On the one dimensional behaviour of atoms in intense homogeneous magnetic fields

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Abstract. We show that the Hamiltonian of an atomic ion with N bosonic electrons which is submitted to an homogeneous magnetic field and when appropriately scaled and zoomed, converges, as the strength of the field tends to infinity and in the norm resolvent sense, to the Hamiltonian of a one dimensional atomic ion where the coulombic interactions have been replaced by "delta potentials".

1 Introduction

By intense magnetic fields we mean magnetic fields which are so strong that the energy to jump between two Landau levels is much larger than the one to ionise a hydrogen atom. The electrons being confined in the lowest Landau level have no freedom in the directions orthogonal to the field and therefore will behave as one dimensional particles. Moreover these particles are expected to interact through delta potentials. This note is a contribution to a rigorous analysis of this phenomena in the particular case of bosonic electrons, see the theorem below.

The strength B of the field which is necessary for such a situation to occur must be larger that 10^5 tesla, a value which is today beyond the possibilities of the laboratories on Earth but seems to be present on the surface of neutron stars. This is why one can find in the astrophysical literature many discussions about possible formations of new atoms and molecules in this context. Among these new species stable atoms with 2Z electrons are conjectured to exist for $B >> Z^3$ where Z is the nuclear charge, see [LSoY, p524].

With the help of the theorem below it is sufficient to study this stability question for a one dimensional atom whose particles interact through delta potentials and whose Hamiltonian reads:

$$h = \sum_{j=1}^{N} -\partial_j^2 - Z\delta(x_j) + \sum_{j<k} \delta(x_j - x_k).$$

Unfortunately these Hamiltonians do not seem to have been much studied in the literature except of course for the trivial case $N = 1$ and the case $N = 2$, see [Ro]. In the latter case it is proven that an L^2-ground state exists as long as $Z > 0.374903$ which shows that the conjecture is true at least up to $N = 2$. This communication is also a call to experts in zero-range interactions to devote efforts to study the above Hamiltonian h.

We are far from being the first to tackle this problem. We recommand the two papers [LSoY], [BaSoY], which deal with fermionic atoms. However these authors use a variational approach which does not directly guarantee that the ground state exists and which does not seem to be able to give information on the structure of this ground state. Two things we can handle with our theorem below (but for bosonic atoms, though).

The Hamiltonian of an N-electron atom in an homogeneous magnetic field, with a fixed centre of mass and without Fermi-statistics acts in $L^2(\mathbb{R}^{3N})$ and reads as

$$\mathbb{H} := \sum_{j=1}^{N} H_j + \sum_{j<k} \frac{1}{|r_j - r_k|} \quad \text{where} \quad H_j := (-i\nabla_{r_j} - B\hat{x} \wedge r_j)^2 - \frac{Z}{|r_j|}.$$

Here $r_j = (x_j, y_j, z_j)$ denotes the position of the j^{th} electron with respect to the nucleus, Z is the charge of the nucleus and B is proportional to the strength of the magnetic field which points towards the x-axis; $\hat{x}$ denotes the unit vector on the x-axis of the ambiant space $\mathbb{R}^3$. We have set the Planck constant $\hbar = 1$ and the mass of the electron $m_e = 1/2$ since their values are irrelevant for our discussion. If we denote by $L_{x_j} := -i\partial_{z_j} y_j + i\partial_{y_j} z_j$ the angular momentum operator of the j^{th} electron with respect to the x-axis and $\mathbb{L} := \sum_{j=1}^{N} L_{x_j}$ then it is straightforward to check that $\mathbb{H}$ commutes with $\mathbb{L}$. Therefore $\mathbb{H}$ is reduced by the spectral projections of $\mathbb{L}$. We shall be concerned here by the part $\mathbb{H}^{(0)}$ of $\mathbb{H}$ which corresponds to a zero total angular momentum with respect to the x-axis. This choice is motivated partly to simplify the analysis and partly because it is believed that the lowest energy is reached in this sector, see [BaSe] and [AHS] for the special case $N = 1$. To state the main theorem of this note we introduce the unitary transformation U associated to the following scaling

$$\forall j = 1, \dots N, \quad r_j \to \left(\frac{x_j}{\log B}, \frac{y_j}{\sqrt{B}}, \frac{z_j}{\sqrt{B}} \right)$$

and the projector $\mathit{\Pi}_{\mathbf{0}}^{(0)}$ on states where all the electrons are in the lowest Landau level and have zero angular momentum with respect to the x-axis. Although these objects are standard we give a precise definition of them in the next section. Then one has the

Theorem. *As B tends to infinity one has, uniformly with respect to ξ in every compact of the resolvent set of h, that:*

$$\left(\frac{1}{(\log B)^2} U \left(\mathbb{H}^{(0)} - 2NB \right) U^{-1} - \xi \right)^{-1} - (h - \xi)^{-1} \otimes \mathit{\Pi}_{\mathbf{0}}^{(0)} = \mathcal{O}((\log B)^{-\frac{1}{3}}).$$

From this theorem we can draw two conclusions. Let Z and N be such that h possesses a true L^2-ground state φ with energy e. Then for B large enough $\mathbb{H}^{(0)}$ has also an L^2-ground state $\Phi(B)$ with energy $E(B)$ such that

$$\lim_{B\to\infty} \frac{E(B)-2NB}{(\log B)^2} = e,$$

and

$$\lim_{B\to\infty} \|\Phi(B) - \sqrt{\log B}\,\varphi(\log B\,\cdot) \otimes \sqrt{B}\chi_{\mathbf{0}}^{(0)}(\sqrt{B}\,\cdot)\| = 0$$

where $\chi_{\mathbf{0}}^{(0)}$ denotes the product of N lowest Landau level functions with zero angular momentum with respect to the x axis.

2 Proof of the theorem

First, we give the definition of h. Let $D : \mathbb{R}^{N-1} \to \mathbb{R}^N$ be a regular parametrisation of a plane in $\mathbb{R}^N$ of codimension 1 and $t_D : \mathcal{H}^1(\mathbb{R}^N) \to \mathbb{C}$ be the quadratic form defined by ($\mathcal{H}^1(\mathbb{R}^N)$ stands for the first L^2 Sobolev space)

$$t_D[\psi] := \int_{\mathbb{R}^{N-1}} |\psi \circ D(u)|^2 |D'(u)| du.$$

If D_j and $D_{j,k}$ denote respectively a parametrisation of $x_j = 0$ and $x_j = x_k$ we let $t := t_0 + t_1$ with $t_0, t_1 : \mathcal{H}^1(\mathbb{R}^N) \to \mathbb{C}$ being the quadratic forms defined by $t_0[\psi] := \sum_{j=1}^N \|\partial_j \psi\|^2$ and

$$t_1 = -Z \sum_{j=1}^N t_{D_j} + \sum_{j<k}^N t_{D_{j,k}}.$$

One can check easily that t is closed symmetric, densely defined and bounded below and therefore t defines a selfadjoint operator that we denote by h.

Next we explain what $I\!\!\varPi_{\mathbf{0}}^{(0)}$ is. Each H_j is unitarily equivalent to:

$$H_j \sim B \bigoplus_{m_j\in\mathbb{Z}} \left(-\partial_{x_j}^2 + H_{\mathrm{osc},j}^{(m_j)} - 2m_j - \lambda Z \frac{1}{|r_j|} \right), \quad \lambda := \frac{1}{\sqrt{B}}$$

where $H_{\mathrm{osc},j} := -\partial_{y_j}^2 - \partial_{z_j}^2 + y_j^2 + z_j^2$ is the harmonic oscillator which describes the Landau levels of the j^{th} electron and $H_{\mathrm{osc},j}^{(m)}$ its restriction to angular momenta in the x-direction of value m. In other words

$$H_{\mathrm{osc},j}^{(m_j)} - 2m_j = \bigoplus_{n_j=0}^{\infty} \left(2(|m_j| - m_j) + 4n_j + 2\right) \varPi_{j,n_j}^{(m_j)}$$

where $\Pi_{j,n_j}^{(m_j)}$ is the eigenprojector on the n_j^{th} level of this harmonic oscillator. Then

$$\mathit{I\!I}_{\mathbf{0}}^{(0)} := \Pi_{1,0}^{(0)} \otimes \ldots \otimes \Pi_{N,0}^{(0)} \quad \text{and} \quad \hat{\mathit{I\!I}}_{\mathbf{0}}^{(0)} := \mathit{I\!I}^{(0)} - \mathit{I\!I}_{\mathbf{0}}^{(0)}$$

where

$$\mathit{I\!I}^{(0)} := \sum_{m_1+\ldots+m_N=0} \sum_{n\in\mathbb{N}^N} \Pi_{1,n_1}^{(m_1)} \otimes \ldots \otimes \Pi_{N,n_N}^{(m_N)}.$$

With these notations one has: $\mathbb{H}^{(0)} := \mathit{I\!I}^{(0)} \mathbb{H}\, \mathit{I\!I}^{(0)}$.

Then we introduce new notations. Let

$$a := 2\lambda \log \lambda^{-1} = \frac{\log B}{\sqrt{B}}$$

and

$$(\log B)^{-2} U \left(\mathbb{H}^{(0)} - 2NB \right) U^{-1} =: T^a - \frac{\lambda}{a^2} \mathcal{V}^a, \quad \mathcal{V} := \sum_{j=1}^{N} \frac{1}{|r_j|} - \sum_{j<k} \frac{1}{|r_j - r_k|}$$

where T^a is the kinetic part i.e. $T^a = \sum_{j=1}^{N} -\partial_{x_j}^2 + a^{-2}(H_{\text{osc},j} - 2 - 2L_{x_j})$ and $\mathcal{V}^a(r) := \mathcal{V}(x/a, y, z)$, both sandwiched between $\mathit{I\!I}^{(0)}$.

We compute the resolvent of this operator at a point $\xi < -NZ^2/4$ since such a point surely belongs to the resolvent set of h. We do this with a Feshbach-type formula using the decomposition of $\operatorname{Ran} \mathit{I\!I}^{(0)}$ as $\operatorname{Ran} \mathit{I\!I}_s \oplus \operatorname{Ran} \mathit{I\!I}_r$ where we have set $\mathit{I\!I}_s := \mathit{I\!I}_{\mathbf{0}}^{(0)}$ and $\mathit{I\!I}_r := \hat{\mathit{I\!I}}_{\mathbf{0}}^{(0)}$, to ease the notations. In the sequel a subscript s or r on an operator indicate that the projection $\mathit{I\!I}_s$ or $\mathit{I\!I}_r$ has been applied on the left or on the right according to the position of the subscript: for example $\mathcal{V}_{s,r}^a := \mathit{I\!I}_s \mathcal{V}^a \mathit{I\!I}_r$, or simply $T_r^a := \mathit{I\!I}_r T^a \mathit{I\!I}_r$ when this projection commutes with this operator. So

$$(T^a - \frac{\lambda}{a^2}\mathcal{V}^a - \xi)^{-1} = \begin{pmatrix} S^a & \frac{\lambda}{a^2} S^a \mathcal{V}_{s,r}^a R^a \\ \frac{\lambda}{a^2} R^a \mathcal{V}_{r,s}^a S^a & R^a + \frac{\lambda^2}{a^4} R^a \mathcal{V}_{r,s}^a S^a \mathcal{V}_{s,r}^a R^a \end{pmatrix}$$

where

$$R^a := (T_r^a - \frac{\lambda}{a^2} \mathcal{V}_{r,r}^a - \xi)^{-1}, \quad S^a := (T_s^a - \frac{\lambda}{a^2} \mathcal{V}_{s,s}^a - \frac{\lambda^2}{a^4} \mathcal{V}_{s,r}^a R^a \mathcal{V}_{r,s}^a - \xi)^{-1}.$$

We inspect all terms step by step.

Bound on $R_0^a := (T_r^a - \xi)^{-1}$. Since $T_r^a \geq 4a^{-2} \mathit{I\!I}_r$ one has $\|R_0^a\| \leq a^2/4$. We denote by $R_0 := (T_r - a^2\xi)^{-1}$ and therefore $\|R_0\| \leq 1/4$. Here T means T^1.

Bound on $|r_j|^{-1}R_0|r_j|^{-1}$. Using the Hardy's inequality, the spectral decomposition of $T := T^1$ and the facts $\sum_j m_j = 0$, $\xi \leq 0$ we get:

$$\begin{aligned}(|r_j|^{-1}R_0|r_j|^{-1})(|r_j|^{-1}R_0|r_j|^{-1}) &= |r_j|^{-1}R_0|r_j|^{-2}R_0|r_j|^{-1}\\ &\leq |r_j|^{-1}R_0(-4\Delta_{r_j})R_0|r_j|^{-1}\\ &\leq 4|r_j|^{-1}R_0(-\Delta_{r_j}+y_j^2+z_j^2)R_0|r_j|^{-1}\\ &\leq 6|r_j|^{-1}R_0|r_j|^{-1}\end{aligned}$$

and solving this quadratic inequality gives

$$\forall j = 1,\ldots,N, \quad \||r_j|^{-1}R_0|r_j|^{-1}\|^{-1} \leq 6.$$

Bound on $\lambda a^{-2}\sqrt{R_0^a}Z|r_j^a|^{-1}\sqrt{R_0^a}$. Here r_j^a denotes $(x_j/a, y_j, z_j)$. Doing the reverse dilation $x_j \to ax_j$, $j = 1,\ldots,N$ we get

$$\begin{aligned}\|\lambda a^{-2}\sqrt{R_0^a}\,Z|r_j^a|^{-1}\sqrt{R_0^a}\| &= \|\lambda Z\sqrt{R_0}\,|r_j|^{-1}\sqrt{R_0}\|\\ &\leq |\lambda Z|\|\sqrt{R_0}\|\||r_j|^{-1}R_0|r_j|^{-1}\|^{\frac{1}{2}}\\ &\leq |\lambda Z|\frac{1}{2}\sqrt{6} = \mathcal{O}(\lambda).\end{aligned}$$

Bound on R^a. Let R^a_{NI} be defined as R^a with the interactions between the electrons removed, then $R^a \leq R^a_{\mathrm{NI}}$ and

$$R^a_{\mathrm{NI}} = \sqrt{R_0^a}\left(1-\frac{\lambda Z}{a^2}\sum_{j=1}^{N}\sqrt{R_0^a}|r_j^a|^{-1}\sqrt{R_0^a}\right)^{-1}\sqrt{R_0^a}$$

which implies $\|R^a\| = \mathcal{O}(a^2)$.

Bound on $|r_j - r_k|^{-1}R_0|r_j - r_k|^{-1}$. Using the unitary transform associated to the change of variables

$$(r_j, r_k) \to \left(t := \frac{r_j - r_k}{\sqrt{2}}, s := \frac{r_j + r_k}{\sqrt{2}}\right) \tag{2.1}$$

$-\Delta_{r_j} - \Delta_{r_k} + y_j^2 + z_j^2 + y_k^2 + z_k^2$ is changed in $-\Delta_s - \Delta_t + s_2^2 + s_3^2 + t_2^2 + t_3^2$, and $L_{x_j} + L_{x_k}$ in $L_{s_1} + L_{t_1}$. Thus we may use the bound on $|r_j|^{-1}R_0|r_j|^{-1}$ to deduce that

$$\forall j \neq k, \quad \||r_j - r_k|^{-1}R_0|r_j - r_k|^{-1}\| \leq 6$$

Bound on $\lambda^2 a^{-4}\mathcal{V}^a_{s,r}R_0^a\mathcal{V}^a_{r,s}$. Doing the reverse scaling $x_j \to ax_j$, $j = 1,\ldots,N$ and expanding gives :

$$\|\lambda^2 a^{-4}\mathcal{V}^a_{s,r}R_0^a\mathcal{V}^a_{r,s}\| = \|\lambda^2 a^{-2}\mathcal{V}_{s,r}R_0\mathcal{V}_{r,s}\|$$

$$\begin{aligned}
&\leq 2N \sum_j \| \lambda^2 a^{-2} |r_j|^{-1} R_0 |r_j|^{-1} \| \\
&\quad + N(N-1) \sum_{j \neq k} \| \lambda^2 a^{-2} |r_j - r_k|^{-1} R_0 |r_j - r_k|^{-1} \| \\
&= \mathcal{O}\left(\frac{\lambda^2}{a^2} \right).
\end{aligned}$$

Bound on $\lambda^2 a^{-4} \mathcal{V}^a_{s,r} R^a \mathcal{V}^a_{r,s}$. Using $R^a \leq R^a_{\mathrm{NI}}$, the formula for R^a_{NI} in term of R^a_0 and the previous bound gives:

$$\lambda^2 a^{-4} \mathcal{V}^a_{s,r} R^a \mathcal{V}^a_{r,s} = \mathcal{O}\left(\frac{\lambda^2}{a^2} \right).$$

Now we need the following

Lemma. *As* B *tends to infinity one has*

$$\Big((\log B)^{-2} I\!\!\Pi_s U (I\!\!H^{(0)} - 2NB) U^{-1} I\!\!\Pi_s - \xi \Big)^{-1} - (h - \xi)^{-1} = \mathcal{O}((\log B)^{-\frac{1}{3}}).$$

Proof. Let W be the following one dimensional potential

$$W(x) := \int_0^\infty \frac{e^{-u}}{\sqrt{x^2 + u}} du.$$

It is proven in [BRu] that: $B^{-1} I\!\!\Pi_s U_{B^{-\frac{1}{2}}} \Big(I\!\!H^{(0)} - 2NB \Big) U^{-1}_{B^{-\frac{1}{2}}} I\!\!\Pi_s$ is equal to

$$\sum_{j=1}^N -\partial^2_{x_j} - \lambda Z W(x_j) + \frac{\lambda}{\sqrt{2}} \sum_{j<k} W(\frac{x_j - x_k}{\sqrt{2}});$$

where $U_{B^{-\frac{1}{2}}}$ denotes the unitary implementation of the scaling $r_j \to B^{-\frac{1}{2}} r_j$, $j = 1, \ldots, N$. This shows that we have to face a weak coupling limit problem since we are interested in the limit $\lambda \to 0$. We use the approach described in the next section. We perform an extra scaling $x_j \to x_j / a$, $j = 1, \ldots, N$ which finally gives:

$$\begin{gathered}
h(B) := (\log B)^{-2} I\!\!\Pi_s U_{B^{-\frac{1}{2}}} (I\!\!H^{(0)} - 2NB) U^{-1}_{B^{-\frac{1}{2}}} I\!\!\Pi_s = \\
\sum_{j=1}^N -\partial^2_{x_j} - \frac{\lambda}{a^2} Z W(\frac{x_j}{a}) + \frac{\lambda}{a^2 \sqrt{2}} \sum_{j<k} W(\frac{x_j - x_k}{a\sqrt{2}}).
\end{gathered}$$

One can check easily that

$$\int_{|x|<R} W(x) dx = 2 \log(2R) + \gamma + \mathcal{O}(R^{-2}), \quad \text{as} \quad R \to \infty$$

where γ is the Euler constant. It is shown in the next section that under the above asymptotic behaviour one has[1]:

$$\|\frac{\lambda}{a^2}W(\frac{\cdot}{a})-\delta\|_{-1,1}=\mathcal{O}((\log B)^{-\frac{1}{3}})\quad\text{as}\quad B\to\infty$$

where $\|\cdot\|_{-1,1}$ means the operator norm from $\mathcal{H}^1(\mathbb{R})$ to $\mathcal{H}^{-1}(\mathbb{R})$; here δ must be understood as the operator $\psi\to\psi(0)\delta$. Obviously this implies

$$\|1\ldots\otimes 1\left(\frac{\lambda}{a^2}W(x_j/a)-\delta(x_j)\right)1\otimes\ldots 1\|_{-1,1}=\mathcal{O}((\log B)^{-\frac{1}{3}})$$

as $B\to\infty$ and as an operator from $\mathcal{H}^1(\mathbb{R}^n)$ to $\mathcal{H}^{-1}(\mathbb{R}^n)$, for all $j=1,\ldots N$.

The analogous statement, for all $j\neq k$,

$$\|1\ldots 1\otimes\left(\frac{\lambda}{a^2\sqrt{2}}W_{0,0}(\frac{x_j-x_k}{a\sqrt{2}})-\delta(x_j-x_k)\right)\otimes 1\ldots 1\|_{-1,1}=\mathcal{O}((\log B)^{-\frac{1}{3}})$$

as $B\to\infty$ follows by using a change of variables of the type (2.1). These two properties are sufficient to imply the statement of the lemma since

$$(h(B)-\xi)^{-1}=\sqrt{(h-\xi)^{-1}}(1+K)^{-1}\sqrt{(h-\xi)^{-1}}$$

with

$$K:=\sqrt{(h-\xi)^{-1}}\left(-Z\sum_j\left(\frac{\lambda}{a^2}W(\frac{x_j}{a})-\delta(x_j)\right)+\right.$$
$$\left.\sum_{j<k}\left(\frac{\lambda}{a^2\sqrt{2}}W(\frac{x_j-x_k}{\sqrt{2}a})-\delta(x_j-x_k)\right)\right)\sqrt{(h-\xi)^{-1}}$$

and $K\to 0$ as $B\to\infty$ thanks to them. ∎

Convergence of S^a to $(h-\xi)^{-1}$. From the above lemma one has: $(T^a_s-\frac{\lambda}{a^2}\mathcal{V}^a_{s,s}-\xi)^{-1}-(h-\xi)^{-1}=\mathcal{O}\left((\log B)^{-\frac{1}{3}}\right)$ whereas using the second resolvent equation one gets $S^a-(T^a_s-\frac{\lambda}{a^2}\mathcal{V}^a_{s,s}-\xi)^{-1}=\mathcal{O}\left(\lambda^2a^{-2}\right)=\mathcal{O}((\log B)^{-2})$.

The last terms. It is now easy to verify that $\mathcal{V}^a_{s,r}R^a=\mathcal{O}(a^2)$ and thus

$$\frac{\lambda}{a^2}S^a\mathcal{V}^a_{s,r}R^a=\mathcal{O}(\lambda),\quad\frac{\lambda^2}{a^4}R^a\mathcal{V}^a_{r,s}S^a\mathcal{V}^a_{s,r}R^a=\mathcal{O}(\lambda^2)=\mathcal{O}(B^{-1}).$$

The theorem is proven.

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[1] we warn the reader that the parameters a and v of the next section are presently changed as $a\to 2a$ and $v\to v/2$; a change which does not affect the following limit as one can see with a simple scaling argument.

3 The weak coupling limit revisited

In this section we address the following question. Let $V : \mathbb{R} \to \mathbb{R}_-$ be a measurable function such that (with $\langle x\rangle := \sqrt{1+x^2}$)

$$\langle x\rangle V \in L^\infty(\mathbb{R}) \qquad \text{(H1)}.$$

How much V is attractive is made more precise by

$$\exists v > 0, \quad \mathcal{V}(R) := -\int_{|x|<R} V(x)dx = v\log(R) + \mathcal{O}(1) \quad \text{as} \quad R \to \infty \qquad \text{(H2)}.$$

Let $H_\lambda := -\frac{d^2}{dx^2} + \lambda V$. *Does H_λ possess negative eigenvalues as λ tends to 0?* The answer is yes as one can see with an easy variational argument. If V obeys slightly stronger assumptions, a perturbation expansion for the lowest eigenvalue in terms of λ is derived in [AHS] which uses a modified Birman-Schwinger technic. We shall rederive part of this result with a different method better suited to this note. This method is based on the following property: H_λ, when appropriately scaled and zoomed, converges in the norm resolvent sense to $h_0 := -\frac{d^2}{dx^2} - v\delta$. Let

$$h(a,\lambda) := -\frac{d^2}{dx^2} + \frac{\lambda}{a^2}V(\frac{\cdot}{a}) = a^{-2}U_{\frac{1}{a}}H_\lambda U_{\frac{1}{a}}^{-1}$$

where $U_{\frac{1}{a}}$ denotes the unitary scaling associated to the change of variable $x \to x/a$, $a > 0$.

Lemma. *Let $V : \mathbb{R} \to \mathbb{R}_-$ satisfy (H1,2) and $a(\lambda) := \lambda\log\lambda^{-1}$. Then $h(a,\lambda)$ converges in the norm resolvent sense to h_0 as $\lambda \downarrow 0$. More precisely for every ξ in the resolvent set of h_0*

$$(h(a,\lambda)-\xi)^{-1} - (h_0-\xi)^{-1} = \mathcal{O}\left(\left(\log\lambda^{-1}\right)^{\frac{1}{3}}\right) \quad \text{as} \quad \lambda \to 0.$$

Proof. We assume without loss of generality that $v = 1$. We split V in two parts:

$$\forall R > 0, \quad V_R(x) := \begin{cases} V(x) & \text{if } |x| < R \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \tilde{V}_R := V - V_R.$$

and we impose the following relation between the three parameters a, R and λ:

$$a = -\lambda\int_{|x|<R} V(x)dx \iff a = \lambda\mathcal{V}(R) \qquad (3.1)$$

so that

$$\frac{\lambda}{a^2}\int_{\mathbb{R}} V_R(\frac{x}{a})dx = \frac{\lambda}{a}\int_{\mathbb{R}} V_R(x)dx = -1.$$

Let $r_0 := (h_0 + 5/4)^{-1}$ thus $\|r_0\| \leq 1$; clearly the norm resolvent convergence of $h(a,\lambda)$ to h_0 follows from the convergence to 0 of

$$\sqrt{r_0}\left(\frac{\lambda}{a^2}V(\frac{\cdot}{a})+\delta\right)\sqrt{r_0} = \sqrt{r_0}\left(\frac{\lambda}{a^2}V_R(\frac{\cdot}{a})+\delta\right)\sqrt{r_0}+\sqrt{r_0}\frac{\lambda}{a^2}\tilde{V}_R(\frac{\cdot}{a})\sqrt{r_0}.$$

The second term is estimated as follows (we use (3.1) for the last inequality):

$$\begin{aligned}\|\sqrt{r_0}\frac{\lambda}{a^2}\tilde{V}_R(\frac{\cdot}{a})\sqrt{r_0}\| &\leq \frac{\lambda}{a^2}\|\tilde{V}_R(\frac{\cdot}{a})\|_\infty \leq \frac{\lambda}{a^2}\sup\{|V(x)|,\, |x|\geq R\}\\ &\leq \frac{\lambda}{a^2}\frac{1}{\sqrt{1+R^2}}\|\langle x\rangle V\|_\infty \leq \frac{\|\langle x\rangle V\|_\infty}{\lambda R\mathcal{V}(R)^2}.\end{aligned}$$

For the first term we have, using also (3.1) and with u, v in $\mathcal{H}^1(\mathbb{R})$ and $\varphi := u\overline{v}$:

$$\begin{aligned}\left|\left\langle\left(\frac{\lambda}{a^2}V_R(\frac{\cdot}{a})+\delta\right)u,v\right\rangle_{-1,1}\right| &= \left|\int_{\mathbb{R}}dx\frac{\lambda}{a^2}V_R(\frac{x}{a})(\varphi(x)-\varphi(0))\right|\\ &= \left|\int_{\mathbb{R}}dx\frac{\lambda}{a}V_R(x)(\varphi(ax)-\varphi(0))\right|\\ &\leq \frac{\lambda}{a}\int_{|x|<R}dx|V(x)\int_0^{ax}dy\varphi'(y)|\\ &\leq \frac{\lambda}{a}\sqrt{aR}\int_{|x|<R}dx|V(x)|\|\varphi'\|\\ &\leq 2\lambda\sqrt{\frac{R}{a}}\int_{|x|<R}dx|V(x)|\|u\|_1\|v\|_1\end{aligned}$$

and consequently

$$\|\sqrt{r_0}\left(\frac{\lambda}{a^2}V_R(\frac{\cdot}{a})+\delta\right)\sqrt{r_0}\| \leq 2\lambda\sqrt{\frac{R}{a}}\,\mathcal{V}(R) = 2\sqrt{\lambda R\mathcal{V}(R)};$$

$\langle\cdot,\cdot\rangle_{-1,1}$ denotes the duality brackets between $\mathcal{H}^1(\mathbb{R})$ and $\mathcal{H}^{-1}(\mathbb{R})$ and $\|\cdot\|_1$ the norm of $\mathcal{H}^1(\mathbb{R})$. Thus with $C_V := \|\langle x\rangle V_\infty\|$

$$\|\sqrt{r_0}\left(\frac{\lambda}{a^2}V_R(\frac{\cdot}{a})+\delta\right)\sqrt{r_0}\| \leq 2\sqrt{\lambda R\mathcal{V}(R)}+\frac{C_V}{\lambda R\mathcal{V}(R)^2}.$$

We choose R as a function of λ so that

$$2\sqrt{\lambda R\mathcal{V}(R)} = \frac{C_V}{\lambda R\mathcal{V}(R)^2} \iff \lambda = \left(\frac{C_V}{2}\right)^{\frac{2}{3}}\frac{1}{R\mathcal{V}(R)^{\frac{5}{3}}} =: \frac{1}{G(R)}.$$

Let $R_0 \geq 0$ be the infimum of the essential support of V in $\mathbb{R}_+$, then the map $G : [R_0,\infty[\to\mathbb{R}$ is strictly increasing since

$$(C_V/2)^{\frac{2}{3}}G'(R) = \mathcal{V}(R)^{\frac{2}{3}}\left(5/3(-V(R)-V(-R))R+\mathcal{V}(R)\right).$$

Therefore for every λ in $]0, 1/G(R_0)]$ there exists a unique R solving $\lambda = 1/G(R)$. With such a choice of R we get

$$\|\sqrt{r_0}\left(\frac{\lambda}{a^2}V_R(\frac{\cdot}{a})+\delta\right)\sqrt{r_0}\| \le \left(\frac{4C_V}{\mathcal{V}(R)}\right)^{\frac{1}{3}}.$$

Using (H3) it is straightforward though tedious to find that

$$R_\lambda = \frac{C}{\lambda(\log\lambda^{-1})^{\frac{5}{3}}}\left(1+\mathcal{O}\left(\frac{\log_2\lambda^{-1}}{\log\lambda^{-1}}\right)\right), \quad a = \lambda\log\lambda^{-1}\left(1+\mathcal{O}\left(\frac{\log_2\lambda^{-1}}{\log\lambda^{-1}}\right)\right).$$

and consequently $\mathcal{V}(R) = \mathcal{O}((\log\lambda)^{-1})$ as $\lambda \to 0$. Finally an easy argument based on the scaling properties of δ shows that one can replace a by any other function b such that $a/b \to 1$ as $\lambda \to 0$. Thus we are allowed to take $a = \lambda\log\lambda^{-1}$. $\blacksquare$

Once the norm resolvent convergence is proven it is standard to show that the infimum of the spectrum is a simple isolated eigenvalue for λ small enough and to derive the perturbation expansion of this eigenvalue in terms of λ.

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