THE CLASSICAL KAM THEORY AT THE DAWN OF THE TWENTY-FIRST CENTURY

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To V. I. Arnol'd on the occasion of his 65th birthday

Abstract. We survey several recent achievements in the KAM theory. The achievements chosen pertain to Hamiltonian systems only and are closely connected with the content of Kolmogorov's original theorem of 1954. They include the weak nondegeneracy conditions, Gevrey smoothness of families of perturbed invariant tori, the "exponential condensation" of perturbed tori, destruction mechanisms of the resonant unperturbed tori, the excitation of the elliptic normal modes of the unperturbed tori, and "atropic" invariant tori (i.e., tori that are neither isotropic nor coisotropic). The exposition is informal and nontechnical, and, as a rule, the methods of proofs are not discussed.

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1. Introduction

The descriptive term "*KAM theory*" was first used in works [61, 152] for the theory of quasi-periodic motions in smooth and analytic dynamical systems founded by A. N. Kolmogorov [67], V. I. Arnol′d [3, 4], and J. Moser [89]. The contribution of each of the three authors is vividly reviewed in [8, 9]. During almost fifty years that have elapsed since Kolmogorov's breakthrough four-paged note [67] appeared, the KAM theory has turned a vast collection of ideas, methods, and results pertaining to quasi-periodic motions and therefore to "small divisors" in dynamical systems. It is often regarded as one of the most important attainments in the qualitative theory of ordinary differential equations over the whole second half of the twentieth century [134]. For instance, according to the KAM theory, a typical (in the sense to be made precise) Hamiltonian or reversible system admits many invariant tori of various dimensions which are organised into rather regular (although Cantor-type) multi-parameter families. This implies, in particular, that the thesis *"A generic Hamiltonian system is ergodic on (almost) every compact and connected energy level hypersurface"* widespread formerly is incorrect [3, 4, 6, 10, 18, 84, 92, 111, 149, 150, 152]. During the last decade and a half, a deep and versatile progress in the KAM theory has taken place, two most significant achievements being perhaps the KAM theory for infinite dimensional systems (see [73, 74, 110] and references therein) and the so called "direct methods" in proving the existence and persistence theorems for quasi-periodic motions (see [34–36, 44, 47, 48] and references therein).

The present survey, however, touches upon none of these impressive discoveries. Either it does not consider the two theories "accompanying" the KAM theory, namely, the Nekhoroshev theory [10, 11, 49, 82, 95–97] and the theory of Arnol′d's diffusion [5, 10, 33, 37, 78, 82, 83, 152]. Instead, our goal is to describe the most substantial results in the KAM theory of the last decade and a half that are closely connected with the content of Kolmogorov's theorem [67] of 1954, i.e., with perturbations of completely integrable finite dimensional Hamiltonian systems. In fact, the studies of quasi-periodic motions in *Hamiltonian* systems constitute the main part of the KAM theory although there are also well developed theories of quasi-periodic motions in *reversible* (see [20–22, 60, 80, 109, 112, 119–122, 126, 127, 144] and references therein), *volume-preserving* (see [19, 21, 22, 60, 129] and references therein), and *dissipative* [19, 21, 22, 60] systems.

The exposition is as non-technical and informal as possible, in particular, we are trying to avoid precise formulations of the statements (which are usually quite long in the KAM theory) and even precise definitions and, as a rule, confine ourselves with references to the original papers. The reader is not assumed to possess any prior knowledge of the theory. Apart from this Introduction, the paper consists of six sections, each being devoted to one of the results we are discussing or a group of close results (a "topic").

Although the literature devoted to the KAM theory is now enormous, there are not so many monographs or expository works. One may mention relatively recent books [22, 78] and memoir [19] which treat some problems of the finite dimensional part of the theory very minutely. For the first acquaintance with the subject, manual [7], book [132], survey [18], and very recent tutorials [81, 111] are highly recommended. A detailed survey of the Hamiltonian finite dimensional KAM theory including the latest results is presented in [10] (the first Russian edition of this book of 1985 and the English editions of 1988, 1993, and 1997 are much briefer). Reviews of various special aspects of the KAM theory [21, 31, 92, 125, 127] appeared recently. Works [10, 22, 81] contain an extensive bibliography.

All the Hamiltonian systems in the sequel will be assumed to be autonomous.

2. The first topic: weak nondegeneracy

Recall the set-up of original Kolmogorov's theorem [67] of 1954. Consider a Hamiltonian system with $n \geq 2$ degrees of freedom and the Hamilton function of the form

$$H(I, \varphi) = H_0(I) + H_1(I, \varphi), \qquad |H_1| \ll 1 \tag{1}$$

where the "action" variable $I = (I_1, \ldots, I_n)$ ranges over a bounded and connected open domain $D \subset \mathbb{R}^n$ while the "angle" variable $\varphi = (\varphi_1, \ldots, \varphi_n)$ ranges over the standard n-torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. The symplectic structure is assumed to be $dI \wedge d\varphi = dI_1 \wedge d\varphi_1 + \cdots + dI_n \wedge d\varphi_n$. The dynamics of the *unperturbed* system governed by the Hamilton function H_0 is very simple: the whole phase space $D \times \mathbb{T}^n$ is smoothly foliated into invariant n-tori $\{I = \text{const}\}$, and the motion on each torus $\{I = I^*\}$ is determined by the equation

$$\dot{\varphi} = \omega(I^*) := \partial H_0(I^*)/\partial I.$$

One says that the system with the Hamilton function H_0 (which is a function of I only) is *completely integrable*. We will suppose that $\omega(I) \neq 0$ in D.

In the sequel, the following standard terms will be used. An invariant n-torus $\mathcal{T}$ of a given flow is said to carry *conditionally periodic motions* if in some coordinates $\phi \in \mathbb{T}^n$ on this torus, the induced dynamics is afforded by the equation $\dot{\phi} = \text{const} = \varpi$. For instance, each invariant torus $\{I = \text{const}\}$ of a completely integrable Hamiltonian system carries conditionally periodic motions. The quantity $\varpi \in \mathbb{R}^n$ [defined uniquely up to an action of $\mathrm{GL}(n, \mathbb{Z})$] is called the *frequency vector*, and its components $\varpi_1, \ldots, \varpi_n$ are called the *frequencies*. If these frequencies are incommensurable (i.e., rationally independent), the torus $\mathcal{T}$ in question is said to carry *quasi-periodic motions*.

How does a small φ-dependent perturbation H_1 affect the completely integrable dynamics? H. Poincaré called this question *the main problem of dynamics* [104]. It did not yield to the mathematicians' efforts until Kolmogorov's landmark note [67]. To put Kolmogorov's result into a context suitable for the further discussion, introduce the following fundamental definition [122].

Definition. A function H_0 is said to be *KAM-stable* if it possesses the following two properties:

1) for any sufficiently small perturbation H_1, the corresponding perturbed system governed by (1) admits many invariant n-tori close to the unperturbed tori $\{I = \mathrm{const}\}$, and these tori carry quasi-periodic motions;

2) the Lebesgue measure of the complement to the union $\mathcal{W}$ of the perturbed tori vanishes as the perturbation magnitude tends to zero.

As far as the author knows, the term "*KAM-stable*" was first used by S. B. Kuksin [72] in 1992 [in a more general context of unperturbed Hamiltonian systems of the form (18) (see Section 6 below)].

Kolmogorov's famous theorem [67] of 1954 asserts essentially that in the analytic category (i.e., if both H_0 and H_1 are analytic and the smallness of H_1 is understood in the real analytic topology), the following condition is *sufficient* for the KAM-stability of H_0:

$$\det \frac{\partial^2 H_0}{\partial I^2} = \det \frac{\partial \omega}{\partial I} \neq 0 \quad \text{everywhere in } D. \tag{2}$$

This condition called the *Kolmogorov nondegeneracy* means that the unperturbed frequency map $I \mapsto \omega(I)$ is a local diffeomorphism of D. Another condition on H_0 guaranteeing KAM-stability is the so called *isoenergetic nondegeneracy* [3, 4]:

$$\det \begin{pmatrix} \partial^2 H_0/\partial I^2 & \partial H_0/\partial I \\ \partial H_0/\partial I & 0 \end{pmatrix} = \det \begin{pmatrix} \partial \omega/\partial I & \omega \\ \omega & 0 \end{pmatrix} \neq 0 \quad \text{everywhere in } D. \tag{3}$$

This condition means that the map

$$I \mapsto (\omega_1(I) : \omega_2(I) : \ldots : \omega_n(I)) \in \mathbb{RP}^{n-1}$$

is a local diffeomorphism of each unperturbed energy level hypersurface $\{H_0(I) = \mathrm{const}\}$ in D. For isoenergetically nondegenerate H_0, perturbed systems with the Hamilton functions (1) admit many invariant tori *on each energy level hypersurface* $\{H(I, \varphi) = \mathrm{const}\}$ in $D \times \mathbb{T}^n$ [3, 4]. The two conditions (2) and (3) are independent, i.e., there are Kolmogorov nondegenerate functions H_0 for which the determinant (3) (called sometimes the *Arnol'd determinant* [22, 82]) is identically zero, and there are isoenergetically nondegenerate functions H_0 for which the determinant (2) (the Hessian of H_0) is identically zero. Explicit examples for any n are presented in, e.g., [22, 132]. Both the Kolmogorov and isoenergetic nondegeneracy conditions can be interpreted in terms of the Lie algebras of the symmetries of the unperturbed system [14–16].

The analyticity requirement in Kolmogorov's theorem can be relaxed greatly, namely, it can be replaced by C^r-smoothness of H for $r < \infty$ sufficiently large [89, 91, 93, 108, 109, 117]. The best result here known by now is that any r greater than $2n$ is enough [91, 108]. To be more precise, J. Moser [91] and J. Pöschel [108] proved the KAM-stability of H_0 (under the Kolmogorov nondegeneracy condition) for analytic H_0 and C^r-smooth H_1 for any $r > 2n$, while D. Salamon [117] showed that one can allow H_0 to be C^r-smooth as well. Note that r here is not necessarily an integer, and C^r-smoothness is to be understood here and henceforth in the Hölder sense for non-integer r.

What is much more important for us is that the nondegeneracy condition (2) in Kolmogorov's theorem can be relaxed greatly as well. The ultimate result here (essentially due to H. Rüssmann [115, 116]) is as follows:

Theorem 1. *For analytic H_0, the following condition is* necessary and sufficient *for KAM-stability: the image $\omega(D) \subset \mathbb{R}^n$ of the unperturbed frequency map $\omega\colon D \to \mathbb{R}^n$ does not lie in any hyperplane passing through the origin.*

This condition called the *Rüssmann nondegeneracy* is very weak: for example, the image $\omega(D)$ of an unperturbed frequency map ω nondegenerate in the sense of Rüssmann can be a smooth submanifold of $\mathbb{R}^n$ of any prescribed dimension s from 1 to n.

Example. For $n = s$ one may choose H_0 to be an arbitrary Kolmogorov nondegenerate function (e.g., a nondegenerate quadratic form of I). For $1 \le s \le n-1$, denote by $u = u(I_2, \dots, I_{n-s+1})$ the solution of the equation

$$\sum_{i=2}^{n-s+1} (i-1)u^{i-2} I_i = u$$

that is defined and analytic in $I_2, \dots, I_{n-s+1}$ near the point $I_2 = \cdots = I_{n-s+1} = 0$ and vanishes at that point. The local existence and uniqueness of such a solution are ensured by the Implicit Function Theorem. One has $u(I_2, 0, \dots, 0) \equiv I_2$. Consider the Hamilton function

$$H_0(I) = I_1 + \int_0^{I_2} u(x, I_3, \dots, I_{n-s+1})\, dx + \tfrac{1}{2} \sum_{i=n-s+2}^{n} I_i^2 \tag{4}$$

(domain D here is a neighbourhood of the origin). Since

$$\frac{\partial u}{\partial I_i} = \frac{(i-1)u^{i-2}}{1 - \sum_{j=3}^{n-s+1} (j-1)(j-2)u^{j-3} I_j} = (i-1)u^{i-2}\frac{\partial u}{\partial I_2} = \frac{\partial (u^{i-1})}{\partial I_2}$$

for each $2 \le i \le n-s+1$ and $u = 0$ for $I_2 = 0$, one sees that

$$\omega_i(I) = \frac{\partial H_0(I)}{\partial I_i} = \left[\begin{array}{ll} (u(I_2, \dots, I_{n-s+1}))^{i-1} & \text{for} \quad 1 \le i \le n-s+1 \\ I_i & \text{for} \quad n-s+2 \le i \le n. \end{array} \right. \tag{5}$$

Thus, the Hamilton function (4) is Rüssmann nondegenerate and the image of its frequency map is of dimension s.

This example (for the particular case of $s = 1$, $n = 3$) was first presented in [25]. The case of $s = 1$ and arbitrary n was considered in [123]. A similar example (for any s and n) is contained in [22, 122].

As far as the *sufficiency* of the Rüssmann nondegeneracy condition is concerned, analyticity of H_0 cannot be relaxed to C^∞-smoothness (see below). For the *necessity* of the Rüssmann nondegeneracy condition, on the other hand, analyticity is not important at all. If the unperturbed Hamilton function H_0 (of any smoothness class) does not meet the Rüssmann condition then there are arbitrarily small perturbations H_1 (of the same smoothness class) that remove *all* the invariant n-tori of the unperturbed system [22, 122]. The corresponding perturbed systems admit no invariant n-tori (not just no invariant tori carrying quasi-periodic motions but no invariant tori at all).

Example [22, 122]**.** Let $\langle v, \omega(I)\rangle \equiv 0$ for some vector $v \in \mathbb{R}^n \setminus \{0\}$ (here and henceforth, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$). One can choose a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ arbitrarily close to the identity $n \times n$ matrix in such a way that the vector Av will be proportional to an integer vector $k \in \mathbb{Z}^n \setminus \{0\}$. Now set

$$H_1(I, \varphi) = H_0(A^{-1}I) - H_0(I) + \epsilon \cos\langle k, \varphi\rangle \tag{6}$$

with $\epsilon \in \mathbb{R} \setminus \{0\}$ arbitrarily small. Of course, this function H_1 is defined in $(D \cap AD) \times \mathbb{T}^n$ rather than in $D \times \mathbb{T}^n$, but this is of no importance if one uses the precise definition of KAM-stability [122] we would not like to dwell upon here (this definition takes care of the behaviour of the systems near the boundary of D). Denote $\partial H_0(A^{-1}I)/\partial I$ by $\widetilde{\omega}(I)$, then $\langle Av, \widetilde{\omega}(I)\rangle = \langle v, \omega(A^{-1}I)\rangle$ whence $\langle k, \widetilde{\omega}(I)\rangle \equiv 0$. The Hamilton function $H = H_0 + H_1$ determines the equations of motion

$$\dot{I} = \epsilon k \sin\langle k, \varphi\rangle, \qquad \dot{\varphi} = \widetilde{\omega}(I). \tag{7}$$

One has $d\langle k, \varphi\rangle/dt = \langle k, \widetilde{\omega}(I)\rangle \equiv 0$. Consequently, $\dot{I}$ is an integral of motion, and if $\sin\langle k, \varphi(0)\rangle = c \neq 0$ then $I(t) = I(0) + \epsilon ctk$. Thus, system (7) has no compact invariant manifolds containing points (I, φ) with $\sin\langle k, \varphi\rangle \neq 0$. On the other hand, function H_1 (6) can be made arbitrarily small: of course, the length of vector k, generally speaking, tends to infinity as Av tends to v but we can paralyse this by a suitable choice of ϵ.

Although this argument is very simple, the necessity of the Rüssmann nondegeneracy for KAM-stability was first observed, as far as the author knows, no earlier than in 1995 [122].

Thus, for analytic functions H_0, one has the following alternative. If the image of the gradient map $\omega = \partial H_0/\partial I$ does not lie in any hyperplane passing through the origin of the frequency space, then any Hamiltonian system sufficiently close to the completely integrable system with Hamilton function H_0 admits many invariant n-tori carrying quasi-periodic motions. In fact, perturbations here are allowed to be no smoother than C^r (with r large enough). On the other hand, if the image of ω lies in a hyperplane of $\mathbb{R}^n$ passing through the origin, then there are analytic Hamiltonian systems arbitrarily close to the completely integrable system with Hamilton function H_0 that possess no invariant n-tori at all.

For highly degenerate functions H_0, one can achieve even more. Suppose that H_0 is linear: $H_0(I) = \varpi_1 I_1 + \varpi_2 I_2 + \cdots + \varpi_n I_n$, so that $\omega(I) = (\varpi_1, \varpi_2, \ldots, \varpi_n) = \text{const}$. A. B. Katok [65, 66] showed in 1969–73 for certain domains D that for linear H_0 with positive ϖ_i the following holds. For any $r < \infty$ there exist C^∞-perturbations H_1 arbitrarily small in the C^r-topology and such that the Hamiltonian system with the Hamilton function H (1) is *ergodic on each energy level hypersurface* (after an appropriate compactification of the phase space).

A minor modification of example (6)–(7) enables one to explain why analyticity of H_0 is essential for the sufficiency of the Rüssmann nondegeneracy condition. Let $D_1 \subset D$ be an open subset of D such that the interior of $D \setminus D_1$ is not empty. It is very easy to construct a Rüssmann nondegenerate C^∞-function $H_0 \colon D \to \mathbb{R}$ whose restriction to D_1 is degenerate in the sense of Rüssmann. Such a function will not be KAM-stable.

The history of the proof of the "hard" part of Theorem 1 (that Rüssmann nondegenerate analytic functions H_0 are KAM-stable) is rather dramatic, this statement was proven independently five times during a decade. The first proof was given by H. Rüssmann himself in the mid eighties, presented by him in a number of talks (e.g., in the well-known talk "On twist-Hamiltonians" at the *Colloque international*: *Mécanique céleste et systèmes hamiltoniens* in Marseille in 1990) and announced in [115]. A detailed written account of the proof, however, appeared no earlier than in a 1998 Johannes Gutenberg-Universität preprint (Mainz) which was published in 2001 [116]. In fact, memoir [116] (as well as notes [115]) treats not only the context of Kolmogorov's theorem but also the more general context of unperturbed Hamiltonian systems of the form (18) (see Section 6 below). Another proof was found by M. R. Herman and presented in his talk at an international conference on dynamical systems in Lyons in 1990 but (as far as the author knows) has remained unpublished. The first published proof appeared in 1994 in a paper by Ch.-Q. Cheng and Y.-S. Sun [27]. Other sources are a 1994 ETH-Zürich preprint by J. Xu, J. You, and Q. Qiu (some extract from this preprint without a proof was published in 1997 [147]) and M. B. Sevryuk's papers [122, 123] of 1995–96 (see also [128]). The proofs by Herman and Sevryuk are similar and differ drastically from other proofs (see surveys [21, 22]).

Interesting topological conditions (in terms of the so called topological Conley index) for the presence of invariant n-tori in nearly integrable Hamiltonian systems with n degrees of freedom were found by P. I. Plotnikov [103]. His results imply, in particular, the existence of invariant tori under rather weak nondegeneracy conditions on H_0 (close to the Rüssmann nondegeneracy).

Let $F\colon H_0(D) \to \mathbb{R}$ be a smooth function whose derivative vanishes nowhere in the interval $H_0(D)$. The Hamiltonian flows governed by the Hamilton functions H (1) and $F(H)$ coincide up to a reparametrisation. The problems of looking for invariant tori in these two Hamiltonian systems are therefore equivalent. It is sometimes useful to consider the unperturbed Hamilton function $F \circ H_0$ with suitable F instead of H_0. If $\partial H_0(I)/\partial I = \omega(I)$ then $\partial F(H_0(I))/\partial I = F'(H_0(I))\omega(I)$. Consequently, $F \circ H_0$ is isoenergetically (Rüssmann) nondegenerate if and only if H_0 is isoenergetically (respectively Rüssmann) nondegenerate. On the other hand, $F \circ H_0$ can be Kolmogorov nondegenerate for Kolmogorov degenerate H_0. For instance, set

$$H_0(I) = I_1 + \tfrac{1}{2}\sum_{i=2}^{n} I_i^2, \qquad F(H) = e^H. \tag{8}$$

Then the Hessian of H_0 is identically zero whereas the Hessian of $F \circ H_0$ is equal to $e^{nH_0(I)}$. Note that H_0 in (8) is isoenergetically nondegenerate and coincides with the function (4) for $s = n-1$. However, there are Rüssmann nondegenerate completely integrable Hamilton functions that cannot be reduced to Kolmogorov nondegenerate ones by this trick. For example, consider the function H_0 (4) for $n \geq 3$ and any $1 \leq s \leq n-2$. This function is nondegenerate in the sense of Rüssmann and the image of its frequency map (5) is a smooth submanifold of the frequency space of codimension $n - s \geq 2$. Both the determinants (2) and (3) for this H_0 are identically zero. The dimension of the image of the frequency map of $F \circ H_0$ is no greater than $s+1$ for every F. Consequently, the Hessian of $F \circ H_0$ is still identically zero for any choice of F.

By the way, consider a Hamilton function of the form $H_0(I) = F(\varpi_1 I_1 + \varpi_2 I_2 + \cdots + \varpi_n I_n)$. For such a function, Katok's constructions [65, 66] mentioned above are still applied while its frequency map ω is, generally speaking, no longer a constant (although it is a constant on each unperturbed energy level hypersurface $\{H_0(I) = \mathrm{const}\}$).

Although the Rüssmann nondegeneracy condition implies KAM-stability for analytic unperturbed Hamilton functions only, it admits smooth analogues. Namely, the following condition guarantees the KAM-stability of a C^∞-function H_0 [122, 123]: for every $I \in D$, there is a positive integer $\mathcal{N} = \mathcal{N}(I)$ such that the collection of the $(n+\mathcal{N})!/(n!\mathcal{N}!)$ partial derivatives

$$\frac{\partial^{|\alpha|}\omega(I)}{\partial I^\alpha}, \qquad \alpha \in \mathbb{Z}_+^n, \quad 0 \leq |\alpha| \leq \mathcal{N}, \tag{9}$$

span $\mathbb{R}^n$ [i.e., the linear hull of the vectors (9) coincides with $\mathbb{R}^n$]. Here $\mathbb{Z}_+$ denotes the set of non-negative integers and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

For analytic maps ω, this condition is equivalent to the Rüssmann nondegeneracy (and is therefore also necessary for KAM-stability). But for C^∞-maps ω, this condition is much stronger than the Rüssmann nondegeneracy. Note that the Kolmogorov nondegeneracy condition (2) is tantamount to that for every $I \in D$, the collection of the n partial derivatives $\partial^{|\alpha|}\omega(I)/\partial I^\alpha$ with $|\alpha| = 1$ span $\mathbb{R}^n$. The isoenergetic nondegeneracy condition (3) implies that for every $I \in D$, the collection of the $n+1$ partial derivatives $\partial^{|\alpha|}\omega(I)/\partial I^\alpha$ with $0 \le |\alpha| \le 1$ span $\mathbb{R}^n$.

For any number $r < \infty$ large enough, there is an integer $\mathcal{F} = \mathcal{F}(r) \le \lfloor r \rfloor - 1$ such that the following condition guarantees the KAM-stability of a C^r-function H_0 [122, 123]: for every $I \in D$, the collection of the partial derivatives (9) with $\mathcal{N} = \mathcal{F}$ span $\mathbb{R}^n$. Here $\lfloor r \rfloor$ denotes the integer part of r, and $\mathcal{F}(r) \to \infty$ monotonously as $r \to \infty$.

The "optimal" (simultaneously sufficient and necessary) conditions of KAM-stability in the C^r-categories ($r \le \infty$) are not known yet.

There are two main differences in the behaviour of Hamiltonian systems close to Kolmogorov (or isoenergetically) nondegenerate completely integrable ones and the behaviour of those close to general Rüssmann nondegenerate completely integrable ones. The first difference pertains to the measure of the *resonant set*, i.e., the complement to the union $\mathcal{W}$ of the perturbed tori. Suppose that the perturbation H_1 in (1) has the form

$$H_1(I, \varphi) = \varepsilon h(I, \varphi, \varepsilon) \tag{10}$$

where $\varepsilon \ge 0$ is a small parameter. If the unperturbed part H_0 of the Hamilton function (1) is Kolmogorov or isoenergetically nondegenerate, then the measure of the resonant set is at most of the order of $\varepsilon^{1/2}$ in any smoothness category [94, 109]. However, for general Rüssmann nondegenerate functions H_0, the measure of the resonant set can be larger. If for some positive integer $\mathcal{N}$ the collection of partial derivatives (9) span $\mathbb{R}^n$ for every $I \in D$, then the measure of the resonant set is at most of the order of $\varepsilon^{1/(2\mathcal{N})}$ [122].

The second difference concerns the set of the perturbed frequency vectors. For Kolmogorov nondegenerate Hamilton functions H_0, the set of the frequency vectors of the perturbed tori is essentially the same for all the sufficiently small perturbations H_1 and is known "beforehand". For isoenergetically nondegenerate Hamilton functions H_0, the same is true for the set of the *ratios* of the frequencies of the perturbed tori on each energy level hypersurface. The point is that the frequencies of the perturbed tori in the KAM theory are not just incommensurable but *strongly* incommensurable (Diophantine), and families constituted by these tori are therefore Cantor-type.

Definition. Let $\tau > 0$ and $\gamma > 0$. A vector $\varpi \in \mathbb{R}^n$ is said to be (τ, γ)-*Diophantine* if for each $k \in \mathbb{Z}^n \setminus \{0\}$ the inequality

$$|\langle k, \varpi \rangle| \geq \gamma |k|^{-\tau} \tag{11}$$

holds, where $|k| := |k_1| + \cdots + |k_n|$.

For any fixed $\tau > n - 1$ and for any bounded open domain $\mathcal{D} \subset \mathbb{R}^n$, the relative measure of the set of (τ, γ)-Diophantine vectors $\varpi \in \mathcal{D}$ tends to 1 as $\gamma \to 0$ [135]. For some generalisations of the arithmetical condition (11) on the frequencies of quasi-periodic motions see, e.g., [115, 116].

Before formulating the second difference between Kolmogorov/isoenergetic nondegeneracy and Rüssmann nondegeneracy, remark that for φ-independent terms H_1 in (1), the perturbed system is still completely integrable and its invariant n-tori $\{I = \text{const}\}$ coincide with the unperturbed n-tori (but have different frequencies). In fact, "generic" nearly integrable Hamiltonian systems are *not* integrable (see [10, 68, 84, 104] and references therein).

The second difference consists in the following. Let the unperturbed Hamilton function H_0 be Kolmogorov nondegenerate. Then, for any fixed $\tau > 0$ and $\gamma > 0$, the perturbed system with Hamilton function (1) possesses invariant n-tori with all the (τ, γ)-Diophantine frequency vectors $\varpi \in \omega(D)$ not very close to the boundary $\partial\omega(D)$, provided that the perturbation H_1 is sufficiently small [3, 4, 67, 94, 109, 117] (the "sufficient smallness" of H_1 depends on τ and γ). This is true in any smoothness category. For isoenergetically nondegenerate unperturbed Hamilton functions H_0, an analogous statement holds for the ratios $(\varpi_1 : \varpi_2 : \ldots : \varpi_n) \in \mathbb{RP}^{n-1}$ of the components of (τ, γ)-Diophantine frequency vectors $\varpi \in \omega(D)$ on each energy level hypersurface. On the other hand, one can choose a Rüssmann nondegenerate analytic function H_0 possessing the following two properties:

1) there are arbitrarily small analytic φ-independent perturbations H_1 for which the set of the ratios of the unperturbed frequencies and the set of the ratios of the perturbed ones are disjoint;

2) for each $\varpi \in \mathbb{R}^n \setminus \{0\}$, there is an arbitrarily small analytic φ-independent perturbation H_1 for which the perturbed system has no invariant n-torus with frequency ratio $(\varpi_1 : \varpi_2 : \ldots : \varpi_n)$.

Example. Let $n \geq 3$ and consider function (4) for any $1 \leq s \leq n - 2$. For this function, $\omega_1(I) \equiv 1$, $\omega_2(I) = u$, and $\omega_3(I) = u^2$ with $u = u(I_2, \ldots, I_{n-s+1})$ [see (5)]. Consequently, for $H_1(I, \varphi) = \varepsilon I_3$, the ratio of the first three frequencies of any n-torus $\{I = \text{const}\}$ is $(1 : u : u^2 + \varepsilon)$. For any $\varepsilon \neq 0$, the set of the unperturbed frequency ratios and that of the perturbed frequency ratios will be disjoint. Let now $(\varpi_1 : \varpi_2 : \ldots : \varpi_n)$ be an arbitrary point in $\mathbb{RP}^{n-1}$. If $\varpi_1 = 0$, then the set of the perturbed frequency ratios does not contain this point for any ε. If $\varpi_1 \neq 0$, then the set of the perturbed frequency ratios does not contain this point for any $\varepsilon \neq \varpi_3 \varpi_1^{-1} - \varpi_2^2 \varpi_1^{-2}$.

Actually, in this example, the sets of the perturbed frequency ratios corresponding to $\varepsilon = \varepsilon_1$ and to $\varepsilon = \varepsilon_2$ are disjoint whenener $\varepsilon_1 \neq \varepsilon_2$ (cf. [25]).

Thus, for a Rüssmann nondegenerate completely integrable Hamiltonian system, one may assert, generally speaking, just that for any sufficiently small Hamiltonian perturbation of this system, the perturbed system admits many invariant tori close to the unperturbed ones. The set of the frequency vectors of these perturbed tori and even the set of the ratios of the perturbed frequencies depend on the perturbation and cannot be predicted "beforehand". One should not say that the unperturbed tori "survive" the perturbation or "persist".

It would be interesting to apply the KAM theory with weak nondegeneracy conditions to the problems of planetary motions in celestial mechanics connected with the so called "wild resonance" discovered by M. R. Herman in 1997 (see [1, 10]).

The science surveyed in this section is straightforwardly carried over to Hamiltonian systems with discrete time, i.e., exact symplectic diffeomorphisms, as explained in [22, 122, 126] (of other works devoted—at least in part—to the discrete time Hamiltonian KAM theory, one may mention [6, 75–78, 89, 93, 131, 132, 137]). The analogue of the Rüssmann nondegeneracy condition for completely integrable exact symplectic diffeomorphisms

$$(I, \varphi) \mapsto (I, \varphi + \omega(I)), \qquad \omega(I) = \partial H_0(I)/\partial I$$

is as follows: the image of the unperturbed frequency map ω does not lie in any affine hyperplane of the frequency space (in other words, the hyperplanes that do not pass through the origin are forbidden as well). For analytic maps ω, this condition is necessary and sufficient for KAM-stability [122]. In the discrete time case, one should consider collections of partial derivatives $\partial^{|\alpha|}\omega(I)/\partial I^\alpha$ [cf. (9)] with $1 \le |\alpha| \le \mathcal{N}$ instead of $0 \le |\alpha| \le \mathcal{N}$. The definition of (τ, γ)-Diophantine vectors $\varpi \in \mathbb{R}^n$ in the discrete time case involves inequalities

$$|\langle k, \varpi \rangle - 2\pi k_0| \ge \gamma |k|^{-\tau}$$

for any $k_0 \in \mathbb{Z}$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ instead of (11), and the set of such vectors is of full measure for any fixed $\tau > n$ [135].

For some local versions of the theorems and examples reviewed above, see [65, 66, 126] and references therein.

Theorem 1 is carried over *mutatis mutandis* to reversible [21, 22, 121, 122], volume-preserving [21, 22], and dissipative [21, 22] systems.

3. The second topic: Gevrey smoothness

For this topic, we will need two important concepts: Gevrey smoothness and Whitney smoothness.

Gevrey smoothness is a notion intermediate between C^∞-smoothness and analyticity. Gevrey smooth functions are C^∞-functions with a moderate growth of the Taylor coefficients. The formal definition is as follows.

Definition. Let $D \subset \mathbb{R}^n$ be an open domain. A function $f\colon D \to \mathbb{R}^N$ is said to belong to the *Gevrey class* G^μ of index μ ($\mu \geq 1$ is a real number) if f is C^∞-smooth and for any compact set $X \subset D$ there exists a constant $c = c(X) > 0$ such that

$$\left|\frac{\partial^{|\alpha|} f(I)}{\partial I^\alpha}\right| \leq c^{|\alpha|+1}(\alpha_1!\alpha_2!\ldots\alpha_n!)^\mu$$

for each $I \in X$ and $\alpha \in \mathbb{Z}_+^n$.

For the general theory of Gevrey smooth functions, see, e.g., [56]. Gevrey smooth functions of index 1 are just analytic functions. That every analytic function is Gevrey smooth of index 1 is obvious; the converse follows immediately from any explicit form of the remainder in the Taylor formula. By the way, the statement that every Gevrey smooth function of index 1 is analytic is a trivial particular case of the following deep and rather hard theorem [69]: if the convergence radii of the Taylor series of a C^∞-function at all the points are bounded away from zero then this function is analytic. The one-dimensional case ($n = 1$) of this theorem is called the Pringsheim–Boas theorem [12, 13, 69].

For any $\mu > 1$, however, the space G^μ of Gevrey smooth functions of index μ contains nonzero functions vanishing on some set with non-empty interior and, in particular, nonzero functions with compact support. For $1 < \mu_1 < \mu_2$, one has

$$\mathcal{A} = G^1 \subset G^{\mu_1} \subset G^{\mu_2} \subset C^\infty,$$

where $\mathcal{A}$ denotes the class of analytic functions $D \to \mathbb{R}^N$. All the inclusions here are strict. The space of Gevrey smooth functions is "negligible" (in the sense to be made precise) in the space of all the C^∞-functions [26], but it is still "rich" enough for many applications (see, e.g., [56, 69, 118]).

Gevrey-type functions and asymptotic expansions are widely used in the theory of dynamical systems (see, e.g., [82] and references therein).

Whitney smoothness is a concept of entirely different nature. Whereas Gevrey smoothness refers to the smoothness *degree*, Whitney smoothness refers to the *regularity*: it means the possibility of a smooth interpolation. We will give here an informal definition of a Whitney smooth family of invariant tori of a dynamical system (for the precise definition, see [21, 22, 121]). The general theory of Whitney smoothness (far from being confined to families of invariant tori) is expounded in, e.g., [56, 69].

Consider a dynamical system with phase space $\mathcal{M}$. Suppose that this system has a family $\{\mathcal{T}_\xi \subset \mathcal{M}\}_\xi$ of invariant n-tori labelled by index $\xi \in \Xi \subset \mathbb{R}^q$ and carrying conditionally periodic motions with frequency vectors $\varpi_\xi \in \mathbb{R}^n$. It is assumed that Ξ is a set of positive measure. The structure of Ξ can be "bad" (e.g., Ξ may be Cantor-like). Each torus $\mathcal{T}_\xi$ is the image of an embedding $\mathbb{T}^n \to \mathcal{M}$.

Let $\mathcal{C}$ be any smoothness class (C^r with $r < \infty$, C^∞, or G^μ with $\mu > 1$) admitting nonzero functions with compact support.

Definition. The family $\{\mathcal{T}_\xi\}_{\xi\in\Xi}$ is said to be *Whitney smooth* of smoothness class $\mathcal{C}$ if there are an open set B such that $\Xi \subset B \subset \mathbb{R}^q$ and an embedding $\Psi\colon \mathbb{T}^n \times B \to \mathcal{M}$ $\mathcal{C}$-smooth in $\xi \in B$ and such that for each $\xi \in \Xi$

a) $\Psi(\mathbb{T}^n, \xi) = \mathcal{T}_\xi$,

b) the pullback dynamics on $\mathbb{T}^n \times \{\xi\}$ is given by the vector field $(\varpi_\xi, 0)$.

So, invariant n-tori $\mathcal{T}_\xi$ with $\xi \in \Xi$ can be $\mathcal{C}$-smoothly "interpolated" by the sets $\Psi(\mathbb{T}^n, \xi)$ with $\xi \in (B \setminus \Xi)$. In most the cases, these sets are also n-tori but not necessarily invariant.

It turns out that Cantor-type families of invariant tori in dynamical systems carrying quasi-periodic motions with Diophantine frequencies are always Whitney smooth (at least, no counterexample is known and in almost all the settings, Whitney smoothness has been proven). This refers not only to invariant n-tori in Hamiltonian systems with n degrees of freedom but also to invariant tori of other dimensions and invariant tori of systems which are not Hamiltonian. Whitney smoothness of families of invariant tori was discovered by V. F. Lazutkin [75, 76] for the case of invariant curves of finitely smooth area-preserving mappings of an annulus and was then extensively studied by many authors in various situations, important subsequent references are [19–22, 32, 60, 70, 71, 77, 78, 105–107, 109, 121, 131, 137].

Consider a Hamiltonian system with $n \geq 2$ degrees of freedom and the Hamilton function H of the form (1). Suppose that the unperturbed Hamilton function H_0 is analytic and Kolmogorov nondegenerate [i.e., that it meets condition (2)]. J. Pöschel [109] proved that, depending on how smooth H_1 (and, consequently, H) is, the smoothness class of an individual perturbed invariant n-torus and the *Whitney* smoothness class of the family of the perturbed tori are as follows:

H	a perturbed torus	the family of tori
C^r, $3n-1 < r < \infty$	$C^{r'}$, any $r' < r - 2n + 1$	$C^{r''}$, any $r'' < (r - 2n + 1)/n$
C^∞	C^∞	C^∞
$\mathcal{A}$	$\mathcal{A}$	C^∞

Note that the Whitney smoothness of the family of the perturbed tori is never higher than the smoothness of individual tori.

We would like to comment the C^r-case here. As a matter of fact, Pöschel [109] showed the following. Let τ, λ, and β be some numbers subject to the inequalities

$$\tau > n-1, \quad \lambda > \tau + 1, \quad \beta > 1, \tag{12}$$

and

$$\beta \neq i\lambda^{-1} + j \quad \text{for any integers } i,\, j \geq 0. \tag{13}$$

Then, if the unperturbed Hamilton function H_0 is analytic and Kolmogorov nondegenerate and a perturbation H_1 is $C^{\beta\lambda+\lambda+\tau}$-smooth, then each perturbed invariant n-torus with the (τ, γ)-Diophantine frequency vector (for suitable $\gamma > 0$ dependent on the perturbation magnitude) is $C^{\beta\lambda}$-smooth while the family of such tori is C^{β}-smooth in the sense of Whitney. Now the data pointed out in the table above for the finite differentiable case is implied by the following simple lemma. Let τ, λ, and β satisfy (12) and let $\beta\lambda + \lambda + \tau = r$, $\beta\lambda = r'$, $\beta = r''$. Then

$$r > 3n-1, \quad r' < r - 2n + 1, \quad r'' < (r - 2n + 1)/n. \tag{14}$$

Conversely, let r, r', and r'' be positive numbers satisfying (14). Then there exist numbers τ, λ, and β meeting (12)–(13) and such that $r = \beta\lambda + \lambda + \tau$, $r' \leq \beta\lambda$, $r'' \leq \beta$.

However, the case most interesting for us now is the case of analytic perturbations H_1. If the Hamilton function H is analytic then each individual perturbed n-torus is also analytic and the family of these tori is infinitely differentiable (in the sense of Whitney). Pöschel noted in [109] (cf. also [10]) that *"this is probably the most one can hope to get"*. Indeed, the family of the perturbed tori is generically not analytic, otherwise this family would be continuous rather than Cantor-type and the perturbed system would be still completely integrable.

Nevertheless, very recently, G. S. Popov announced in [105] and showed in [106] that *one does get more*. Namely, he proved that if both H_0 and H_1 are analytic and H_0 is Kolmogorov nondegenerate, then the family of the perturbed tori is not just infinitely differentiable in the sense of Whitney but Gevrey smooth in the sense of Whitney. To be more precise, the last line of the table above should be corrected as follows:

H	a perturbed torus	the family of tori
$\mathcal{A}$	$\mathcal{A}$	G^{μ}, any $\mu > \max(9/2, n+1)$

As a matter of fact, Popov [106] showed that for every number $\tau > n-1$, the family of the perturbed n-tori with (τ, γ)-Diophantine frequency vectors (for suitable $\gamma > 0$ dependent on the perturbation magnitude) is G^{μ}-smooth in the sense of Whitney for any $\mu > \max(9/2, \tau + 2)$.

Thus, the families of invariant n-tori of analytic Hamiltonian systems with n degrees of freedom turn out to be *much smoother* (in the sense of Whitney) than one thought before. Gevrey smoothness of those families is important for semiclassical asymptotics of eigenvalues and eigenfunctions of Schrödinger-type operators [105–107] (for the previous results on the connection between invariant tori in classical Hamiltonian systems and short wave approximations of the corresponding quantum systems, see [77, 78] and references therein).

It would be interesting to examine the smoothness of the perturbed n-tori and the Whitney smoothness of their family in the case where the perturbation is Gevrey smooth (of some index $\mu > 1$). Conjecturally, individual perturbed tori will be Gevrey smooth of some index $\mu' > \mu$ in this case, whereas the family of the tori will be Gevrey smooth of some index $\mu'' > \mu'$ (in the sense of Whitney).

Actually, in the context of Kolmogorov's theorem, a perturbed system not only admits a Whitney smooth family of invariant n-tori but also is integrable (in a certain sense) on the union of these tori. For precise formulations and the proofs of this fundamental result, see [109].

4. The third topic: "exponential condensation"

Consider again a Hamiltonian system with $n \geq 2$ degrees of freedom and the Hamilton function H of the form (1). Suppose that both the unperturbed Hamilton function H_0 and its perturbation H_1 are analytic. Assume also that H_0 is Kolmogorov nondegenerate [i.e., condition (2) is satisfied] and *quasi-convex*. The latter means that

$$\left\langle \frac{\partial \omega(I)}{\partial I} \eta, \eta \right\rangle \neq 0$$

whenever $\eta \in \mathbb{R}^n \setminus \{0\}$ and $\langle \eta, \omega(I) \rangle = 0$. The concept of quasi-convex Hamilton functions was introduced by N. N. Nekhoroshev [96]. Quasi-convexity of H_0 means strict convexity of the corresponding unperturbed energy level hypersurfaces $\{H_0(I) = \mathrm{const}\} \subset D$. One can easily verify that the conditions of Kolmogorov nondegeneracy and quasi-convexity are independent [22] but quasi-convexity implies isoenergetic nondegeneracy [82]. In particular, the Kolmogorov degenerate and isoenergetically nondegenerate Hamilton function (8) is quasi-convex. For $n = 2$, quasi-convexity is equivalent to isoenergetic nondegeneracy [82, 96].

Under the hypotheses of analyticity of H_0 and H_1 together with Kolmogorov nondegeneracy and quasi-convexity of H_0, A. Morbidelli and A. Giorgilli [86] proved in 1995 the following.

Theorem 2. *Let H_1 be fixed and sufficiently small. Denote by $\mathcal{W}$ the union of the perturbed invariant n-tori and let $\mathcal{T}$ be any fixed perturbed torus. Then the measure of $\mathcal{U}_\rho(\mathcal{T}) \setminus \mathcal{W}$ is* *at most of the order of* $\exp(-c/\rho)$ *as* $\rho \to 0$ *where* $\mathcal{U}_\rho(\mathcal{T})$ *is the* ρ*-neighbourhood of* $\mathcal{T}$ *in the phase space and* c *is a certain constant. Moreover, if the frequency vector of torus* $\mathcal{T}$ *is* (τ, γ)*-Diophantine then all the trajectories starting at a distance* $\rho < \rho^*$ *from* $\mathcal{T}$ (ρ^* *being a certain constant*) *remain close to* $\mathcal{T}$ *for an exceedingly long time of the order of*

$$\exp\left\{\exp\left[\left(\frac{\rho^*}{\rho}\right)^{1/(\tau+1)}\right]\right\}.$$

One says that the perturbed tori "*exponentially condense*" to each of them [125] and that the rate of moving away from each torus is *superexponentially small* with respect to the distance from the torus (the tori are "*superexponentially sticky*"). For these results, analyticity of the Hamilton function is very essential.

Every torus among the perturbed tori that "condense" to $\mathcal{T}$ is, in turn, a "condensation point" of other perturbed tori, and so on. This hierarchy (not completely understood yet, as far as the author knows) is described and discussed in papers [49, 87, 88].

The "exponential condensation" of invariant tori proven in [86] was confirmed numerically (for a model problem of area-preserving mappings of the plane) in [46, 79].

"Exponential condensation" of invariant n-tori near elliptic equilibria (i.e., equilibria with nonzero purely imaginary eigenvalues) of analytic Hamiltonian systems with n degrees of freedom was established by A. Delshams and P. Gutiérrez [41] in 1996 (see also [42]). Here and henceforth, an eigenvalue of an equilibrium means an eigenvalue of the linearisation of the corresponding vector field around this equilibrium.

5. The fourth topic: destruction of resonant tori

This topic also concerns Hamiltonian systems with $n \geq 2$ degrees of freedom and the Hamilton functions H of the form (1). If the unperturbed Hamilton function H_0 meets condition (2) (i.e., is Kolmogorov nondegenerate), then, as was explained at the end of Section 2, all the unperturbed invariant n-tori $\{I = I^*\}$ with Diophantine frequency vectors $\omega(I^*)$ "survive" small perturbations H_1. What is the "fate" of a torus $\{I = I^*\}$ in the opposite case, where the frequencies $\omega_1(I^*)$, $\omega_2(I^*)$, $\ldots$, $\omega_n(I^*)$ are rationally dependent (such tori $\{I = I^*\}$ are said to be *resonant* and are foliated into invariant tori of a smaller dimension)? To be more precise, suppose that among the n components of the vector $\omega(I^*) = \omega^*$, there are $n - l$ $(1 \leq l \leq n-1)$ strongly incommensurable numbers

$$\omega^*_{i_1}, \omega^*_{i_2}, \ldots, \omega^*_{i_{n-l}} \tag{15}$$

whereas the remaining l components $\omega^*_{j_1}$, $\omega^*_{j_2}$, $\ldots$, $\omega^*_{j_l}$ are rational combinations of numbers (15) (so that the frequencies of the torus $\{I = I^*\}$ satisfy l independent resonance relations). Then the following statement holds.

Theorem 3. *Let H_0 be Kolmogorov nondegenerate. Then for* generic *sufficiently small perturbations H_1, the n-torus $\{I = I^*\}$ breaks up into a finite collection of invariant $(n-l)$-tori carrying quasi-periodic motions.*

We will not attach any exact meaning to the word "generic" here (precise formulations of Theorem 3 can be found in [30, 40, 138, 142]). Instead, let us note the following. Near the torus $\{I = I^*\}$, a certain averaging and truncation procedure described in detail in, e.g., [10] reduces the Hamilton function (1) to

$$\widehat{\mathcal{H}}(K, J, \chi) = \langle \widehat{\omega}^*, K \rangle + \mathcal{H}(J, \chi), \qquad \mathcal{H}(J, \chi) = \tfrac{1}{2}\langle AJ, J \rangle + V(\chi),$$

where $\widehat{\omega}^* \in \mathbb{R}^{n-l}$ is the vector with components (15), $K \in \mathbb{R}^{n-l}$, $J \in \mathbb{R}^l$, $\chi \in \mathbb{T}^l$, A is a real symmetric $l \times l$ matrix ($\det A \neq 0$), and $|V| \ll 1$. It turns out that generically to each *nondegenerate* critical point $\chi^\star$ of the function V, there "corresponds" (in the sense to be made precise) an invariant $(n-l)$-torus (lying near $\{I = I^*\}$ and carrying quasi-periodic motions) of the system with the Hamilton function (1). The word "nondegenerate" here means that

$$\det \frac{\partial^2 V(\chi^\star)}{\partial \chi^2} \neq 0.$$

On the other hand, $\chi^\star \in \mathbb{T}^l$ is a nondegenerate critical point of the potential V if and only if the point $(0, \chi^\star)$ is a nondegenerate equilibrium of the Hamiltonian system with l degrees of freedom and the Hamilton function $\mathcal{H}$ (an equilibrium is said to be nondegenerate if all its eigenvalues are other than zero). The symplectic structure here is assumed to be $dJ \wedge d\chi = dJ_1 \wedge d\chi_1 + \cdots + dJ_l \wedge d\chi_l$. In fact, the eigenvalues of the equilibrium $(0, \chi^\star)$ are those of the $2l \times 2l$ matrix

$$\begin{pmatrix} 0 & -\partial^2 V(\chi^\star)/\partial \chi^2 \\ A & 0 \end{pmatrix}. \tag{16}$$

If λ is an eigenvalue of an equilibrium of a Hamiltonian system, so is $-\lambda$. Thus, a nondegenerate equilibrium of a Hamiltonian system can be *elliptic* (all the eigenvalues are purely imaginary), *hyperbolic* (all the eigenvalues lie outside the imaginary axis), and *of mixed type*. According to the type of equilibria $(0, \chi^\star)$ of the system with Hamilton function $\mathcal{H}$, the corresponding invariant $(n-l)$-tori of the original system with Hamilton function H are also said to be *elliptic*, *hyperbolic*, and *of mixed type*. It turns out that the case of hyperbolic $(n-l)$-tori in Theorem 3 is much easier than the case of nonhyperbolic tori (i.e., elliptic tori and tori of mixed type).

In particular, suppose that the perturbation H_1 in (1) has the form (10). Then $V(\chi) = \varepsilon v(\chi) + O(\varepsilon^2)$, and for every sufficiently small $\varepsilon > 0$, the original system possesses a hyperbolic invariant $(n-l)$-torus "emerging" from a given hyperbolic equilibrium $(0, \chi^\star)$

of the system with the Hamilton function

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathcal{H}(\varepsilon^{1/2}J,\chi)=\tfrac{1}{2}\langle AJ,J\rangle+v(\chi) \tag{17}$$

(provided that h is generic). In the analytic category this torus depends on ε analytically [138]. A nonhyperbolic equilibrium $(0,\chi^\star)$ of the system with the Hamilton function (17) gives rise to a nonhyperbolic invariant $(n-l)$-torus of the original system only for the most values (in the Lebesgue measure sense) of the perturbation parameter ε.

The case $l=n-1$ of "maximal" resonance (where the tori in question are in fact circles) in Theorem 3 was considered by H. Poincaré [104] (for a modern presentation see, e.g., [22]). This classical result is outside the KAM theory because it does not involve "small divisors". The case of arbitrary l was proven no earlier than in 1989 by D. V. Treshchëv [138] (see also a discussion in [22, 68]). But Treshchëv treated only hyperbolic invariant $(n-l)$-tori. The hyperbolic $l=1$ case was also examined independently in subsequent papers [28, 43, 98, 113, 139, 141]. In works [43, 98, 113, 138, 139], a special attention was paid to the n-dimensional separatrix stable and unstable manifolds ("whiskers") of the hyperbolic invariant $(n-l)$-tori one looks for. Such "whiskers" are of great importance in the Arnol′d diffusion mechanism. Nonhyperbolic invariant tori in Theorem 3 were first constructed by Ch.-Q. Cheng [29] for the case $l=1$ (note that if $l=1$ then the tori are either hyperbolic or elliptic). The general case of Theorem 3—an arbitrary l and arbitrary type of the invariant $(n-l)$-tori—was announced by Cheng and Sh. Wang [30, 142]. In fact, Cheng and Wang [30, 142] considered only the case where the eigenvalues of matrix (16) are either real or purely imaginary (quadruplets $\pm a\pm bi$ of complex eigenvalues were excluded). Finally, very recently, F. Cong, T. Küpper, Y. Li, and J. You [40] proved Theorem 3 for arbitrary l and arbitrary type of the invariant $(n-l)$-tori (and arbitrary collections of eigenvalues). Thus, now *we have got the complete picture* of the destruction of resonant tori of integrable Hamiltonian systems under small perturbations.

The papers [28, 29, 40, 43, 98, 113, 138, 139, 141] cited above studied the analytic situation whereas the articles [30, 142] dealt with finitely smooth systems. Theorem 3 admits reversible analogues [80, 144]. Papers [80, 144] consider an arbitrary number of resonance relations and arbitrary type of the tori.

6. The fifth topic: excitation of elliptic normal modes

This topic pertains to a more complicated setting than before. Consider a Hamiltonian system with $n+m$ degrees of freedom ($n\ge 2$, $m\ge 1$) and the Hamilton function of the form

$$\widetilde{H}_0(I,\varphi,z)=H_0(I)+\tfrac{1}{2}\langle M(I,\varphi)z,z\rangle+O(|z|^3) \tag{18}$$

where the variables I, φ, and z range respectively over a bounded and connected open domain $D \subset \mathbb{R}^n$, the standard n-torus $\mathbb{T}^n$, and a neighbourhood of the origin in $\mathbb{R}^{2m}$, while $M(I, \varphi)$ is a real symmetric $2m \times 2m$ matrix for every I and φ. The symplectic structure is supposed to be

$$\sum_{i=1}^{n} dI_i \wedge d\varphi_i + \sum_{j=1}^{m} dz_j \wedge dz_{j+m}. \tag{19}$$

The Hamilton function (18) determines the equations of motion

$$\dot{I} = O(|z|^2), \qquad \dot{\varphi} = \omega(I) + O(|z|^2), \qquad \dot{z} = \Omega(I, \varphi)z + O(|z|^2) \tag{20}$$

where

$$\omega = \frac{\partial H_0}{\partial I}, \qquad \Omega = \begin{pmatrix} 0_m & -E_m \\ E_m & 0_m \end{pmatrix} M, \tag{21}$$

0_m denotes the zero $m \times m$ matrix, and E_m denotes the identity $m \times m$ matrix. Thus, the $2n$-dimensional surface $\{z = 0\}$ is smoothly foliated into n-tori $\{z = 0,\ I = \mathrm{const}\}$ invariant under the flow of the system governed by the Hamilton function (18), and these tori carry conditionally periodic motions with frequency vectors $\omega(I)$. The restriction of the system to the surface $\{z = 0\}$ is still Hamiltonian (with respect to the symplectic structure $dI \wedge d\varphi$) and completely integrable.

As soon as one proceeds to small Hamiltonian perturbations of the Hamilton function (18), two problems arise.

The first problem: the existence of invariant n-tori (carrying quasi-periodic motions) near the surface $\{z = 0\}$ in perturbed systems.

This problem has been addressed in very many works starting with V. K. Mel′nikov's article [85] of 1965. An extensive bibliography is presented in [10, 22], these two books contain also detailed reviews of the results obtained. Of the most recent papers, one can mention [17, 36, 39, 57, 114, 116, 128, 148, 151] (see also the mini-survey [143]). It turns out that invariant n-tori in a perturbed system do exist under rather general conditions. To be more precise, the following statement is valid.

Theorem 4. *Assume that at least one of the following two hypotheses holds*:

a) *for every I and φ, the matrix $\Omega(I, \varphi)$ [see (21)] has no purely imaginary eigenvalues,*

b) *the matrix M (and, consequently, Ω) does not depend on φ.*

Suppose also that the functions ω and Ω satisfy certain nondegeneracy and nonresonance conditions. Then any Hamiltonian system with the Hamilton function sufficiently close to $\widetilde{H}_0$ (18) admits many invariant n-tori which are close to the unperturbed n-tori $\{z = 0,\ I = const\}$ and carry quasi-periodic motions. The Lebesgue measure of the complement to the union of the images of the perturbed tori under the projection $\pi \colon (I, \varphi, z) \mapsto (I, \varphi, 0)$

onto the surface $\{z = 0\}$ *vanishes as the perturbation magnitude tends to zero. If the hypothesis* a) *is valid then each perturbed* n*-torus is attached* $(n+m)$*-dimensional separatrix stable and unstable manifolds ("whiskers").*

The numerous versions of Theorem 4 differ mainly in the set of nondegeneracy and nonresonance conditions to be imposed on the unperturbed Hamilton function (18). Of the sources cited above, works [10, 22, 39, 116, 128, 148, 151] use Rüssmann-like nondegeneracy conditions on the map ω (21). The "optimal" nondegeneracy and nonresonance conditions on ω and Ω (like the Rüssmann nondegeneracy of analytic completely integrable Hamiltonian systems) are not known yet even in the analytic category.

Now assume that the $2m \times 2m$ matrix $\Omega(I, \varphi) = \Omega(I)$ is φ-independent and possesses κ pairs of purely imaginary eigenvalues for every I $(1 \le \kappa \le m)$. These eigenvalues are sometimes called the *elliptic normal modes* of the unperturbed tori $\{z = 0,\ I = \text{const}\}$ and yield the following problem.

The second problem: the existence of invariant $(n + \nu)$-tori (carrying quasi-periodic motions) near the surface $\{z = 0\}$ in the unperturbed system and perturbed systems for each ν, $1 \le \nu \le \kappa$.

If such tori exist one sometimes says that the elliptic normal modes of the unperturbed n-tori $\{z = 0,\ I = \text{const}\}$ *excite*.

It turns out that under certain nondegeneracy and nonresonance conditions, invariant tori of all the dimensions $n + 1,\ n + 2,\ \ldots,\ n + \kappa$ *do exist* in the unperturbed system as well as in perturbed systems. We will not formulate the corresponding theorem even vaguely and will confine ourselves with relevant references. Up to now, the excitation of the elliptic normal modes of the unperturbed n-tori has been explored for analytic Hamiltonian systems only (although it undoubtedly takes place for C^∞- and finitely differentiable systems as well). The first excitation results were obtained in 1962–63 by V. I. Arnol′d [2, 4] who considered the particular case $\nu = \kappa = m$. In 1974, A. D. Bruno [23] examined the general case of arbitrary κ and ν and constructed analytic families of invariant $(n + \nu)$-tori (preprints [23] were translated into English as the second part of book [24]). General theorems describing Whitney smooth Cantor-type families of invariant $(n + \nu)$-tori (for arbitrary κ and ν) were proven independently by H. W. Broer, G. B. Huitema, and M. B. Sevryuk in 1996–97 [22, 124] and by À. Jorba and J. Villanueva in 1997 [63, 140] (see also [64]). Analytic families of tori found by Bruno are subfamilies of these Cantor-type families. Jorba and Villanueva [63, 64, 140] also established the "exponential condensation" of invariant $(n + \nu)$-tori (which, of course, does not take place in the C^r-categories for any $r \le \infty$). The "exponential condensation" of perturbed invariant n-tori in the context of Theorem 4 was verified by Jorba and Villanueva in a separate paper [62]. Various versions of the excitation theorem have been surveyed in detail in review [125].

The phenomenon of the excitation of elliptic normal modes is also known in reversible [22, 112, 119–121, 127] and volume-preserving [129] set-ups. The works [22, 119, 121, 127] consider the excitation of elliptic normal modes in reversible flows and papers [112, 120, 121], in reversible diffeomorphisms.

7. The sixth topic: "atropic" invariant tori

This last topic concerns the most important, in the author's opinion, and simultaneously least understood results in the Hamiltonian KAM theory for the last five years. First of all, recall the following definition.

Definition. Let $\mathcal{M}^{2N}$ be a symplectic manifold of dimension $2N$. A smooth n-dimensional submanifold $\mathcal{L}^n$ of $\mathcal{M}$ is said to be *isotropic* (*Lagrangian* for $n = N$) if the tangent space $T_P\mathcal{L}$ to $\mathcal{L}$ at any point $P \in \mathcal{L}$ lies in its skew-orthogonal complement: $T_P\mathcal{L} \subset (T_P\mathcal{L})^{\complement}$, i.e., if the restriction of the symplectic structure to $\mathcal{L}$ vanishes. A smooth n-dimensional submanifold $\mathcal{L}^n$ of $\mathcal{M}$ is said to be *coisotropic* (*Lagrangian* for $n = N$) if the tangent space $T_P\mathcal{L}$ to $\mathcal{L}$ at any point $P \in \mathcal{L}$ contains its skew-orthogonal complement: $(T_P\mathcal{L})^{\complement} \subset T_P\mathcal{L}$.

For instance, every point ($n = 0$) and curve ($n = 1$) in a symplectic manifold are isotropic, and any hypersurface ($n = 2N - 1$) is coisotropic. Any symplectic manifold is a coisotropic submanifold of itself ($n = 2N$). If $\mathcal{L}^n \subset \mathcal{M}^{2N}$ is isotropic then $n \le N$. If $\mathcal{L}^n \subset \mathcal{M}^{2N}$ is coisotropic then $n \ge N$.

All the invariant tori of Hamiltonian systems we have spoken of by now are *isotropic*. The reason is the following theorem due to M. R. Herman of 1988–89 [51, 52]:

Theorem 5. *Any invariant torus of a Hamiltonian system carrying quasi-periodic motions is isotropic provided that the symplectic structure is* exact.

Proof. Let $\mathcal{T}$ be an invariant n-torus carrying quasi-periodic motions. In some coordinate $\phi \in \mathbb{T}^n$ on this torus, the induced dynamics is given by the equation $\dot{\phi} = \varpi$ where the frequencies $\varpi_1, \varpi_2, \ldots, \varpi_n$ are rationally independent. Since a Hamiltonian flow preserves the symplectic structure, the flow $g^t \colon \phi \mapsto \phi + t\varpi$ preserves the restriction

$$\sum_{1 \le i_1 < i_2 \le n} f_{i_1 i_2}(\phi)\, d\phi_{i_1} \wedge d\phi_{i_2}$$

of the symplectic structure to $\mathcal{T}$. This means that all the coefficients $f_{i_1 i_2}$ are constants along the orbits of the flow g^t. As any orbit of a quasi-periodic flow on a torus is everywhere dense, each coefficient $f_{i_1 i_2}$ is a constant on $\mathcal{T}$. Now it suffices to note that the only exact differential form on a torus with constant coefficients (of any degree) is zero. $\square$

In fact, Herman [51, 52] proved Theorem 5 for a particular case of invariant n-tori of symplectic diffeomorphisms of $2n$-dimensional symplectic manifolds, but the general case (verified in [22]) of invariant tori of arbitrary dimensions is not harder at all.

A certain version of the KAM theory can nevertheless be developed also for *coisotropic* invariant tori of dimensions greater than the number of degrees of freedom. Of course, the symplectic structure in this case should be nonexact. Moreover, it turns out that the periods of the symplectic structure (its integrals over the two-dimensional cycles within the tori in question) should satisfy certain Diophantine-like conditions: all the theorems on coisotropic tori proven by now include such Diophantine hypotheses. The coisotropic Hamiltonian KAM theory was founded by I. O. Parasyuk [99] in 1984, see also subsequent papers [70, 71, 100–102] by Parasyuk and his co-worker A. A. Kubichka. Coisotropic invariant n-tori of Hamiltonian systems with $N < n$ degrees of freedom were also studied by Herman [53, 54] (see also [92, 149, 150]) and by F. Cong and Y. Li [38]. The papers [53, 54] are devoted to the particular case $n = 2N - 1$.

In Parasyuk's theory, one starts with an "unperturbed" Hamiltonian system with $N \geq 2$ degrees of freedom whose phase space is smoothly foliated into coisotropic invariant n-tori carrying conditionally periodic motions ($N + 1 \leq n \leq 2N - 1$). Then, as in Kolmogorov's setting, one can prove that, under certain conditions on the symplectic structure and the unperturbed Hamilton function, perturbed systems still admit many coisotropic invariant n-tori carrying quasi-periodic motions. The measure of the complement to the union of the perturbed tori vanishes as the perturbation magnitude tends to zero. The symplectic structure here is usually supposed to be fixed, as in the "conventional" isotropic Hamiltonian KAM theory. However, in their latest papers [71, 102], Kubichka and Parasyuk considered the case where the symplectic structure is perturbed as well (both the unperturbed and perturbed structures being assumed to meet certain Diophantine conditions).

The most important application of the coisotropic Hamiltonian KAM theory is counterexamples to the so called quasi-ergodic conjecture [10, 22, 92, 149, 150]. The ergodic conjecture (to be more precise, one of the versions of this conjecture) says that a generic Hamiltonian system is ergodic on (almost) every compact and connected energy level hypersurface. This conjecture is wrong [3, 4, 6, 10, 18, 84, 92, 111, 149, 150, 152]: a Hamiltonian system close to a KAM-stable completely integrable one does not possess such an ergodic property. Moreover, a system sufficiently close to an isoenergetically nondegenerate completely integrable one admits many Lagrangian invariant tori on each energy level hypersurface [3, 4] and is therefore ergodic on *no* energy level hypersurfaces. The quasi-ergodic conjecture says that on (almost) every compact and connected energy level hypersurface of a generic Hamiltonian system, there is an everywhere dense trajectory. For the case of two degrees of freedom, this second conjecture is also wrong: the two-dimensional

Lagrangian invariant tori divide the three-dimensional energy level hypersurfaces and exclude everywhere dense trajectories. But M. R. Herman noticed (see [92, 149, 150]) that for some nonexact symplectic structures, the quasi-ergodic conjecture is wrong for the case of $N \geq 3$ degrees of freedom as well: $(2N-2)$-dimensional coisotropic invariant tori divide the $(2N-1)$-dimensional energy level hypersurfaces. Moreover, in both the cases, under appropriate conditions, everywhere dense trajectories will occur on *no* energy level hypersurfaces. To the best of the author's knowledge, whether the quasi-ergodic conjecture is valid for $N \geq 3$ degrees of freedom and exact symplectic structures is still an open question.

Formulate now what may be called *the Hamiltonian KAM paradigm*: what families of invariant tori carrying quasi-periodic motions can one expect to find in a *typical* Hamiltonian system with $N \geq 1$ degrees of freedom? The word "typical" here means that one deals with Hamilton functions constituting an open set in the space of all the functions on a given symplectic manifold (in other words, that the Hamiltonian systems we are studying possess no additional symmetries). The answer is as follows [21, 22, 112].

Proposition 1. *In a typical Hamiltonian system with $N \geq 1$ degrees of freedom, there are n-parameter families of invariant* isotropic *n-tori for each $0 \leq n \leq N$ and, if $N \geq 2$ and the symplectic structure is not exact (and meets some Diophantine conditions), also $(2N-n)$-parameter families of invariant* coisotropic *n-tori for each $N+1 \leq n \leq 2N-1$. For $n = 0$, 1, and $2N-1$ these families are smooth, otherwise they are Cantor-type and Whitney smooth.*

It was widely believed until 2000 that this paradigm embraces all the possible finite dimensional Hamiltonian KAM statements (so that the tori in the Hamiltonian KAM theory should be either isotropic or coisotropic, just as the tori treated in the reversible KAM theory are invariant under the reversing involution). However, in 2000–01, Q. Huang, F. Cong, and Y. Li [58, 59] obtained some KAM-type results (in the analytic category) for invariant tori that are neither isotropic nor coisotropic. We will call such tori *"atropic"*. Of course, the symplectic structure in [58, 59] is not exact. Taking into account the discovery of Huang, Cong, and Li, Proposition 1 above can be supplemented as follows.

Proposition 2. *In a typical Hamiltonian system with $N \geq 3$ degrees of freedom, there are, if the symplectic structure is not exact (and meets some Diophantine conditions), also p-parameter families of invariant* "atropic" *n-tori for each $3 \leq n \leq 2N-3$ and $1 \leq p \leq \min(n-2, 2N-n-2)$ such that $n+p$ is even. These families are Cantor-type and most probably Whitney smooth.*

Unfortunately, papers [58, 59] contain serious inaccuracies (see [130]; in fact, the criticism of [58] in review [130] applies mutatis mutandis to articles [38, 59] as well). The main idea of [58, 59] is nevertheless correct: perturbing a system possessing an analytic

$(n+p)$-dimensional invariant surface Π_0^{n+p} smoothly foliated into invariant n-tori carrying conditionally periodic motions and coisotropic (but not isotropic) *within* Π_0 (one assumes that the restriction of the symplectic structure to Π_0 is a symplectic structure on Π_0). In the ambient phase space, these tori are "atropic". Roughly speaking, Π_0 plays the rôle of the surface $\{z = 0\}$ in (18). In a perturbed system, one has to find invariant n-tori carrying quasi-periodic motions and close to the unperturbed n-tori. Note that n and p in this construction should satisfy the conditions

$$n \geq 3, \qquad 1 \leq p \leq n-2, \qquad n+p \text{ is even}, \qquad 2N \geq n+p+2$$

where N is the number of degrees of freedom. One easily sees that these conditions are equivalent to the conditions pointed out in Proposition 2:

$$N \geq 3, \qquad 3 \leq n \leq 2N-3, \qquad 1 \leq p \leq \min(n-2,\, 2N-n-2), \qquad n+p \text{ is even}.$$

We will conclude this section (and the whole survey) by discussing a method of the reduction of some complicated KAM settings to simpler ones.

Theorem 6. *Let* $\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}\right)$ *be a symplectic manifold and* $\Pi_0 \subset \widetilde{\Pi}$ *a submanifold of* $\widetilde{\Pi}$. *Suppose that the restriction* $\boldsymbol{\omega}_0 = \widetilde{\boldsymbol{\omega}}|_{\Pi_0}$ *is a symplectic structure on* Π_0. *Let also* $\widetilde{H}_0\colon \widetilde{\Pi} \to \mathbb{R}$ *be a Hamilton function on* $\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}\right)$ *for which* Π_0 *is a normally hyperbolic invariant manifold of the corresponding Hamiltonian system. Assume also that the restriction* $H_0 = \widetilde{H}_0|_{\Pi_0}\colon \Pi_0 \to \mathbb{R}$ *behaves as an unperturbed Hamilton function in a certain KAM-type theorem for the symplectic manifold* $(\Pi_0, \boldsymbol{\omega}_0)$. *Namely,* Π_0 *is smoothly foliated into invariant* n*-tori of the Hamiltonian flow on* Π_0 *with the Hamilton function* H_0, *and any Hamiltonian system on* $(\Pi_0, \boldsymbol{\omega}_0)$ *with the Hamilton function sufficiently close to* H_0 *admits many invariant* n*-tori close to the unperturbed ones. Then any Hamiltonian system on* $\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}\right)$ *with the Hamilton function sufficiently close to* $\widetilde{H}_0$ *admits many invariant* n*-tori close to the unperturbed* n*-tori on* Π_0.

Sketch of a proof. Let $\widetilde{H}\colon \widetilde{\Pi} \to \mathbb{R}$ be a Hamilton function close to $\widetilde{H}_0$. According to the general theory of normally hyperbolic invariant manifolds [45, 55, 146], the Hamiltonian system on $\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}\right)$ with the Hamilton function $\widetilde{H}$ possesses an invariant manifold Π close to Π_0. The restriction $\boldsymbol{\omega} = \widetilde{\boldsymbol{\omega}}|_{\Pi}$ is a symplectic structure on Π. Denote by H the restriction $H = \widetilde{H}|_{\Pi}\colon \Pi \to \mathbb{R}$. So, we have the unperturbed objects

$$\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}, \widetilde{H}_0\right), \quad (\Pi_0, \boldsymbol{\omega}_0, H_0)$$

and the perturbed ones

$$\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}, \widetilde{H}\right), \quad (\Pi, \boldsymbol{\omega}, H).$$

The main idea of the proof is to verify that the Hamiltonian system on $(\Pi, \boldsymbol{\omega})$ with the Hamilton function H admits many invariant n-tori close to the unperturbed n-tori on Π_0. This does not follow directly from the hypothesis of the theorem because $\Pi \neq \Pi_0$ and $\boldsymbol{\omega} \neq \boldsymbol{\omega}_0$.

Consider a diffeomorphism $\alpha\colon \Pi_0 \to \Pi$ close to the identity mapping $\iota\colon \Pi_0 \to \Pi_0$ (the closeness of α and ι is to be understood as that of two embeddings $\Pi_0 \to \widetilde{\Pi}$). Then $\alpha^*\boldsymbol{\omega}$ is a symplectic structure on Π_0 close to $\boldsymbol{\omega}_0$ and $H \circ \alpha$ is a function on Π_0 close to H_0. Actually, $\boldsymbol{\omega}_0$ and $\alpha^*\boldsymbol{\omega}$ are also of the same cohomology class. Indeed, let Γ be any two-dimensional cycle in Π_0, then

$$\int_\Gamma \alpha^*\boldsymbol{\omega} = \int_{\alpha(\Gamma)} \boldsymbol{\omega} = \int_{\alpha(\Gamma)} \widetilde{\boldsymbol{\omega}} = \int_\Gamma \widetilde{\boldsymbol{\omega}} = \int_\Gamma \boldsymbol{\omega}_0.$$

We have used here the fact that Γ and $\alpha(\Gamma)$ are close and, consequently, homologous to each other in $\widetilde{\Pi}$. Now we need the following lemma (essentially due to J. Moser [90], for more recent presentations see, e.g., [133, 145]):

Lemma. *Let $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ be two close symplectic structures of the same cohomology class on a certain manifold Π_0. Then there is a diffeomorphism $\beta\colon \Pi_0 \to \Pi_0$ close to the identity transformation and such that $\beta^*\boldsymbol{\omega}_1 = \boldsymbol{\omega}_0$.*

This lemma (which will be discussed below) provides us with a diffeomorphism $\beta\colon \Pi_0 \to \Pi_0$ close to the identity transformation and such that $\beta^*\alpha^*\boldsymbol{\omega} = \boldsymbol{\omega}_0$. It suffices finally to apply the hypothesis of the theorem to the Hamiltonian system on $(\Pi_0, \boldsymbol{\omega}_0)$ with the Hamilton function $H \circ \alpha \circ \beta$. □

Of course, the power of the reduction approach described in Theorem 6 is rather limited. First of all, this approach applies to normally hyperbolic manifolds Π_0 only, but at the same time gives no tools to watch over separatrix stable and unstable manifolds ("whiskers") of the perturbed tori. What is more serious is that the perturbed normally hyperbolic invariant manifolds are, generally speaking, finitely smooth only even in the analytic and C^∞-categories [45, 55, 136, 146]. Consequently, even if all the "input" objects ($\widetilde{\Pi}$, Π_0, $\widetilde{\boldsymbol{\omega}}$, $\widetilde{H}_0$, and $\widetilde{H}$) in Theorem 6 are analytic (C^∞) and the perturbed invariant tori will most probably be analytic (respectively C^∞) as well, the surface Π containing these tori will be in general finitely differentiable only, and the proof of Theorem 6 outlined above enables one to establish only finite smoothness of the perturbed tori.

Example 1 is the context of Theorem 4 in Section 6. Here $\widetilde{\Pi}$ is the phase space for the Hamiltonian system with the Hamilton function $\widetilde{H}_0$ (18), $\widetilde{\boldsymbol{\omega}}$ is the symplectic structure (19), Π_0 is the surface $\{z = 0\}$, and $\boldsymbol{\omega}_0 = dI \wedge d\varphi$. The surface Π_0 considered as an invariant manifold of system (20) is normally hyperbolic if and only if for every I and φ, the matrix $\Omega(I, \varphi)$ [see (21)] has no purely imaginary eigenvalues. If this hyperbolicity condition is met and the function H_0 in (18) is KAM-stable in the sense of Section 2, we immediately arrive at the conclusion of Theorem 4.

That one can obtain this way invariant n-tori in the framework of Theorem 4 "gratis" was realised long ago [50] (see also [22] for a recent discussion). However, due to the reasons pointed out above, various versions of Theorem 4 are usually proven in the literature by entirely different methods. The author is aware of the only paper [57] where some particular case of Theorem 4 was established by an argument of the same kind as that of Theorem 6 (however, see also a discussion in [17]). On the other hand, Theorem 6 above in its full generality seems to be new.

Now construct "atropic" invariant tori in typical Hamiltonian systems with the help of Theorem 6.

Example 2. Let a symplectic manifold $(\Pi_0, \boldsymbol{\omega}_0)$ and a function $H_0 \colon \Pi_0 \to \mathbb{R}$ determine an unperturbed Hamiltonian system in the Parasyuk theory [99]. In other words, suppose that Π_0 is smoothly foliated into coisotropic invariant n-tori (of codimension $p < n$) of the Hamiltonian system with the Hamilton function H_0, and that any close Hamiltonian system on $(\Pi_0, \boldsymbol{\omega}_0)$ admits many invariant n-tori close to the unperturbed ones. Of course, the symplectic structure $\boldsymbol{\omega}_0$ is not exact and $n+p$ is even. For an arbitrary integer $m \geq 1$, consider the manifold

$$\widetilde{\Pi} = \Pi_0 \times \mathcal{O}(0),$$

where $\mathcal{O}(0)$ is a neighbourhood of the origin in $\mathbb{R}^{2m}$. Equip $\widetilde{\Pi}$ with the symplectic structure

$$\widetilde{\boldsymbol{\omega}} = \pi^* \boldsymbol{\omega}_0 + \sum_{j=1}^{m} dz_j \wedge dz_{j+m}$$

where $(z_1, z_2, \ldots, z_{2m})$ are the coordinates in $\mathcal{O}(0) \subset \mathbb{R}^{2m}$ and $\pi \colon \widetilde{\Pi} \to \Pi_0$ is the natural projection. On $\widetilde{\Pi}$, consider an arbitrary function of the form

$$\widetilde{H}_0(\zeta, z) = H_0(\zeta) + \tfrac{1}{2}\langle M(\zeta) z, z\rangle + O(|z|^3)$$

where $(\zeta_1, \zeta_2, \ldots, \zeta_{n+p})$ are the coordinates in Π_0, while $M(\zeta)$ for every ζ is a real symmetric $2m \times 2m$ matrix such that the matrix $\Omega(\zeta)$ defined by (21) has no purely imaginary eigenvalues. Then $\Pi_0 \times \{0\}$ is a normally hyperbolic invariant manifold of the Hamiltonian system on $\widetilde{\Pi}$ with the Hamilton function $\widetilde{H}_0$, and the restrictions of $\widetilde{\boldsymbol{\omega}}$ and $\widetilde{H}_0$ to this manifold coincide respectively with $\boldsymbol{\omega}_0$ and H_0. Now Theorem 6 guarantees that any Hamiltonian system on $\left(\widetilde{\Pi}, \widetilde{\boldsymbol{\omega}}\right)$ with a Hamilton function close to $\widetilde{H}_0$ admits many invariant n-tori close to the unperturbed n-tori on $\Pi_0 \times \{0\}$. These tori will be "atropic".

In original papers [58, 59], the existence of "atropic" tori was proven by completely different methods. However, the argument of Example 2 shows that Proposition 2 above is valid indeed (at least in the finitely smooth category) independently of the errors in [58, 59].

Now return to Lemma in the proof of Theorem 6.

Sketch of a proof of the Lemma. Since the symplectic structures $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ are close and of the same cohomology class, their difference $\boldsymbol{\omega}_0 - \boldsymbol{\omega}_1$ is small and exact. Hence, in Π_0 there is a small 1-form σ such that

$$\boldsymbol{\omega}_0 - \boldsymbol{\omega}_1 = d\sigma. \tag{22}$$

Consider the family of 2-forms

$$\boldsymbol{\omega}_t = (1-t)\boldsymbol{\omega}_0 + t\boldsymbol{\omega}_1, \qquad 0 \le t \le 1.$$

Since the forms $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ are close and nondegenerate, the form $\boldsymbol{\omega}_t$ is nondegenerate for each t. Consequently, for each t, the equality $i_{v_t}\boldsymbol{\omega}_t = \sigma$ defines a vector field v_t on Π_0 which is small with σ [recall that this equality means that $\boldsymbol{\omega}_t(v_t, \xi) \equiv \sigma(\xi)$ for any vector field ξ on Π_0]. Denote by g_0^t the phase flow map of the nonautonomous vector field v for the time interval from 0 to t. It is not hard to verify (see [90, 133, 145]) that $(g_0^t)^*\boldsymbol{\omega}_t = \boldsymbol{\omega}_0$ for each t, $0 \le t \le 1$. Thus, $\beta = g_0^1$ is the desired diffeomorphism of Π_0. It is close to the identity transformation because v_t are small. $\square$

There are two delicate points in this proof. The first one is that Π_0 in our situation is not compact. In fact, this difficulty arises throughout the proof of Theorem 6. However, all the troubles connected with the noncompactness of Π_0 are of purely technical nature and can be dealt with rather easily. The second point is more fundamental: why, after all, can the form σ be chosen to be small? This question is addressed neither in Moser's original paper [90] nor in subsequent works I know (e.g., [133, 145], actually, all those works contain somewhat different versions of the Lemma). Note that, again, we omit all the technical issues here, in particular, we do not point out in what smoothness class the smallness of all the objects involved is to be understood. In the context of Theorem 6, Π_0 is always diffeomorphic to $\mathbb{T}^n \times \mathbb{R}^p$ for some p. For such Π_0, the 1-forms σ satisfying condition (22) may be explicitly expressed in terms of $\boldsymbol{\omega}_0 - \boldsymbol{\omega}_1$, and one can easily see that σ can be chosen to be small indeed. But, as a matter of fact, under rather general conditions, for any small exact differential form Λ (of any degree) on a smooth manifold $\mathcal{M}$, a form λ subject to the equality $d\lambda = \Lambda$ can be chosen to be small with Λ. I am grateful to M. A. Shubin who has explained to me that this statement follows from the theory of pseudodifferential operators.

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