Generalized relativistic billiards in external force fields

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Abstract

In this paper we study generalized billiards, in which, as the particle reflects from the boundary of the domain, its velocity is transformed as in an elastic collision with a moving wall, considered in the framework of the special theory of relativity. Inside the domain the particle moves under the influence of some gravitational and non-gravitational force fields.

We study both periodic and general non-periodic action of the boundary, and also the relativistic analogue of the accelerating model in the gravitational field. We prove that under some general conditions the invariant manifold in the velocity phase space of the generalized billiard, where the point velocity equals the velocity of light, is an exponential attractor, and for an open set of initial conditions the particle energy tends to infinity.

Dedicated to Prof. Ph.Blanchard
on the occasion of his 60th birthday

1 Introduction

Following G.Birkhoff [1], a *billiard* is a dynamical system, generated by uniform linear motion of a mass point (a particle) with a constant velocity inside a closed domain $\Pi \in \mathbb{R}^n$ with piece-wise smooth boundary Γ, and with reflection at the boundary Γ, such that the normal component of the velocity changes sign while the tangential component remains the same. Thus the length of the velocity vector is constant. For an arbitrary Riemannian manifold with boundary,

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the billiards are geodesic flows, with the reflection from the boundary according to the law "the angle of incidence equals the angle of reflection". Classical billiards have been studied in many works, see, e.g., [3], and references therein. For a wide class of classical billiards (Sinai billiards and their generalizations) strong stochastic properties have been proved (ergodicity, mixing, etc.), see [11], [2].

This work is devoted to *generalized billiards*, that were introduced in the general case in [7] and, in the case when Π is a parallelepiped, in [8], Chapter 1. From the physical point of view, a generalized billiard describes a gas, consisting of finitely many particles moving in a vessel, while the walls of the vessel heat up or cool down. The essence of the generalization is that, as the point hits the boundary Γ, the projection of its velocity to the normal to Γ transforms with the help of a given function $f(\gamma, t)$, defined on the direct product $\Gamma \times \mathbb{R}^1$ (where $\mathbb{R}^1$ is the real line, $\gamma \in \Gamma$ is a point of the boundary and $t \in \mathbb{R}^1$ designates time), according to the following law. Suppose that the trajectory of the particle, which moves with the velocity v, intersects Γ at the point $\gamma \in \Gamma$ at time t^*. Then at time t^* the particle acquires the velocity v^*, as if it underwent an elastic push from the infinitely-heavy plane γ^*, which is tangent to Γ at γ and at time t^* moves along the normal to Γ at the point γ according to the law $f^*(t) = f(\gamma, t)$. Here we take the positive direction of motion of the plane γ^* to be towards the *interior* of Π.

If the velocity v^*, acquired by the particle as the result of the above reflection law, is directed to the interior of the domain Π, then the particle will continue moving uniformly with this velocity until the next collision with the boundary Γ. If the velocity v^* is directed towards the outside of Π, then the particle remains motionless at γ until at some time $\tilde{t} > t^*$ the interaction with γ^* will force it to change the direction of its velocity. If the function $f(\gamma, t)$ does not depend on time t, the generalized billiard coincides with the classical one.

It is possible to consider the reflection from the boundary γ^* both in the framework of Newtonian mechanics and the special theory of relativity ([7], [8]). For the classical billiard (when $\frac{\partial f(\gamma,t)}{\partial t} \equiv 0$) there is no difference between these to cases: it is the same dynamical system. However for the generalized billiard ($\frac{\partial f(\gamma,t)}{\partial t} \neq 0$) there is

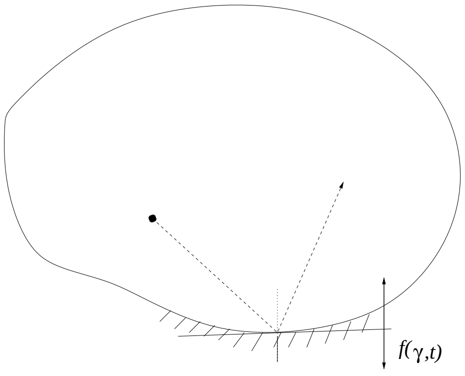

Figure 1: A generalized billiard

an enormous and principal difference: in the Newtonian case the corresponding dynamical system is conservative (i.e. during some time interval the measure, equivalent to the Lebesgue measure, is preserved), and in the relativistic case the corresponding system is dissipative. If Π is a parallelepiped, these facts were proved in [8], moreover, here, in the Newtonian case, an invariant measure, equivalent to the Lebesgue measure, exists and was found explicitly.

This principal difference leads to the fact that in the Newtonian case the Gibbs entropy is constant, while in the relativistic case it increases [8].

The present work is devoted to the relativistic billiards with the relativistic reflection law in the most general case, when there is an external force field acting on the particle. A similar problem in the Newtonian case was studied in [4].

In the relativistic case the system always has an invariant manifold $\mathcal{M}$ in velocity phase space, where the particle velocity v equals the velocity of light c: $\mathcal{M} = \{(x, v) : \|v\| = c\}$. Here $x \in \Pi$ is the spatial coordinate of the particle. We measure velocities relative to the velocity of light, that is, $c = 1$.

It is well-known that in the theory of relativity one should separate the gravitational forces from the non-gravitational ones, as their influence on a particle is essentially different. A "usual" force equals the impuls time derivative. The main postulate of the general theory of relativity is that a gravitational field is a nonflat metric on space-time itself, of signature $(+,-,-,-)$ in every point. In the absence of the gravitational forces the metric is "flat":

$$ds^2 = c^2dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

We consider the following cases:

i. The motion of the particles in the vessel occurs under the influence of an external non-gravitational force.

ii. The particle moves in a gravitational field. We will assume that the metric, which defines the gravitational field, is stationary, thus it does not depend on the motion of the particles themselves.

In both cases we prove that under some natural conditions the trajectories tend to the invariant manifold $\mathcal{M} = \{(x,v) : \|v\| = c\}$ exponentially fast. A consequence of this fact is the system Gibbs entropy grows and the exponential growth of the energy of the particles.

We study the particles motion both in an arbitrary vessel with a general non-periodic action of its boundary on the particles (Sec. 4), and a particular, but very important case, when the vessel is a parallelepiped, while the action of the boundary on the particles is given by a periodic function of time, see Fig. 2.

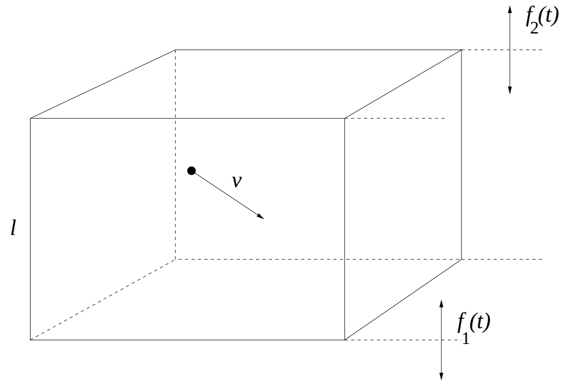

Figure 2: Generalized relativistic billiard in a parallelepiped

The latter case is the relativistic version of a classic Poincaré model

[6], which was studied in [8] under the condition of no external forces acting on the system. It was proved that under some integral condition both the energy and the entropy will grow. The physical meaning of this condition is that the walls are "hotter" than the gas, and it plays the same role in the model as Bolzman collision integral. For a vessel of an arbitrary shape these results were obtained in [7]. Note that the same results are true for a gas consisting of rigid balls.

We consider also a generalization of the accelerating model in the constant force field: a point mass falls vertically on an infinitely-heavy horisontal wall, see Fig.3. We suppose that the relativistic factor appears only at the impact with the wall; the wall itself is motionless, but it acts on the particle by the generalized billiard law, given by some periodic function of time with period 1. Above the wall the particle moves with the constant acceleration g, directed orthogonal to the wall (as in the classical case of the free fall).

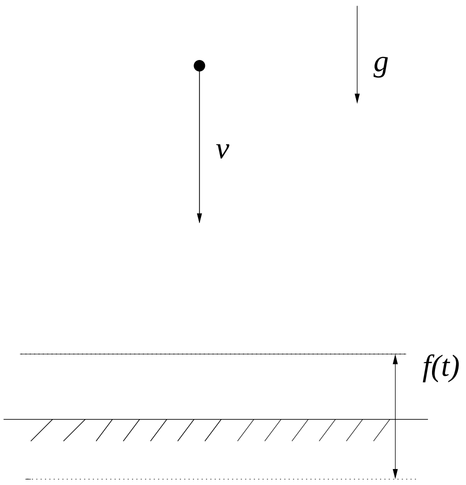

Figure 3: Accelerating model with one wall

If we give the wall position by the equation $x = 0$, then the manifold $\mathcal{M}' = \{x, v : \|v\| = c \text{ at } x = 0\}$ is invariant, moreover, the motion on this manifold is periodic, with some period t_g (which depends only on the acceleration constant). Two cases are possible:

1. $t_g \in \mathbb{Q}$ is a rational number. We prove that under some natural conditions on the set of initial conditions $\{x = 0, \|v\| \leq c\}$ there is a sub-set of positive Lebesgue measure such that the corresponding

trajectories tend to the invariant manifold $\mathcal{M}'$.

2. $t_g \in \mathbb{R}\backslash\mathbb{Q}$ is irrational. In this case, again, under some additional conditions, the set $\mathcal{M}'$ will be attracting: there is $v_0 > 0$, such that if $\|v\| \geq v_0$ is the initial velocity, then the trajectory tends to $\mathcal{M}'$ exponentially fast (but not necessarily monotonously!).

Note that this is the first study of this kind of systems in the relativistic case, as the similar Newtonian systems were studied in [8], [9].

Although the generalized relativistic billiard is a dissipative system, the ergodic theory plays an important role: it is applied to a certain dynamical system, which preserves Lebesgue measure, and is connected to the generalized relativistic billiard in the limit case, when the particle velocity tends to the velocity of light.

The main results for the generalizations of Poincaré models and the accelerating model in the second case follow from the general theorem on the discrete dynamical systems on a cylinder $S^1 \times D^n$ (or a "solid torus", $D^n \in \mathbb{R}^n$ is an n-dimensional disk). On the base S^1 this dynamical system reduces to the circle rotation. In the linear approximation the system is a skew product (see, e.g., [12]) of the rotation of S^1 and some linear automorphism of D^n. If the rotation number is irrational, then the action of this dynamical system on the fibres D^n in logarithmic coordinates is an ergodic sum, applied to some function, which takes part in Birkhoff's ergodic theorem. As the circle rotation is uniquely ergodic, the ergodic sum converges in the mean uniformly and everywhere to some positive constant. As the result, the trajectories tend exponentially fast (but not necessarily monotonously) to the base circle, also when the nonlinear terms are taken into account. Moreover, we prove that this is true for rational rotation numbers as well, provided the period of the periodic trajectories on the base S^1 is large enough.

The same ideas have been used to prove the exponential growth of the energy for Poincaré models and its generalizations in case of no external forces (and an irrational rotation number) [8], while here for the first time they are applied to the accelerating model. Note that the idea of using the ergodic theory to the justification of the second law of thermodynamics belongs to Boltzman, however he suggested that in another context.

The work does not assume any background in the general and special theory of relativity of the reader. The necessary results are presented in Section 1. The main information that we use here, is the law of the velocity vector transformation for a particle, as it reflects from an infinitely-heavy wall. We deduce it in Section 1 using the momentum conservation law for these kind of collisions, which was obtained in [8].

The main feature of the velocity transformation law in the relativistic elastic collision, differing from the Newtonian case, is that although the wall acts along the normal to the vessel boundary, the projection of the particle velocity vector to the tangent plane to the boundary at this point changes also. The reason is that in the relativistic case the projection of the momentum vector (but not velocity vector) to the tangent plane is preserved. The vectors of the velocity and momentum are expressed through each other using their moduli, which change under the impact.

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2 Equations of the energy, momentum and velocity transformation at the relativistic elastic collision with the wall for a generalized billiard

In the Newtonian case the law of the particle momentum and velocity transformation at the elastic collision with an infinitely heavy wall is well-known: if the mass equals 1, and the particle falls vertically to the horizontal wall, then after the impact its velocity equals

$$v' = -v + 2V,$$

where v is the particle velocity before the impact, and V is the wall velocity.

We assume further that all the velocities are measured in the proportion to the velocity of light, i.e., $c = 1$.

The transformation law for the relativistic elastic collision should obviously differ from the Newtonian one. The laws of the particle energy and momentum transformation at the impact we first obtained in [8], [10]. Let the particle of mass 1 fall to the infinitely-heavy horizontal wall, which in turn moves in the vertical direction with the velocity V. Let

$$p = \frac{1}{\sqrt{1 - \|v\|^2}} v$$

be the particle momentum. After the impact the projection of the momentum to the tangent plane to the wall remains the same: $p'_\tau = p_\tau$, while the projection to the normal to the wall p'_ν after the impact equals

$$p'_\nu = (-p_\nu)\frac{1+V}{1-V} + \frac{2V}{1-V^2}\Big(\sqrt{\|p\|^2 + 1} + p_\nu\Big). \qquad (1)$$

Here $\|p\| = \|p_\tau + p_\nu\|$ is the length of the momentum vector. We also suppose that $p_\nu \leq 0$, which means that the particle falls to the wall.

The energy E' after the impact depends both on the energy E before the impact and the tangential component of the momentum (or velocity):

$$E' = E\frac{1+V}{1-V} + \frac{2V}{1-V^2}\Big(\sqrt{E^2 - \Delta} - E\Big), \qquad (2)$$

Here $\Delta = \|p_\tau\|^2 + 1$. Note that $E \geq \sqrt{\Delta}$ for all momenta p.

The relations (1), (2) appeared first in [8], [10] and were proved for the one-dimensional case. Here we give a general proof.

Following [8], we assume first that the mass M of the wall is finite, an it moves with in the vertical direction with the velocity V. The momentum of the wall equals MP, where $P = \frac{V}{\sqrt{1-V^2}}$. The tangential component of the wall momentum equals zero, as the velocity V is directed along the normal to the wall.

Using the laws of conservations of momentum and energy, we obtain the following equalities:

$$p_\nu + MP = p'_\nu + MP', \tag{3}$$

$$\sqrt{\|p\|^2+1} + M\sqrt{P^2+1} = \sqrt{\|p'\|^2+1} + M\sqrt{P'^2+1} \tag{4}$$

Using (4) to find P', we obtain:

$$MP' = M\left\{ \Big(\frac{\sqrt{1+\|p\|^2} - \sqrt{1+\|p'\|}\|^2}{M} + \sqrt{1+P^2} \Big)^2 - 1 \right\}^{\frac{1}{2}} =$$

$$MP + \frac{\sqrt{1+P^2}}{P}(\sqrt{1+\|p\|^2} - \sqrt{1+\|p'\|^2}) + O\left(\frac{1}{M}\right).$$

Substituting the last equality in (3) and going to the limit $M \to \infty$, we obtain the equality $\sqrt{\|p'\|^2+1} - \sqrt{\|p\|^2+1} = V(p'_\nu - p_\nu)$, that leads to the equalities

$$p'^{\,2}_\nu - p^2_\nu = V(p'_\nu - p_\nu)(\sqrt{1+\|p\|^2} + \sqrt{1+\|p'\|^2}),$$

$$p'_\nu + p_\nu = V(\sqrt{1+\|p\|^2} + \sqrt{1+\|p'\|^2}) \tag{5}$$

$$-V^2(p'_\nu - p_\nu) = -V(\sqrt{1+\|p'\|^2} - \sqrt{1+\|p\|^2}). \tag{6}$$

Adding (5) and (6), we obtain the equality

$$p'_\nu(1-V^2) = -p_\nu(1+V^2) + 2V\sqrt{\|p\|^2+1},$$

from which the equality for p' in (1) follows.

To prove the energy transformation law, we do the following trick. Consider $\tilde{p} = p_\nu/\Delta$, $\Delta = \|p_\tau\|^2 + 1$. The transformation formulas (1) in terms of $\tilde{p}$ are exactly the same as for one dimensional case [8]. Thus the "energy" $\tilde{E} = \sqrt{\tilde{p}^2+1}$ is transformed exactly as in [8]

$$\tilde{E}' = \tilde{E}\frac{1+V}{1-V} + \frac{2V}{1-V^2}\Big(\sqrt{\tilde{E}^2-1} - \tilde{E}\Big).$$

But Δ is constant under the transformation, and one can easily show that the "real" energy $E = \sqrt{\Delta}\tilde{E}$, thus the relation (2) is proved. $\square$

If the particle moves up and hits the horizontal wall, which, again, moves in the vertical direction, then the momentum transformation will be [8]

$$p'_\nu = -p_\nu \frac{1-V}{1+V} + \frac{2V}{1-V^2}\Big(\sqrt{||p||^2+1} - p_\nu\Big) \tag{7}$$

Here V is the velocity of the wall directed upwards.

We use the equalities (1), (2) and (7) to find the relation between the particle velocity v before the collision with its velocity v' after the collision.

Lemma 2.1 *Let the particle fall to the horizontal infinitely-heavy wall with the velocity v, directed towards the wall (i.e., the normal component $v_\nu < 0$), and at the impact the wall moves along its normal with the velocity V. Then after the impact the particle velocity v' equals*

$$v'_\nu = -\frac{v_\nu - 2V + V^2 v_\nu}{1 - 2Vv_\nu + V^2}, \quad v'_\tau = \frac{v_\tau(1-V^2)}{1-2Vv_\nu + V^2}, \tag{8}$$

v_τ is the projection of the velocity to the horizontal plane.

Proof. Suppose that before the collision the particle has the momentum p, and after the collision it obtains the momentum p'. As

$$p = \frac{1}{\sqrt{1-||v||^2}} v,$$

the velocity equals

$$v = \frac{1}{\sqrt{1+||p||^2}} p$$

Suppose first that projection of the particle momentum to the vertical direction both before and after the collision is not zero: $p_\nu \neq 0$, $p'_\nu \neq 0$. Then the normal velocity components v_ν, v'_ν are not zeroes. From (1) we get

$$\frac{p'_\nu}{p_\nu} = -\frac{1+V}{1-V} - \frac{2V}{1-V^2}\Big(\frac{1}{v_\nu} + 1\Big) \tag{9}$$

Form (7) we have

$$\frac{p_\nu}{p'_\nu} = -\frac{1-V}{1+V} - \frac{2V}{1-V^2}\Big(-\frac{1}{v'_\nu}+1\Big) \tag{10}$$

We multiply equalities (9) and (10):

$$1 = 1 + \frac{2V}{1-V^2}\frac{1+V}{1-V}\Big(1-\frac{1}{v'_\nu}\Big) - \frac{2V}{1-V^2}\frac{1-V}{1+V}\Big(1+\frac{1}{v_\nu}\Big) -$$

$$-\frac{4V^2}{(1-V^2)^2}\Big(1-\frac{1}{v'_\nu}\Big)\Big(1+\frac{1}{v_\nu}\Big)$$

and express v'_ν through v_ν and V:

$$1-\frac{1}{v'_\nu} = \Big(\frac{1-V}{1+V}\Big(1+\frac{1}{v_\nu}\Big)\Big)\Big(\frac{1+V}{1-V} - \frac{2V}{1-V^2}\Big(1+\frac{1}{v_\nu}\Big)\Big)^{-1},$$

$$v'_\nu = -\frac{v_\nu - 2V + V^2 v_\nu}{V^2 - 2Vv_\nu + 1}$$

As the tangential component of the velocity equals

$$v_\tau = \frac{1}{\sqrt{1+\|p\|^2}}p_\tau,$$

and $p'_\tau = p_\tau$, we get using (2)

$$v'_\tau = \frac{1}{\sqrt{1+\|p'\|^2}}p_\tau = \frac{E}{E'}v_\tau = \frac{1-V^2}{1-2Vv_\nu+V^2}v_\tau. \tag{11}$$

The momenta p_ν and p'_ν cannot equal zeto simultaneously, see (7. Suppose that $p_\nu = 0$. Then $v_\nu = 0$. Using (2), we get

$$v'_\nu = \frac{p'_\nu}{\sqrt{1+{p'_\nu}^2+p_\tau^2}} = \frac{p'_\nu}{E'} = \frac{2V}{1+V^2},$$

which is exactly (8) for $v_\nu = 0$.

Let now $p'_\nu = 0$. We have to check, that the relation (8) gives $v'_\nu = 0$. But this follows immediately from (9.

The expression (11) is also true for both cases, as it is derived using the energy relations, rather than the momenta. □

Lemma 2.2 *Suppose that the particle hits the boundary with some velocity* v. *Then after the impact the particle velocity vector length* $\|v'\|$ *can be expressed as*

$$1 - \|v'\|^2 = \frac{(1-V^2)^2}{(1+V^2-2Vv_\nu)^2}(1-\|v\|^2), \tag{12}$$

V *being the "boundary velocity" at the moment of the impact.*

Proof. It follows from (8) that

$$\begin{gathered} 1 - \|v'\|^2 = \\ = \frac{(1-2Vv_\nu+V^2)^2 - (v_\nu - 2V + V^2 v_\nu)^2 - \|v_\tau\|^2(1-V^2)^2}{(1-2Vv_\nu+V^2)^2} = \\ = \frac{1 + 2V^2v_\nu^2 + V^4 - v_\nu^2 - u^4v_\nu^2 - \|v\|^2 - \|v_\tau\|^2V^4 + 2\|v_\tau\|^2V^2}{(1-2Vv_\nu+V^2)^2} = \\ = \frac{(V^4-2V^2+1)(1-\|v\|^2)}{(1-2Vv_\nu+V^2)^2} \end{gathered}$$

$\square$

We now give some basic results from the special theory of relativity. Let a particle of mass 1 move under the influence of the (non-gravitational) potential force field $F(x) = -grad\, U(x)$, x is the spatial coordinate of the particle. We assume that the motion occurs in some compact domain, thus functions $U(x), F(x)$ are uniformly bounded. The dynamics can be described by the Lagrange equations with the Lagrange function

$$L(\dot{x}, x) = -\sqrt{1 - \|\dot{x}\|^2} - U(x),$$

see, e.g., [5], Section 16. Again, we assume that the velocity of light equals $c = 1$.

The equations of motion can be written as

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\|\dot{x}\|^2}} = -\frac{\partial U}{\partial x} \tag{13}$$

Equations (13) have a first integral

$$\frac{\partial L}{\partial \dot{x}}\dot{x} - L = \frac{1}{\sqrt{1-\|\dot{x}\|^2}} + U = const,$$

as the Lagrange function does not depend explicitly on time (and thus the energy is preserved).

We denote $1 - ||\dot{x}||^2 = w$, $w > 0$. Differentiating the energy integral, we get

$$-\frac{\dot{w}}{2w^{3/2}} + \frac{\partial U}{\partial x}\dot{x} = 0 \tag{14}$$

Suppose now that the particle velocity value $||\dot{x}||$ is close to 1, or, which is the same, $w << 1$. As both the velocities and the coordinates are bounded (as we suppose that the motion occurs in some compact region), the time derivative $\dot{w}$ is bounded by

$$\dot{w} \leq Cw^{3/2} \tag{15}$$

for some constant C.

Equations (13) can be re-written as

$$\begin{aligned} \ddot{x} - \frac{\dot{w}}{2w}\dot{x} &= -\sqrt{w}\frac{\partial U}{\partial x}, \\ \ddot{x} &= \sqrt{1 - ||\dot{x}||^2}G(\dot{x}, x), \end{aligned} \tag{16}$$

where the functions G are smooth and bounded.

If we formally substitute $w = 1 - ||\dot{x}||^2 = 0$ into these equations, then we get $\ddot{x} = 0$, $||\dot{x}|| = 1$, which corresponds to the motion of the light particles in the special theory of relativity.

Remark. The same considerations work also for non-potential forces and the electro-magnetic fields. Indeed, our estimates were based on the energy relation

$$\frac{dE}{dt} = F \cdot v,$$

(where $E = 1/\sqrt{1 - ||v||^2}$ is the particle kinetic energy and F is the force), which is also true in general for non-potential forces. The electro-magnetic field is defined by adding some terms to the Lagrange function that are linear in the velocity v, but one can easily show that the equations of the particle motion are still of the form (16), see, e.g., [5], Section 17.

Now let K be the velocity phase-space of the generalized billiard.

Lemma 2.3 *The set* $\mathcal{M} = \{(x,v) \in K : \|v\| = 1\}$ *is invariant under the dynamics of the generalized relativistic billiard.*

Proof As the particle velocity equals the velocity of light, an external force cannot change the velocity vector (spatial) length. It follows from the velocity transformation relations (12) that $|v_\nu|^2 + \|v_\tau\|^2 = 1$ if and only if $|v'_\nu|^2 + \|v'_\tau\|^2 = 1$, which proves the lemma. $\square$

When a particle moves along a segment, the motion on the invariant manifold $\mathcal{M}$ coincides with the corresponding classical billiard. This is also true for N identical particles moving in a segment: dynamics on the invariant manifold

$$\hat{\mathcal{M}} = \{(x_1, \dots, x_N, v_1, \dots v_N) : \|v_1\| = 1, \dots, \|v_N\| = 1\}$$

is exactly the motion of a classical billiard in an N-simplex.

As soon as the dimension of the domain Π becomes greater than one, the situation turns out to be more complicated: the dynamics on $\mathcal{M}$ is in some sense dissipative, for example, it does not admit a smooth invariant measure. We give a proof for a very simple situation.

Proposition 2.1 *Suppose that the boundary* Γ *contains two parallel segments (thus there is a continuum of periodic trajectories on* $\mathcal{M}$*), and suppose that the "wall velocity" function* V *does not depend on time, and is positive on these segments. Then the generalized billiard on* $\mathcal{M}$ *does not admit a smooth invariant measure.*

Proof. Consider a mapping $\bar{T} : (\gamma, v) \to (\bar{\gamma}, \bar{v})$ consisting of two successfull reflections from the boundary, which is defined for almost any point in $\Gamma \times S$, where $S = \{\|v\| = 1, v_\nu > 0\}$ is the set of "admissible velocities". Here, as above, we denote by v_ν the normal component of the particle velocity as it hits the boundary, and by v_τ the tangential component. The periodic trajectories correspond to the *stationary points* of the mapping $\bar{T}$. Let $v_\tau = v << 1$. Then

$$v_\nu = 1 - \frac{v^2}{2} + O(v^4).$$

Suppose that the particle reflects from these parallel segments of the boundary. It follows from (8) that

$$\bar{v}_\nu = v_\nu + O(v^2), \quad \bar{v}_\tau = \sigma v + O(v^2), \quad \bar{x} = x + O(v),$$

where x is the coordinate on one of these segments and

$$0 < \sigma \leq \Big(\frac{1-V}{1+V}\Big)^2 < 1.$$

The Jacobian of this mapping in a stationary point is obviously less than 1, thus there cannot exist an invariant metrics (see, e.g. [12]). □

As we have already mentioned, the main postulate of the general theory of relativity is that the gravitational field is defined by the metric of the space-time, i.e., a quadratic form

$$ds^2 = g_{ij}dx^i dx^j, \quad i, j = 0 \dots 3$$

of signature $(+, -, -, -)$ in every point, see, e.g., [5], Sec. 87. The first coordinate, $x^0 = t$, will be reffered to as the "world time". We assume that $g_{00} > 0$.

The motion of a particle in the gravitational field is described by the geodesics of this metric, thus the length of the particle velocity 4-vector is a constant of motion:

$$\Big(\frac{dx}{ds}, \frac{dx}{ds}\Big) = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1,$$

where s is a natural parameter (called the "particle time") along the geodesic.

The light particles move along so called isotropic geodesics, along which $(\frac{dx}{d\lambda}, \frac{dx}{d\lambda}) = 0$. Here λ is some affine parameter on the isotropic geodesic. Note that the natural parameter is identically constant along isotropic curves, as $ds^2 = g_{ij}dx^i dx^j = 0$.

The time interval in the general theory of relativity depends not only on the velocity of the reference frame (as in the special theory of relativity), but also on the position of the clocks in space. The "proper time" τ in a fixed point in space is defined by

$$\tau = \int \sqrt{g_{00}}dx^0.$$

We shall assume that the gravitational field is constant, i.e., the functions g_{ij} do not depend on the world time x^0. As in the case of the "flat" metric $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, one can define the three-dimensional particle velocity v as

$$v^\alpha = \frac{dx^\alpha}{\sqrt{g_{00}}(dx^0 + g_\alpha dx^\alpha)}, \quad \alpha = 1, 2, 3, \tag{17}$$

see [5], Sec. 89. Here we denoted $g_\alpha = g_{0\alpha}/g_{00}$. This velocity is measured in terms of the "proper time", i.e., the clocks are syncronized along the trajectory of the particle. If the gravitational field is static, i.e., the functions $g_{0\alpha} = 0$ and $ds^2 = g_{00}dx^0 - dl^2$, where $-dl^2 = g_{ab}dx^a dx^b$, $a, b = 1, 2, 3$, then this expression for the velocity gives $v = dl/d\tau$.

The four-dimensional interval ds^2 can be expressed as

$$ds^2 = g_{00}(dx^0 + g_\alpha dx^\alpha)^2 \Big(1 - v^2\Big),$$

where $v^2 = -g_{ab}v^a v^b$, $a, b = 1, 2, 3$.

The particle energy given by

$$\mathcal{E}_0 = \frac{\sqrt{g_{00}}}{\sqrt{1 - v^2}} \tag{18}$$

is a constant of motion.

The definition of the generalized billiard reflection from the boundary remains almost the same: the normal should be taken in the metric g_{ij}, and to obtain the new velocity one should make a coordinate transformation in the neighbourhood of the collision point, such that in this point the metric becomes "flat":

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

and use formulas (8).

3 The main theorem

Here we formulate and prove a general theorem, which is the main tool in the proofs of the unbounded energy grouth for generalized relativistic billiards.

Let M be a smooth n-dimensional manifold, which is (locally) a direct product of a circle S^1 and an $n-1$-dimensional disk D^{n-1}. For example one can consider a smooth fiber bundle $\pi : M \to B$, $D^{n-1} \in B$, and the fibers are the one-dimensional circles. Let t, w, x be the local coordinates on M, where $t \pmod 1 \in S^1$, $w \in \mathbb{R}^1$, $x \in \mathbb{R}^{n-2}$.

Consider a smooth mapping $T : t, w, x \to t', w', x'$ of the compact domain $S^1 \times D^{n-1}$ to itself. Suppose that the mapping leaves the manifold $w = 0$ invariant, and on this manifold the restriction of T to the fibers S^1 is the rotation of the circle:

$$T : (t, 0, x) \to (t + l \pmod 1), 0, x').$$

We also assume that in the neighbourhood of the manifold $w = 0$ the mapping T can be writen as

$$\begin{aligned} w' = A(t)w + B_1(t, w, x), \quad |B_1| \leq c_1|w|^{1+\alpha}, \\ t' = t + l + B_2(t, w, x), \quad |B_2| \leq c_2|w|, \end{aligned} \tag{19}$$

where the constants c_1, c_2 and α are positive. All the functions are 1-periodic in t.

We denote by $T^n(w)$, $T^n(t)$ and $T^n(x)$ the w-, $t-$ and $x-$coordinate of the n^{th} power of the mapping $T(t, w, x) \to (t', w', x')$.

Theorem 3.1 *Let*

$$\int_0^1 \ln A(t)dt = -\delta < 0.$$

Then for any $\tilde{\delta} > 0$, $\tilde{\delta} < \delta$ there exists a constant $N \in \mathbb{N}$ such that for any $l \neq \frac{p}{q}$, $p, q \in \mathbb{N}$, $q \leq N$ there is a constant $\omega > 0$ such that for any w, $|w| \leq \omega$, the estimate

$$|T^n(w)| \leq Ce^{-\tilde{\delta}n}|w|$$

holds for some constant C for any $n \in \mathbb{N}$.

Proof. Consider the "linearized" mapping $\tilde{T} : (w, t) \to (\tilde{w}, \tilde{t})$, where

$$\tilde{w} = A(t)w, \quad \tilde{t} = t + l,$$

which is a skew product of a circle rotation an a linear mapping.

Lemma 3.1 *For any $\hat{\delta} > 0$, $\hat{\delta} < \delta$ there exists such constant $N \in \mathbb{N}$, that for any $l \neq \frac{p}{q}$, $p, q \in \mathbb{N}$, $q \leq N$, there exists $m \in \mathbb{N}$ such that*

$$|\tilde{T}^m(w)| \leq e^{-\hat{\delta} m}|w|$$

for any w.

Proof of Lemma 3.1 If l is an irrational number, then Lemma 3.1 follows from the corresponding result in [8] (Chapter 1, Sec.3, Lemma 3): for an irrational rotation number the rotation of the circle $t \to t + l \pmod 1$ is a uniquely ergodic mapping, and by the ergodic theorem, the sum (in the logarithmic coordinates) converges uniformly to this δ. Note that in this is a stronger result than the one of the lemma 3.1: for any sufficiently large $m \in \mathbb{N}$

$$|\tilde{T}^m(w)| \leq e^{-\delta m + o(m)}|w|,$$

where the function $o(m)/m \to 0$, as $m \to \infty$.

Suppose now that $l = \frac{p}{q}$.

Consider the sum

$$\sum_{k=0}^{q-1} \ln A(t + \frac{kp}{q})\frac{1}{q}.$$

It is easy to see that it is exactly an integral sum

$$\sum_{k=0}^{q-1} \ln A(t + \frac{k}{q})\frac{1}{q},$$

of the integral $\int_0^1 \ln A(t)dt$, as all the q points $t + pk/q \pmod 1$ are different on the cycle S^1 $(k = 0 \ldots q-1)$. Thus choosing q large enough, we can approximate the integral

$$\int_0^1 \ln A(t)dt = -\delta$$

with any given precision, i.e. for any $0 < \hat{\delta} < \delta$ there is such $N \in \mathbb{N}$ that

$$\sum_{k=0}^{q-1} \ln A(t + \frac{kp}{q})\frac{1}{q} \leq -\hat{\delta}$$

for any $t \in S^1$, any $q > N$.

Now for a rational $l = \frac{p}{q}$, $q > N$, take $m = q$. Then for any w

$$|\tilde{T}^m(w)| \le e^{-\hat{\delta} m}|w|.$$

□

Consider now our original mapping T.

Lemma 3.2 *For any $n \in \mathbb{N}$ and for any $|w| \le 1$*

$$T^n(w) = A(t+(n-1)l)A(t+(n-2)l) \cdot \ldots \cdot A(t)w + B_n(t,w,x),$$
$$|B_n(t,w,x)| \le C_n |w|^{1+\alpha},$$
$$T^n(t) = t + nl + B_n^t(t,w,x), \quad |B_n^t(t,w,x)| \le C_n|w|$$

for some constant C_n.

Proof of Lemma 3.2 Proof by induction. The statement is true for $n = 1$. Suppose that for $n = k$

$$T^k(w) = A(t+(k-1)l) \cdot \ldots \cdot A(t)w + B_k(t,w,x),$$
$$T^k(t) = t + kl + B_k^t(t,w,x),$$

and the corresponding estimates hold. Then

$$T^{k+1}(w) = T(T^k(w)) = A(T^k(t))T^k(w) + B_1(T^k(t), T^k(w), T^k(x)) =$$
$$= A(t+kl+B_k^t(t,w,x))(A(t+(k-1)l) \cdot \ldots \cdot A(t)w + B_k(t,w,x)) +$$
$$+ B_1(T^k(t), T^k(w), T^k(x)) =$$
$$= A(t+kl)A(t+(k-1)l) \cdot \ldots \cdot A(t)w + \frac{dA}{dt}|_{t=\xi} B_k^t(t,w,x)T^k(w) +$$
$$+ A(t+kl)B_k(t,w,x) + B_1(T^k(t), T^k(w), T^k(x)).$$

We now denote

$$B_{k+1}(t,w,x) = \frac{dA}{dt}|_{t=\xi} B_k^t(t,w,x)T^k(w) +$$
$$+ A(t+kl)B_k(t,w,x) + B_1(T^k(t), T^k(w), T^k(x))$$

As $\frac{d}{dt}A$ is bounded ($t \in S^1$) and $|w|$, $\|x\|$ are bounded, there exists a constant $\tilde{C}_{k+1}$ such that

$$|B_{k+1}(t, w, x)| \leq \tilde{C}_{k+1}|w|^{1+\alpha}.$$

The same way we show that

$$|B^t_{k+1}(t, w, x)| \leq \tilde{\tilde{C}}_{k+1}|w|$$

and take $C_{k+1} = max\{\tilde{C}_{k+1}, \tilde{\tilde{C}}_{k+1}\}$. □

We now take $\hat{\delta} = (\delta + \tilde{\delta})/2$. By Lemmas 1 and 2, there are $m \in \mathbb{N}$ and C_m such that

$$|T^m(w)| \leq e^{-\hat{\delta}m}|w| + C_m|w|^{1+\alpha}.$$

We choose $\tilde{\omega} > 0$ such that for $|w| \leq \tilde{\omega}$ the second term is so small that

$$|T^m(w)| \leq e^{-\tilde{\delta}m}|w|.$$

The following lemma is obvious (for example, it can be proved by induction):

Lemma 3.3 *For any $k \in \mathbb{N}$, any w, $|w| \leq 1$ there is a constant $C^*_k > 0$ such that $|T^j(w)| \leq C^*_k|w|$ for any $j = 0, \ldots, k$*

Now take $\omega < 1$ such that $|T^j(w)| \leq \tilde{\omega}$ for $j \leq m$, $|w| \leq \omega$. By Lemma 3, we can find a constant $C > 0$ such that $|T^j(w)| \leq Ce^{-\tilde{\delta}j}|w|$ for $j = 0, \ldots, m$, $|w| \leq 1$.

Consider $|T^{mn+j}(w)|$, $|w| \leq \omega$. As $|T^j(w)| \leq \tilde{\omega}$,

$$|T^{m+j}(w)| = |T^m(T^j(w))| \leq e^{-\tilde{\delta}m}|T^j(w)|.$$

But this is again less than $\tilde{\omega}$, thus

$$|T^{2m+j}(w)| = |T^m(T^{m+j}(w))| \leq e^{-\tilde{\delta}m}|T^{m+j}(w)| \leq$$

$$\leq e^{-\tilde{\delta}2m}|T^j(w)|,$$

and so on. Thus

$$|T^{mn}(T^j(w))| \leq e^{-\tilde{\delta}mn}|T^j(w)| \leq Ce^{-\tilde{\delta}(mn+j)}|w|$$

for all $k, n \in \mathbb{N} \cup 0$. □

Remark The theorem can be generalized to the case when the fibers S^1 and the base B are replaced by some manifolds (for example, one can consider the m-dimensional tori $\mathbb{T}^m$ instead of S^1), and demand that on the fiber $w = 0$ the transformation of this manifold is uniquely ergodic. The "linearized" system will still be a skew product of this uniquely ergodic mapping of the manifold $w = 0$ and some linear mapping.

Consider now a more general situation, when on the manifold $w = 0$

$$T : t \to t' = t + l + G(t, x),$$

and in its neighbourhood

$$w' = (A(t) + A_1(t, x))w + B_1(t, w, x), \quad |B_1(t, w, x)| \leq c_1|w|^{1+\alpha},$$

$$t' = t + l + G(t, x) + B_2(t, w, x), \quad |B_2(t, w, x)| \leq c_2|w|.$$

We suppose that the functions G, A_1 are small: $|G| \leq \mu$, $|A| \leq \mu$ for some $\mu > 0$.

Theorem 3.2 *Let*

$$\int_0^1 \ln A(t)dt \leq -\delta < 0$$

for any $u \in D^m$ Then for any $\tilde{\delta} > 0$, $\tilde{\delta} < \delta$ there exists a constant $N \in \mathbb{N}$ such that for any $l \neq \frac{p}{q}$, $p, q \in \mathbb{N}$, $q \leq N$ there are constants $\omega > 0$, $\tilde{\mu} > 0$ such that for any w, $|w| \leq \omega$, for any $\mu \leq \tilde{\mu}$ the estimate

$$|T^n(w)| \leq Ce^{-\tilde{\delta}n}|w|$$

holds for some constant C for any $n \in \mathbb{N}$.

Both theorems claim that the invariant manifold $\{(t, w, x) : w = 0\}$ is attracting, and that under certain conditions the trajectories tend to it exponentially fast. Theorem 3.2 shows that this situation is structurally stable (in some appropriate topology).

The proof of Theorem 3.2 repeats the proof of Theorem 3.1.

4 The relativistic billiards in non-gravitational force fields

We consider first the case, when the particle of mass 1 moves inside a parallelepiped $\Pi \in \mathbb{R}^3$. We assume that the particle reflects from the upper and lower boundaries of Π according to the generalized billiard law, while the reflections from the other sides are the classical elastic ones. Let v be the particle velocity. We denote by v_ν the velocity component orthogonal to the upper and lower boundaries, and by v_τ its tangential component. Note that while v_ν is a number (one-dimensional), v_τ is a two-dimensional vector.

We suppose that in the interior of Π the particle moves under the influence of some non-gravitational force field, given by a smooth bounded vector function $F(x, \dot{x}, t)$. We assume for simplicity that the tangential component F_τ of the force vector admits the following estimate:

$$||F_\tau|| = O(||v_\tau||)$$

for small values of $||v_\tau||$.

Examples 1. The force field F is orthogonal to the upper and lower boundaries of the parallelepiped Π.
2. The particle moves under the action of the magnetic field H, which is directed orthogonal to the upper and lower boundaries of Π. In this case

$$F = e(v \times H), \quad ||F|| = e||v_\tau||||H||,$$

e is the charge of the particle.

Consider the mapping $\bar{T}$, which consists of two successfull reflections of the particle from the upper and lower boundaries of Π:

$$\bar{T} : (v, t, x) \to (\bar{v}, \bar{t}, \bar{x})$$

The velocity $v_\nu < 0$ as it hits the lower wall, thus $\bar{v_\nu} < 0$.

Let V_1 be the velocity function of the lower wall, and V_2 be the velocity function of the upper wall, and let l the hight of the parallelepiped Π.

Theorem 4.1 *Suppose that*

$$\int_0^1 \ln\frac{(1-V_1(t))(1+V_2(t+l))}{(1+V_1(t))(1-V_2(t+l))}dt = -\delta/2 < 0$$

Then for any $\tilde\delta > 0$, $\tilde\delta < \delta$ there exists a constant $N \in \mathbb{N}$ such that for any $l \neq \frac{p}{q}$, $p, q \in \mathbb{N}$, $q \leq N$ there is a constant $\omega > 0$ such that for any v, $1 - |v_\nu|^2 \leq \omega$, the estimate

$$1 - \|T^n(v)\|^2 \leq Ce^{-\tilde\delta n}(1 - |v_\nu|^2)$$

holds for some constant C for any $n \in \mathbb{N}$.

Remark An example of functions satisfying this integral inequality is given in [8], Chapter 1, Sec.3.

Proof. Denote $w_\nu = 1 - v_\nu^2$. Then $w_\nu = w + \|v_\tau\|^2$, where, as before, $w = 1 - \|v\|^2$. Differentiating with respect to time t we get

$$\dot w_\nu = \dot w + 2(v_\tau, \dot v_\tau) = O(w^{3/2}) + O(\|v_\tau\|^2\sqrt{w}) = O(w_\nu^2)$$

as $\|F_\tau\| = O(\|v_\tau\|)$, and (15), (16).

Consider the motion of the particle in $\mathbb{R}^3$ under the influence of the force field F. On any time interval $[t_0, t_0 + 2l]$

$$w_\nu(t) = w_\nu(t_0) + O(w_\nu(t_0)^{3/2})$$

As $v_\nu(t) = 1 - \frac{1}{2}w_\nu(t) + O(w_\nu(t)^2)$, the particle velocity on the time interval $t \in [t_0, t_0 + 2l]$ is estimated as

$$v_\nu(t) = 1 - \frac{1}{2}w_\nu(t_0) + O(w_\nu(t_0)^{3/2})$$

Thus the normal component of the particle velocity $1 - v_\nu^2$ does not change in the linear approximation when the particle moves in the interior of Π. The mapping $\bar T$ sends the point w_ν, t to

$$\bar w_\nu = \Big(\frac{1-V_1(t)}{1+V_1(t)}\Big)^2\Big(\frac{1+V_2(\tilde t)}{1-V_2(\tilde t)}\Big)^2 w_\nu + O(w_\nu^{3/2}),$$

$$\bar t = t + l(1 + O(w_\nu)),$$

see (8).

The proof is completed by applying Theorem 3.1. □

Remark. The teorem is also true for the force field F such that the tangential component $\|F_\tau\| \leq \mu$ for some sufficiently small constant μ, cf Theorem 3.2, however the proof is more technical.

We consider now the general case, when the particle moves inside a compact domain Π of an arbitrary form. The action of its boundary Γ on the particles is given by some function $f(\gamma, t)$ (see the Introduction). As before, the normal to the boundary is directed inside the vessel Π. Suppose that in the interior of Π the particle moves under the influence of some force field F.

We assume that for all points $\gamma \in \Gamma$ and for all $t \in \mathbb{R}^1$ the velocity

$$\frac{\partial f}{\partial t}(\gamma, t) > 0,$$

(cf [7]), which means that the gas is heating.

Theorem 4.2 *There is a constant $u > 0$, $u < 1$, such that if the particle velocity $\|v\| \geq u$, then there exist constants $\delta > 0$, $C > 0$, such that*

$$1 - \|v(t)\|^2 \leq Ce^{-\delta t}$$

Both Theorems 4.1 and 4.2 claim that under the corresponding conditions the invariant manifold $\mathcal{M} = \{(x, v) : \|v\| = 1\}$ is attracting already in the linear approximation. This means that the particle energy grows exponentially fast.

Proof. Let L be the "diameter" of Π, i.e. $L = max_{\gamma_1, \gamma_2}(\rho(\gamma_1, \gamma_2)$, $\rho(x, y)$ is the distance between the two points x and y. As Π is compact, $L < \infty$. We will need the following

Lemma 4.1 *There is a constant $\tilde{u} > 0$, $\tilde{u} < 1$, s.t. if the particle velocity $\|v\| \geq \tilde{u}$, then for some constant $\hat{t} > 0$ the time interval Δt between any two successive collisions of the particle with the boundary Γ is $\Delta t \leq \hat{t}$.*

Proof of Lemma 4.1. The motion of the particle under the influence of the external force field is described by equations (16). Let the particle velocity $\|v\|$ be close to 1: $1-\|v\|^2 = w << 1$. Consider the time interval $[t_0, t_0+2L]$. If $w(t_0)$ is small enough, then on this time interval

$$w(t) \leq w(t_0) + O(w(t_0)^{3/2}) \leq 2w(t_0).$$

The particle velocity v can be estimated as

$$v(t) = v(t_0) + O(w(t)) = v(t_0) + O(w(t_0)).$$

Thus if the particle velocity v is close to 1, then the trajectory is close to the straight line. More precisely, there is a constant $u' > 0, u' < 1$ such that if $\|v(t_0)\| > u'$ then $\rho(x(t_0), x(t)) > L$ for some $t \in [t_0, t_0+2L]$, $x(t)$ is the particle coordinate.

Now take $\tilde{u} = max(w, u')$. $\square$

In the linear approximation on a time interval $[t, t+\hat{t}]$ the velocity $w = 1 - \|v\|^2$ is constant. Consider a mapping $A : (t, \gamma, v) \rightarrow (t', \gamma', v)$, which consists of one reflection from the boundary and a free motion till the next collision with the boundary.

As for all points $\gamma \in \Gamma$ and for all $t \in \mathbb{R}^1$ the velocity $\frac{\partial f}{\partial t}(\gamma, t) > 0$, there exists a constant $V > 0$ such that $\frac{\partial f}{\partial t}(\gamma, t) \geq V$. Thus at the reflection from the boundary the particle velocity changes as

$$\bar{w} = W(\frac{\partial f}{\partial t}(\gamma, t), v_\nu)w,$$

where $W \leq \tilde{\delta} = \left(\frac{1-V^2}{1+V^2}\right)^2 < 1$, see(12). The mapping A can now be written as

$$w' = W(w, t, \gamma)w + o(w).$$

There is a constant $w_0 > 0$, such that if $w < w_0$, then the n^{th} power of the mapping A can be estimated as

$$A^n(w) \leq C_w \tilde{\delta}^n w$$

for some constant C_w.

To complete the proof of the theorem take $u = max(\tilde{u}, \sqrt{1-w_0})$. $\square$

5 The generalized billiards in the gravitational field

We consider the particle free motion in the 4-dimensional "space-time" with some metric $g_{ij}, i, j = 0, \ldots, 3$ of signature $(+, -, -, -)$ in every point. We suppose that the particle moves inside a bounded domain Π in the "space" variables. The action of its boundary Γ on the particle is given by some vector-function $V(\gamma, t)$ (see Section 2), which is directed along the normal to the boundary Γ at a point γ (as above, $t = x^0$ is the "world time"). The normal to the boundary is directed to the interior of the vessel Π. We suppose that $V(\gamma, t) \geq V_0 = const > 0$ is also directed to the interior of Π.

Let the metric be such that for some $\epsilon > 0$ every geodesic, such that $1 - v^2 \leq \epsilon$, (the three-dimensional velocity v is defined by (17), see Section 2), which starts inside the domain Π, crosses the boundary Γ, and there is a constant $\hat{t} > 0$ such that $|t_0 - t_c| \leq \hat{t}$. Here t_0 is the initial moment of time, and t_c is the moment of time of the collision.

In case of non-gravitational forces this condition is always fulfilled (see Lemma 4.1), but it is not true for an arbitrary metric, i.e., for an arbitrary gravitational field.

Theorem 5.1 *Let the particle velocity at* $t = 0$ *satisfy the following inequality:* $1 - v(0)^2 \leq \epsilon$. *Then there are the constants* $\delta > 0$, $C > 0$, *such that for all* $t > 0$

$$1 - v(t)^2 \leq Ce^{-\delta t}.$$

Proof. The proof is similar to [7]. The particle energy $\mathcal{E}_0$ (18) is constant when the particle moves in the interior of the domain Π. As the particle collides with the boundary, the energy transformation can be estimated using relation (2):

$$\mathcal{E}_0' \geq \frac{1 + V^2}{1 - V^2} \mathcal{E}_0,$$

$\mathcal{E}_0'$ being the particle energy after the collision: indeed, the energy is independent of the coordinate transformations, thus relation (2) is also valid for curvilinear coordinates (cf Section 2). As the gravitaional field is constant and Π is compact, g_{00} is bounded. Thus the estimate of the theorem follows, as the time interval between every two successfull collisions is bounded. $\Box$

6 An accelerating model in the constant force field

We consider a generalization of the accelerating model in the constant force field: a particle of mass 1 falls vertically on an infinitely-heavy horisontal wall. We suppose that the relativistic factor appears only at the impact with the wall; the wall itself is motionless, but it acts on the particle by the generalized billiard law, given by periodic function $f(t)$ with period 1. Above the wall the particle moves with the constant acceleration $g > 0$, directed orthogonal to the wall (as in the classical case of the free fall): $\ddot{x} = -g$.

Let the wall position be given by the equation $x = 0$.

Lemma 6.1 *The manifold* $\mathcal{M}' = \{x, v : \|v\| = 1 \text{ at } x = 0\}$ *is invariant.*

Proof. Let the particle leave the wall with some velocity $|v|$. As the wall is motionless, the particle hits the wall again with the same velocity (as the system is one-dimensional and the energy is conserved). Let $|v| = 1$. Then after the impact the velocity still equals 1. Thus $\mathcal{M}'$ is invariant. $\Box$

If $|v| = 1$, then the time between two successive collisions with the wall equals $2/g$.

We introduce the mapping $T : (v, t) \to (v', t')$ in the following way. Suppose that at the time moment t the particle leaves the wall with the velocity $v > 0$. Then t' is the moment of time of the next collision and $v' > 0$ is the velocity that the particle equires after the next collision with the wall:

$$t' = t + 2v/g \pmod 1, \quad v' = \frac{v - 2V + V^2 v}{1 - 2Vv + V^2},$$

where $V = \frac{df}{dt} t'$.

As before, we denote by $T^n(v)$, $T^n(t)$ the $v-$ and $t-$coordinate of the n^{th} power of the mapping $T : (v,t) \to (v',t')$.

Theorem 6.1 *Suppose that $2/g$ is a rational number and that there is a moment τ, such that $\frac{df}{dt}(\tau) > 0$. Then on the set of the initial conditions ($|v| \leq 1, t \pmod 1$) there is a subset of positive Lebesgue measure, such that $|T^n(v)| \to 1$*

Proof. We assume for simplicity that $2/g = n$, $n \in \mathbb{N}$. We first prove the following lemma. Consider a smooth mapping $A : (x,y) \to (x',y')$ of the plane $\mathbb{R}^2$ to itself, such that all the points $y = 0$ are the stationary points for this mapping:

$$x' = x + B_1(x,y), \quad y' = a(x)y + B_2(x,y),$$

where the functions $|B_1| \leq C_1|y|$, $|B_2| \leq C_2|y|^2$ as $|x|, |y| < 1$.

Lemma 6.2 *Let on the interval $x \in (x_1,x_2) \in [-1,1]$ the function $|a(x)| \leq \sigma < 1$. Then the equilibria $(x,0)$, $x \in (x_1,x_2)$ are Lyapunov stable and asymptotically stable with respect to y.*

Proof of Lemma 6.2. Let $0 \in (x_1,x_2)$. We show that the equilibrium (0,0) is Lyapunov stable. Consider a δ-neighbourhood of zero, which lies inside the interval (x_1,x_2). Let $x \in (-\delta,\delta)$. Consider the iterations of the mapping A. For any $\tilde{\sigma} > \sigma$, $\tilde{\sigma} < 1$ there is a $\tilde{\delta} > 0$ such that when $|y| < \tilde{\delta}$ and $x \in (x_1,x_2)$, $|y'| < \tilde{\sigma}|y|$. Let $|y| < \tilde{\delta}$.

Suppose that at $n \leq N$ the n^{th} power of the mapping $A^n(x) \in (x_1,x_2)$, and, consequently, $|A^n(y)| < \tilde{\sigma}^n|y|$, and as $n = N+1$ $A^{N+1}(x) \notin (x_1,x_2)$. Then

$$|A^{N+1}(x)| \leq |x| + C_1|y| + \ldots + C_1|A^N(y)| \leq$$

$$\leq \delta + C_1|y|(1 + \tilde{\sigma} + \ldots + \tilde{\sigma}^N) \leq \delta + C_1\tilde{\delta}\frac{1}{1-\tilde{\delta}}$$

This estimate does not depend on N. If the constants $\delta, \tilde{\delta}$ are small enough, then $A^n(x) \in (x_1,x_2)$ for all $n \in \mathbb{N}$, if $x \in (-\delta,\delta)$. But in this case $|A^n(y)|$ decays exponentially fast: $|A^n(y)| \leq \tilde{\sigma}^n|y|$ (which means asymptotic stability), and $|A^n(x)| \leq \delta + C_1\frac{\tilde{\delta}}{1-\tilde{\delta}}$, which proves Lyapunov stability. □

Consider now the "light" trajectory with the initial conditions $v = 1$, $t = \tau$. This trajectory is a *stationary point* of the mapping T, as we assumed, that $2/g = n$, moreover, all the points $(v = -1, t)$ are stationary.

There is an interval $(\tau - \sigma, \tau + \sigma)$ such that $df/dt(t) > 0$ for all $t \in (\tau - \sigma, \tau + \sigma)$. But this means that the mapping T is asymptotically stable with respect to v in the linear approximation for any fixed t in this interval. Thus the conditions of Lemma 1 are fulfilled, and the "light" trajectories $v = 1$, $t \in (\tau - \sigma, \tau + \sigma)$ are *asymptotically* stable in v. Obviously the "basin of attraction" has positive Lebesgue measure on $(|v| \leq 1, \quad t \pmod 1)$. □

Remark. Under the conditions of Theorem 6.1 the invariant set $\mathcal{M}'$ may not be attracting! Indeed, if there is such moment of time τ_1, that $\frac{df}{dt}(\tau_1) < 0$, then the equilibria $v = 1, t = t_1 \in (\tau_1 - \sigma_1, \tau_1 + \sigma_1)$ of the mapping T are *unstable* (the constant σ_1 is chosen such that $\frac{df}{dt}(t) < 0$ for $t \in (\tau_1 - \sigma_1, \tau_1 + \sigma_1)$).

Theorem 6.2 *Let $2/g$ satisfy the following condition: $2/g \neq p/q$ for any integers $p, q > 0$, such that $q \leq N$. Suppose that*

$$\int_0^1 \ln \frac{1 - df/dt(t)}{1 + df/dt(t)} dt < 0.$$

Then, if N is sufficiently large, the invariant set $\mathcal{M}' = \{(v, t \pmod 1) : v = 1\}$ is attracting.

Proof. Let $w = 1 - v^2$. The transformation T takes (w, t) to

$$w' = \left(\frac{1 - V}{1 + V}\right)^2 w + O(w^2), \quad t' = t + 2/g + O(w) \pmod 1,$$

where $V = \frac{df}{dt}(t')$.

Now Theorem 6.2 follows from Theorem 3.1. □

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