential on scale* 0, is given by an expression like (2.12) below and E_0 is defined by the condition $\mathcal{V}^{(0)}(0) = 0$.

The analysis of the ultraviolet integration is far easier than the infrared one. It can be done by the same procedure applied below for the infrared problem, by making a multiscale expansion of the u.v. propagator $g^{(1)}(\mathbf{x})$, based on an obvious smooth partition of the interval $\{|k_0| > 1\}$. In this way, one can build a tree expansion for $\mathcal{V}^{(0)}$, with endpoints on scale $M > 0$, similar to the infrared tree expansion, to be described below, see Fig. 1 and following items 1)-6). It is easy to see that there is no relevant or marginal term on any scale > 0, except those which are obtained by contracting two fields associated with the same space-time point in a vertex located between an endpoint and the first non-trivial vertex following it (*i.e.* the *tadpoles*). However the sum over the scales of this type of terms, which is not absolutely convergent for $M \to +\infty$, can be controlled by using the explicit expression of the single scale propagator, since there is indeed no divergence, but only a discontinuity at $x_0 = 0$ for $\vec{x} = 0$. We shall omit the details, which are of the same type of those used below for the infrared part of the model.

Let us now consider the infrared integration; it will be performed, as usual, by an iterative procedure. Note first that we can write

$$H_0(t) = \sum_{h=-\infty}^{0} \tilde{f}_h(t) \ , \tag{2.5}$$

where $\tilde{f}_h(t) = H_0(\gamma^{-h}t) - H_0(\gamma^{-h+1}t)$ is a smooth function, with support in the interval $[\gamma^{h-2}e_0, \gamma^h e_0]$, and $\gamma > 1$ is the scaling parameter. In order to simplify some calculations, we will put in the following $\gamma = 4$, but this choice is not essential.

Since $|k_0| \geq \pi/\beta$, $\forall \mathbf{k} \in \mathcal{D}$, if we define

$$h_\beta = \max\{h \leq 0 : \gamma^{h-1}e_0 < \pi/\beta\} \ , \tag{2.6}$$

we have the identity

$$C_0^{-1}(\mathbf{k}) = \sum_{h=h_\beta}^{0} f_h(\mathbf{k}) \quad , \quad f_h(\mathbf{k}) \equiv \tilde{f}_h\left(\sqrt{k_0^2 + [\varepsilon(\vec{k}) - \mu]^2}\right) \ . \tag{2.7}$$

We associate with the decomposition (2.7) a sequence of constants E_h, $h = h_\beta, ..., 0$, and a sequence of *effective potentials* $\mathcal{V}^{(h)}(\psi)$ such that $\mathcal{V}^{(h)}(0) = 0$ and

$$e^{-L^2\beta E_{L,\beta}} = e^{-L^2\beta E_h} \int P(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})} \ , \tag{2.8}$$

where $P(d\psi^{\leq h})$ is the fermionic integration with propagator

$$g^{(\leq h)}(\mathbf{x}) = \frac{1}{L^2\beta} \sum_{\mathbf{k}\in\mathcal{D}} \frac{C_h^{-1}(\mathbf{k})}{-ik_0 + \varepsilon(\vec{k}) - \mu} e^{-i\mathbf{k}\mathbf{x}} \ , \tag{2.9}$$

with

$$C_h^{-1}(\mathbf{k}) = \sum_{j=h_\beta}^{h} f_j(\mathbf{k}) = C_{h-1}^{-1}(\mathbf{k}) + f_h(\mathbf{k}) \ . \tag{2.10}$$

The definition (2.8) implies that

$$E_{L,\beta} = E_{h_\beta} - \frac{1}{L^2\beta} \log \int P(d\psi^{(\le h_\beta)}) e^{-\mathcal{V}^{(h_\beta)}(\psi^{(\le h_\beta)})} \; . \tag{2.11}$$

If we neglect the spin indices and we put $\varepsilon_1 = \ldots = \varepsilon_n = +$, $\varepsilon_{n+1} = \ldots = \varepsilon_{2n} = -$, we can write the effective potentials in the form

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=1}^{\infty} \int d\mathbf{x}_1 ... d\mathbf{x}_{2n} \left[\prod_{i=1}^{2n} \psi_{\mathbf{x}_i}^{(\le h)\varepsilon_i} \right] W_{2n}^{(h)}(\mathbf{x}_1, ..., \mathbf{x}_{2n}) \; . \tag{2.12}$$

Remark - The terms in the r.h.s. of (2.12) are well defined at finite M and L, as elements of a finite Grassmanian algebra, but have only a formal meaning for $M = L = \infty$. However, one can prove that the kernels, as well as $E_{L,\beta}$, have well defined limits as M and L go to infinity. Such result is achieved by studying a suitable perturbative expansion of these quantities and by proving that they are uniformly (in M and L) convergent and, in the case of the kernels, that they have fast decaying properties in the $\mathbf{x}$ variables; see [BM] for a complete analysis of this type in the one dimensional case. However, since this procedure is cumbersome and difficult to describe rigorously without making obscure the main ideas, which have nothing to do with the details related with the finite values of M and L, we shall discuss in the following only the formal limit of our expansions and we shall prove that the kernels as well as the free energy constants E_h are well defined. For similar reasons, we shall also consider k_0 as a continuous variable and we shall take into account the essential infrared cut-off related with the finite temperature value, by preserving the definition (2.10) of the cut-off functions. This means, in particular that, from now on

$$\frac{1}{L^2\beta} \sum_{\mathbf{k} \in \mathcal{D}} \to \frac{1}{(2\pi)^3} \int_{\mathcal{D}} d\mathbf{k} \; . \tag{2.13}$$

Moreover, we shall suppose that the space coordinates are continuous variables, both in the continuum and lattice models. This means that, from now on, $\int d\mathbf{x}$ will denote the integral over $\mathbb{R}^3$. Finally, we shall still use the symbol $L^2\beta$ to denote the formally infinite space-time volume in the extensive quantities like $L^2\beta E_h$.

2.2 The localization procedure

Let us now describe our expansion, which is produced by using an inductive procedure. First of all, we define an $\mathcal{L}$ operator acting on the kernels in the following way:

1. $\mathcal{L}W_{2n}^{(h)} = 0$ if $n \geq 3$.

2. If $n = 2$ and we put $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_4)$, $\mathbf{x}_i = (x_{i,0}, \vec{x}_i)$, $\tilde{\mathbf{x}}_i = (\tilde{x}_{i,0}, \vec{x}_i)$, $\delta(\underline{x}_0) = \delta(x_{1,0} - x_{2,0})\delta(x_{1,0} - x_{3,0})\delta(x_{1,0} - x_{4,0})$
$$\mathcal{L}W_4^{(h)}(\underline{\mathbf{x}}) = \delta(\underline{x}_0) \int d(\underline{\tilde{x}}_0 \backslash \tilde{x}_{1,0}) W_4^{(h)}(\underline{\tilde{\mathbf{x}}}) \; . \tag{2.14}$$
Note that, because of translation invariance, this definition is independent of the choice of the *localization point*, that is the point whose time coordinate is not integrated ($\mathbf{x}_1$ in (2.14)).

3. If $n = 1$ and we put (by using translation invariance) $W_2^{(h)}(\mathbf{x}_1, \mathbf{x}_2) = \tilde{W}_2^{(h)}(\mathbf{x}_1 - \mathbf{x}_2)$,
$$\begin{aligned}\mathcal{L}W_2^{(h)}(\mathbf{x}_1, \mathbf{x}_2) &= \delta(x_{1,0} - x_{2,0}) \int dt \; \tilde{W}_2^{(h)}(t, \vec{x}_1 - \vec{x}_2) + \\ &+ \partial_{x_{2,0}} \delta(x_{1,0} - x_{2,0}) \int dt \; t \; \tilde{W}_2^{(h)}(t, \vec{x}_1 - \vec{x}_2) \; . \end{aligned} \tag{2.15}$$

The definition of $\mathcal{L}$ is extended by linearity to $\mathcal{V}^{(h)}$, so that we can write
$$\begin{aligned}\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) &= \int d\mathbf{x}_1 d\mathbf{x}_2 \delta(x_{1,0} - x_{2,0}) \gamma^h \nu_h(\vec{x}_1 - \vec{x}_2) \psi_{\mathbf{x}_1}^{(\leq h)+} \psi_{\mathbf{x}_2}^{(\leq h)-} + \\ &+ \int d\mathbf{x}_1 d\mathbf{x}_2 \delta(x_{1,0} - x_{2,0}) z_h(\vec{x}_1 - \vec{x}_2) \psi_{\mathbf{x}_1}^{(\leq h)+} \partial_{x_{2,0}} \psi_{\mathbf{x}_2}^{(\leq h)-} + \\ &+ \int d\underline{\mathbf{x}} \lambda_h(\underline{\vec{x}}) \delta(\underline{x}_0) \psi_{\mathbf{x}_1}^{(\leq h)+} \psi_{\mathbf{x}_2}^{(\leq h)+} \psi_{\mathbf{x}_3}^{(\leq h)-} \psi_{\mathbf{x}_4}^{(\leq h)-} \; , \end{aligned} \tag{2.16}$$
where $\lambda_h(\underline{\vec{x}}) = \int d(\underline{x}_0 \backslash x_{1,0}) W_4^{(h)}(\underline{\mathbf{x}})$, $\gamma^h \nu_h(\vec{x}_1 - \vec{x}_2) = \int dt \tilde{W}_2^{(h)}(t, \vec{x}_1 - \vec{x}_2)$ and $z_h(\vec{x}_1 - \vec{x}_2) = -\int dt \; t \; \tilde{W}_2^{(h)}(t, \vec{x}_1 - \vec{x}_2)$.

Note that, in the term containing $z_h(\vec{x}_1 - \vec{x}_2)$, we can substitute $\psi_{\mathbf{x}_1}^{(\leq h)+} \partial_{x_{2,0}} \psi_{\mathbf{x}_2}^{(\leq h)-}$ with $-[\partial_{x_{1,0}} \psi_{\mathbf{x}_1}^{(\leq h)+}] \psi_{\mathbf{x}_2}^{(\leq h)-}$.

The functions λ_h, ν_h and z_h will be called the *running coupling functions of scale h* or simply the coupling functions.

It is useful to consider also the representation of $\mathcal{L}V^{(h)}(\psi^{(\leq h)})$ in terms of the Fourier transforms, defined so that, for example,
$$W_2^{(h)}(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)} \hat{W}_2^{(h)}(\mathbf{k}) \; , \tag{2.17}$$
$$W_4^{(h)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int \prod_{i=1}^{3} \left[\frac{d\mathbf{k}_i}{(2\pi)^3} e^{-i\varepsilon_i \mathbf{k}_i(\mathbf{x}_i - \mathbf{x}_4)} \right] \hat{W}_4^{(h)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \; . \tag{2.18}$$

We can write

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) &= \int \frac{d\mathbf{k}}{(2\pi)^3}[\gamma^h\hat{\nu}_h(\vec{k}) - ik_0\hat{z}_h(\vec{k})]\psi^{(\le h)+}_{\mathbf{k}}\psi^{(\le h)-}_{\mathbf{k}} + \\ &+ \int\prod_{i=1}^{4}\frac{d\mathbf{k}_i}{(2\pi)^3}\psi^{(\le h)+}_{\mathbf{k}_1}\psi^{(\le h)+}_{\mathbf{k}_2}\psi^{(\le h)-}_{\mathbf{k}_3}\psi^{(\le h)-}_{\mathbf{k}_4}\;\cdot \qquad (2.19)\\ &\cdot\;\hat{\lambda}_h(\vec{k}_1,\vec{k}_2,\vec{k}_3)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\;,\end{aligned}$$

where $\hat{\lambda}_h(\vec{k}_1,\vec{k}_2,\vec{k}_3) = \hat{W}^{(h)}_4((0,\vec{k}_1),(0,\vec{k}_2),(0,\vec{k}_3))$, $\gamma^h\hat{\nu}_h(\vec{k}) = \hat{W}^{(h)}_2(0,\vec{k})$, $\hat{z}_h(\vec{k}) = i\partial_{k_0}\hat{W}^{(h)}_2(0,\vec{k})$.

We also define $\mathcal{R}\equiv 1-\mathcal{L}$; by using (2.15), we get:

$$\begin{aligned}\mathcal{R}W^{(h)}_2(\mathbf{x}_1,\mathbf{x}_2) &= \tilde{W}^{(h)}_2(\mathbf{x}_1-\mathbf{x}_2) - \delta(x_{1,0}-x_{2,0})\bar{W}^{(h)}_2(0,\vec{x}_1-\vec{x}_2) - \\ &- i\partial_{x_{1,0}}\delta(x_{1,0}-x_{2,0})\partial_{k_0}\bar{W}^{(h)}_2(0,\vec{x}_1-\vec{x}_2)\;, \qquad (2.20)\end{aligned}$$

where $\bar{W}^{(h)}_2(k_0,\vec{x}) = \int dt\, e^{ik_0t}\tilde{W}^{(h)}_2(t,\vec{x})$. Furthermore

$$\mathcal{R}W^{(h)}_4(\underline{\mathbf{x}}) = W^{(h)}_4(\underline{\mathbf{x}}) - \delta(\underline{x}_0)\bar{W}^{(h)}_4(\underline{0},\underline{\vec{x}})\;, \qquad (2.21)$$

where $\bar{W}^{(h)}_4(\underline{k}_0,\underline{\vec{x}})$ is the Fourier transform of $W^{(h)}_4(\underline{\mathbf{x}})$ with respect to the time coordinates.

2.3 The sector decomposition

We now further decompose the field $\psi^{(\le h)}$, by slicing the support of $C_h^{-1}(\mathbf{k})$ as in [FMRT]. Let $H_2(t)$ be a smooth function on the interval $[-1,+1]$, such that

$$H_2(t) = \begin{cases} 1 & \text{if } |t|<1/4 \\ 0 & \text{if } |t|>3/4; \end{cases} \quad , \quad H_2(t)+H_2(1-t) = 1 \;\text{ if }\; 1/4<t<3/4, \qquad (2.22)$$

and let us define, if ω is an integer in the set $O_h\equiv\{0,1,\dots,\gamma^{-(h-1)/2}-1\}$ (recall that $\gamma=4$) and $h\le 0$,

$$\bar{\zeta}_{h,\omega}(t) = H_2\left(\frac{\gamma^{-\frac{h}{2}}}{\pi}(t-\theta_{h,\omega})\right) \quad , \quad \theta_{h,\omega} = \pi(\omega+\frac{1}{2})\gamma^{\frac{h}{2}}\;. \qquad (2.23)$$

It is easy to see that $\bar{\zeta}_{h,\omega}(t)$ can be extended to the real axis as a periodic function of period 2π, that we can use to define a smooth function on the one-dimensional torus $\mathbb{T}^1$, to be called $\zeta_{h,\omega}(\theta)$; moreover

$$\sum_{\omega\in O_h}\zeta_{h,\omega}(\theta) = 1 \quad , \quad \forall\theta\in\mathbb{T}^1\;. \qquad (2.24)$$

On the other hand, the properties of $\varepsilon(\vec{k})$ assumed in §1.2 imply that, if $C_h^{-1}(\mathbf{k}) \neq 0$, $\vec{k} = u(\theta, e)\vec{e}_r(\theta)$ with $e = \varepsilon(\vec{k}) - \mu$. Hence, we can write

$$\psi_{\mathbf{x}}^{(\leq h)\pm} \equiv \sum_{\omega \in O_h} e^{\pm i \vec{p}_F(\theta_{h,\omega})\vec{x}} \psi_{\mathbf{x},\omega}^{(\leq h)\pm} \quad , \quad P(d\psi^{(\leq h)}) = \prod_{\omega \in O_h} P(d\psi_\omega^{(\leq h)}) \; , \tag{2.25}$$

where $P(d\psi_\omega^{(\leq h)})$ is the Grassmanian integration with propagator

$$g_\omega^{(\leq h)}(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\left[\mathbf{k}(\mathbf{x}-\mathbf{y}) - \vec{p}_F(\theta_{h,\omega})(\vec{x}-\vec{y})\right]} \frac{C_h^{-1}(\mathbf{k})\zeta_{h,\omega}(\theta)}{-ik_0 + \varepsilon(\vec{k}) - \mu} \; . \tag{2.26}$$

If we insert the l.h.s. of (2.25) in (2.12), we get

$$\begin{aligned} \mathcal{V}^{(h)} \left(\sum_{\omega \in O_h} e^{\varepsilon i \vec{p}_F(\theta_{h,\omega})\vec{x}} \psi_\omega^{(\leq h))\varepsilon} \right) &= \sum_{n=1}^{\infty} \sum_{\omega_1,\ldots,\omega_{2n} \in O_h} \cdot \\ \cdot \int d\mathbf{x}_1 \ldots d\mathbf{x}_{2n} \left[\prod_{i=1}^{2n} e^{\varepsilon_i i \vec{p}_F(\theta_{\omega_{h,i}})\vec{x}_i} \psi_{\mathbf{x}_i,\omega_i}^{(\leq h)\varepsilon_i} \right] & W_{2n}^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}). \end{aligned} \tag{2.27}$$

By using (2.10), we can write

$$\begin{aligned} &\int \prod_{\omega \in O_h} P(d\psi_\omega^{(\leq h)}) e^{-(\mathcal{L}+\mathcal{R})\mathcal{V}^{(h)}\left(\sum_{\omega \in O_h} e^{\varepsilon i \vec{p}_F(\theta_{h,\omega})\vec{x}} \psi_\omega^{(\leq h)\varepsilon}\right)} = \\ &= \int P(d\psi^{(\leq h-1)}) \int \prod_{\omega \in O_h} P(d\psi_\omega^{(h)}) \cdot \\ &\cdot e^{-(\mathcal{L}+\mathcal{R})\mathcal{V}^{(h)}\left(\psi_{\mathbf{x}}^{(\leq h-1)\varepsilon} + \sum_{\omega \in O_h} e^{\varepsilon i \vec{p}_F(\theta_{h,\omega})\vec{x}} \psi_{\mathbf{x},\omega}^{(h)\varepsilon}\right)} \; , \end{aligned} \tag{2.28}$$

where $P(d\psi_\omega^{(h)})$ is the integration with propagator

$$g_\omega^{(h)}(\mathbf{x}) \equiv \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\left(\mathbf{k}\mathbf{x} - \vec{p}_F(\theta_{h,\omega})\vec{x}\right)} \frac{F_{h,\omega}(\mathbf{k})}{-ik_0 + \varepsilon(\vec{k}) - \mu} \; , \tag{2.29}$$

$$F_{h,\omega}(\mathbf{k}) = f_h(\mathbf{k})\zeta_{h,\omega}(\theta) \; . \tag{2.30}$$

The support of $F_{h,\omega}(\mathbf{k})$ will be called the *sector of scale h and sector index ω.*

In order to compute the asymptotic behavior of $g_\omega^{(h)}(\mathbf{x})$ it is convenient to introduce a coordinate frame adapted to the Fermi surface in the point $\vec{p}_F(\theta_{h,\omega})$. By using the definitions of §1.2 and putting $\vec{e}_t(\theta) = (-\sin\theta, \cos\theta)$, we define

$$\begin{aligned} \vec{\tau}(\theta) &= \frac{d\vec{p}_F(\theta)}{d\theta} \left| \frac{d\vec{p}_F(\theta)}{d\theta} \right|^{-1} = \frac{u'(\theta)\vec{e}_r(\theta) + u(\theta)\vec{e}_t(\theta)}{\sqrt{u'(\theta)^2 + u(\theta)^2}} \; , \\ \vec{n}(\theta) &= \frac{u(\theta)\vec{e}_r(\theta) - u'(\theta)\vec{e}_t(\theta)}{\sqrt{u'(\theta)^2 + u(\theta)^2}} \; . \end{aligned} \tag{2.31}$$

Moreover, given any $\mathbf{k}$ belonging to the support of $F_{h,\omega}(\mathbf{k})$, we put

$$\vec{k} = \vec{p}_F(\theta_{h,\omega}) + k'_1 \vec{n}(\theta_{h,\omega}) + k'_2 \vec{\tau}(\theta_{h,\omega}) = \vec{p}_F(\theta_{h,\omega}) + \vec{k}' \ ; \tag{2.32}$$

it is easy to verify that $|k'_1| \le C\gamma^h$, $|k'_2| \le C\gamma^{\frac{h}{2}}$, see Lemma 7.3 in §7 for details. By using (2.32), we can rewrite (2.29) as

$$g^{(h)}_\omega(\mathbf{x}) \equiv \frac{1}{(2\pi)^3} \int dk_0 d\vec{k}' e^{-i\left(k_0 x_0 + \vec{k}'\vec{x}\right)} \frac{F_{h,\omega}(k_0, \vec{p}_F(\theta_{h,\omega}) + \vec{k}')}{-ik_0 + \varepsilon(\vec{p}_F(\theta_{h,\omega}) + \vec{k}') - \mu} \ . \tag{2.33}$$

Let us now put

$$\vec{x} = x'_1 \vec{n}(\theta_{h,\omega}) + x'_2 \vec{\tau}(\theta_{h,\omega}) \ ; \tag{2.34}$$

the following lemma gives a bound on the asymptotic behavior of $g^{(h)}_\omega(\mathbf{x})$, which is very important in our analysis, as in [FMRT]. It will be proved in §7.

Lemma 2.1 *Given the integers $N, m, n_0, n_1, n_2 \ge 0$, with $m = n_0 + n_1 + n_2$, there exists a constant $C_{N,m}$ such that*

$$|\partial^{n_0}_{x_0} \partial^{n_1}_{x'_1} \partial^{n_2}_{x'_2} g^{(h)}_\omega(\mathbf{x})| \le \frac{C_{N,m} \gamma^{\frac{3}{2}h} \gamma^{(n_0 + n_1 + \frac{1}{2}n_2)h}}{1 + (\gamma^h |x_0| + \gamma^h |x'_1| + \gamma^{\frac{1}{2}h} |x'_2|)^N} \ . \tag{2.35}$$

Remark Lemma 2.1 holds also for non C^∞ Fermi surfaces: it is sufficient the condition that the derivatives of $\varepsilon(\vec{k})$ diverge "not too fast" (*i.e.* that $\partial^n / \partial k'^{n_1}_1 \partial k'^{n_2}_2 [\varepsilon(\vec{k}) - \mu] = O(\gamma^{-h(n_1 + \frac{1}{2}n_2 - 1)})$).

2.4 The tree expansion

Our expansion of $\mathcal{V}^{(h)}$, $0 \ge h \ge h_\beta$ is obtained by integrating iteratively the field variables of scale $j \ge h+1$ and sector index $\omega = 1, \ldots, \gamma^{-h/2}$ and by applying at each step the *localization procedure* described above, which has the purpose of summing together the relevant contributions of the same type. The result can be expressed in terms of a *tree expansion*, similar to that described, for example, in [BM].

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \ge 1$ points, the *endpoints* of the *unlabeled tree* (see Fig. 1), so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

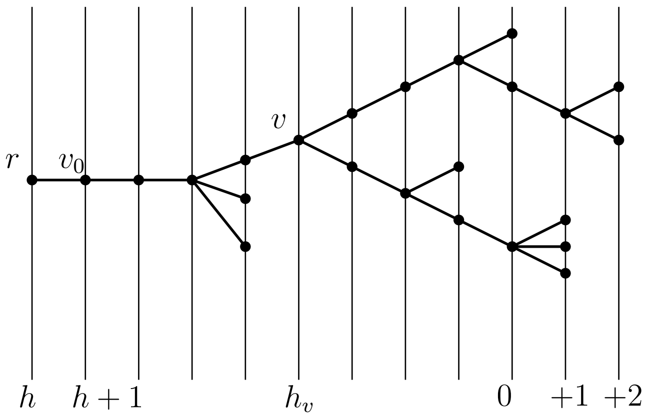

Figure 1: A possible tree of the expansion for the effective potentials.

Two unlabelled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n.

We shall consider also the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \le 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labelled trees with n endpoints. Moreover, we introduce a family of vertical lines, labelled by an integer taking values in $[h,2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h+1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices.* The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint (see above); its scale is $h+1$.

Finally, if there is only one endpoint, its scale must be equal to $h+2$.

3) With each endpoint v of scale $h_v = +2$ we associate one of the two contributions to $\mathcal{V}^{(1)}(\psi^{(\le 1)})$, and a set $\mathbf{x}_v$ of space-time points (the two corresponding integration variables); we shall say that the endpoint is of type λ or ν, respectively. With each endpoint v of scale $h_v \le 1$ we associate one

of the three terms appearing in (2.16) and the set $\mathbf{x}_v$ of the corresponding integration variables; we shall say that the endpoint is of type ν, z or λ, respectively.

Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

Moreover, we impose the constraint that, if v is an endpoint, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

4) If v is not an endpoint, the *cluster* L_v with scale h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) The trees containing only the root and an endpoint of scale $h+1$ will be called the *trivial trees*; note that they do not belong to $\mathcal{T}_{h,1}$, if $h \le 0$ (see the end of item 3 above), and can be associated with the three terms in the local part of $\mathcal{V}^{(h)}$.

6) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$ and $\varepsilon(f)$ will denote the space-time point and the ε index, respectively, of the field variable with label f.

If $h_v \le +1$, one of the field variables belonging to I_v carries also a time derivative ∂_0 if the corresponding local term is of type z, see (2.16). Hence we can associate with each field label f an integer $m(f) \in \{0,1\}$, denoting the order of the time derivative. Note that $m(f)$ is not uniquely determined, since we are free to choose on which field exiting from a vertex of type z the derivative falls, see comment after (2.16); we shall use this freedom in the following.

If $h \le 0$, the effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + L\beta \tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) \;, \tag{2.36}$$

where, if v_0 is the first vertex of τ and $\tau_1, .., \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ is defined inductively by the relation

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{(\le h+1)}); \ldots ; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{(\le h+1)})] \;, \tag{2.37}$$

and $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$

a) is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$ if the subtree τ_i is not trivial;

b) if τ_i is trivial and $h \le -1$, it is equal to one of the three terms in $\mathcal{L}\mathcal{V}^{(h+1)}(\psi^{(\le h+1)})$ or, if $h = 0$, to one of the two terms contributing to $\mathcal{V}^{(1)}(\psi^{\le 1})$.

$\mathcal{E}^T_{h+1}$ denotes the truncated expectation with respect to the measure $\prod_\omega P(d\psi^{(h+1)}_\omega)$, that is

$$\begin{aligned}&\mathcal{E}^T_{h+1}(X_1;\ldots;X_p)\equiv\\ &\equiv\frac{\partial^p}{\partial\lambda_1\ldots\partial\lambda_p}\log\int\prod_\omega P(d\psi^{(h+1)}_\omega)e^{\lambda_1X_1+\cdots\lambda_pX_p}\Big|_{\lambda_i=0}\ .\end{aligned}\tag{2.38}$$

This means, in particular, that, in (2.37), one has to use for the field variables the sector decomposition (2.25).

We can write (2.37) in a more explicit way, by a procedure very similar to that described, for example, in [BM]. Note first that, if $h=0$, the r.h.s. of (2.37) can be written more explicitly in the following way. Given $\tau\in\mathcal{T}_{0,n}$, there are n endpoints of scale 2 and only another one vertex, v_0, of scale 1; let us call $v_1,\ldots,v_n$ the endpoints. We choose, in any set I_{v_i}, a subset Q_{v_i} and we define $P_{v_0}=\cup_iQ_{v_i}$; then we associate a sector index $\omega(f)\in O_0$ with any $f\in P_{v_0}$ and we put $\Omega_{v_0}=\{\omega(f):f\in P_{v_0}\}$. We have

$$\mathcal{V}^{(0)}(\tau,\psi^{(\le0)})=\sum_{P_{v_0},\Omega_{v_0}}\mathcal{V}^{(0)}(\tau,P_{v_0},\Omega_{v_0})\ ,\tag{2.39}$$

$$\mathcal{V}^{(0)}(\tau,P_{v_0},\Omega_{v_0})=\int d\mathbf{x}_{v_0}\tilde\psi^{\le0}_{\Omega_{v_0}}(P_{v_0})K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})\ ,\tag{2.40}$$

$$K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})=\frac{1}{n!}\mathcal{E}^T_1[\bar\psi^{(1)}(P_{v_1}\backslash Q_{v_1}),\ldots,\bar\psi^{(1)}(P_{v_n}\backslash Q_{v_n})]\prod_{i=1}^nK^{(2)}_{v_i}(\mathbf{x}_{v_i})\ ,\tag{2.41}$$

where we use the definitions (∂_0 is from now on the time derivative)

$$\tilde\psi^{(\le h)}_{\Omega_v}(P_v)=\prod_{f\in P_v}e^{i\varepsilon(f)\vec p_F(\theta_{h,\omega(f)})\vec x(f)}\partial_0^{m(f)}\psi^{(\le h)\varepsilon(f)}_{\mathbf{x}(f),\omega(f)}\quad,\quad h\le0\ ,\tag{2.42}$$

$$\bar\psi^{(1)}(P_v)=\prod_{f\in P_v}\psi^{(1)\varepsilon(f)}_{\mathbf{x}(f)}\ ,\tag{2.43}$$

$$K^{(2)}_{v_i}(\mathbf{x}_{v_i})=\begin{cases}\lambda v(\vec x-\vec y)\delta(x_0-y_0) & \text{if } v_i \text{ is of type } \lambda \text{ and } \mathbf{x}_{v_i}=(\mathbf{x},\mathbf{y}),\\ \nu(\vec x-\vec y)\delta(x_0-y_0) & \text{if } v_i \text{ is of type } \nu,\end{cases}\tag{2.44}$$

and we suppose that the order of the (anticommuting) field variables in (2.43) is suitable chosen in order to fix the sign as in (2.41). Note that the terms with $P_{v_0}=\emptyset$ in the r.h.s. of (2.39) contribute to $L\beta\tilde E_1$, while the others contribute to $\mathcal{V}^{(0)}(\psi^{(\le0)})$.

We now write $\mathcal{V}^{(0)}$ as $\mathcal{L}\mathcal{V}^{(0)}+\mathcal{R}\mathcal{V}^{(0)}$, with $\mathcal{L}\mathcal{V}^{(0)}$ defined as in §2.2 (it represent, in the usual RG language, the relevant and marginal contributions to $\mathcal{V}^{(0)}(\psi^{(\le0)})$), and we write for $\mathcal{R}\mathcal{V}^{(0)}$ a decomposition similar to the previous one, with $\mathcal{R}\mathcal{V}^{(0)}(\tau, P_{v_0}, \Omega_{v_0})$ in place of $\mathcal{V}^{(0)}(\tau, P_{v_0}, \Omega_{v_0})$; this means that we modify, according to the representation (2.20), (2.21) of the $\mathcal{R}$ operation, the kernels

$$W^{(0)}_{\tau,P_{v_0}}(\mathbf{x}_{P_{v_0}}) = \int d(\mathbf{x}_{v_0} \backslash \mathbf{x}_{P_{v_0}}) K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0}) \; , \tag{2.45}$$

where $\mathbf{x}_{P_{v_0}} = \cup_{f \in P_{v_0}} \mathbf{x}(f)$. In order to remember this choice, we write

$$\mathcal{R}\mathcal{V}^{(0)}(\tau, P_{v_0}, \Omega_{v_0}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq 0)}_{\Omega_{v_0}}(P_{v_0}) [\mathcal{R} K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})] \; . \tag{2.46}$$

It is not hard to see that, by iterating the previous procedure, one gets for $\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)})$, for any $\tau \in \mathcal{T}_{h,n}$, the representation described below.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union $\mathcal{I}_v$ will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$.

Moreover, we associate with any $f \in \mathcal{I}_v$ a scale label $h(f) = h_v$ and, if $h(f) \leq 0$, an index $\omega(f) \in O_{h(f)}$, while, if $h(f) = +1$, we put $\omega(f) = 0$. Note that, if $h(f) \leq 0$, $h(f)$ and $\omega(f)$ single out a sector of scale $h(f)$ and sector index $\omega(f)$ associated with the field variable of index f. In this way we assign $h(f)$ and $\omega(f)$ to each field label f, except those which correspond to the set P_{v_0}; we associate with any $f \in P_{v_0}$ the scale label $h(f) = h$ and a sector index $\omega(f) \in O_h$. We shall also put, for any $v \in \tau$, $\Omega_v = \{\omega(f), f \in P_v\}$.

Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Analogously, we shall call $\mathcal{O}_\tau$ the family of possible values of $\Omega = \{\omega(f), f \in \cup_v I_v\}$.

Then we can write

$$\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau, \Omega \in \mathcal{O}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega) \; . \tag{2.47}$$

$\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega)$ can be represented as

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq h)}_{\Omega_{v_0}}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P},\Omega}(\mathbf{x}_{v_0}) \; , \tag{2.48}$$

with $K^{(h+1)}_{\tau,\mathbf{P},\Omega}(\mathbf{x}_{v_0})$ defined inductively (recall that $h_{v_0} = h+1$) by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P},\Omega}(\mathbf{x}_v) = \frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})]\ \ \mathcal{E}^T_{h_v}[\tilde\psi^{(h_v)}_{\Omega_1}(P_{v_1}\backslash Q_{v_1}),\ldots,\tilde\psi^{(h_v)}_{\Omega_{s_v}}(P_{v_{s_v}}\backslash Q_{v_{s_v}})]\;, \tag{2.49}$$

where $\Omega_i = \{\omega(f), f \in P_{v_i}\backslash Q_{v_i}\}$ and $\tilde\psi^{(h_v)}_{\Omega_i}(P_{v_i}\backslash Q_{v_i})$ has a definition similar to (2.42), if $h_v \le 0$, while, if $h_v = +1$, is defined as in (2.43).

Moreover, if v is an endpoint, $K^{(2)}_v(\mathbf{x}_v)$ is defined as in (2.44) if $h_v = 2$, otherwise

$$K^{(h_v)}_v(\mathbf{x}_v) = \begin{cases} \lambda_{h_v-1}(\underline{\vec x})\delta(\underline{x}_0) & \text{if } v \text{ is of type } \lambda, \\ \gamma^{h_v-1}\nu_{h_v-1}(\vec x-\vec y)\delta(x_0-y_0) & \text{if } v \text{ is of type } \nu, \\ z_{h_v-1}(\vec x-\vec y)\delta(x_0-y_0) & \text{if } v \text{ is of type } z, \end{cases} \tag{2.50}$$

where $\mathbf{x}_v = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ if v is of type λ, and $\mathbf{x}_v = (\mathbf{x}, \mathbf{y})$ in the other two cases. If v_i is not an endpoint,

$$K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i}) = \mathcal{R}K^{(h_v+1)}_{\tau_i,\mathbf{P}^{(i)},\Omega^{(i)}}(\mathbf{x}_{v_i})\;, \tag{2.51}$$

where τ_i is the subtree of τ starting from v and passing through v_i (hence with root the vertex immediately preceding v), $\mathbf{P}^{(i)}$ and $\Omega^{(i)}$ are the restrictions to τ_i of $\mathbf{P}$ and Ω. The action of $\mathcal{R}$ is defined using the representations (2.20), (2.21) of the regularization operation, as in (2.45), (2.46).

Remark - In order to simplify (2.42) and the following discussion, we now decide to use the freedom in the choice of the field that carries the ∂_0 derivative in the endpoints of type z, so that, given any vertex v, which is not an endpoint of type z, $m(f) = 0$ for all $f \in P_v$.

(2.47) is not the final form of our expansion, since we further decompose $\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega)$, by using the following representation of the truncated expectation in the r.h.s. of (2.49). Let us put $s = s_v$, $P_i \equiv P_{v_i}\backslash Q_{v_i}$; moreover we order in an arbitrary way the sets $P_i^\pm \equiv \{f \in P_i, \varepsilon(f) = \pm\}$, we call $f^\pm_{ij}$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f\in P_i^+}\mathbf{x}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f^-_{i,j})$, $\mathbf{y}_{ij} = \mathbf{x}(f^+_{i,j})$. Note that $\sum_{i=1}^s |P_i^-| = \sum_{i=1}^s |P_i^+| \equiv n$, otherwise the truncated expectation vanishes. A couple $l \equiv (f^-_{ij}, f^+_{i'j'}) \equiv (f^-_l, f^+_l)$ will be called a line joining the fields with labels $f^-_{ij}, f^+_{i'j'}$ and sector indices $\omega^-_l = \omega(f^-_l)$, $\omega^+_l = \omega(f^+_l)$ and connecting the points $\mathbf{x}_l \equiv \mathbf{x}_{i,j}$ and $\mathbf{y}_l \equiv \mathbf{y}_{i'j'}$, the *endpoints* of l. Moreover, we shall put $m_l = m(f^-_l) + m(f^+_l)$ and, if $\omega^-_l = \omega^+_l$, $\omega_l \equiv \omega^-_l = \omega^+_l$. Then, it is well known (see [Le], [BGPS], [GM] for example) that, up to a sign, if $s > 1$,

$$\begin{aligned} &\mathcal{E}^T_h(\tilde\psi^{(h)}_{\Omega_1}(P_1), ..., \tilde\psi^{(h)}_{\Omega_s}(P_s)) = \\ &= \sum_T \prod_{l\in T} \partial_0^{m_l}\tilde g^{(h)}_{\omega_l}(\mathbf{x}_l - \mathbf{y}_l)\delta_{\omega^-_l,\omega^+_l}\int dP_T(\mathbf{t})\det G^{h,T}(\mathbf{t})\;, \end{aligned} \tag{2.52}$$

where

$$\tilde{g}^{(h)}_\omega(\mathbf{x}) = e^{-i\vec{p}_F(\theta_{h,\omega})\vec{x}} g^{(h)}_\omega(\mathbf{x}) \;, \tag{2.53}$$

T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n-s+1) \times (n-s+1)$ matrix, whose elements are given by $G^{h,T}_{ij,i'j'} = t_{i,i'} \partial_0^{m(f^-_{ij})+m(f^+_{i'j'})} \tilde{g}^{(h)}_{\omega_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) \delta_{\omega_l^-,\omega_l^+}$ with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T.

In the following we shall use (2.52) even for $s = 1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1| = 0$, otherwise as equal to $\det G^h = \mathcal{E}^T_h(\tilde{\psi}^{(h)}(P_1))$.

If we apply the expansion (2.52) in each non trivial vertex of τ, we get an expression of the form

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega) &= \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\le h)}_{\Omega_{v_0}}(P_{v_0}) W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_{v_0}) \\ &\equiv \sum_{T \in \mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega, T) \;, \end{aligned} \tag{2.54}$$

where $\mathbf{T}$ is a special family of graphs on the set of points $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v. Note that any graph $T \in \mathbf{T}$ becomes a tree graph on $\mathbf{x}_{v_0}$, if one identifies all the points in the sets x_v, for any vertex v which is also an endpoint.

Remarks - An important role in this paper, as in [FMRT], will have the remark that, thanks to momentum conservation and compact support properties of propagator Fourier transforms, $\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega)$ vanishes for some choices of Ω. This constraint will be made explicit below in a suitable way, see (2.79).

Note also that $W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_{v_0})$, as underlined in the notation, is independent of Ω_{v_0}, so that $\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega, T)$ depends on Ω_{v_0} only through the external fields sector indices.

2.5 Detailed analysis of the $\mathcal{R}$ operation

The kernels $W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_{v_0})$ in (2.54) have a rather complicated expression, because of the presence of the operators $\mathcal{R}$ acting on the tree vertices, which are not endpoints. In order to clarify their structure, we have to further expand each term in the r.h.s. of (2.54), by a procedure similar to that explained in [BM].

We start this analysis by supposing that $|P_{v_0}| > 0$ (otherwise there is no $\mathcal{R}$ operation acting on v_0) and by considering the action of $\mathcal{R}$ on a single contribution to the sum in the r.h.s. of (2.54). This action is *trivial*, that is $\mathcal{R} = I$, by definition, if $|P_{v_0}| > 4$ or, since $\mathcal{R}^2 = \mathcal{R}$, if v_0 is a trivial vertex ($s_{v_0} = 1$) and $|P_{v_0}|$ is equal to $|P_{\bar v}|$, $\bar v$ being the vertex (of scale $h+2$) immediately following v_0. Hence there is nothing to discuss in these cases.

Let us consider first the case $|P_{v_0}| = 4$ and note that, by the remark following (2.51), $m(f) = 0$ for all $f \in P_{v_0}$. If $P_{v_0} = (f_1, f_2, f_3, f_4)$, with $\varepsilon(f_1) = \varepsilon(f_2) = + = -\varepsilon(f_3) = -\varepsilon(f_4)$, and we put $\mathbf{x}(f_i) = \mathbf{x}_i$, $\tilde{\mathbf{x}}_i = (x_{1,0}, \vec{x}_i)$, $\omega(f_i) = \omega_i$, $\vec{p}_{F,i} = \vec{p}_F(\theta_{h,\omega_i})$, we can write, by using (2.21),

$$\begin{aligned}
&\mathcal{R}\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega, T) = \int d\underline{\mathbf{x}}\; e^{\sum_{i=1}^4 \varepsilon_i \vec{p}_{F,i} \vec{x}_i} W_4(\underline{\mathbf{x}}) \;\cdot \\
&\cdot\; \Big\{ (x_{2,0} - x_{1,0}) \psi^{(\le h)+}_{\mathbf{x}_1,\omega_1} [\hat\partial^1(x_{1,0}) \psi^{(\le h)+}_{\mathbf{x}_2,\omega_2}] \psi^{(\le h)-}_{\mathbf{x}_3,\omega_3} \psi^{(\le h)-}_{\mathbf{x}_4,\omega_4} + \\
&+ (x_{3,0} - x_{1,0}) \psi^{(\le h)+}_{\mathbf{x}_1,\omega_1} \psi^{(\le h)+}_{\tilde{\mathbf{x}}_2,\omega_2} [\hat\partial^1(x_{1,0}) \psi^{(\le h)-}_{\mathbf{x}_3,\omega_3}] \psi^{(\le h)-}_{\mathbf{x}_4,\omega_4} + \\
&+ (x_{4,0} - x_{1,0}) \psi^{(\le h)+}_{\mathbf{x}_1,\omega_1} \psi^{(\le h)+}_{\tilde{\mathbf{x}}_2,\omega_2} \psi^{(\le h)-}_{\tilde{\mathbf{x}}_3,\omega_3} [\hat\partial^1(x_{1,0}) \psi^{(\le h)-}_{\mathbf{x}_4,\omega_4}] \Big\} \;, \qquad (2.55)
\end{aligned}$$

where $W_4(\underline{\mathbf{x}})$ is the integral of $W^{(h)}_{\tau,\mathbf{P},\Omega,T}(\mathbf{x}_{v_0})$ over the variables $\mathbf{x}_{v_0} \backslash \underline{\mathbf{x}}$, up to a sign, and $\hat\partial^1(x_0)$ is an operator defined by

$$\hat\partial^1(x_0) F(\mathbf{y}) = \int_0^1 ds \partial_0 F(\xi_0(s), \vec{y}) \quad , \quad \xi_0(s) = x_0 + s(y_0 - x_0) \; . \qquad (2.56)$$

Similar expressions are obtained, if the localization point (see comment after (2.14)) is changed.

Let us now consider the case $|P_{v_0}| = 2$. If only one of the external fields of v_0 carries a ∂_0 derivative, the action of $\mathcal{R}$ would not be trivial. However, we can limit this possibility to the contribution corresponding to the tree with $n = 1$, whose only endpoint is of type z, which gives no contribution to $\mathcal{R}\mathcal{V}^{(h)}$. In fact, if there is more than one endpoint, at most one of the fields of any endpoint of type z can belong to P_{v_0}, so that we can use the freedom in the choice of the field which carries the derivative so that $m(f) = 0$ for both $f \in P_{v_0}$ (see remark after (2.51)).

Hence, we have to discuss only the case $m(f) = 0$ for both $f \in P_{v_0}$; if we put $P_{v_0} = (f_1, f_2)$, $\mathbf{x}(f_i) = \mathbf{x}_i$, $\omega(f_i) = \omega_i$, $\vec{p}_{F,i} = \vec{p}_F(\theta_{h,\omega_i})$, we can write

$$\begin{aligned}
&\mathcal{R}\mathcal{V}^{(h)}(\tau, \mathbf{P}, \Omega, T) = \qquad (2.57) \\
&= \int d\mathbf{x} d\mathbf{y} e^{i(\vec{p}_{F,1}\vec{x} - \vec{p}_{F,2}\vec{y})} (y_0 - x_0)^2 W(\mathbf{x} - \mathbf{y}) \psi^{(\le h)+}_{\mathbf{x},\omega_1} [\hat\partial^2(x_0) \psi^{(\le h)-}_{\mathbf{y},\omega_2}] \;,
\end{aligned}$$

where $W(\mathbf{x}_1 - \mathbf{x}_2)$ is the integral of $W^{(h)}_{\tau,\mathbf{P},\Omega,T}(\mathbf{x}_{v_0})$ over the variables

$\mathbf{x}_{v_0}\backslash(\mathbf{x}_1,\mathbf{x}_2)$, up to a sign, and $\hat\partial^2(x_0)$ is an operator defined by

$$\hat\partial^2(x_0)\psi^{(\le h)\varepsilon}_{\mathbf{y},\omega_2} = \int_0^1 ds(1-s)\partial_0^2\psi^{(\le h)\varepsilon}_{\xi_0(s),\vec y,\omega} \quad , \quad \xi_0(s) = x_0 + s(y_0-x_0)\ . \tag{2.58}$$

Instead of (2.57), one could also use a similar expression with $[\hat\partial^2(y_0)\psi^{(\le h)+}_{\mathbf{x},\omega_1}]$ $\psi^{(\le h)-}_{\mathbf{y},\omega_2}$ in place of $\psi^{(\le h)+}_{\mathbf{x},\omega_1}[\hat\partial^2(x_0)\psi^{(\le h)-}_{\mathbf{y},\omega_2}]$. We shall distinguish these two different choices by saying that we have taken $\mathbf{x}$, in the case of (2.57), or $\mathbf{y}$, in the other case, as the *localization point*.

By using (2.49) and (2.52), we can also write

$$\begin{aligned}
&\mathcal{R}\mathcal{V}^{(h)}(\tau,\mathbf{P},\Omega,T) = \\
&= \frac{1}{s_{v_0}!}\sum_{\alpha\in A}\int d\mathbf{x}_{v_0}\int dP_{T_{v_0}}(\mathbf{t})\ \mathcal{R}[\tilde\psi^{(\le h)}_{\Omega_{v_0},\alpha}(P_{v_0})]\ (y_{\alpha,0}-x_{\alpha,0})^{b(|P_{v_0}|)}\ \cdot \\
&\cdot\Big[\prod_{l\in T_{v_0}}\partial_0^{m_l}\tilde g^{(h+1)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)\delta_{\omega_l^-,\omega_l^+}\Big]\det G^{h+1,T_{v_0}}(\mathbf{t})\prod_{i=1}^{s_{v_0}}[K^{(h+2)}_{v_i}(\mathbf{x}_{v_i})]\ ,
\end{aligned} \tag{2.59}$$

where A is a set of indices containing only one element, except in the case $|P_{v_0}| = 4$, when $|A| = 3$, and $\mathbf{x}_\alpha,\mathbf{y}_\alpha$ are two points of $\mathbf{x}_{v_0}$. Moreover, $\mathcal{R}[\tilde\psi^{(\le h)}_{\Omega_{v_0},\alpha}(P_{v_0})] = \tilde\psi^{(\le h)}_{\Omega_{v_0}}(P_{v_0})$, except if $|P_{v_0}| = 4$ or $|P_{v_0}| = 2$ and $m(f) = 0$ for both $f\in P_{v_0}$; in these case, its expression can be easily deduced from (2.55) and (2.57). Finally, $b(p)$ is an integer, equal to 1, if $p = 4$, equal to 2, if $p = 2$, and equal to 0 otherwise.

We would like to apply iteratively equation (2.55) and (2.57), starting from v_0 and following the partial order of the tree τ, in all the τ vertices with $|P_v| = 4$ or $|P_v| = 2$ and $m(f) = 0$ for $f\in P_v$. However, in order to control the combinatorics, it is convenient to decompose the factor $(y_{\alpha,0} - x_{\alpha,0})^{b(|P_{v_0}|)}$ in the following way. Let us consider the unique subset $(l_1,\dots,l_m)$ of T_{v_0}, which selects a path joining the cluster containing $\mathbf{x}_\alpha$ with the cluster containing $\mathbf{y}_\alpha$, if one identifies all the points in the same cluster; if this subset is empty (since $\mathbf{x}_\alpha$ and $\mathbf{y}_\alpha$ belong to the same cluster), we put $m = 0$. If $m > 0$, we call $(\bar v_{i-1},\bar v_i)$, $i = 1,m$, the couple of vertices whose clusters of points are joined by l_i. We shall put $\mathbf{x}_{2i-1}$, $i = 1,m$, equal to the endpoint of l_i belonging to $\mathbf{x}_{\bar v_{i-1}}$, $\mathbf{x}_{2i}$ equal to the endpoint of l_i belonging to $\mathbf{x}_{\bar v_i}$, $\mathbf{x}_0 = \mathbf{x}_\alpha$ and $\mathbf{x}_{2m+1} = \mathbf{y}_\alpha$. These definitions imply that there are two points of the sequence $\mathbf{x}_r$, $r = 0,\dots,\bar m = 2m+1$, possibly coinciding, in any set $\mathbf{x}_{\bar v_i}$, $i = 0,\dots,m$; these two points are the space-time points of two different fields belonging to $P_{\bar v_i}$. Then, we can write

$$y_{\alpha,0} - x_{\alpha,0} = \sum_{r=1}^{\bar m}(x_{r,0} - x_{r-1,0})\ . \tag{2.60}$$

If we insert (2.60) in (2.59), the r.h.s. can be written as the sum over a set B_{v_0} of different terms, that we shall distinguish with a label α_{v_0}; note that $|B_{v_0}| \le 3(2s_{v_0}-1)^2$. We get an expression of the form

$$\begin{aligned}\mathcal{R}\mathcal{V}^{(h)}(\tau,\mathbf{P},\Omega,T) &= \frac{1}{s_{v_0}!}\sum_{\alpha_{v_0}\in B_{v_0}}\int d\mathbf{x}_{v_0}\int dP_{T_{v_0}}(\mathbf{t})\mathcal{R}[\tilde\psi^{(\le h)}_{\Omega_{v_0},\alpha}(P_{v_0})] \;\cdot\\ &\cdot \Big[\prod_{l\in T_{v_0}}(x_{l,0}-y_{l,0})^{b_l(\alpha_{v_0})}\partial_0^{m_l}\tilde g^{(h+1)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)\delta_{\omega_l^-,\omega_l^+}\Big] \;\cdot\\ &\cdot\; \det G^{h+1,T_{v_0}}(\mathbf{t})\prod_{i=1}^{s_{v_0}}[(x_0^{(i)}-y_0^{(i)})^{b_{v_i}(\alpha_{v_0})}K^{(h+2)}_{v_i}(\mathbf{x}_{v_i})]\;,\end{aligned} \tag{2.61}$$

where we called $(\mathbf{x}^{(i)},\mathbf{y}^{(i)})$ the couple of points which, in the previous argument, belong to $\mathbf{x}_{v_i}$ and $b_l(\alpha_{v_0})$, $b_{v_i}(\alpha_{v_0})$ are integers with values in $\{0,1,2\}$, such that their sum is equal to $b(|P_{v_0}|)$.

Let us now see what happens, if we iterate the argument leading to (2.61). Let us suppose, for example, that $|P_{v_1}|=2$, that the action of $\mathcal{R}$ is not trivial on v_1 and that $b\equiv b_{v_1}(\alpha_{v_0})>0$. In this case, if we exploit the action of $\mathcal{R}$ in the form of (2.57), we have an overall factor $(x_0^{(1)}-y_0^{(1)})^m$, $m=2+b$, which multiplies $K^{(h+2)}_{v_1}(\mathbf{x}_{v_1})$. Hence, if we expand this factor, by using an equation similar to (2.60), we get terms with some propagator multiplied by a factor $(x_{l,0}-y_{l,0})^{b_l}$, with $b_l>2$. If we further iterate this procedure, we can end up with an expansion, where some propagator is multiplied by a factor $(x_{l,0}-y_{l,0})^{b_l}$ with b_l of order $|h|$, which would produce bad bounds. However, we can avoid very simply this difficulty, by noticing that, if we insert (2.20) in an expression like

$$J_b=\int d\mathbf{x}d\mathbf{y}F_1(\mathbf{x})F_2(\mathbf{y})(y_0-x_0)^b\mathcal{R}W(\mathbf{x}-\mathbf{y})\;, \tag{2.62}$$

we get, by a simple integration by part, if $b=2$,

$$J_2=\int d\mathbf{x}d\mathbf{y}F_1(\mathbf{x})F_2(\mathbf{y})(y_0-x_0)^2W(\mathbf{x}-\mathbf{y})\;, \tag{2.63}$$

that is the $\mathcal{R}$ operation can be substituted by the identity, while, if $b=1$, we get

$$J_1=\int d\mathbf{x}d\mathbf{y}F_1(\mathbf{x})[\hat\partial^1(x_0)F_2(\mathbf{y})](y_0-x_0)^2W(\mathbf{x}-\mathbf{y})\;, \tag{2.64}$$

where $\hat\partial^1(x_0)$ is the operator defined by (2.56). This means that, if $b=1$, the action of $\mathcal{R}$ only increases the power of (y_0-x_0) by one unit. Note that, in (2.64) one could substitute $F_1(\mathbf{x})[\hat\partial^1(x_0)F_2(\mathbf{y})]$ with $-[\hat\partial^1(y_0)F_1(\mathbf{x})]F_2(\mathbf{y})$; we shall again distinguish these two different choices by saying that we have taken $\mathbf{x}$, in the case of (2.64), or $\mathbf{y}$, in the other case, as the localization point.

Even simpler is the situation, when $|P_{v_1}| = 4$. In fact, if we insert (2.21) in an expression like $\int d\underline{\mathbf{x}} F(\underline{\mathbf{x}})(y_0^* - x_0^*)\mathcal{R}W_4(\underline{\mathbf{x}})$, y^* and x^* being two points of $\underline{\mathbf{x}}$, we get

$$\int d\underline{\mathbf{x}} F(\underline{\mathbf{x}})(y_0^* - x_0^*)\mathcal{R}W_4(\underline{\mathbf{x}}) = \int d\underline{\mathbf{x}} F(\underline{\mathbf{x}})(y_0^* - x_0^*)W_4(\underline{\mathbf{x}}) \ , \tag{2.65}$$

so that, even in this case, the power of the "zero" can not increase.

There are in principle two other problems. First of all, one could worry that there is an accumulation of the operators $\hat{\partial}^q$ (dimensionally equivalent to a derivative of order q) on a same line, if this line is affected many times by the $\mathcal{R}$ operation in different vertices. Moreover, since the definition of the $\hat{\partial}^q(x_0)$ operators depends on the choice of the localization point $\mathbf{x}$, it could happen that there is an "interference" between the $\mathcal{R}$ operations in two different vertices, which would make more involved the expansion. However, one can show, by the same arguments given in §3.3 and §3.4 of [BM] in the one dimensional case, that these problems can be avoided by using the freedom in the choice of the localization point and, mainly, the fact that some regularization operations are not really present. Let us consider, for example, the first term in the r.h.s. of (2.55) and note that, if we sum it over the sector indices, we get, in terms of Fourier transforms, an expression of the type

$$\begin{aligned}\int d\underline{\mathbf{k}} \prod_{i=1}^{4} \hat{\psi}_{\mathbf{k}_i}^{(\le h),\varepsilon_i} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \cdot \\ \cdot \left[\hat{W}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - \hat{W}_4(\mathbf{k}_1, (0, \vec{k}_2), \mathbf{k}_3)\right] \ .\end{aligned} \tag{2.66}$$

However, if $\bar{f}$ is the label of the field $\psi_{\mathbf{x}_2}^{(\le h),+}$, it is easy to see that $\hat{W}_4(\mathbf{k}_1, (0, \vec{k}_2), \mathbf{k}_3) = 0$, if there is a vertex $\bar{v} > v_0$ with four external legs, such that $f \in P_{\bar{v}}$ and f is affected by the $\mathcal{R}$ operation in $\bar{v}$. Hence, in this case, we can substitute the first term in the braces of (2.55) with $\prod_i \psi_{\mathbf{x}_i,\omega_i}^{(\le h),\varepsilon_i}$.

We refer to §3.3 and §3.4 of [BM] for a complete analysis of this problem, whose final result is that the action of $\mathcal{R}$ on all the vertices of τ will produce terms where the propagators related with the lines of T are multiplied by a factor $(x_{l,0} - y_{l,0})^{b_l}$ with $b_l \le 2$ and (after that) are possibly subject to one or two operators $\hat{\partial}^q$, $q = 1, 2$. Moreover, some of the external lines belonging to P_{v_0} can be affected from one operator $\hat{\partial}^q$, as a consequence of the action of $\mathcal{R}$ on v_0 or some other vertex $v > v_0$. Finally, the lines involved in the determinants may be affected from one operator $\hat{\partial}^q$. We introduce an index α

to distinguish these different terms and, given α, we shall denote by $\hat{\partial}^{q_\alpha(f)}$ the differential operators acting on the external lines of P_{v_0} or the propagators belonging to T, as a consequence of the regularization procedure.

All the previous considerations imply that $\mathcal{RV}^{(h)}(\tau, \mathbf{P}, \Omega, T) = 0$, if $|P_{v_0}| = 4$ and $n = 1$ (that is there is only an endpoint of type λ and no internal line associated with v_0) or $P_{v_0} = (f_1, f_2)$ and $m(f_1) + m(f_2) = 1$ (since this can happen only if $n = 1$ and the endpoint is of type z, as a consequence of the freedom in the choice of the field carrying the derivative in the endpoints of type z) or $m(f_1) + m(f_2) = 0$ and $n = 1$. In all the other cases, we can write $\mathcal{RV}^{(h)}(\tau, \mathbf{P}, \Omega, T)$ in the form

$$\mathcal{RV}^{(h)}(\tau, \mathbf{P}, \Omega, T) = \sum_{\alpha \in A_T} \int d\mathbf{x}_{v_0} W_{\tau, \mathbf{P}, \Omega \backslash \Omega_{v_0}, T, \alpha}(\mathbf{x}_{v_0}) \mathcal{R}[\tilde{\psi}^{(\leq h)}_{\Omega_{v_0}, \alpha}(P_{v_0})] \ , \tag{2.67}$$

where

$$\mathcal{R}[\tilde{\psi}^{(\leq h)}_{\Omega_{v_0}, \alpha}(P_{v_0})] = \prod_{f \in P_{v_0}} e^{i\varepsilon(f) \vec{p}_F(\theta_{h,\omega(f)}) \vec{x}(f)} [\hat{\partial}^{q_\alpha(f)} \psi]^{(\leq h)\varepsilon(f)}_{\mathbf{x}_\alpha(f), \omega(f)}, \tag{2.68}$$

and, up to a sign,

$$\begin{aligned} & W_{\tau, \mathbf{P}, \Omega \backslash \Omega_{v_0}, T, \alpha}(\mathbf{x}_{v_0}) = \\ & = \left[\prod_{i=1}^{n} K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*}) \right] \Bigg\{ \prod_{\substack{v \\ \text{not e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v, T_v}_\alpha(\mathbf{t}_v) \cdot \\ & \cdot \Big[\prod_{l \in T_v} \delta_{\omega_l^+, \omega_l^-} \hat{\partial}^{q_\alpha(f_l^-)}(x'_{l,0}) \hat{\partial}^{q_\alpha(f_l^+)}(y'_{l,0}) [(x_{l,0} - y_{l,0})^{b_\alpha(l)} \partial_0^{m_l} \tilde{g}^{(h_v)}_{\omega_l}(\mathbf{x}_l - \mathbf{y}_l)] \Big] \Bigg\} \ , \end{aligned} \tag{2.69}$$

where “e.p.” is an abbreviation of “endpoint” and, together with the definitions used before, we are using the following ones:

1. A_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations and the iterative decomposition of the zeros;

2. $v_1^*, \ldots, v_n^*$ are the endpoints of τ and $h_i = h_{v_i^*}$;

3. $b_\alpha(v)$, $b_\alpha(l)$, $q_\alpha(f_l^-)$ and $q_\alpha(f_l^+)$ are positive integers ≤ 2;

4. if $q_\alpha(f_l^-) > 0$, $x'_{l,0}$ denote the time coordinate of the point involved, together with $\mathbf{x}_l$, in the corresponding $\mathcal{R}$ operation, see (2.58) and (2.56), otherwise $\hat{\partial}^0(x'_{l,0}) = I$;

5. if v is a non trivial vertex (so that $s_v > 1$),the elements $G^{h_v,T_v}_{\alpha,ij,i'j'}$ of $G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ are of the form
$$\begin{aligned} G^{h_v,T_v}_{\alpha,ij,i'j'} &= t_{i,i'} \cdot \\ &\cdot \hat\partial_0^{q_\alpha(f^-_{ij})}(x'_{l,0}) \hat\partial_0^{q_\alpha(f^+_{i'j'})}(y'_{l,0}) \partial_0^{m(f^-_l)} \partial_0^{m(f^-_l)} \tilde g^{(h_v)}_{\omega_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) \delta_{\omega^-_l,\omega^+_l} \; ; \end{aligned} \tag{2.70}$$
if v is trivial, T_v is empty and $\int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ has to be interpreted as 1, if $|\mathcal{I}_v| = 0$ ($\mathcal{I}_v$ is the set of internal fields of v), otherwise it is the determinant of a matrix of the form (2.70) with $t_{i,i'} = 1$.

2.6 Modification of the running coupling functions

We want now to introduce a different representation of the running coupling functions λ_h, ν_h, z_h, which will be useful in the following, in order to perform the bounds. This new representation is suggested by the remark that, if we substitute (2.69) in (2.67) and we express the whole integral in Fourier space, the Fourier transform of $K^{h_i}_{v^*_i}(\mathbf{x}_{v^*_i})$ is multiplied by the factor

$$\prod_{f \in P_{v^*_i} \cap P_{v_0}} \hat\psi^{\le h_{v_0}, \varepsilon(f)}_{\mathbf{k}(f),\omega(f)} \prod_{f \in P_{v^*_i} \setminus P_{v_0}} F_{h(f),\omega(f)}(\mathbf{k}(f)) \; . \tag{2.71}$$

In order to use this property, we define, for any $h \le 0$ and $\omega \in O_h$, the *s-sector* $S_{h,\omega}$ (see §2.1 and §2.3 for related definitions) as

$$S_{h,\omega} = \{\vec k = \rho \vec e_r(\theta) \in \mathbb{R}^2 : |\varepsilon(\vec k) - \mu| \le \gamma^h e_0, \; \zeta_{h,\omega}(\theta) \neq 0\} \; . \tag{2.72}$$

Note that the definition of s-sector has the property, to be used extensively in the following, that the s-sector $S_{h+1,\omega}$ of scale $h+1$ contains the union of two s-sectors of scale h: $S_{h+1,\omega} \supseteq \{S_{h,2\omega} \cup S_{h,2\omega+1}\}$, as follows from the definition of $\zeta_{h,\omega}$, see (2.23).

We now observe that the field variables $\hat\psi^{\le h_{v_0},\varepsilon(f)}_{\mathbf{k}(f),\omega(f)}$ have the same supports as the functions $C^{-1}_{h_{v_0}}(\mathbf{k}(f)) \; \zeta_{h_{v_0},\omega(f)}(\theta(f))$ and $h(f) \le h_i - 1$, $\forall f \in P_{v^*_i}$; hence in the expression (2.69), for any i, we can freely multiply $\hat K^{h_i}_{v^*_i}(\mathbf{k}_{v^*_i})$ by $\prod_{f \in P_{v^*_i}} \tilde F_{h_i-1,\tilde\omega(f)}(\vec k)$, where $\tilde F_{h,\omega}(\vec k)$ is a smooth function $= 1$ on $S_{h,\omega}$ and with a support slightly greater than $S_{h,\omega}$, while $\tilde\omega(f) \in O_{h_i-1}$ is the unique sector index such that $S_{h(f),\omega(f)} \subseteq S_{h_i-1,\tilde\omega(f)}$. In order to formalize this statement, it is useful to introduce the following definition.

Let $G(\underline{\vec x})$ be a function of $2p$ variables $\underline{\vec x} = (\vec x_1, \ldots, \vec x_{2p})$ with Fourier transform $\hat G(\underline{\vec k})$, defined so that $G(\underline{\vec x}) = \int d\underline{\vec k} (2\pi)^{-4p} \exp(-i \sum_{l=1}^{2p} \varepsilon_i \vec k_i \vec x_i) \hat G(\underline{\vec k})$, where $\varepsilon_1, \ldots, \varepsilon_p = -\varepsilon_{p+1} = \ldots = -\varepsilon_{2p} = +1$. Then, we define, given $h \le 0$ and a family $\underline{\sigma} = \{\sigma_i \in O_h, i = 1, \ldots, 2p\}$ of sector indices,

$$(\mathfrak{F}_{2p,h,\underline{\sigma}} * G)(\underline{\vec{x}}) = \int \frac{d\underline{\vec{k}}}{(2\pi)^{4p}} e^{-i\sum_{l=1}^{2p} \varepsilon_i \vec{k}_i \vec{x}_i} \left[\prod_{i=1}^{2p} \tilde{F}_{h,\sigma_i}(\vec{k}_i)\right] \hat{G}(\underline{\vec{k}}) \ . \tag{2.73}$$

In order to extend this definition to the case $h = 1$, when the sector index can take only the value 0, we define $\tilde{F}_{1,0}(\vec{k})$ as a smooth function of compact support, equal to 1 on the support of $\bar{C}_0^{-1}(\vec{k})$, defined in §1.2.

Hence, if we put $p_i = |P_{v_i^*}|$, $\tilde{\Omega}_i = \{\tilde{\omega}(f), f \in P_{v_i^*}\}$ and we define, for any family $\underline{\sigma} = \{\sigma(f) \in O_{h_i-1}, f \in P_{v_i^*}\}$ of sector indices of scale $h_i - 1$, labelled by the set $P_{v_i^*}$ ($\tilde{\Omega}_i$ is a particular example of such a family),

$$\tilde{K}^{h_i}_{v_i^*,\underline{\sigma}}(\mathbf{x}_{v_i^*}) = \left(\mathfrak{F}_{p_i,h_i-1,\underline{\sigma}} * K^{h_i}_{v_i^*}\right)(\mathbf{x}_{v_i^*}) \ , \tag{2.74}$$

we can substitute in (2.69) each $K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*})$ with $\tilde{K}^{h_i}_{v_i^*,\tilde{\Omega}_i}(\mathbf{x}_{v_i^*})$. If v_i^* is of type ν, z or λ, $\tilde{K}^{h_i}_{v_i^*,\underline{\sigma}}(\mathbf{x}_{v_i^*})$ can be written as $\gamma^{h_i-1}\delta(x_{0,v_i^*})\tilde{\nu}_{h_i-1,\underline{\sigma}}(\vec{x}_{v_i^*})$, $\delta(x_{0,v_i^*})\tilde{z}_{h_i-1,\underline{\sigma}}(\vec{x}_{v_i^*})$ or $\delta(x_{0,v_i^*})\ \tilde{\lambda}_{h_i-1,\underline{\sigma}}(\vec{x}_{v_i^*})$ respectively. $\tilde{\nu}_{h_i-1,\underline{\sigma}}(\vec{x}_i - \vec{y}_i)$, $\tilde{z}_{h_i-1,\underline{\sigma}}(\vec{x}_i - \vec{y}_i)$ and $\tilde{\lambda}_{h_i-1,\underline{\sigma}}(\vec{x}_{v_i^*})$ will be called the *modified coupling functions.*

We shall call $W^{(mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})$ the expression we get from $W_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0})$ by the substitution of the running coupling functions with the modified ones. Note that $W^{(mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})$ is not independent of Ω_{v_0}, unlike $W_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0})$, and that $W^{(mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})$ is equal to $W_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0})$, only if $|P_{v_0}| = 0$; however, the previous considerations imply that, if $p_0 = |P_{v_0}| > 0$,

$$\left(\mathfrak{F}_{p_0,h,\Omega_{v_0}} * W^{(mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}\right)(\mathbf{x}_{v_0}) = \left(\mathfrak{F}_{p_0,h,\Omega_{v_0}} * W_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}\right)(\mathbf{x}_{v_0}) \ , \tag{2.75}$$

a trivial remark which will be important in the discussion of the running coupling functions flow in §4.

2.7 Bounds for the effective potentials and the free energy

Given a vertex v of a tree τ and an arbitrary family $\bar{\mathcal{S}} = \{S_{j_f,\sigma_f}, f \in P_v\}$ of s-sectors labelled by P_v, we define

$$\chi_v(\bar{\mathcal{S}}) = \chi\left(\forall f \in P_v, \exists \vec{k}(f) \in S_{j_f,\sigma_f} : \sum_{f\in P_v} \varepsilon(f)\vec{k}(f) = 0\right) \ , \tag{2.76}$$

where $\chi(condition)$ is the function $= 1$ when *condition* is verified, and $= 0$ in the opposite case. Moreover, given a set P of field labels, we denote by $\mathcal{S}(P)$ the special family of s-sectors labelled by P, defined as

$$\mathcal{S}(P) = \{S_{h(f),\omega(f)}\ ,\ f \in P\}\ . \tag{2.77}$$

The previous considerations imply that

$$E_{L,\beta} \le \sum_{h=h_\beta-1}^{0} \sum_{n=1}^{\infty} J_{h,n}(0,0)\ , \tag{2.78}$$

with

$$\begin{aligned} J_{h,n}(2l_0,q_0) \quad = \quad & \sum_{\tau\in\mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:|P_{v_0}|=2l_0,\\ \sum_{f\in P_{v_0}} q_\alpha(f)=q_0}} \sum_{T\in\mathbf{T}} \sum_{\alpha\in A_T} \sideset{}{^*}\sum_{\Omega\in\mathcal{O}_\tau} \left[\prod_v \chi_v(\mathcal{S}(P_v))\right] \cdot \\ \cdot \quad & \int d(\mathbf{x}_{v_0}\backslash\mathbf{x}^*) \left|W^{(mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})\right|\ , \end{aligned} \tag{2.79}$$

where $\mathbf{x}^*$ is an arbitrary point in $\mathbf{x}_{v_0}$, l_0 is a non negative integer and $\sum^*_{\Omega\in\mathcal{O}_\tau}$ differs from $\sum_{\Omega\in\mathcal{O}_\tau}$ since one ω index, arbitrarily chosen among the $2l_0$ ω's in Ω_{v_0}, is not summed over, if $l_0 > 0$, otherwise it coincides with $\sum_{\Omega\in\mathcal{O}_\tau}$.

Remarks - Note that we could freely insert $[\prod_v \chi_v(\mathcal{S}(P_v))]$ in (2.79), because of the constraints following from momentum conservation and the compact support properties of propagator's Fourier transform.

Note also that, if $l_0 = 0$, given $\tau \in \mathcal{T}_{h,n}$, the number of internal lines in the lowest vertex v_0 (of scale $h+1$) has to be different from zero.

Hence, in order to prove that the free energy and the effective potentials are well defined (in the limit $L \to \infty$ and β not "too large"), we need a "good" bound of $J_{h,n}(2l_0,q_0)$.

In order to get this bound, we shall extend the procedure used in [BM] for the analysis of the one dimensional Fermi systems, which we shall refer to for some details (except for the sum over the sector indices, which is a new problem).

An important role has the following bound for the determinants appearing in (2.69):

$$\begin{aligned} &|\det G^{h_v,T_v}_\alpha(\mathbf{t}_v)| \le c^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\ \cdot \\ &\cdot\ \gamma^{h_v\frac{3}{4}\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}\gamma^{h_v\sum_{i=1}^{s_v}\left[q_\alpha(P_{v_i}\backslash Q_{v_i})+m(P_{v_i}\backslash Q_{v_i})\right]}\ \cdot \\ &\cdot\ \gamma^{-h_v\sum_{l\in T_v}\left[q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^+)+m(f_l^-)\right]}\ , \end{aligned} \tag{2.80}$$

where, if $P \subset I_{v_0}$, we define $q_\alpha(P) = \sum_{f\in P} q_\alpha(f)$ and $m(P) = \sum_{f\in P} m(f)$.

The proof of (2.80) is based on the well known *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} =< A_i, B_j >$, where A_i, B_j are vectors in a Hilbert space $\mathcal{H}$ with scalar product $< \cdot, \cdot >$ and induced norm $||\cdot||$, then

$$|\det M| \le \prod_i ||A_i|| \cdot ||B_i|| \ . \tag{2.81}$$

Let $\mathcal{H} = \mathbb{R}^{|O_h|} \otimes \mathbb{R}^s \otimes L^2(\mathbb{R}^3)$; it can be shown that

$$G^{h_v,T_v}_{\alpha,ij,i'j'} =< \mathbf{v}_{\omega_l^-} \otimes \mathbf{u}_i \otimes A^{(h_v)}_{\mathbf{x}(f^-_{ij}),\omega_l^-}, \mathbf{v}_{\omega_l^+} \otimes \mathbf{u}_{i'} \otimes B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega_l^+} > , \tag{2.82}$$

where $\mathbf{v}_\omega \in \mathbb{R}^{|O_h|}$, $\omega \in O_h$, and $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \ldots, s$, are unit vectors such that $\mathbf{v}_\omega \cdot \mathbf{v}_{\omega'} = \delta_{\omega,\omega'}$, $\mathbf{u}_i \cdot \mathbf{u}_{i'} = t_{i,i'}$; moreover, $A^{(h_v)}_{\mathbf{x}(f^-_{ij}),\omega_l}$, $B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega_l}$ are defined so that:

$$\begin{aligned} &\hat\partial_0^{q_\alpha(f^-_{ij})}(x'_{l,0})\hat\partial_0^{q_\alpha(f^+_{i'j'})}(y'_{l,0})\partial_0^{m(f^-_l)}\partial_0^{m(f^+_l)}\tilde g^{(h_v)}_{\omega_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) = \\ &=< A^{(h_v)}_{\mathbf{x}(f^-_{ij}),\omega_l}, B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega_l} >\equiv \int \frac{d\mathbf{k}}{(2\pi)^3} A^{*(h_v)}_{\mathbf{x}(f^-_{ij}),\omega_l}(\mathbf{k}) B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega_l}(\mathbf{k}) \ , \end{aligned} \tag{2.83}$$

with $||A_i|| \cdot ||B_i||$ satisfying the same dimensional bound as the left side of (2.83). For example, if $q_\alpha(f^-_{ij}) = q_\alpha(f^+_{i'j'}) = 0$, one can put,

$$\begin{aligned} A^{(h_v)}_{\mathbf{x},\omega_l}(\mathbf{k}) &= e^{i\mathbf{k}\mathbf{x}} \frac{\sqrt{F_{h_v,\omega_l}}}{k_0^2 + \left(\varepsilon(\vec k) - \mu\right)^2} (ik_0)^{m(f^-_l)} (ik_0)^{m(f^+_l)} \\ B^{(h_v)}_{\mathbf{x},\omega_l}(\mathbf{k}) &= e^{i\mathbf{k}\mathbf{x}} \sqrt{F_{h_v,\omega_l}} \left[ik_0 + \varepsilon(\vec k) - \mu\right] \ . \end{aligned} \tag{2.84}$$

Using Lemma 2.1 and (2.81), we easily get (2.80).

The next step is to bound by 1 the integrals over the probability measures dP_{T_v} appearing in (2.69). After that, we bound the integral

$$\begin{aligned} &\int d(\mathbf{x}_{v_0} \backslash \mathbf{x}^*) \Bigg| \prod_{i=1}^n \left[\tilde K^{h_i}_{v_i^*,\tilde\Omega_i}(\mathbf{x}_{v_i^*})\right] \prod_{v\,\text{not e.p.}} \frac{1}{s_v!} \ \cdot \\ &\cdot \prod_{l\in T_v} \left\{\hat\partial_0^{q_\alpha(f^-_l)}(x'_{l,0})\hat\partial_0^{q_\alpha(f^+_l)}(y'_{l,0})[(x_{l,0} - y_{l,0})^{b_\alpha(l)}\partial_0^{m_l}\tilde g^{(h_v)}_{\omega_l}(\mathbf{x}_l - \mathbf{y}_l)]\right\}\Bigg| \ . \end{aligned} \tag{2.85}$$

We can take from §3.15 of [BM] the identity (independent of the dimension):

$$d(\mathbf{x}_{v_0} \backslash \mathbf{x}^*) = \prod_{l\in T^*} d\mathbf{r}_l \ , \tag{2.86}$$

where T^* is a tree graph obtained from $T = \cup_v T_v$, by adding in a suitable (obvious) way, for each endpoint v_i^*, $i = 1, \ldots, n$, one or more lines connecting the space-time points belonging to $\mathbf{x}_{v_i^*}$. Moreover $\mathbf{r}_l = (\xi_0(t_l) - \eta_0(s_l), \vec{x}_l - \vec{y}_l)$ (see (2.56)), if $l \in \cup_v T_v$, and $\mathbf{r}_l = \mathbf{x}_l - \mathbf{y}_l$, if $l \in T^* \setminus \cup_v T_v$.

Hence (2.85) can be written as

$$J_{\tau,\mathcal{P},T,\alpha} \int \prod_{l \in T^* \setminus \cup_v T_v} d\mathbf{r}_l \left| \prod_{i=1}^{n} \tilde{K}^{h_i}_{v_i^*,\tilde{\Omega}_i}(\mathbf{x}_{v_i^*}) \right| , \tag{2.87}$$

with

$$\begin{aligned} J_{\tau,\mathcal{P},T,\alpha} = & \prod_{v \,\text{not e.p.}} \frac{1}{s_v!} \int \prod_{l \in T_v} d\mathbf{r}_l \cdot \\ & \cdot \left| \hat{\partial}_0^{q_\alpha(f_l^-)}({x'}_{l,0}) \hat{\partial}_0^{q_\alpha(f_l^+)}({y'}_{l,0}) [(x_{l,0} - y_{l,0})^{b_\alpha(l)} \partial_0^{m_l} \tilde{g}^{(h_v)}_{\omega_l}(\mathbf{x}_l - \mathbf{y}_l)] \right| . \end{aligned} \tag{2.88}$$

By using Lemma 2.1, we can bound each propagator, each derivative and each zero by a dimensional factor, so finding

$$\begin{aligned} J_{\tau,\mathbf{P},T,\alpha} \le c^n & \prod_{v \,\text{not e.p.}} \Big[\frac{1}{s_v!} c^{2(s_v-1)} \gamma^{-h_v \sum_{l \in T_v} b_\alpha(l)} \cdot \\ & \cdot \gamma^{-h_v(s_v-1)} \gamma^{h_v \sum_{l \in T_v} \left[q_\alpha(f_l^+) + q_\alpha(f_l^-) + m(f_l^+) + m(f_l^-) \right]} \Big] . \end{aligned} \tag{2.89}$$

Let us now define, for any set of field indices P, $\mathbf{O}_h(P) = \otimes_{f \in P} O_h$. The next step is to use the following lemma, to be proved in §3.

Lemma 2.2 *Suppose that there exist two constants C_1 and C_ν such that the modified coupling functions satisfy the following conditions:*
i) if $|P_v| = 4$, then

$$\sum_{\underline{\sigma} \in \mathbf{O}_{h_v - 1}}^{*} \int d(\vec{x}_v \backslash \vec{x}^*) |\tilde{\lambda}_{h_v-1,\underline{\sigma}}(\vec{x}_v)| \le 2 C_1 |\lambda| \gamma^{-\frac{1}{2}(h_v - 1)} , \tag{2.90}$$

where $\sum^*$ means that one of the sector indices is not summed over;
ii) if $|P_v| = 2$ and $\mathbf{x}_v = (\mathbf{x}_1, \mathbf{x}_2)$, then

$$\sum_{\underline{\sigma} \in \mathbf{O}_{h_v - 1}}^{*} \int d\vec{x}_1 |\tilde{\nu}_{h_v-1,\underline{\sigma}}(\vec{x}_1 - \vec{x}_2)| \le 2 C_1 C_\nu |\lambda| , \tag{2.91}$$

$$\sum_{\underline{\sigma} \in \mathbf{O}_{h_v - 1}}^{*} \int d\vec{x}_1 |\tilde{z}_{h_v-1,\underline{\sigma}}(\vec{x}_1 - \vec{x}_2)| \le C_1 |\lambda| . \tag{2.92}$$

Consider a tree $\tau \in \mathcal{T}_{h,n}$, a graph $T \in \mathbf{T}$ and the corresponding tree graph T^*, defined as after (2.86). Then

$$\sum_{\Omega\in\mathcal{O}_\tau}^{*}\left[\prod_{v\in\tau}\left(\chi_v(\mathcal{S}(P_v))\prod_{l\in T_v}\delta_{\omega_l^+,\omega_l^-}\right)\right]\int\prod_{l\in T^*\backslash T}d\mathbf{r}_l\left|\prod_{i=1}^n\tilde{K}^{h_i}_{v_i^*,\tilde{\Omega}_i}(\mathbf{x}_{v_i^*})\right|\leq$$

$$\leq\quad c^n|\lambda|^n\gamma^{-\frac12 h[m_4(v_0)+\chi(P_{v_0}=\emptyset)]}\prod_{i=1}^n\gamma^{(h_i-1)\chi(v\text{ is of type }\nu)}\;\cdot$$

$$\cdot\prod_{v\text{ not e. p.}}\gamma^{\left[-\frac12 m_4(v)+\frac12(|P_v|-3)\chi(4\leq|P_v|\leq 8)+\frac12(|P_v|-1)\chi(|P_v|\geq 10)\right]}\;,\qquad(2.93)$$

where $m_4(v)$ denotes the number of endpoints of type λ following the vertex v.

Since $\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\leq 4n$, $\sum_v(s_v-1)=n-1$ and $|A_T|\leq c^n$, (2.80), (2.88) and Lemma 2.2 imply that

$$|J_{h,n}(2l_0,q_0)|\leq(c|\lambda|)^n\;\cdot\qquad(2.94)$$

$$\cdot\sum_{\tau\in\mathcal{T}_{h,n}}\;\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:|P_{v_0}|=2l_0,\\ \sum_{f\in P_{v_0}}q_\alpha(f)=q_0}}\;\sum_{T\in\mathbf{T}}\gamma^{-\frac12 h(m_4(v_0)+\chi(l_0=0))}\prod_{i=1}^n\gamma^{(h_i-1)\chi(v\text{ is of type }\nu)}\;\cdot$$

$$\cdot\prod_{v\text{ not e. p.}}\Big[\frac{1}{s_v!}\gamma^{h_v\frac34\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}\gamma^{h_v\sum_{i=1}^{s_v}\left[q_\alpha(P_{v_i}\backslash Q_{v_i})+m(P_{v_i}\backslash Q_{v_i})\right]}\;\cdot$$

$$\cdot\,\gamma^{-h_v\sum_{l\in T_v}b_\alpha(l)}\gamma^{-h_v(s_v-1)}\gamma^{\left[-\frac12 m_4(v)+\frac12(|P_v|-3)\chi(4\leq|P_v|\leq 8)+\frac12(|P_v|-1)\chi(|P_v|\geq 10)\right]}\Big]\;.$$

Note now that the constraints on the values of $q_\alpha(f)$ and $b_\alpha(l)$ imply, as shown in detail in §3.11 of [BM], that

$$\sum_{v\text{ not e. p.}}h_v\sum_{i=1}^{s_v}q_\alpha(P_{v_i}\backslash Q_{v_i})+h\;q_0=\sum_{f\in I_{v_0}}h(f)q_\alpha(f)\;,\qquad(2.95)$$

$$\Big[\prod_{f\in I_{v_0}}\gamma^{h(f)q_\alpha(f)}\Big]\Big[\prod_{l\in T}\gamma^{-h_\alpha(l)b_\alpha(l)}\Big]\leq\prod_{v\text{ not e.p.}}\gamma^{-z(v)}\qquad(2.96)$$

where

$$z(v)=\begin{cases}2 & \text{if }|P(v)|=2\;,\\ 1 & \text{if }|P(v)|=4\;,\\ 0 & \text{otherwise.}\end{cases}\qquad(2.97)$$

Moreover, since the freedom in the choice of the field carrying the derivative in the endpoints of type z was used (see remark a few lines before (2.57)) so that $m(P_v)=0$, if v is not an endpoint, and the field with $m(f)=1$ belonging to the endpoint v is contracted in the vertex immediately preceding v, whose scale is $h_v - 1$, we have the identity

$$\prod_{v \text{ not e. p.}} \gamma^{h_v \sum_{i=1}^{s_v} \left[m(P_{v_i} \backslash Q_{v_i})\right]} = \prod_{i=1}^{n} \gamma^{(h_i-1)\chi(v \text{ is of type } z)} \; . \tag{2.98}$$

Putting together the previous bounds and supposing that the hypothesis (2.90), (2.91), (2.92) of Lemma 2.2 are verified, we find that

$$\begin{aligned} J_{h,n}(2l_0, q_0) \le (c|\lambda|)^n \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau : |P_{v_0}| = 2l_0, \\ \sum_{f \in P_{v_0}} q_{\alpha}(f) = q_0}} \sum_{T \in \mathbf{T}} \gamma^{-h[\frac{1}{2} m_4(v_0) + \frac{1}{2}\chi(l_0 = 0) + q_0]} \; \cdot \\ \cdot \left[\prod_{i=1}^{n} \gamma^{(h_i-1)\chi(|P(v)|=2)}\right] \prod_{v \text{ not e. p.}} \left[\frac{1}{s_v!} \gamma^{h_v \left[\frac{3}{4}\left(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|\right) - \frac{5}{2}(s_v - 1)\right]} \; \cdot \right. \\ \left. \cdot \gamma^{\left[-z(v) - \frac{1}{2} m_4(v) + \frac{1}{2}(|P_v| - 3)\chi(4 \le |P_v| \le 8) + \frac{1}{2}(|P_v| - 1)\chi(|P_v| \ge 10)\right]}\right] \; . \end{aligned} \tag{2.99}$$

On the other hand, if $m_2(v)$ denotes the number of endpoints of type ν or z following v, we have, if $\tilde{v}$ is not an endpoint, the identities

$$\begin{aligned} \sum_{\substack{v \ge \tilde{v} \\ v \text{ not e. p.}}} \left(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|\right) &= 4 m_4(\tilde{v}) + 2 m_2(\tilde{v}) - |P_{\tilde{v}}| \; , \\ \sum_{v \ge \tilde{v}} (s_v - 1) &= m_4(\tilde{v}) + m_2(\tilde{v}) - 1 \; , \end{aligned} \tag{2.100}$$

which, together with (2.99) imply that

$$\begin{aligned} & J_{h,n}(2l_0, q_0) \le \\ & \le (c|\lambda|)^n \gamma^{h[-q_0 + \delta_{\text{ext}}(2l_0)]} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \\ |P_{v_0}| = 2l_0}} \sum_{T \in \mathbf{T}} \prod_{\substack{v \\ \text{not e. p.}}} \frac{1}{s_v!} \gamma^{\delta(|P_v|)} \; , \end{aligned} \tag{2.101}$$

where

$$\begin{aligned} \delta(p) = & -\chi(2 \le p \le 4) + \\ & + \left(1 - \frac{p}{4}\right) \chi(6 \le p \le 8) + \left(2 - \frac{p}{4}\right) \chi(p \ge 10) \; , \end{aligned} \tag{2.102}$$

$$\delta_{\text{ext}}(p) = \frac{5}{2} - \frac{3}{4} p - \frac{1}{2} \chi(p = 0) \; . \tag{2.103}$$

Since $\delta(|P_v|) < 0$, for any vertex v, which is not an endpoint, a standard argument, see [BM] or [GM], allows to show that

$$\sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \\ |P_{v_0}| = 2l_0}} \sum_{T \in \mathbf{T}} \prod_{\substack{v \\ \text{not e. p.}}} \frac{1}{s_v!} \gamma^{\delta(|P_v|)} \le c^n \; . \tag{2.104}$$

The bounds (2.101) and (2.104) imply the following theorem.

Theorem 2.1 *If conditions (2.90), (2.91), (2.92) are satisfied, then*

$$J_{h,n}(2l_0,q_0) \le (c|\lambda|)^n \gamma^{h[-q_0+\delta_{\rm ext}(2l_0)]}\ . \tag{2.105}$$

Remark - We will prove in §4 that, if $|\lambda|$ is small enough and $C_{1,2}\log\beta|\lambda| \le 1$, where $C_{1,2}$ is a constant depending only on first and second order contributions of perturbation theory, it is possible to choose $\tilde\nu_1(\vec{\underline{x}})$ so that the modified running coupling functions satisfy the hypotheses of Lemma 2.2, (2.90), (2.91) and (2.92). So, in that case, we see from Theorem 2.1 that $\lim_{L\to\infty} E_{L,\beta}$ does exist and is of order λ.

3 Proof of Lemma 2.2

3.1 The sector counting Lemma

In order to present the proof of Lemma 2.2, we need to introduce some new definitions.

1. Given a tree τ and $\mathbf{P}\in\mathcal{P}_\tau$, we shall call χ-vertices the vertices v of τ, such that $\mathcal{I}_v$ (the set of internal lines, that is the lines contracted in v) is not empty. We shall also call V_χ the family of all χ-vertices, whose number is of order n.
2. Given $h\le 0$ and a set of field indices P, we define $\mathbf{O}_h(P)=\otimes_{f\in P}O_h$ and we shall call $\underline{\sigma}=\{\sigma_f\in O_h,\ f\in P\}$ the elements of $\mathbf{O}_h(P)$.
3. Given $h\le 0$ and $\underline{\sigma}\in\mathbf{O}_h(P)$, we define $\mathfrak{S}_h(\underline{\sigma})=\{S_{h,\sigma_f},\sigma_f\in\underline{\sigma}\}$.
4. Given a set of field indices P and two families of s-sectors labelled by P, $\mathcal{S}^{(i)}=\{S_{j_f^{(i)},\sigma_f^{(i)}},\ f\in P\}$, $i=1,2$, we shall say that $\mathcal{S}^{(1)}\prec\mathcal{S}^{(2)}$, if $S_{j_f^{(1)},\sigma_f^{(1)}}\subset S_{j_f^{(2)},\sigma_f^{(2)}}$, for any $f\in P$.

The main point in the proof is the following lemma, which is an extension of that proved in [FMRT] in the jellium case; see §7 for a proof.

Lemma 3.1 *Let h',h,L be integers such that $h'\le h\le 0$. Let v be a vertex of a tree τ, such that $|P_v|=L$ and f_1 a fixed element of P_v. Then, given the sector index $\sigma_{f_1}\in O_{h'}$, and a set $\underline{\sigma}\in\mathbf{O}_h(P_v\backslash f_1)$, the following bound holds:*

$$\sum_{\substack{\underline{\sigma}'\in\mathbf{O}_{h'}(P_v\backslash f_1)\\ \mathfrak{S}_{h'}(\underline{\sigma}')\prec\mathfrak{S}_h(\underline{\sigma})}} \chi_v\left(\{S_{h',\sigma_{f_1}}\}\cup\mathfrak{S}_{h'}(\underline{\sigma}')\right) \le \begin{cases} c^L\gamma^{\frac{h-h'}{2}(L-3)} & ,\ \mathit{if}\ L\ge 4, \\ c & ,\ \mathit{if}\ L=2\ . \end{cases} \tag{3.1}$$

3.2 Proof of Lemma 2.2

First of all, we note that

$$\prod_v \chi_v(\mathcal{S}(P_v)) = \prod_{v\in V_\chi} \chi_v(\mathcal{S}(P_v)) \; . \tag{3.2}$$

Let us consider first the case $P_{v_0} \neq \emptyset$ and let $\tilde{v}_0$ be the first χ-vertex following the root (possibly equal to v_0); note that $P_{\tilde{v}_0} = P_{v_0}$ and that $h(f) = h$ for any $f \in P_{\tilde{v}_0}$. In the following it will also very important to remember that Ω is the family of all sector indices $\omega(f)$ associated with the field labels f and that $\omega(f) \in O_{h(f)}$, $h(f)$ being the scale of the propagator connected to the corresponding field variable, see §2.4. In agreement with this definition, if $\bar{\Omega}$ is a subset of Ω, $\sum_{\bar{\Omega}}$ will denote the sum over $\omega(f) \in O_{h(f)}$, for any $f \in \bar{\Omega}$.

Let us call f_0 the field whose sector index $\omega(f_0) \in O_h$ is fixed in the sum over Ω. We rewrite the sector sum in the l.h.s. of (2.93) as:

$$\sum_{\Omega}^{*} = \sum_{\Omega_{\tilde{v}_0}}^{*} \sum_{\Omega\backslash\Omega_{\tilde{v}_0}} = \sum_{\underline{\sigma}_{\tilde{v}_0} \in \mathbf{O}_{h_{\tilde{v}_0}}(P_{\tilde{v}_0}\backslash f_0)} \; \sum^{*}_{\substack{\Omega_{\tilde{v}_0}: \\ \mathcal{S}(P_{\tilde{v}_0}\backslash f_0) \prec \mathfrak{S}_{h_{\tilde{v}_0}}(\underline{\sigma}_{\tilde{v}_0})}} \; \sum_{\Omega\backslash\Omega_{\tilde{v}_0}} \; . \tag{3.3}$$

Then, for any fixed $\underline{\sigma}_{\tilde{v}_0} \in \mathbf{O}_{h_{\tilde{v}_0}}(P_{\tilde{v}_0}\backslash f_0)$, we bound the product of χ_v functions as

$$\prod_{v\in V_\chi} \chi_v(\mathcal{S}(P_v)) \le \chi_{\tilde{v}_0}(\mathcal{S}(P_{\tilde{v}_0})) \prod_{v\in \{V_\chi \backslash \tilde{v}_0\}} \chi_v(\tilde{\mathcal{S}}_{v,\tilde{v}_0}) \; , \tag{3.4}$$

where

$$\tilde{\mathcal{S}}_{v,\tilde{v}_0} = \mathcal{S}\Big(P_v \backslash (P_{\tilde{v}_0}\backslash f_0)\Big) \cup \Big\{ S_{h_{\tilde{v}_0},\sigma_f} \in \mathfrak{S}_{h_{\tilde{v}_0}}(\underline{\sigma}_{\tilde{v}_0}), f \in P_v \cap (P_{\tilde{v}_0}\backslash f_0) \Big\} \; . \tag{3.5}$$

In other words, for any $v \neq \tilde{v}_0$, we relax the sector condition by allowing the external fields of v, which are also external fields of $\tilde{v}_0$ and are not equal to f_0, to have a momentum varying, instead than in the original sector, of scale h, in that of scale $h_{\tilde{v}_0}$ containing it.

Let us now observe that the modified running coupling functions do not depend on $\Omega_{\tilde{v}_0}$, if $\underline{\sigma}_{\tilde{v}_0}$ is fixed, as it follows from definition (2.74); hence the only remaining dependence on $\Omega_{\tilde{v}_0}$ is in $\chi_{\tilde{v}_0}(\mathcal{S}(P_{\tilde{v}_0}))$. It follows, by using Lemma 3.1 for $|P_{\tilde{v}_0}| \le 8$ and the trivial bound

$$\sum^{*}_{\substack{\Omega_{\tilde{v}_0} \\ \mathcal{S}(P_{\tilde{v}_0}\backslash f_0) \prec \mathfrak{S}_{h_{\tilde{v}_0}}(\underline{\sigma}_{\tilde{v}_0})}} 1 \le c\gamma^{\frac{1}{2}(h_{\tilde{v}_0}-h)(|P_{\tilde{v}_0}|-1)} \; , \tag{3.6}$$

for $|P_{\tilde v_0}| \geq 10$, that we can bound the sum over $\Omega_{\tilde v_0}$, for any $\mathfrak{S}_{h_{\tilde v_0}}(\underline{\sigma}_{\tilde v_0})$, as

$$\sum^{*}_{\substack{\Omega_{\tilde v_0} \\ \mathcal{S}(P_{\tilde v_0}\setminus f_0) \prec \mathfrak{S}_{h_{\tilde v_0}}(\underline{\sigma}_{\tilde v_0})}} \chi_{\tilde v_0}(\mathcal{S}(P_{\tilde v_0})) \leq$$

$$\leq c\gamma^{(h_{\tilde v_0}-h)\left[\frac{1}{2}(|P_{\tilde v_0}|-3)\chi(4\leq|P_{\tilde v_0}|\leq 8)+\frac{1}{2}(|P_{\tilde v_0}|-1)\chi(|P_{\tilde v_0}|\geq 10)\right]} . \tag{3.7}$$

We are thus left with the problem of bounding a sum similar to the initial one, but with all the external sector indices on scale $h_{\tilde v_0}$ instead of h. We shall do that by iterating the previous procedure, in a way which depends on the structure of the tree τ and of the graph T; the iteration stops at the endpoints, where we can use the hypotheses (2.90), (2.91) and (2.92).

To describe this inductive procedure, we establish, for any vertex $v \in V_\chi$, a partial ordering of the s_v vertices $v_1, \ldots, v_{s_v} \in V_\chi$ immediately following v on τ, by assigning a root to the tree graph T^* and to each anchored tree graph T_v. We decide that the root of T^* is the space-time point containing f_0; then we assign a direction to the lines of the tree graph T^*, the one which goes from the root towards the leaves. Finally we decide that the root of T_v is the vertex which the line of $T_{v'}$ enters, where v' is the χ-vertex immediately preceding $v \in V_\chi$, if $f_0 \notin P_v$; otherwise, the root of T_v is the vertex containing the root of T^*, see Fig. 2.

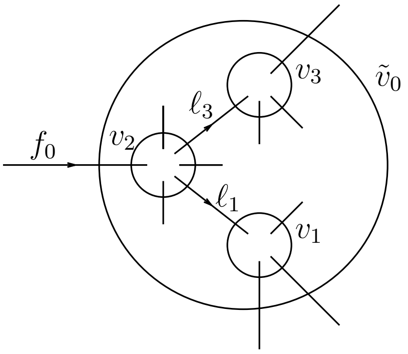

Figure 2: A possible cluster structure corresponding to a tree τ of the expansion for the effective potentials such that $s_{\tilde v_0} = 3$. The set $T_{\tilde v_0}$ is formed by the lines ℓ_1 and ℓ_3. The lines different from ℓ_1 and ℓ_3 and not belonging to $P_{\tilde v_0}$ have to be contracted into the Lesniewski determinants.

The l.h.s. of (2.93) is bounded by the product of the r.h.s. of (3.7) and the following quantity:

$$\left[\prod_{v>\tilde v_0, v\in V_\chi}\ \sum_{\tilde\Omega_{v,\tilde v_0}}\chi_v\left(\tilde{\mathcal S}_{v,\tilde v_0}\right)\right]\left[\prod_{l\in T}\delta_{\omega_l^+,\omega_l^-}\right]\int\prod_{l\in T^*\backslash T} d\mathbf r_l \prod_{i=1}^n\left|\tilde K^{h_i}_{v_i^*,\tilde\Omega_i}(\mathbf x_{v_i^*})\right| , \qquad (3.8)$$

where

$$\tilde\Omega_{v,\tilde v_0}=\{\Omega_v\backslash\Omega_{\tilde v_0}\}\cup\{\sigma_f\in O_{h_{\tilde v_0}}, f\in P_v\cap(P_{\tilde v_0}\backslash f_0)\}\ . \qquad (3.9)$$

Note that there is no sector index associated with f_0 in $\tilde\Omega_{v,\tilde v_0}$ and that, if v is a χ-vertex immediately following $\tilde v_0$ on τ, all the sector indices included in $\tilde\Omega_{v,\tilde v_0}$ belong to $O_{h_{\tilde v_0}}$, since in this case the fields associated with $P_v\backslash P_{\tilde v_0}$ are contracted on scale $h_{\tilde v_0}$.

We now consider the $s_{\tilde v_0}$ χ-vertices immediately following $\tilde v_0$ and we reorder the expression (3.8) in the following way:

$$\begin{aligned}(3.8)\ =\ &\prod_{j=1}^{s_{\tilde v_0}}\left[\sum_{\cup_{v\geq v_j}\tilde\Omega_{v,\tilde v_0}}\left(\chi_{v_j}(\tilde{\mathcal S}_{v_j,\tilde v_0})\prod_{\substack{v>v_j\\ v\in V_\chi}}\chi_v(\tilde{\mathcal S}_{v,\tilde v_0})\prod_{l\in\cup_{v\geq v_j}T_v}\delta_{\omega_l^+,\omega_l^-}\right)\cdot\right.\\ &\left.\cdot\prod_{v_i^*\geq v_j}\int d\mathbf r_{v_i^*}\left|\tilde K^{h_i}_{v_i^*,\tilde\Omega_i}(\mathbf x_{v_i^*})\right|\right]\prod_{l\in T_{\tilde v_0}}\delta_{\omega_l^+,\omega_l^-}\ , \qquad (3.10)\end{aligned}$$

where: i) $\int d\mathbf r_{v_i^*}$ is equal to $\int\prod_{l\in T_{v_i^*}}d\mathbf r_l$, where $T_{v_i^*}$ denotes the subset of the tree graph T^* connecting the set $x_{v_i^*}$; ii) if $s_v=1$, $\prod_{l\in T_v}\delta_{\omega_l^+,\omega_l^-}$ has to be thought as equal to 1.

We now choose a leave of T_{v_0} (v_1 or v_3 in Fig. 2), say v^*, and we consider the factor in the product $\prod_{j=1}^{s_{\tilde v_0}}$ appearing in the r.h.s. of (3.10) corresponding to v^*, together with the line $l^*\in T_{v_0}$ entering v^* (ℓ_1 or ℓ_3 in Fig. 2). We can associate with v^* the following quantity, which is independent of all the other leaves and of the sector indices associated with the lines of T_{v_0}:

$$\begin{aligned}[v^*]\ =\ &\left[\sum^*_{\tilde\Omega_{v^*,\tilde v_0}}\left(\chi_{v^*}(\tilde{\mathcal S}_{v^*,\tilde v_0})\sum^*_{\cup_{v>v^*}\tilde\Omega_{v,\tilde v_0}\backslash\tilde\Omega_{v^*,\tilde v_0}}\ \prod_{\substack{v>v^*\\ v\in V_\chi}}\chi_v(\tilde{\mathcal S}_{v,\tilde v_0})\prod_{l\in\cup_{v\geq v^*}T_v}\delta_{\omega_l^+,\omega_l^-}\right)\cdot\right.\\ &\left.\cdot\prod_{i:v_i^*\geq v^*}\int d\mathbf r_{v_i^*}\left|\tilde K^{h_i}_{v_i^*,\tilde\Omega_i}(\mathbf x_{v_i^*})\right|\right]\ , \qquad (3.11)\end{aligned}$$

where $\sum^*_{\tilde\Omega_{v^*,\tilde v_0}}$ means that we do not sum over the sector index associated with l^*.

In order to bound the expression in the r.h.s. (3.11), we have to distinguish two cases (a) v^* is an endpoint. In this case $\sum^*_{\tilde\Omega_{v^*,\tilde v_0}}=\sum^*_{\underline\sigma\in\Omega_{h_{v^*}-1}}$ and

$\chi_{v^*}(\tilde{\mathcal{S}}_{v^*,\tilde v_0}) = 1$, since the corresponding constraint is already included in the definition of the modified coupling functions, so that the expression to bound is simply:

$$\sum_{\underline{\sigma}\in\Omega_{h_{v^*}-1}}^{*} \int d\mathbf{r}_{v^*} \left|\tilde K^{h_{v^*}}_{v^*,\underline{\sigma}}(\mathbf{x}_{v^*})\right| . \tag{3.12}$$

Hence, conditions (2.90), (2.91), (2.92) imply that

$$[v^*] \le c|\lambda|\gamma^{-\frac{1}{2}(h_{v^*}-1)\chi(|P_{v^*}|=4)}\gamma^{(h_{v^*}-1)\chi(v^* \text{ is of type } \nu)} . \tag{3.13}$$

(b) v^* is not an end point. In this case, by the remark following (3.9), the expression in the r.h.s. of (3.11) has exactly the same structure as the l.h.s. of (2.93), which we started the iteration from; one has only to substitute $\tilde v_0$ with v^*, h with $h_{\tilde v_0}$ and $h_{\tilde v_0}$ with h_{v^*}. Hence we can bound the r.h.s. of (3.11) by extracting a factor

$$c\gamma^{(h_{v^*}-h_{\tilde v_0})\left(\frac{1}{2}(|P_{v^*}|-3)\chi(4\le|P_{v^*}|\le 8)+\frac{1}{2}(|P_{v^*}|-1)\chi(|P_{v^*}|\ge 10)\right)} \tag{3.14}$$

and we end up with an expression similar to (3.8), the line l^* acting now as an external field, since there is only one sector sum associated with it, thanks to the factor $\delta_{\omega^+_{l^*},\omega^-_{l^*}}$ present in the r.h.s. of (3.10).

It is now completely obvious that we can iterate the previous procedure, for each leave of $T_{\tilde v_0}$, ending up with a bound of the l.h.s. of (2.93) of the form

$$(c|\lambda|)^n \left[\prod_{v\in V_\chi} \gamma^{(h_v-h_{v'})\left(\frac{1}{2}(|P_v|-3)\chi(4\le|P_v|\le 8)+\frac{1}{2}(|P_v|-1)\chi(|P_v|\ge 10)\right)}\right] \cdot$$
$$\cdot \left[\prod_{i=1}^{n} \gamma^{-\frac{1}{2}(h_i-1)\chi(v^*_i \text{ is of type } \lambda)}\gamma^{(h_i-1)\chi(v^*_i \text{ is of type } \nu)}\right] , \tag{3.15}$$

where v' is the χ-vertex immediately preceding v on τ, if $v > \tilde v_0$, or the root, if $v = \tilde v_0$. On the other hand, given $v \in V_\chi$, $P_{\bar v} = P_v$ if $v' < \bar v \le v$. Moreover,

$$\prod_{i=1}^{n} \gamma^{-\frac{1}{2}(h_i-1)\chi(v^*_i \text{ is of type } \lambda)} = \gamma^{-\frac{1}{2}hm_4(v_0)} \prod_{v \text{ not e. p.}} \gamma^{-\frac{1}{2}m_4(v)} , \tag{3.16}$$

where $m_4(v)$ is the number of end points of type λ following vertex v on τ. It follows that (3.15) can be written in the form

$$(c|\lambda|)^n\gamma^{-\frac{1}{2}hm_4(v_0)} \left[\prod_{i=1}^{n} \gamma^{(h_i-1)\chi(v^*_i \text{ is of type } \nu)}\right] \cdot$$
$$\cdot \prod_{v \text{ not e. p.}} \gamma^{\left[-\frac{1}{2}m_4(v)+\frac{1}{2}(|P_v|-3)\chi(4\le|P_v|\le 8)+\frac{1}{2}(|P_v|-1)\chi(|P_v|\ge 10)\right]} , \tag{3.17}$$

which proves Lemma 2.2 in the case $|P_{v_0}| > 0$.

The case $P_{v_0} = \emptyset$ is treated in a similar way. The only real difference is that one has to sum over all sector indices. However, since the set of internal fields $\mathcal{I}_{v_0}$ is necessarily not empty (our definitions imply that, in this case, $\tilde v_0 = v_0$), we can choose in an arbitrary way one field $f_0 \in \mathcal{I}_{v_0}$ and let it play the same role of the selected external field of v_0 in the previous iterative procedure. Of course, the first iteration step, which produced before the "scale jump" factor in the r.h.s. of (3.7), is now missing, but this is irrelevant, since that factor is equal to 1 if $|P_{v_0}| = 0$. All the other steps are absolutely identical, but, at the end of the iteration, we end up with the sector sum related with f_0; this produces a factor $\gamma^{-\frac{1}{2}h_{v_0}} = \gamma^{-\frac{1}{2}(h+1)}$. This completes the proof. ▌

4 The flow of running coupling functions

4.1 The expansion for $\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})$

By using (2.36), (2.47) and (2.54), we get

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=2,4}} \sum_{\Omega\in\mathcal{O}_\tau} \sum_{T\in\mathbf{T}} \cdot$$
$$\cdot \int d\mathbf{x}_{v_0} \tilde\psi^{(\le h)}_{\Omega_{v_0}}(P_{v_0}) \mathcal{L}W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_{v_0}) , \tag{4.1}$$

where, if $P_{v_0} = (f_1, \ldots, f_4)$ and we put $\mathbf{x}(f_i) = \mathbf{x}_i = (x_{i,0}, \vec{x}_i)$, $\tilde{\mathbf{x}}_i = (\tilde x_{i,0}, \vec{x}_i)$ and $\mathbf{x}^*$ is any point in $\mathbf{x}_{v_0}$,

$$\mathcal{L}W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\underline{\mathbf{x}}) = \delta(\underline{x}_0) \int d(\underline{\tilde x}_0 \backslash \tilde x_0^*) W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\underline{\tilde{\mathbf{x}}}) , \tag{4.2}$$

while, if $P_{v_0} = (f_1, f_2)$ and $m(P_{v_0}) = m(f_1) + m(f_2) = 0$,

$$\mathcal{L}W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_1, \mathbf{x}_2) = \delta(x_{1,0} - x_{2,0}) \int d\tilde x_{1,0} W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) , \tag{4.3}$$

and finally, if $P_{v_0} = (f_1, f_2)$, $m(P_{v_0}) = 1$,

$$\mathcal{L}W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_1, \mathbf{x}_2) = \delta(x_{1,0} - x_{2,0}) \cdot$$
$$\int d\tilde x_{1,0} (\tilde x_{1,0} - \tilde x_{2,0}) W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) . \tag{4.4}$$

Note that there is no other case to consider, since, as a consequence of the freedom in the choice of the field carrying the derivative in the endpoints of type z, there is no contribution to the effective potential with $n \geq 2$ and a derivative acting on the external fields of v_0, before the application of the $\mathcal{L}$ operator.

Let us consider first the contributions to the r.h.s. of (4.1) coming from the trees with $n = 1$. These trees have only two vertices, v_0 (of scale $h+1$) and the endpoint v^*, whose scale has to be equal to $h+2$. If we impose the further condition that $P_{v^*} = P_{v_0}$, the sum of these terms is equal to $\mathcal{L}\mathcal{V}^{(h+1)}(\tau, \psi^{(\leq h)})$. In order to control the flow of the running coupling functions, we need a "good bound" of the remaining terms.

Let us consider a contribution to the r.h.s. of (4.1), such that $n \geq 2$ or $n = 1$ and $P_{v^*} \neq P_{v_0}$. By proceeding as in §2.5, it is easy to show that

$$\mathcal{L}W^{(h)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T}(\mathbf{x}_{v_0}) = \sum_{\alpha\in A_T} W^{(L)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0}) \;, \tag{4.5}$$

where A_T is a suitable set of indices and $W^{(L)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0})$ can be represented as in (2.69). There is indeed a small difference, because of the delta function and the integral appearing in (4.2), (4.3) and (4.4), but it can be treated without any new problem. Moreover, by the considerations of §2.6, if we insert (4.5) in the r.h.s. of (4.1), we can substitute $W^{(L)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\mathbf{x}_{v_0})$ with $W^{(L,mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})$, obtained by using the modified running coupling functions in place of the original ones. As before, these modified functions are not constant with respect to Ω_{v_0}. We can prove the following Theorem, analogous to Theorem 2.1.

Theorem 4.1 *If conditions (2.90), (2.91), (2.92) are satisfied, given a couple of integers* (p, m) *equal to* $(2,0)$, $(2,1)$ *or* $(4,0)$, *we have:*

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}_{h,n}} \sideset{}{^{**}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau: \\ |P_{v_0}|=p, m(P_{v_0})=m}} \sum_{T\in\mathbf{T}} \sum_{\alpha\in A_T} \sideset{}{^{*}}\sum_{\Omega\in\mathcal{O}_\tau} \int d(\mathbf{x}_{v_0}\backslash\mathbf{x}^*) \left|W^{(L,mod)}_{\tau,\mathbf{P},\Omega,T,\alpha}(\mathbf{x}_{v_0})\right| \leq \\ &\leq\; (c|\lambda|)^n \gamma^{h[\delta_{ext}(p)-m]} \;, \end{aligned} \tag{4.6}$$

with $\delta_{ext}(p)$ *defined by (2.103) and* $\sum^{**}$ *means that, if* $n = 1$ *and* v^* *is the endpoint,* $P_{v^*} \neq P_{v_0}$.

PROOF - We can repeat step by step the proof of Theorem 2.1 and use the remark that, in the identity (2.95), $q_0 = m$. ▮

4.2 The beta function

The discussion of §4.1 and the definition of modified running coupling functions (MRCF in the following) of §2.6 imply that

$$\tilde\lambda_{h,\underline{\sigma}}(\underline{\vec x}) = (\mathfrak{F}_{4,h,\underline{\sigma}} * \lambda_{h+1})(\underline{\vec x}) + (\mathfrak{F}_{4,h,\underline{\sigma}} * \beta^{4,0}_{h+1})(\tilde{\mathbf{v}}_{h+1}, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \tag{4.7}$$

$$\tilde\nu_{h,\underline{\sigma}}(\underline{\vec x}) = \gamma \; (\mathfrak{F}_{2,h,\underline{\sigma}} * \nu_{h+1})(\underline{\vec x}) + (\mathfrak{F}_{2,h,\underline{\sigma}} * \beta^{2,0}_{h+1})(\tilde{\mathbf{v}}_{h+1}, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \tag{4.8}$$

$$\tilde z_{h,\underline{\sigma}}(\underline{\vec x}) = (\mathfrak{F}_{2,h,\underline{\sigma}} * z_{h+1})(\underline{\vec x}) + (\mathfrak{F}_{2,h,\underline{\sigma}} * \beta^{2,1}_{h+1})(\tilde{\mathbf{v}}_{h+1}, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \tag{4.9}$$

where $\mathbf{v}_h \equiv (\lambda_h, \nu_h, z_h)$, $\tilde{\mathbf{v}}_h$ is the set of the corresponding MRCF, $\underline{\vec x} = (\vec x_1, \ldots, \vec x_p)$, $\underline{\sigma} = (\sigma_1, \ldots, \sigma_p)$, with $\sigma_i \in O_{h-1}$ and $p = 4$ in (4.7), $p = 2$ in (4.8) and (4.9). Finally, the *beta function* $\beta^{p,m}_{h+1}(\mathbf{v}_h, .., \mathbf{v}_1; \underline{\vec x})$ is defined by the equation

$$\begin{aligned} \beta^{p,m}_{h+1}(\mathbf{v}_h, .., \mathbf{v}_1; \underline{\vec x}) = \gamma^{-h\cdot\chi(p=2,m=0)} \; \cdot \\ \cdot \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \sum^{**}_{\substack{\mathbf{P}:|P_{v_0}|=p \\ m(P_{v_0})=m}} \sum_{T\in\mathbf{T}} \sum_{\Omega\backslash\Omega_{v_0}} \sum_{\alpha\in A_T} \int d(\underline{x}_0\backslash x_0^*) W^{(L)}_{\tau,\mathbf{P},\Omega\backslash\Omega_{v_0},T,\alpha}(\underline{\mathbf{x}}) \; . \end{aligned} \tag{4.10}$$

Note that, given a tree contributing to the r.h.s. of (4.10), we can substitute the RCF with the MRCF in all endpoints except those containing one of the external fields of v_0. However $(\mathfrak{F}_{p,h,\underline{\sigma}} * \beta^{p,m}_{h+1})$ is indeed a function of the MRCF, as we made explicit in the r.h.s. of (4.7)-(4.9) and

$$\begin{aligned} (\mathfrak{F}_{p,h,\underline{\sigma}} * \beta^{p,m}_{h+1})(\tilde{\mathbf{v}}_{h+1}, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) = \gamma^{-h\cdot\chi(p=2,m=0)} \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \sum^{**}_{\substack{\mathbf{P} \\ |P_{v_0}|=p, \; m(P_{v_0})=m}} \cdot \\ \cdot \sum_{T\in\mathbf{T}} \sum_{\substack{\Omega: \\ \Omega_{v_0}=\underline{\sigma}}} \sum_{\alpha\in A_T} \int d(\underline{x}_0\backslash x_0^*) \; (\mathfrak{F}_{p,h,\underline{\sigma}} * W^{(L,mod)}_{\tau,\mathbf{P},\Omega,T,\alpha})(\underline{\mathbf{x}}) \; . \end{aligned} \tag{4.11}$$

Iterating (4.7), (4.8) and (4.9) we find, for $h \le 0$,

$$\tilde\lambda_{h,\underline{\sigma}}(\underline{\vec x}) = (\mathfrak{F}_{4,h,\underline{\sigma}} * \tilde\lambda_1)(\underline{\vec x}) + \sum_{j=h+1}^{1} (\mathfrak{F}_{4,h,\underline{\sigma}} * \beta^{4,0}_j)(\tilde{\mathbf{v}}_j, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \tag{4.12}$$

$$\begin{aligned} \tilde\nu_{h,\underline{\sigma}}(\underline{\vec x}) &= \gamma^{-h+1}(\mathfrak{F}_{2,h,\underline{\sigma}} * \tilde\nu_1)(\underline{\vec x}) + \\ &+ \sum_{j=h+1}^{1} \gamma^{-h+j-1}(\mathfrak{F}_{2,h,\underline{\sigma}} * \beta^{2,0}_j)(\tilde{\mathbf{v}}_j, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \end{aligned} \tag{4.13}$$

$$\tilde z_{h,\underline{\sigma}}(\underline{\vec x}) = (\mathfrak{F}_{2,h,\underline{\sigma}} * \tilde z_1)(\underline{\vec x}) + \sum_{j=h+1}^{1} (\mathfrak{F}_{2,h,\underline{\sigma}} * \beta^{2,1}_j)(\tilde{\mathbf{v}}_j, .., \tilde{\mathbf{v}}_1; \underline{\vec x}) \; , \tag{4.14}$$

where, ignoring in the notation the spin dependence of $v(\vec{x})$, see (1.8), $\tilde{\lambda}_1(\underline{\vec{x}}) = (\mathfrak{F}_{4,1,\underline{0}} * \lambda_1)(\underline{\vec{x}})$, with $\lambda_1(\underline{\vec{x}}) = -\lambda v(\vec{x}_1 - \vec{x}_2)\delta(\vec{x}_3 - \vec{x}_1)\delta(\vec{x}_4 - \vec{x}_2)$, and $\tilde{\nu}_1(\underline{\vec{x}}) = (\mathfrak{F}_{2,1,(0,0)} * \nu_1)(\underline{\vec{x}})$. Furthermore $\tilde{z}_1(\underline{\vec{x}}) = z_1(\underline{\vec{x}}) = 0$ and $\nu_1(\underline{\vec{x}})$ must be suitably chosen.

We note that it is possible to choose the functions $\tilde{F}_{h,\sigma}(\vec{k})$ appearing in the definition of the operators $(\mathfrak{F}_{p,h,\underline{\sigma}} * \cdot)$, see (2.73), in such a way that, if $h \le 0$,

$$|\varepsilon(\vec{k}) - \mu| \le e_0 \gamma^h \Rightarrow \frac{1}{2} \sum_{\sigma \in O_h} \tilde{F}_{h,\sigma}(\vec{k}) = 1 \; . \tag{4.15}$$

In order to simplify the following discussion, we shall suppose that the property (4.15) is satisfied. Moreover we define $\mathbf{O}_{h,p} = \otimes_{i=1}^p O_h$.

Theorem 4.1 implies that, given $\bar{h} < 0$, the MRCF are well defined for $\bar{h} \le h \le 1$, if λ and $\tilde{\nu}_1(\underline{\vec{x}})$ are small enough. We want to show that, given λ small enough and $\log \beta \le c_0 |\lambda|^{-1}$, it is possible to choose $\tilde{\nu}_1(\underline{\vec{x}})$ so that the MRCF are well defined for $h_\beta \le h$, with h_β defined by (2.6). We shall try to fix $\tilde{\nu}_1(\underline{\vec{x}})$ in such a way that

$$\gamma^{-h_\beta+1}\tilde{\nu}_1(\underline{\vec{x}}) + \sum_{j=h_\beta+1}^{1} \gamma^{-h_\beta+j-1} \frac{1}{4} \sum_{\underline{\sigma}_j \in \mathbf{O}_{j,2}} (\mathfrak{F}_{2,j,\underline{\sigma}_j} * \beta_j^{2,0})(\tilde{\mathbf{v}}_j, .., \tilde{\mathbf{v}}_1; \underline{\vec{x}}) = 0 \; , \tag{4.16}$$

so that (4.13) becomes:

$$\begin{aligned} \tilde{\nu}_{h,\underline{\sigma}}(\underline{\vec{x}}) = &- \sum_{j=h_\beta+1}^{h} \gamma^{-h+j-1} \cdot \\ &\cdot \frac{1}{4} \sum_{\underline{\sigma}_j \in \mathbf{O}_{j,2}}^{*} \left(\mathfrak{F}_{2,h,\underline{\sigma}} * \mathfrak{F}_{2,j,\underline{\sigma}_j} * \beta_j^{2,0} \right) (\tilde{\mathbf{v}}_j, .., \tilde{\mathbf{v}}_1; \underline{\vec{x}}) \; , \end{aligned} \tag{4.17}$$

where, given $\underline{\sigma} = (\sigma_1, \sigma_2) \in \mathbf{O}_{h,2}$, $\sum^*_{\underline{\sigma}_j \in \mathbf{O}_{j,2}}$ is the sum restricted to the $\underline{\sigma}_j = (\sigma'_1, \sigma'_2) \in \mathbf{O}_{j,2}$ such that $S_{h,\sigma_i} \cap S_{j,\sigma'_i} \ne \emptyset$, $i = 1, 2$.

In order to present our results, we have to introduce a few other definitions. Given $h \le 1$ and $\omega \in O_h$, we denote by $D_{h,\sigma} \in \mathbb{R}^2$ the support of $\tilde{F}_{h,\sigma}(\vec{k})$. Moreover, if $p = 2, 4$, we call $\mathcal{M}_{h,p}$ the space of functions $G_{\underline{\sigma}}(\underline{\vec{x}}) : \mathbf{O}_{h,p} \times \mathbb{R}^{2p} \to \mathbb{R}$, such that

1) for any $\underline{\sigma} \in \mathbf{O}_{h,p}$, $G_{\underline{\sigma}}(\underline{\vec{x}})$ is translation invariant;

2) for any $\underline{\sigma} \in \mathbf{O}_{h,p}$, the Fourier transform $\hat{G}_{\underline{\sigma}}(\underline{\vec{k}})$ of $G_{\underline{\sigma}}(\underline{\vec{x}})$, defined so that,

$$G_{\underline{\sigma}}(\underline{\vec{x}}) = \int \frac{d\underline{\vec{k}}}{(2\pi)^{2p}} \, e^{-i\underline{\vec{k}} \cdot \underline{\vec{x}}} \, \hat{G}_{\underline{\sigma}}(\underline{\vec{k}}) \, \delta(\sum_{i=1}^{p} \varepsilon_i \vec{k}_i) \; , \tag{4.18}$$

with $\varepsilon_1 = -\varepsilon_2 = +$, if $p = 2$, and $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = +$, if $p = 4$, is a continuous function with support in the set $\otimes_{i=1}^p D_{h,\sigma_i}$.

Given $G \in \mathcal{M}_{h,p}$, we shall say that $G_{\underline{\sigma}}(\underline{\vec{x}})$ is the $\underline{\sigma}$-component of G. These definitions are such that $\tilde{\nu}_{h,\underline{\sigma}}(\vec{x})$ and $\tilde{z}_{h,\underline{\sigma}}(\vec{x})$ are the $\underline{\sigma}$-components of two functions $\tilde{\nu}_h$ and $\tilde{z}_h$ belonging to $\mathcal{M}_{h,2}$, while $\tilde{\lambda}_{h,\underline{\sigma}}(\vec{x})$ is the $\underline{\sigma}$-component of a function $\tilde{\lambda}_h \in \mathcal{M}_{h,4}$.

We shall define a norm on the set $\mathcal{M}_{h,p}$ by putting

$$||G||_{h,p} = \sup_{\substack{i,\sigma_i \in O_h \\ j,\vec{x}_j \in \mathbb{R}^2}} \sum_{\underline{\sigma} \backslash \sigma_i \in \mathbf{O}_{h,p-1}} \int d(\underline{\vec{x}} \backslash \vec{x}_j) |G_{\underline{\sigma}}(\underline{\vec{x}})| \; . \tag{4.19}$$

Finally we shall define $\mathfrak{M}_p$ as the set of sequences $G = \{G_h \in \mathcal{M}_{h,p}, h_\beta \leq h \leq 1\}$, such that the norm

$$||G||_p = \max_{h_\beta \leq h \leq 1} ||G_h||_{h,p} \tag{4.20}$$

is finite. We want to prove that the sequence $\tilde{\lambda} \equiv \{\tilde{\lambda}_h, h_\beta \leq h \leq 1\}$ is well defined as an element of $\mathfrak{M}_4$, while the sequences $\tilde{\nu} \equiv \{\tilde{\nu}_h, h_\beta \leq h \leq 1\}$ and $\tilde{z} \equiv \{\tilde{z}_h, h_\beta \leq h \leq 1\}$ are two elements of $\mathfrak{M}_2$.

We begin our analysis by "decoupling" equations (4.12) and (4.14) from (4.13), that is we imagine that, in the r.h.s. of (4.12) and (4.14), $\tilde{\nu}$ is an arbitrary element of $\mathfrak{M}_2$, acting as a parameter. We want to look for a solution ($\tilde{\lambda}(\tilde{\nu}) \in \mathfrak{M}_4$, $\tilde{z}(\tilde{\nu}) \in \mathfrak{M}_2$). We shall prove the following lemma.

Lemma 4.1 *There exist positive constants C_1 and C_2, depending only on first and second order terms in our expansion, such that, given two positive constants $C_3 \geq C_1$ and C_4, there exists λ_0 so that, if $|\lambda| \leq \lambda_0$,*

$$2C_2 C_3 \max\{1, C_4^{-1}\} |\lambda| |h_\beta| \leq 1 \tag{4.21}$$

and $||\tilde{\nu}||_2, ||\tilde{\nu}'||_2 \leq C_3 |\lambda|$, then, for $h_\beta \leq h \leq 1$,

$$||\tilde{\lambda}(\tilde{\nu})_h||_{h,4} \leq 2C_1 |\lambda| \gamma^{-\frac{1}{2}h} \quad , \quad ||\tilde{z}(\tilde{\nu})_h||_{h,2} \leq C_1 |\lambda| \; , \tag{4.22}$$

$$\begin{aligned} ||\tilde{\lambda}(\tilde{\nu})_h - \tilde{\lambda}(\tilde{\nu}')_h||_{h,4} &\leq C_4 \gamma^{-\frac{1}{2}h} \max_{j>h} ||\tilde{\nu}_h - \tilde{\nu}'_h||_{h,2} \; , \\ ||\tilde{z}(\tilde{\nu})_h - \tilde{z}(\tilde{\nu}')_h||_{h,2} &\leq C_4 \max_{j>h} ||\tilde{\nu}_h - \tilde{\nu}'_h||_{h,2} \; . \end{aligned} \tag{4.23}$$

Proof - Note that, if $\bar{F}_{h,\omega}(\vec{x})$ is the Fourier transform of $\tilde{F}_{h,\omega}(\vec{k})$, then

$$\left| \bar{F}_{h,\omega}(\vec{x}) \right| \leq \frac{C_N \gamma^{\frac{3}{2}h}}{1 + \left(\gamma^h |x'_1| + \gamma^{\frac{h}{2}} |x'_2| \right)^N} \; , \tag{4.24}$$

so that $\int d\vec{x} \left|\bar{F}_{h,\omega}(\vec{x})\right| \le c_F$ for some constant c_F independent of h and ω. It follows that there exists a constant C_1, such

$$||\tilde{\lambda}_1||_{1,4} \le 2C_1|\lambda|\gamma^{-1/2} \quad , \quad ||\mathfrak{F}_{4,h,\underline{\sigma}} * \tilde{\lambda}_1||_{h,4} \le C_1|\lambda|\gamma^{-h/2} \ , \tag{4.25}$$

having used also Lemma 3.1 for the second inequality.

We shall prove inductively that, if $||\tilde{\nu}||_2 \le C_3|\lambda|$, with $C_3 \ge C_1$, then $||\tilde{\lambda}(\tilde{\nu})_h||_{h,4} \le 2C_1|\lambda|\gamma^{-\frac{1}{2}h}$ and $||\tilde{z}(\tilde{\nu})_h||_{h,2} \le C_1|\lambda|$. This bound is satisfied for $h = 1$, by the first inequality of (4.25) and the fact that $\tilde{z}_1 = 0$; let us suppose that it is true for any $j > h$. Then, by using (4.12), (4.14), the second inequality of (4.25), Theorem 4.1 and the fact that $\beta_j^{(4,0)}$ and $\beta_j^{(2,1)}$ do not have first order contributions, we find

$$\begin{aligned} ||\tilde{\lambda}(\tilde{\nu})_h||_{h,4} &\le C_1|\lambda|\gamma^{-\frac{1}{2}h} + \gamma^{-\frac{1}{2}h} \sum_{j=h+1}^{1} \Big[C_{2,\lambda}C_1C_3|\lambda|^2 + \sum_{n=3}^{\infty}(c|\lambda|)^n\Big] \ , \\ ||\tilde{z}(\tilde{\nu})_h||_{h,2} &\le \sum_{j=h+1}^{1} \Big[C_{2,z}C_1C_3|\lambda|^2 + \sum_{n=3}^{\infty}(c|\lambda|)^n\Big] \ . \end{aligned} \tag{4.26}$$

Hence, if λ small enough and $2|\lambda||h_\beta|C_3 \max\{C_{2,\lambda}, C_{2,z}\} \le 1$, then $||\tilde{\lambda}(\tilde{\nu})_h||_{h,4} \le 2C_1|\lambda|\gamma^{-\frac{1}{2}h}$ and $||\tilde{z}(\tilde{\nu})_h||_{h,2} \le C_1|\lambda|$, up to $h = h_\beta$.

We still have to prove that, if $||\tilde{\nu}||_2, ||\tilde{\nu}'||_2 \le C_3|\lambda|$, then the bounds (4.23) are verified. We shall again proceed by induction, by using that $\tilde{\lambda}(\tilde{\nu})_1 - \tilde{\lambda}(\tilde{\nu}')_1 = 0$, since $\tilde{\lambda}_1$ is independent of $\tilde{\nu}$, and that $\tilde{z}(\tilde{\nu})_1 = 0$. Then, if we suppose that the bound is true for any $j > h$, we find

$$\begin{aligned} ||\tilde{\lambda}(\tilde{\nu})_h - \tilde{\lambda}(\tilde{\nu}')_h||_{h,4} &\le \gamma^{-\frac{1}{2}h} \max_{j>h} ||\tilde{\nu}_j - \tilde{\nu}'_j||_{j,2} \cdot \\ &\quad \cdot \sum_{j=h+1}^{1} \left[\tilde{C}_{2,\lambda}C_3 \max\{1, C_4\}|\lambda| + \sum_{n=3}^{\infty} c^n|\lambda|^{n-1}\right] \\ ||\tilde{z}(\tilde{\nu})_h - \tilde{z}(\tilde{\nu}')_h||_{h,2} &\le \max_{j>h} ||\tilde{\nu}_j - \tilde{\nu}'_j||_{j,2} \cdot \\ &\quad \cdot \sum_{j=h+1}^{1} \left[\tilde{C}_{2,z}C_3 \max\{1, C_4\}|\lambda| + \sum_{n=3}^{\infty} c^n|\lambda|^{n-1}\right] \ . \end{aligned} \tag{4.27}$$

Hence, if λ small enough and $2|\lambda||h_\beta|C_3 \max\{1, C_4^{-1}\} \max\{\tilde{C}_{2,\lambda}, \tilde{C}_{2,z}\} \le 1$, the bound is verified up to $h = h_\beta$. ▮

We want now to show that there is indeed a solution of the full set of equations (4.12)-(4.14), satisfying condition (4.16).

Theorem 4.2 *If $|\lambda|$ is small enough and $C_{1,2}\log\beta|\lambda| \le 1$, where $C_{1,2}$ is a constant depending only on first and second order contributions of perturbation theory, it is possible to choose $\tilde\nu_1(\underline{\vec x})$ so that the MRCF satisfy the hypothesis of Lemma 2.2, (2.90), (2.91) and (2.92).*

Proof - In order to prove the Theorem, it is sufficient to look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_2 \to \mathfrak{M}_2$, defined in the following way, if $\tilde\nu' \equiv \mathbf{T}(\tilde\nu)$:

$$\tilde\nu'_h = -\sum_{j=h_\beta+1}^{h} \gamma^{-h+j-1}$$
$$\frac{1}{4}\sum^{*}_{\underline\sigma_j\in\mathbf{O}_{j,2}} \left(\mathfrak{F}_{2,h,\underline\sigma} * \mathfrak{F}_{2,j,\underline\sigma_j} * \beta_j^{2,0}\right)\left(\tilde{\mathbf{v}}_j(\tilde\nu),..,\tilde{\mathbf{v}}_1(\tilde\nu);\underline{\vec x}\right) , \qquad (4.28)$$

where $\tilde{\mathbf{v}}_j(\tilde\nu) = (\tilde\lambda(\tilde\nu), \tilde\nu, \tilde z(\tilde\nu))$.

We want to prove that it is possible to choose the constant $C_\nu \ge 1$, so that, if C_1 is the constant defined in Lemma 4.1 and $|\lambda|$ is small enough, the set $\mathcal{F} = \{\tilde\nu \in \mathfrak{M}_2 : ||\tilde\nu||_2 \le 2C_1C_\nu|\lambda|\}$ is invariant under $\mathbf{T}$ and that $\mathbf{T}$ is a contraction on it. This is sufficient to prove the Theorem, since $\mathfrak{M}_2$ is a Banach space, as one can easily show.

By using Theorem 4.1 and Lemma 4.1 (with $C_3 = 2C_1C_\nu$), we see that, if $|\lambda|$ is small enough and $4C_2C_1C_\nu\max\{1, C_4^{-1}\}|\lambda||h_\beta| \le 1$ (C_4 will be chosen later),

$$||\tilde\nu'_h||_{h,2} \le \sum_{j=h_\beta+1}^{h} \gamma^{-h+j-1}\gamma^{\frac{h-j}{2}}\left[C_{1,\nu}C_1|\lambda| + \sum_{n=2}^{\infty} c^n|\lambda|^n\right] , \qquad (4.29)$$

where $\gamma^{\frac{h-j}{2}}$ is, up to a constant, a bound for the number of sectors $\sigma' \in O_j$ with non empty intersection with a given $\sigma \in O_h$, $h \ge j$ and $C_{1,\nu}$ is a constant depending on the first order contribution (*i.e.* the *tadpole*). So, if $C_\nu \ge \frac{C_{1,\nu}}{\gamma-\sqrt{\gamma}}$ and $\sum_{n=2}^{\infty} c^n|\lambda|^n \le C_{1,\nu}C_1|\lambda|$, then $||\tilde\nu'_h|| \le 2C_1C_\nu|\lambda|$.

We then show that $\mathbf{T}$ is a contraction on $\mathcal{F}$. In fact, given $\tilde\nu_1, \tilde\nu_2 \in \mathcal{F}$, by using again Theorem 4.1 and Lemma 4.1, we see that, under the same conditions supposed above,

$$||\tilde\nu'_{1,h} - \tilde\nu'_{2,h}|| \le \sum_{j=h_\beta+1}^{h} \gamma^{-h+j-1}\gamma^{\frac{h-j}{2}} \cdot$$
$$\cdot \max_{i\ge j} ||\tilde\nu_{1,i} - \tilde\nu_{2,i}|| \left[C_{1,\nu}C_4 + \sum_{n=2}^{\infty} c^n|\lambda|^{n-1}\right] , \qquad (4.30)$$

so that, if $\sum_{n=2}^{\infty} c^n|\lambda|^{n-1} \le C_{1,\nu}C_4/2$ and $C_4 = (2C_\nu)^{-1}$, then $||\tilde{\nu}_1' - \tilde{\nu}_2'|| \le \frac{3}{4}||\tilde{\nu}_1 - \tilde{\nu}_2||$, if $8C_2C_1C_\nu^2|\lambda||h_\beta| \le 1$. $\blacksquare$

Remark In [DR], where only the rotational invariant case is considered, the localization acts only on the kernels of the effective potential with $n = 2$. The consequence of this choice is that the effective potential is bounded at order n by $(c|\lambda||h_\beta|)^n$. Hence in [DR] the effective potentials are found to be convergent only for $T \ge O(e^{-\frac{1}{c|\lambda|}})$, c being a "bad" constant, so that such value is very far from the true critical temperature, which is supposed to be driven by the second order contribution to the effective potential, for λ small enough. Note in fact that c depend on bounds at every order in λ, while $C_{1,2}$ only depends on a few lower orders.

5 The two point Schwinger function

In this section we prove Theorem 1.1.

The Schwinger functions can be derived by the *generating function* defined as

$$\mathcal{W}(\phi) = \log \int P(d\psi) e^{-\mathcal{V}(\psi)-\mathcal{N}(\psi)+\int d\mathbf{x}\left[\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}\right]} \;, \tag{5.1}$$

where the variables $\phi^\sigma_{\mathbf{x}}$ are defined to be Grassmanian variables, anticommuting with themselves and $\psi^\sigma_{\mathbf{x}}$. In particular the two point Schwinger function is given by

$$S(\mathbf{x}-\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\mathcal{W}(\phi)\Big|_{\phi=0} \;. \tag{5.2}$$

We can get a multiscale expansion for $\mathcal{W}(\phi)$, by a procedure very similar to that used for the free energy, by taking into account that the interaction contains a new term, linear in ψ and ϕ. This novelty has the consequence that new terms appear in the expansion, containing one or more ϕ fields linked to the corresponding graphs through a single scale propagator. In order to study $S(\mathbf{x}-\mathbf{y})$, it is sufficient to analyze the structure of the terms with one or two ϕ fields.

Let us consider first the terms produced after integrating the scales greater or equal to $h+1$ and linear in ϕ. These terms can be obtained by taking one of the contributions $\mathcal{V}^{(h)}(\tau,\mathbf{P}) \equiv \sum_\Omega \mathcal{V}^{(h)}(\tau,\mathbf{P},\Omega)$ to the effective potential on scale h and by linking one of its external lines, say $\bar{f}$, with the ϕ field through a propagator of scale $j \ge h+1$, to be called the *external propagator*. However, one has to be careful in the choice of the localization point in the vertices v such that $\bar{f} \in P_v$ and $|P_v| \le 4$ (so that the action of $\mathcal{R}$ in v is not trivial); we choose it as that one which connects $\bar{f}$ with the ϕ field (hence no derivative can act on the external propagator, when one exploits the effect of the $\mathcal{R}$ operations as in §2.5). This choice has the aim of preserving the regularizing effect of the $\mathcal{R}$ operation, based on the fact that, if a field acquires a derivative as a consequence of the $\mathcal{R}$ operation on scale i, then it has to be contracted on a scale $j < i$, so producing an improvement of order $\gamma^{-(i-j)}$ in the bounds. Note also that, because of the localization operation, the scale j of the external propagator can be higher of the scale of the endpoint $\bar{v}$, such that $\bar{f} \in P_{\bar{v}}$.

The situation is different in the terms with two ϕ fields, connected through two external propagators of scale j_x and j_y greater than h and involving two ψ fields, of labels f_x and f_y. There are two different type of contributions. The first type is associated with trees τ satisfying the following conditions:

1. the root has scale $h_r \geq h$,
2. $\mathcal{I}_{v_0}$ (the set of internal lines in the vertex immediately following the root) is not empty,
3. 3) there is no external line in v_0, except f_x and f_y, the lines contracted in the external propagators.

These terms are produced, in the iterative integration procedure, at scale $h_r + 1$ and, after that scale are constant with respect to the integration process. The other type of terms is associated with trees such that

1. the root has scale h,
2. $|P_{v_0}| > 2$.

These terms depend on the integration field $\psi^{(\leq h)}$, so that are involved in the subsequent integration steps.

Given a tree τ (of any type) with two ϕ fields, the corresponding contributions to $\mathcal{W}(\phi)$ are obtained in a way slightly different from that described in the case of the effective potential. Given j_x and j_y, larger or equal to $h+1$, select two field labels f_x and f_y and call $\bar{v}$ the higher vertex, of scale $\bar{h}$, such that

1. $\bar{h} \leq \min\{j_x, j_y\}$,
2. f_x and f_y belong to $P_{\bar{v}}$.

Let $\mathcal{C}$ be the path on τ connecting $\bar{v}$ with v_0. Given $v \in \mathcal{C}$, we avoid to apply there the localization procedure, because the $\mathcal{R}$ operation, no matter we choose the localization point, would give rise to terms with a derivative acting on the external propagators (which is not convenient, see above). In all other vertices of τ the localization procedure is defined as in the case of the free energy expansion, by suitably choosing the localization point in the vertices following $\bar{v}$ and containing f_x or f_y, as explained above. Then we substitute f_x and f_y with two external propagators of scale j_x and j_y, respectively. Note that these propagators can acquire a derivative, as a consequence of the $\mathcal{R}$ operation acting on a vertex v, only if h_v is greater or equal to their scale (j_x or j_y).

The previous considerations imply that $S(\mathbf{x}-\mathbf{y})$ is given by the following sum:

$$S(\mathbf{x}-\mathbf{y}) = g(\mathbf{x}-\mathbf{y}) + \sum_{\bar{h}=h_\beta}^{1} \sum_{h_r=h_\beta-1}^{\bar{h}-1} \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_n^{\bar{h},h_r}} \sum_{\mathbf{P}} S_{\tau,\mathbf{P}}(\mathbf{x}-\mathbf{y}) \ , \tag{5.3}$$

where the family of labelled trees $\mathcal{T}_n^{\bar{h},h_r}$ and the families of external lines P_v can be described as in §2, with the following modifications (see Fig. 3).

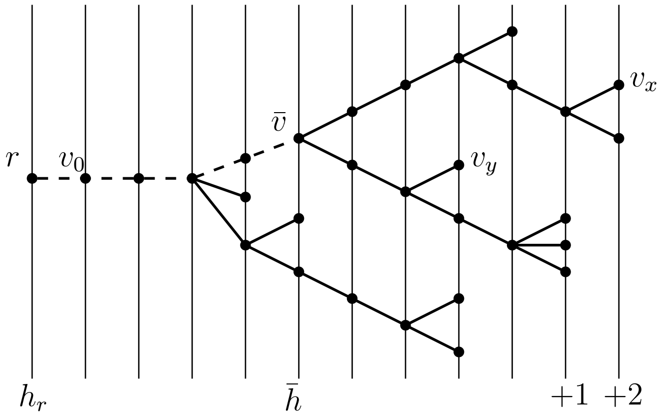

Figure 3: An example of tree contributing to $S(\mathbf{x}-\mathbf{y})$.

1) There are two field labels, f_x and f_y, two scale labels $j_x \geq \bar{h}$ and $j_y \geq \bar{h}$, and a vertex $\bar{v}$ such that $h_{\bar{v}} = \bar{h}$, $f_x, f_y \in P_{\bar{v}}$ and there is no other vertex $v > \bar{v}$ such that $h_v \leq \min\{j_x, j_y\}$ and $f_x, f_y \in P_v$; we shall call v_x and v_y the endpoints (possibly coinciding) that f_x and f_y belong to. Note that we are not introducing the sector decomposition for the external propagators and that the vertex $\bar{v}$ can be lower than the higher vertex preceding both v_x and v_y (opposite to what happens in Fig. 3).

2) Given f_x and f_y, let $\mathcal{C}$ be the path on the tree (see dashed line in Fig. 3), connecting $\bar{v}$ with the lowest vertex v_0, of scale h_r+1. If $v \in \mathcal{C}$ and $v \neq v_0$, $|P_v| \geq 4$, while $|P_{v_0}| = 2$.

Given $\tau \in \mathcal{T}_n^{\bar{h},h_r}$ and $\mathbf{P}$, we have

$$S_{\tau,\mathbf{P}}(\mathbf{x}-\mathbf{y}) = \left[g^{(j_x)} * W_{\tau,\mathbf{P},j_x,j_y} * g^{(j_y)}\right](\mathbf{x}-\mathbf{y}) \;, \tag{5.4}$$

where $*$ means the convolution in $\mathbf{x}$ space and $W_{\tau,\mathbf{P},j_x,j_y}$ differs from the kernel $K_{\tau,\mathbf{P}}^{(h_r+1)} = \sum_{\Omega\backslash\Omega_{v_0}} K_{\tau,\mathbf{P},\Omega}^{(h_r+1)}$ of $\mathcal{V}^{(h_r)}(\tau,\mathbf{P},\Omega)$ (see (2.48) and note that $K_{\tau,\mathbf{P},\Omega}^{(h_r+1)}$ does not depend on Ω_{v_0}) only because no $\mathcal{R}$ operation acts on the vertices of $\mathcal{C}$.

We now consider the Fourier transform $\hat{S}(\mathbf{k})$ of $S(\mathbf{x}-\mathbf{y})$, which can be written in the form:

$$\hat{S}(\mathbf{k}) = \hat{g}(\mathbf{k})\left(1+\lambda\hat{S}_1(\mathbf{k})\right) \;, \tag{5.5}$$

where $\hat{g}(\mathbf{k})$ is the free propagator. In order to prove Theorem 1.1, we have to show that $\hat{S}_1(\mathbf{k})$ is a bounded function.

Let us define $h_{\mathbf{k}} = \max\{h : \hat{g}^{(h)}(\mathbf{k}) \neq 0\}$. By using (5.4), it is easy to see that

$$\lambda\hat{g}(\mathbf{k})\hat{S}_1(\mathbf{k}) = \sum_{j_x,j_y=h_{\mathbf{k}}-1}^{h_{\mathbf{k}}} \sum_{n=1}^{\infty} \sum_{\bar{h}=h_\beta}^{\min\{j_x,j_y\}} \sum_{h_r=h_\beta-1}^{\bar{h}-1} \sum_{\tau\in\mathcal{T}_n^{\bar{h},h_r}} \sum_{\mathbf{P}} \hat{S}_{\tau,\mathbf{P},j_x,j_y}(\mathbf{k}) \;, \tag{5.6}$$

implying that

$$\begin{aligned} &|\hat{S}_1(\mathbf{k})| \leq c|\lambda|^{-1}\gamma^{h_{\mathbf{k}}} \cdot \\ &\cdot \sup_{j_x,j_y=h_{\mathbf{k}}-1,h_{\mathbf{k}}} \sum_{n=1}^{\infty} \sum_{\bar{h}=h_\beta}^{\min\{j_x,j_y\}} \sum_{h_r=h_\beta-1}^{\bar{h}-1} \sum_{\tau\in\mathcal{T}_n^{\bar{h},h_r}} \sum_{\mathbf{P}} ||S_{\tau,\mathbf{P},j_x,j_y}||_1 \;, \end{aligned} \tag{5.7}$$

where $||.||_1$ denotes the L_1 norm.

We can bound $\sum_{\tau\in\mathcal{T}_n^{\bar{h},h_r}} \sum_{\mathbf{P}} ||S_{\tau,\mathbf{P},j_x,j_y}||_1$ by proceeding as in §2.7. Since the combinatorial problems are of the same nature, we can describe in a simple way the result by dimensional arguments. We can take as a reference the bound of $J_{h_r,n}(2,0)$, see (2.79) and (2.101), that is the bound of the L_1 norm of the effective potential terms with two external lines on scale h_r and no external derivative, and multiply it by a factor $\gamma^{-j_x-j_y}$, which comes from the external propagators (the derivatives possibly acting on them are absorbed in the "gain factors" $\gamma^{-(h-h')}$, produced by the localization procedure, so that they do not give any contribution to the final bound). There are two relevant differences.

1) There is no regularization on the vertices with four external lines belonging to $\mathcal{C}$. This implies that one "looses" a factor γ^{-1}, with respect to the bound (2.101), for each vertex $v \in \mathcal{C}$ such that $|P_v| = 4$.

2) The external propagators sectors are not on the scale h_r, but they are exactly fixed. Hence, we have to modify the momentum conservation constraint (2.76) in the tree vertices v such that f_x or f_y belong to P_v, in order to remember this condition when we bound the sector sums. Then, we have to prove a lemma similar to Lemma 3.1, by substituting one sector sum with the constraint that one momentum is exactly fixed. It is not hard to see, by using Lemma 7.5 and by proceeding as in §7.4, that we get a bound of the same type of that of Lemma 3.1.

The previous considerations, together with the bound (2.101), allow to prove that

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}_n^{\bar h,h_r}} \sum_{\mathbf{P}} ||S_{\tau,\mathbf{P},j_x,j_y}||_1 \le (c|\lambda|)^n \gamma^{h_r-2h_{\mathbf{k}}} \cdot \\ &\cdot \sum_{\tau\in\mathcal{T}_n^{\bar h,h_r}} \sum_{\substack{\mathbf{P}\\ |P_{v_0}|=2}} \sum_{T\in\mathbf{T}} \prod_{\substack{v\\ \text{not e. p.}}} \frac{1}{s_v!}\gamma^{\delta_v^*} , \end{aligned} \tag{5.8}$$

where $\delta_v^* = \delta(|P_v|)$, if $v \notin \mathcal{C}$, otherwise $\delta_v^* = \delta(|P_v|) + \chi(|P_v| = 4)$. By using (2.102), it is easy to see that, if we define $\tilde\delta_v = \delta_v^* - 1/2$, if $v \in \mathcal{C}$ and $\tilde\delta_v = \delta_v^*$ otherwise, $\tilde\delta_v < 0$ for all $v \in \tau$. Hence, the bound (2.104) is still valid, if we put $\tilde\delta_v$ in place of $\delta(|P_v|)$, and we get

$$\begin{aligned} |\hat S_1(\mathbf{k})| &\le c\gamma^{-h_{\mathbf{k}}} \sum_{\bar h=h_\beta}^{h_{\mathbf{k}}} \sum_{h_r=h_\beta-1}^{\bar h-1} \gamma^{h_r}\gamma^{(\bar h-h_r)/2} \le \\ &\le c\sum_{\bar h=h_\beta}^{h_{\mathbf{k}}} \gamma^{-(h_{\mathbf{k}}-\bar h)} \sum_{h_r=h_\beta-1}^{\bar h-1} \gamma^{-(\bar h-h_r)/2} \le c\ . \end{aligned} \tag{5.9}$$

▮

6 The rotation-invariant case

We consider now the *Jellium model*, which is defined in the continuum with $\varepsilon(\vec k) = |\vec k|^2/(2m)$ and $v(\vec x - \vec y) = \tilde v(|\vec x - \vec y|)$, implying rotation invariance symmetry. In particular, $p_F = |\vec p_F(\theta)|$ does not depend on θ and the two point contribution to the effective potential $\hat W_2^{(h)}(k_0, \vec k) = \int d\mathbf{x}\tilde W_2^{(h)}(\mathbf{x})\exp(i\mathbf{k}\mathbf{x})$, see (2.17) and the line before (2.15), is of the form $\mathcal{W}_2^{(h)}(k_0, |\vec k|)$, where

$\mathcal{W}_2^{(h)}(k_0,\rho)$ is a function of two variables. We show that in such a case we can choose the counterterm $\hat{\nu}(\vec{k})$ as a constant ν, if the temperature T is big enough, *i.e.* $T \geq e^{-\frac{1}{c_1|\lambda|}}$, where c_1 is a constant depending on a bound to all orders of multiscale perturbation theory.

In order to get this result, we must change the localization definition, so that

1. $\mathcal{L}W_{2n}^{(h)} = 0$ if $n \geq 2$;
2. if $n = 1$, $\mathcal{L}\hat{W}_2^{(h)}(k_0,\vec{k}) = \mathcal{W}_2^{(h)}(0,p_F) \equiv \gamma^h \nu_h$.

We want now to analyze the properties of the $\mathcal{R}$ operator. If we put, as in (2.32), for any $\vec{k} \in S_{h,\omega}$, $\omega \in O_h$, $\vec{k} = \vec{k}' + \vec{p}_F(\theta_{h,\omega})$, we can write

$$\begin{aligned}
\mathcal{R}\hat{W}_2^{(h)}(k_0,\vec{k}) &= \int_0^1 dt \frac{d}{dt}\mathcal{W}_2^{(h)}\left(tk_0, |t\vec{k}' + \vec{p}_F(\theta_{h,\omega})|\right) = \\
&= \int_0^1 dt \Big[k_0 \partial_{k_0}\mathcal{W}_2^{(h)}(tk_0,\rho(t)\ ) + \\
&+ \frac{(tk_1' + p_F)k_1' + t(k_2')^2}{\rho(t)} \partial_\rho \mathcal{W}_2^{(h)}(tk_0,\rho(t)\ )\Big] \ ,
\end{aligned} \tag{6.1}$$

where, for any vector $\vec{v}$, we are defining $v_1 \equiv \vec{v}\cdot\vec{n}(\theta_{h,\omega})$, $v_2 \equiv \vec{v}\cdot\vec{\tau}(\theta_{h,\omega})$ (see (2.31), recalling that now $\vec{e}_r(\theta) = \vec{n}(\theta)$ and $\vec{e}_t(\theta) = \vec{\tau}(\theta)$) and $\rho(t) \equiv \sqrt{(tk_1' + |\vec{p}_F|)^2 + (tk_2')^2}$.

It is easy to see that the term $\partial_\rho \mathcal{W}_2^{(h)}(tk_0,\rho(t)\ )$ in (6.1) can be rewritten in the following form:

$$\begin{aligned}
&\partial_\rho \mathcal{W}_2^{(h)}(tk_0,\rho(t)\ ) = \cos\theta(t)\partial_{k_1}\hat{W}_2^{(h)}(tk_0, t\vec{k}' + \vec{p}_F(\theta_{h,\omega})) + \\
&+ \sin\theta(t)\partial_{k_2}\hat{W}_2^{(h)}(tk_0, t\vec{k}' + \vec{p}_F(\theta_{h,\omega})) = \\
&= \int d\mathbf{y}\left(iy_1 \frac{tk_1' + p_F}{\rho(t)} + iy_2\frac{tk_2'}{\rho(t)}\right)\tilde{W}_2^{(h)}(\mathbf{y})e^{itk_0y_0 + i(t\vec{k}' + \vec{p}_F(\theta_{h,\omega}))\vec{y}} \ ,
\end{aligned} \tag{6.2}$$

where $\theta(t)$ is the angle between $\vec{n}(\theta_{h,\omega})$ and $t\vec{k}' + \vec{p}_F(\theta_{h,\omega})$. Substituting (6.2) in (6.1) we get, if $\mathbf{p}_\omega = (0, \vec{p}_F(\theta_{h,\omega}))$,

$$\begin{aligned}
&\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{\sigma,\omega\in O_h}\int \frac{d\mathbf{k}}{(2\pi)^3}\hat{\psi}^{(\leq h)+}_{\mathbf{k}-\mathbf{p}_\sigma,\sigma}\hat{\psi}^{(\leq h)-}_{\mathbf{k}-\mathbf{p}_\omega,\omega}\mathcal{R}\hat{W}_2^{(h)}(\mathbf{k}) = \\
&= \sum_{\sigma,\omega\in O_h}\int_0^1 dt\int \frac{d\mathbf{k}'}{(2\pi)^3}\hat{\psi}^{(\leq h)+}_{\mathbf{k}'+\mathbf{p}_\omega-\mathbf{p}_\sigma,\sigma}\hat{\psi}^{(\leq h)-}_{\mathbf{k}',\omega}\int d\mathbf{y}\tilde{W}_2^{(h)}(\mathbf{y})e^{i(t\mathbf{k}'+\mathbf{p}_\omega)\mathbf{y}} \cdot \\
&\cdot\left[ik_0y_0 + \frac{(tk_1' + p_F)k_1' + t(k_2')^2}{\rho(t)}\left(\frac{tk_1' + p_F}{\rho(t)}iy_1 + \frac{tk_2'}{\rho(t)}iy_2\right)\right] \ .
\end{aligned} \tag{6.3}$$

Let us define the operators $D_i(t)$, $i=1,2$, so that

$$\begin{aligned}
D_1(t)\psi^{(\le h)\varepsilon}_{\mathbf{x},\omega} &= i\varepsilon \int \frac{d\mathbf{k}'}{(2\pi)^3} e^{i\varepsilon \mathbf{k}'\mathbf{x}} \frac{tk_1'+p_F}{\rho(t)} \frac{(tk_1'+p_F)k_1'+t(k_2')^2}{\rho(t)} \hat{\psi}^{(\le h)\varepsilon}_{\mathbf{k}',\omega} \;, \\
D_2(t)\psi^{(\le h)\varepsilon}_{\mathbf{x},\omega} &= i\varepsilon \int \frac{d\mathbf{k}'}{(2\pi)^3} e^{i\varepsilon \mathbf{k}'\mathbf{x}} \frac{tk_2'}{\rho(t)} \frac{(tk_1'+p_F)k_1'+t(k_2')^2}{\rho(t)} \hat{\psi}^{(\le h)\varepsilon}_{\mathbf{k}',\omega} \;. \qquad (6.4)
\end{aligned}$$

Hence, (6.3) can be written as

$$\begin{aligned}
&\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)}) = -\sum_{\sigma,\omega\in O_h} \int_0^1 dt \int d\mathbf{x} \int d\mathbf{y}\; e^{i\mathbf{p}_\sigma \mathbf{y} - i\mathbf{p}_\omega \mathbf{x}} \tilde{W}_2^{(h)}(\mathbf{y}-\mathbf{x}) \psi^{(\le h)+}_{\mathbf{y},\sigma} \cdot \\
&\cdot \left[(y_0-x_0)\partial_0 + (y_1-x_1)D_1(t) + (y_2-x_2)D_2(t)\right] \psi^{(\le h)-}_{\xi(t),\omega} = \\
&= \sum_{\sigma,\omega\in O_h} \int_0^1 dt \int d\mathbf{x} \int d\mathbf{y}\; e^{i\mathbf{p}_\sigma \mathbf{y} - i\mathbf{p}_\omega \mathbf{x}} \tilde{W}_2^{(h)}(\mathbf{y}-\mathbf{x}) \cdot \qquad (6.5) \\
&\cdot \left[(y_0-x_0)\partial_0 + (y_1-x_1)D_1(t) + (y_2-x_2)D_2(t)\right] \psi^{(\le h)+}_{\eta(t),\sigma} \psi^{(\le h)-}_{\mathbf{x},\omega} \;,
\end{aligned}$$

where

$$\begin{aligned}
\eta(t) &\equiv \mathbf{y} + t(\mathbf{x}-\mathbf{y}) \\
\xi(t) &\equiv \mathbf{x} + t(\mathbf{y}-\mathbf{x}) \;. \qquad (6.6)
\end{aligned}$$

It is easy to prove the following dimensional bound.

Lemma 6.1 *Given non negative integers N, n_0, n_1, n_2, $m = n_0+n_1+n_2$, there exists a constant $C_{N,m}$, such that*

$$|\partial_0^{n_0} D_1^{n_1} D_2^{n_2} g_\omega^{(h)}(\mathbf{x})| \le C_{N,m} \frac{\gamma^{h(\frac{3}{2}+n_0+n_1+\frac{3}{2}n_2)}}{1+\left[(\gamma^h x_0)^2 + (\gamma^h x_1)^2 + (\gamma^{\frac{h}{2}} x_2)^2\right]^N} \;, \qquad (6.7)$$

where D_i^n denotes the product of n factors $D_i(t_j)$, $j=1,\dots,n$.

Remark - Each operator $D_i(t)$ improves the bound of the covariance by a factor at least γ^h; this is what we need to obtain the right dimensional gain from renormalization operations, which also produce a factor $\gamma^{-h'}$ on a scale $h'>h$. This is a consequence of rotational invariance; in fact a naive Taylor expansion would apparently produce a term of the form $(y_2-x_2)\partial_{x_2}$, which would give rise to a "bad factor" $\gamma^{-h'+h/2}$ in the bounds.

We can now repeat the analysis of the previous sections, in a much more simple context. In fact it is easy to see that it is possible to fix ν_1 in such a way that ν_h stay bounded for $h_\beta \le h \le 1$. Furthermore we can easily perform the bounds for the n^{th}-order contributions to the kernel of the effective potentials or to the two point Schwinger functions. In both cases we find that, unless for the external dimensional factors, the n^{th}-order contributions are bounded by $(c|\lambda|)^n (\log\beta)^{n-1}$, where the diverging factor $(\log\beta)^{n-1}$ is due to the choice of not localizing the four-legs clusters and of localizing the two-legs clusters only at the first order. So the result of Theorem 1.1 in the rotational invariant case easily follows.

7 Some technical lemmata.

7.1 Geometrical properties of the dispersion relation

Let $\mathcal{B} = \{\vec p \in \mathbb{R}^2 : |\varepsilon(\vec p) - \mu| \le e_0\}$; the hypotheses on $\varepsilon(\vec p)$ described in §1.2 imply that there is a C^∞ diffeomorphism between $\mathcal{B}$ and the compact set $\mathcal{A} = \mathbb{T}^1 \times [-e_0, e_0]$, defined by

$$\vec p = \vec q(\theta, e) = u(\theta, e)\vec e_r(\theta) \quad , \quad (\theta, e) \in \mathcal{A} \ . \tag{7.1}$$

Moreover, the symmetry property (1.11) implies that

$$\vec q(\theta + \pi, e) = -\vec q(\theta, e) \ , \tag{7.2}$$

a property that will have an important role in the following.

Let us now introduce some more geometrical definitions, which we shall need in the following. For any fixed e, we can locally define the arc length $s(\theta, e)$ on $\Sigma(e)$; we shall denote $\partial/\partial s$ the partial derivative with respect to s, at fixed e, and we shall sometime use the prime to denote the partial derivative with respect to θ. If $\vec\tau(\theta, e) = \partial\vec p(\theta, e)/\partial s$ is the unit tangent vector at $\Sigma(e)$ in $\vec q(\theta, e)$, we have

$$\begin{aligned} s'(\theta, e)\vec\tau(\theta, e) &= \frac{\partial \vec p}{\partial\theta}(\theta, e) = u'(\theta, e)\vec e_r(\theta) + u(\theta, e)\vec e_t(\theta) \ , \\ s'(\theta, e) &= \sqrt{u'(\theta, e)^2 + u(\theta, e)^2} \ , \end{aligned} \tag{7.3}$$

where $\vec e_t(\theta) = (-\sin\theta, \cos\theta)$.

Analogously, if $\vec n(\theta, e)$ is the outgoing unit normal vector at $\Sigma(e)$ in $\vec q(\theta, e)$ and $1/r(\theta, e)$ is the curvature (which satisfies the convexity condition (1.9)), we have

$$\begin{aligned} s'(\theta, e)\vec n(\theta, e) &= u(\theta, e)\vec e_r(\theta) - u'(\theta, e)\vec e_t(\theta) \ , \\ \frac{\partial^2 \vec p}{\partial\theta^2}(\theta, e) &= s''(\theta, e)\vec\tau(\theta, e) - \frac{s'(\theta, e)^2}{r(\theta, e)}\vec n(\theta, e) \ . \end{aligned} \tag{7.4}$$

Lemma 7.1 *The angle $\alpha(\theta,e)$ between $\vec{n}(\theta,e)$ and $\vec{e}_r(0)$ is a monotone increasing function of θ, such that, if $||\theta_1-\theta_2||$ denotes the distance on $\mathbb{T}^1$.*

$$c_1||\theta_2-\theta_1|| \le ||\alpha(\theta_2,e)-\alpha(\theta_1,e)|| \le c_2||\theta_2-\theta_1|| \; ; \tag{7.5}$$

moreover, $\alpha(\theta+\pi,e)-\alpha(\theta,e)=\pi$.

Proof - By using (7.3) and (7.4) and Taylor expansion, one can easily prove that, if $\alpha_i=\alpha(\theta_i,e)$,

$$\begin{aligned}
\sin(\alpha_2-\alpha_1) &= \vec{n}(\theta_2,e)\cdot\vec{\tau}(\theta_1,e) = (\theta_2-\theta_1)\frac{s'(\theta_1,e)}{r(\theta_1,e)} + O(\theta_2-\theta_1)^2 \; , \quad (7.6)\\
\cos(\alpha_2-\alpha_1) &= \vec{n}(\theta_2,e)\cdot\vec{n}(\theta_1,e) = 1-\frac{(\theta_2-\theta_1)^2}{2}\frac{s'^2(\theta_1,e)}{r^2(\theta_1,e)} + O(\theta_2-\theta_1)^3 \; ,
\end{aligned}$$

which implies (7.5) for $|\theta_2-\theta_1|$ small, hence even for any value of $\theta_2-\theta_1$, together with the monotonicity property. The fact that $\alpha(\theta+\pi,e)-\alpha(\theta,e)=\pi$ is a trivial consequence of (7.2). ▌

We denote by $\vec{p}_F(\theta)=\vec{q}(\theta,0)$ the generic point of the Fermi surface $\Sigma_F\equiv\Sigma(0)$. Moreover, to simplify the notation, from now on we shall in general suppress the variable e when it is equal to 0; for example, we shall put $\vec{p}_F(\theta)=u(\theta)\vec{e}_r(\theta)$. Let us consider an s-sector $S_{h,\omega}$, see (2.72).

Lemma 7.2 *If $\vec{p}=\rho\vec{e}_r(\theta)\in S_{h,\omega}$, $h\le 0$, $\omega\in O_h$, then*

$$|\rho-u(\theta)|\le c\gamma^h \quad , \quad |\theta-\theta_{h,\omega}|\le \pi\gamma^{h/2} \; . \tag{7.7}$$

Proof - The bound on θ follows directly from the definition of $S_{h,\omega}$. On the other hand, the identity

$$\varepsilon(\vec{p})-\mu=\varepsilon(\rho\vec{e}_r(\theta))-\varepsilon(u(\theta)\vec{e}_r(\theta))=\int_\rho^{u(\theta)} d\rho'\; \vec{e}_r(\theta)\vec{\nabla}\varepsilon\left(\rho'\vec{e}_r(\theta)\right) \; , \tag{7.8}$$

and the property (1.10) of $\varepsilon(\vec{p})$, easily imply the bound on $\rho-u(\theta,0)$. ▌

The following lemma shows that, if $\vec{p}\in S_{h,\omega}$, the difference between $\vec{p}$ and $\vec{p}_F(\theta_{h,\omega})$ is of order γ^h in the direction normal to Σ_F in the point $\vec{p}_F(\theta_{h,\omega})$, while it is of order $\gamma^{h/2}$ in the tangent direction.

Lemma 7.3 *If* $\vec{p}\in S_{h,\omega}$, $h\le 0$, $\omega\in O_h$, *then*

$$\vec{p}=\vec{p}_F(\theta_{h,\omega})+k_1\vec{n}(\theta_{h,\omega})+k_2\vec{\tau}(\theta_{h,\omega}) \quad , \quad |k_1|\le c\gamma^h \quad , \quad |k_2|\le c\gamma^{h/2} \ . \tag{7.9}$$

Moreover,

$$\left|\frac{\partial}{\partial k_2}\varepsilon(\vec{p}_F(\theta_{h,\omega})+k_1\vec{n}(\theta_{h,\omega})+k_2\vec{\tau}(\theta_{h,\omega}))\right|\le c\gamma^{h/2} \ . \tag{7.10}$$

Proof - If $\vec{p}=\rho\vec{e}_r(\theta)$, by Lemma 7.2 $|\vec{p}-\vec{p}_F(\theta)|\le c\gamma^h$. Hence, to prove (7.9), it is sufficient to prove that $|[\vec{p}_F(\theta)-\vec{p}_F(\theta_{h,\omega})]\vec{n}(\theta_{h,\omega})|\le c\gamma^h$ and $|[\vec{p}_F(\theta)-\vec{p}_F(\theta_{h,\omega})]\vec{\tau}(\theta_{h,\omega})|\le c\gamma^{h/2}$. These bounds immediately follows from the the following ones, which can be easily proved, by using (7.3), (7.4) and some Taylor expansions:

$$[\vec{p}_F(\theta_1)-\vec{p}_F(\theta_2)]\cdot\vec{n}(\theta_2)=O(\theta_1-\theta_2)^2 \ , \tag{7.11}$$

$$[\vec{p}_F(\theta_1)-\vec{p}_F(\theta_2)]\cdot\vec{\tau}(\theta_2)=O(\theta_1-\theta_2) \ . \tag{7.12}$$

It is sufficient to put here $\theta_1=\theta$, $\theta_2=\theta_{h,\omega}$ and to recall that $\theta-\theta_{h,\omega}=O(\gamma^{h/2})$.

Let us now observe that, if we derive with respect to θ the identity $\varepsilon(u(\theta,e)\vec{e}_r(\theta))=e$, we get, for any $\vec{p}\in\mathcal{B}$,

$$[\vec{\nabla}\varepsilon(\vec{p})\vec{\tau}(\theta,e)]s'(\theta,e)=0 \quad\Rightarrow\quad \vec{\nabla}\varepsilon(\vec{p})=a(\theta,e)\vec{n}(\theta,e) \ , \tag{7.13}$$

$a(\theta,e)$ being a smooth function, strictly positive by (1.10). Hence, if $\vec{p}\in S_{h,\omega}$, by using the first line of (7.6), (7.13) and the fact that $|\varepsilon(\vec{p})-\mu|\le c\gamma^h$, $|\theta-\theta_{h,\omega}|\le c\gamma^{h/2}$,

$$\begin{aligned}\frac{\partial\varepsilon(\vec{p})}{\partial k_2} &= \vec{\nabla}\varepsilon(\vec{p})\cdot\vec{\tau}(\theta_{h,\omega})=a(\theta,e)\vec{n}(\theta,e)\cdot\vec{\tau}(\theta_{h,\omega})=\\ &= a(\theta,e)\vec{n}(\theta)\cdot\vec{\tau}(\theta_{h,\omega})+O(\gamma^h)=O(\gamma^{h/2}) \ ,\end{aligned} \tag{7.14}$$

which proves (7.10). ▌

Given $\vec{p}\in S_{h,\omega}$, we shall also consider the projection on the Fermi surface, defined as

$$\vec{p}_\perp=\vec{p}_F(\theta_\perp)=\vec{p}-x\,\vec{n}(\theta_\perp) \ . \tag{7.15}$$

Note that (7.15) has to be thought as an equation for $\theta_\perp$ and x, given $\vec{p}$; it is easy to prove that, as a consequence of the condition (1.9), this equation has a smooth unique solution, if e_0 is small enough, what we shall suppose from now on.

Lemma 7.4 *If $\vec{p} = \rho \vec{e}_r(\theta) \in S_{h,\omega}$, $h \le 0$, and x and $\theta_\perp$ are defined as in (7.15), then $|x| \le c\gamma^h$ and $|\theta_\perp - \theta_{h,\omega}| \le c\gamma^{h/2}$.*

Proof - (7.9) and (7.15) imply that

$$\begin{aligned} k_1 &= [\vec{p}_F(\theta_\perp) - \vec{p}_F(\theta_{h,\omega})] \cdot \vec{n}(\theta_{h,\omega}) + x\vec{n}(\theta_\perp) \cdot \vec{n}(\theta_{h,\omega}) \;, & (7.16) \\ k_2 &= [\vec{p}_F(\theta_\perp) - \vec{p}_F(\theta_{h,\omega})] \cdot \vec{\tau}(\theta_{h,\omega}) + x\vec{n}(\theta_\perp) \cdot \vec{\tau}(\theta_{h,\omega}) \;. & (7.17) \end{aligned}$$

By using the (7.11), (7.12) and (7.6), one can easily complete the proof of the lemma. ▌

7.2 Proof of Lemma 2.1

The bounds on k_1 and k_2 in (7.9) imply that $\int d\mathbf{p} F_{h,\omega}(\mathbf{p}) \le c\gamma^{5h/2}$. On the other hand, if $F_{h,\omega}(\mathbf{p}) \neq 0$, $|-ip_0 + \varepsilon(\vec{p}) - \mu| \ge c\gamma^h$, so that (2.33) implies the bound $|g^{(h)}_\omega(\mathbf{x})| \le c\gamma^{3h/2}$. It is also very easy to prove that $|\partial^n \hat{g}^{(h)}_\omega(\mathbf{p})/\partial p_0^n|$ and $|\partial^n \hat{g}^{(h)}_\omega(\mathbf{p})/\partial k_1^n|$ are bounded by $c\gamma^{-h(n+1)}$. Hence, using simple integration by parts arguments, one can show that $|x_0^n g^{(h)}_\omega(\mathbf{x})| \le c\gamma^{h(3/2-n)}$ and $|{x'_1}^n g^{(h)}_\omega(\mathbf{x})| \le c\gamma^{h(3/2-n)}$. Moreover, it is easy to prove that

$$|\partial^n \hat{g}^{(h)}_\omega(\mathbf{p})/\partial k_2^n| \le c\gamma^{-h}\Big[\gamma^{-h} \sup_{\vec{p}\in S_{h,\omega}} |\frac{\partial \varepsilon(\vec{p})}{\partial k_2}|\Big]^n \;, \tag{7.18}$$

which implies the bound $|{x'_2}^n g^{(h)}_\omega(\mathbf{x})| \le c\gamma^{h(3/2-n/2)}$. Finally, by using Lemma 7.3, it is easy to prove that the previous bounds have to be multiplied by γ^{mh}, if one substitutes $g^{(h)}_\omega(\mathbf{x})$ with $\partial^m g^{(h)}_\omega(\mathbf{x})/\partial x_0^m$ or $\partial^m g^{(h)}_\omega(\mathbf{x})/\partial {x'_1}^m$, while they have to be multiplied by $\gamma^{mh/2}$ if $g^{(h)}_\omega(\mathbf{x})$ is changed in $\partial^m g^{(h)}_\omega(\mathbf{x})/\partial {x'_2}^m$.

The bound (2.35) is a simple consequence of the previous considerations. ▌

7.3 The parallelogram lemma

Let us consider the map F, defined on the two dimensional torus $\mathbb{T}^2$, with values in $\mathbb{R}^2$, such that, if $(\theta_1, \theta_2) \in \mathbb{T}^2$ and $\vec{b} = F(\theta_1, \theta_2)$, then

$$\vec{b} = \vec{p}_F(\theta_1) + \vec{p}_F(\theta_2) \;. \tag{7.19}$$

The differential $J(\theta_1, \theta_2)$ of F is a matrix, whose columns coincide with $s'(\theta_1)\vec{\tau}(\theta_1)$ and $s'(\theta_2)\vec{\tau}(\theta_2)$. Then Lemma 7.1 implies that $\det J \neq 0$, hence F is invertible, around any point $(\theta_1, \theta_2) \in \mathcal{T}$, where

$$\mathcal{T} = \{(\theta_1, \theta_2) \in \mathbb{T}^2 : \sin(\theta_1 - \theta_2) \neq 0\} \;. \tag{7.20}$$

Moreover, if $||\theta_1 - \theta_2|| = \pi$, $\vec{b} = 0$, while, if $\theta_1 = \theta_2 = \theta$, $\vec{b} = 2u(\theta)\vec{e}_r(\theta)$. Finally, $\mathcal{T}$ is the union of two disjoint subsets, which are obtained one from the other by exchanging θ_1 with θ_2, and each one of them is in a one to one correspondence through F with the open set

$$\mathcal{D} = \{\vec{p} = \rho\vec{e}_r(\theta) : \, 0 < \rho < 2u(\theta), \, \theta \in \mathbb{T}^1\} \, . \tag{7.21}$$

The following Lemma will have an important role in the following.

Lemma 7.5 *Let* $(\bar{\theta}_1, \bar{\theta}_2) \in \mathcal{T}$, $\vec{b} = \vec{p}_F(\bar{\theta}_1) + \vec{p}_F(\bar{\theta}_2)$,

$$\phi = \min\{||\bar{\theta}_1 - \bar{\theta}_2||, \pi - ||\bar{\theta}_1 - \bar{\theta}_2|||\} > 0 \, , \tag{7.22}$$

$$\vec{r} = r_1\vec{n}(\bar{\theta}_1) + r_2\vec{\tau}(\bar{\theta}_1) \quad , \quad |r_1| \le c_1\eta\phi \quad , \quad |r_2| \le \eta \le c_2\phi \, . \tag{7.23}$$

Then there exist c_0, $\bar{c}_2$ *and* η_0, *such that, if* $c_2 \le \bar{c}_2$ *and* $\eta \le \eta_0$, *then* $\vec{b}+\vec{r} \in \mathcal{D}$ *and*

$$\vec{b} + \vec{r} = \vec{p}_F(\theta_1) + \vec{p}_F(\theta_2) \quad , \quad |\theta_i - \bar{\theta}_i| \le c_0\eta \, . \tag{7.24}$$

Proof - We shall consider only the case $\phi = ||\bar{\theta}_1 - \bar{\theta}_2||$; the case $\phi = \pi - ||\bar{\theta}_1 - \bar{\theta}_2||$ can be easily reduced to this one, by using the symmetry property (7.2). We shall also choose the sign of $\bar{\theta}_1 - \bar{\theta}_2$, so that $\phi = \bar{\theta}_2 - \bar{\theta}_1$.

Let us define $\delta_i = \theta_i - \bar{\theta}_i$, $\delta = \sqrt{\delta_1^2 + \delta_2^2}$; then we can write, by using (7.19), (7.3) and (7.4), if $\vec{b} + \vec{r} \in \mathcal{D}$ (which is certainly true, if $\vec{r}$ is small enough),

$$\begin{aligned} \vec{r} &= \frac{d\vec{p}_F(\bar{\theta}_1)}{d\theta}\delta_1 + \frac{d\vec{p}_F(\bar{\theta}_2)}{d\theta}\delta_2 + O(\delta^2) = \\ &= \delta_1 s'(\bar{\theta}_1)\vec{\tau}(\bar{\theta}_1) + \delta_2 s'(\bar{\theta}_2)\vec{\tau}(\bar{\theta}_2) + O(\delta^2) \, . \end{aligned} \tag{7.25}$$

Let us now put $\delta_i = \eta x_i$, $r_1 = \eta\phi\tilde{r}_1$, $r_2 = \eta\tilde{r}_2$; condition (7.23) takes the form $|\tilde{r}_1| \le c_1$ and $|\tilde{r}_2| \le 1$. Since, by hypothesis, $\eta \le c_2\phi$, the condition $\vec{b}+\vec{r} \in \mathcal{D}$ is satisfied, together with (7.24), if and only if the following system of two equations in the unknowns x_1, x_2 has a unique solution:

$$\begin{aligned} x_2 &= \frac{\tilde{r}_1\phi}{s'(\bar{\theta}_2)\sin[\alpha(\bar{\theta}_1) - \alpha(\bar{\theta}_2)]} + O(c_2) \, , \\ x_1 &= \frac{\tilde{r}_2}{s'(\bar{\theta}_1)} - \frac{\tilde{r}_1\phi\cos[\alpha(\bar{\theta}_1) - \alpha(\bar{\theta}_2)]}{s'(\bar{\theta}_1)\sin[\alpha(\bar{\theta}_1) - \alpha(\bar{\theta}_2)]} + O(\eta) + O(c_2) \, , \end{aligned} \tag{7.26}$$

where $\alpha(\theta)$ is defined as in Lemma 7.1 and $O(c_2)$, $O(\eta)$ are of second order as functions of the x_i's.

By using Lemma 7.1, we see that the right sides of (7.26) are bounded for $\phi \to 0$. Hence, by Dini Theorem, (7.26) allow to uniquely determine x_1 and x_2 for any $\phi > 0$, given $\vec{r}$, if η and c_2 are small enough, and $|\delta_i| \le c_0\eta$, with c_0 independent of c_2. ▮

7.4 Proof of Lemma 3.1 (sectors counting lemma)

Let h', h, L be integers such that $h' \le h \le 0$. Given $\omega_1 \in O_{h'}$ and $\tilde\omega_i \in O_h$, $i = 2, \dots L$, let $A_{h,h'}(\omega_1; \tilde\omega_2, \dots, \tilde\omega_L)$ be the set of the sequences $(\omega_2, \dots, \omega_L)$, such that i) $S_{h',\omega_i} \subset S_{h,\tilde\omega_i}$ for $i = 2, \dots, L$; ii) there exists, for $i = 1, \dots, L$, a vector $\vec k^{(i)} \in S_{h',\omega_i}$, so that $\sum_{i=1}^{L} \vec k^{(i)} = 0$.

If $L = 2$, the momentum conservation ii) and the symmetry property (1.11) immediately imply that $|A_{h,h'}(\omega_1; \tilde\omega_2, \dots, \tilde\omega_L)| = 1$. Hence, in order to prove Lemma 3.1 it is sufficient to consider the case $L \ge 4$; we have to prove that

$$|A_{h,h'}(\omega_1; \tilde\omega_2, \dots, \tilde\omega_L)| \le c^L \gamma^{\frac{h-h'}{2}(L-3)} . \qquad (7.27)$$

Let $\theta_i \equiv \theta_{h',\omega_i}$, so that θ_i is the center of the θ-interval, which the polar angle of $\vec p$ has to belong to, if $\vec p \in S_{h',\omega_i}$. For any pair (i, j), we define

$$\phi_{i,j} = \min\{||\theta_i - \theta_j||, \pi - ||\theta_i - \theta_j||\} . \qquad (7.28)$$

By a reordering of the sectors, which is unimportant since we are looking for a bound proportional to c^L, we can get the condition (recall that $L \ge 4$):

$$\phi \equiv \phi_{L-1,L} \ge \phi_{i,j} \quad , \quad \forall i, j \in [2, L] . \qquad (7.29)$$

Note that, given $\tilde\omega \in O_h$,

$$|\omega \in O_{h'} : S_{h',\omega} \subset S_{h,\tilde\omega}| = \gamma^{\frac{h-h'}{2}} . \qquad (7.30)$$

Hence, given any positive constant c_0, if we define

$$\mathcal{A}_< = \{(\omega_2, \dots, \omega_L) \in A_{h,h'}(\omega_1, \tilde\omega_2, \dots, \tilde\omega_L) \,:\, \phi \le L c_0^{-1} \gamma^{h'/2}\} , \qquad (7.31)$$

we have:

$$|\mathcal{A}_<| \le \gamma^{\frac{h-h'}{2}(L-3)} (c L c_0^{-1})^2 , \qquad (7.32)$$

where $(cLc_0^{-1})^2$ is a bound on the number of possible choices of ω_{L-1} and ω_L, given $\omega_1, \dots, \omega_{L-2}$. Hence, in order to prove (7.27), it is sufficient to prove that, if c_0 is small enough, a similar bound is valid for the set

$$\mathcal{A}_> = \{(\omega_2, \dots, \omega_L) \in A_{h,h'}(\omega_1, \tilde\omega_2, \dots, \tilde\omega_L) \,:\, \phi \ge L c_0^{-1} \gamma^{h'/2}\} . \qquad (7.33)$$

We have

$$|\mathcal{A}_>| \le m_L \gamma^{\frac{h-h'}{2}(L-3)} , \qquad (7.34)$$

where m_L is a bound on the number of choices of ω_{L-1} and ω_L, given $\omega_1, \dots, \omega_{L-2}$.

In order to get m_L, we consider a particular choice of $\omega_2, \ldots, \omega_{L-2} \in O_{h'}$ and we suppose that the set $\mathcal{E} = \{(\omega_{L-1}, \omega_L) : (\omega_2, \ldots, \omega_L) \in \mathcal{A}_>\}$ is not empty. Moreover, we define

$$\phi_0 = \max_{(\omega_{L-1},\omega_L)\in\mathcal{E}} \phi_{L-1,L} \ , \tag{7.35}$$

By definition, for any choice of $(\omega_{L-1}, \omega_L) \in \mathcal{E}$, we can find L vectors $\vec{k}^{(1)}, \ldots, \vec{k}^{(L)}$, such that $\vec{k}^{(i)} \in S_{h',\omega_i}$ and

$$\sum_{i=1}^{L} \vec{k}^{(i)} = 0 \ . \tag{7.36}$$

Moreover, by Lemma 7.3, for $i = 1, \ldots, L$, we can write

$$\vec{k}^{(i)} = \vec{p}_F(\theta_i) + x_i \vec{n}(\theta_i) + y_i \vec{\tau}(\theta_i) \quad , \quad |x_i| \le c\gamma^{h'} \quad , \quad |y_i| \le c\gamma^{h'/2} \ . \tag{7.37}$$

Hence, since $\phi_0 \ge L c_0^{-1} \gamma^{h'/2}$, we get

$$|\vec{k}^{(i)} \wedge \vec{p}_F(\theta_2)| = |\vec{p}_F(\theta_i)|\,|\vec{p}_F(\theta_2)| \sin\phi_{i,2} + O(\gamma^{\frac{h'}{2}}) \le c\phi_0 \ , \tag{7.38}$$

for $i = 2, \ldots, L$, and, by using (7.36),

$$|\vec{k}^{(1)} \wedge \vec{p}_F(\theta_2)| = \left| \sum_{i=2}^{L} \vec{k}^{(i)} \wedge \vec{p}_F(\theta_2) \right| \le cL\phi_0 \ , \tag{7.39}$$

so that $\phi_{1,2} \le cL\phi_0$.

Lemma 7.1, (7.37) and (7.39) easily imply that

$$\begin{aligned} &\vec{k}^{(i)} = \vec{p}_F(\theta_i) + \bar{x}_i \vec{n}(\theta_2) + \bar{y}_i \vec{\tau}(\theta_2), \\ &|\bar{x}_i| \le \begin{cases} c\phi_0\gamma^{h'/2} & \text{if } i > 1 \\ cL\phi_0\gamma^{h'/2} & \text{if } i = 1 \end{cases} \quad , \quad |\bar{y}_i| \le \begin{cases} c\gamma^{h'/2} & \text{if } i > 1 \\ cL\gamma^{h'/2} & \text{if } i = 1 \end{cases} . \end{aligned} \tag{7.40}$$

Let us now define

$$\vec{a} = -\sum_{i=1}^{L-2} \vec{p}_F(\theta_i) \quad , \quad \vec{b} = \vec{k}_\perp^{(L-1)} + \vec{k}_\perp^{(L)} \quad , \quad \vec{r} = \vec{b} - \vec{a} \ , \tag{7.41}$$

where $\vec{k}_\perp$ denotes the projection on the Fermi surface, see (7.15). By using Lemma 7.4, the momentum conservation (7.36) and (7.40), we get

$$\vec{r} = r_1 \vec{n}(\theta_2) + r_2 \vec{\tau}(\theta_2) \quad , \quad |r_1| \le cL\phi_0\gamma^{h'/2} \quad , \quad |r_2| \le cL\gamma^{h'/2} \ . \tag{7.42}$$

Note now that the vector $\vec{a}$ defined in (7.41) is fixed, if the indices $\omega_1, \ldots, \omega_{L-2}$ are fixed. Hence, if we put $\vec{p}_F(\bar{\theta}_i) = \vec{k}_\perp^{(i)}$, $i = L-1, L$, m_L can be calculated by studying the possible solutions of the equation

$$\vec{p}_F(\bar{\theta}_{L-1}) + \vec{p}_F(\bar{\theta}_L) = \vec{a} + \vec{r}\,, \tag{7.43}$$

as $\vec{r}$ varies satisfying (7.42). Let $(\bar{\theta}_{L-1}^{(0)}, \bar{\theta}_L^{(0)})$ be a particular solution of (7.43), such that $\vec{k}^{(i)} \in S_{h',\omega_i}$, $i = L-1, L$, with $\phi_{L-1,L} = \phi_0$, and put $\vec{b}_0 = \vec{p}_F(\bar{\theta}_{L-1}^{(0)}) + \vec{p}_F(\bar{\theta}_L^{(0)}) = \vec{a} + \vec{r}_0$, so that (7.43) can be written as $\vec{p}_F(\bar{\theta}_{L-1}) + \vec{p}_F(\bar{\theta}_L) = \vec{b}_0 + (\vec{r} - \vec{r}_0)$. The definition of ϕ_0 implies that $\vec{r} - \vec{r}_0$ can be represented as $\vec{r} - \vec{r}_0 = r_1' \vec{n}(\bar{\theta}_L^{(0)}) + r_2' \vec{\tau}(\bar{\theta}_L^{(0)})$, with $|r_1'| \le cL\phi_0\gamma^{h'/2}$ and $|r_2'| \le cL\gamma^{h'/2}$. Hence a simple application of Lemma 7.5 shows that the solutions of (7.43) belong, up to a exchange between $\bar{\theta}_{L-1}$ and $\bar{\theta}_L$, to a connected set and that $m_L \le cL^2$, if $c_0 \le \bar{c}_2/c_1$, where c_1 is the constant c of (7.42) and $\bar{c}_2$ is defined in Lemma 7.5. ▌

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