Some peculiarities of derivative of complex function with respect to complex variable

S. B. Karavashkin
Special Laboratory for Fundamental Elaboration SELF
187 apt., 38 bldg., Prospect Gagarina, Kharkov, 61140, Ukraine
phone +38 (0572) 276624; e-mail: sbkarav@altavista.com
Website of laboratory SELF: http://angelfire.lycos.com/la3/SELFlab
Website of our electronic journal: http://angelfire.lycos.com/la3/selftrans

This paper was published first in SELF Transactions, **1** (1994), pp.77-94 (Eney, Ukraine, English, 118 pp.). It is introducing to a cycle devoted to the new branch of the theory of complex variable − non-conformal mapping and its application to the calculation of complex dynamical fields. It is very simple but necessary as the basic for the following papers. As this edition had a small circulation, we decided to upload this paper to our archive at http://angelfire.lycos.com/la3/selftrans and to the mp_arc, to give the Readers the opportunity to look through the paper which we will often quote in the following papers of this cycle. The more that we will apply this technique in other research directions of laboratory SELF. We will be pleased if you visit our electronic journal SELF Transactions. We will be the more pleased if you find there a responce to your thoughts and research directions. We will be pleased much more, receiving your opinion of our results. We are willing to answer your questions and discuss with you any problems having something in common with our developments.

Abstract

This paper is the introducing for a monograph devoted to the new branch of theory of complex variable – non-conformal mapping. This new original method enables to connect the mathematical models to which the linear modelling is applicable with nonlinear mathematical models, i.e. with the cases when the mapping function is not analytical in a conventional Caushy – Riemann meaning but is analytical in general sense and has all the necessary criterions of the analyticity, except of the direct satisfying to the Caushy – Riemann equations. As an example, the exact analytical solution of the Bessel-type equation in the continuous range of an independent variable has been obtained.

Classification by MSC 2000: 30C62; 30C99; 30G30; 32A30.
Keywords: Theory of complex variable, Non-conformal mapping, Quasi-conformal mapping, Bessel functions

With all outward simplicity and evidence of some statements, this paper tries looking from some unexpected standpoint at the complex plane and transformations realising on it. Rather, not so much the viewpoint will be unexpected, but the concept of a complex function will be extended to the frames of the most general definitions.

First of all, state these definitions:

"It is said that in the set M of points belonging to the plane z the function

$$w = f(z) \tag{1}$$

was assigned, if the regularity has been indicated, by which the relation has been set up so that a definite point or points assemblage w was placed in correspondence with each point z of M" [1, p.17].

"If one puts $z = x+iy$ and $w= u+iv$, then to define the function of a complex variable $w = f(z)$ will be the same as to define two functions of two real variables

$$u = u(x, y)\,; \qquad v = v(x, y)\text{"} \tag{2}$$

[1, p.17].

As one can see from the definitions, the most general concept of the function of a complex variable is not limited by some before-conditioned direct relation

78

between the real variables *u*(*x*, *y*) and *v*(*x*, *y*). In particular, the functions

$$\begin{aligned} &w=u(x,y)+iv(x);\\ &w=u(x)+iv(y);\\ &w=u(y)+iv(x) \end{aligned} \qquad (3)$$

and so on are also the functions of a complex variable, because to set up the correspondence between (3) and (1), it is sufficient to present *x* and *y* in the form

$$x=\operatorname{Re} z;$$

$$y=\operatorname{Im} z.$$

With it the mentioned particular forms of the function of a complex variable fully satisfy the conditions of continuous and one-valued mapping, if this last was understood not in Caushy – Riemann presentation, but more generally, in the Caushy or Heine sense.

Actually, "the function $f(z)$ is continuous at the point z_0 if it was assigned in some vicinity z_0 (including the point z_0 itself) and

$$\lim_{z\to z_0} f(z)=f(z_0)\text{ "} \qquad (4)$$

[1, p.20]. "At $z\to z_0$ the function $w=f(z)$ has the number $w_0\neq\infty$ as its limit in the Caushy sense, if for each $\varepsilon>0$ such $\delta(\varepsilon)>0$ exists that the inequality

79

$$\left|f(z)-w_0<\varepsilon\right| \qquad (5)$$

is true for all $z\in E\cap C(\delta,z_0)$" [2, p.35]. Noting this, we can write (4) as

$$\begin{aligned} &\lim_{\substack{x\to x_0\\ y\to y_0}} \operatorname{Re} f(x,y)=\operatorname{Re} f(x_0,y_0);\\ &\lim_{\substack{x\to x_0\\ y\to y_0}} \operatorname{Im} f(x,y)=\operatorname{Im} f(x_0,y_0) \end{aligned} \qquad (6)$$

because the following statements are true:

$$\left.\begin{aligned} &\delta>|z-z_0|=|(x-x_0)+i(y-y_0)|\geq|x-x_0|\\ &\delta>|z-z_0|=|(x-x_0)+i(y-y_0)|\geq|i(y-y_0)| \end{aligned}\right\}\Rightarrow\left\{\begin{aligned} &|x-x_0|<\delta\ ;\\ &|y-y_0|<\delta\ ; \end{aligned}\right.$$

$$\left.\begin{aligned} &\varepsilon>|w-w_0|=|(u-u_0)+i(v-v_0)|\geq|u-u_0|\\ &\varepsilon>|w-w_0|=|(u-u_0)+i(v-v_0)|\geq|i(v-v_0)| \end{aligned}\right\}\Rightarrow\left\{\begin{aligned} &|u-u_0|<\varepsilon\ ;\\ &|v-v_0|<\varepsilon\ . \end{aligned}\right.$$

Substituting any function from (3) into (6), we yield that in case of continuous functions of real arguments *u* and *v*, the function of complex variable *w* is continuous too. And vice versa, if at least one of functions of real arguments *u* and *v* was discontinuous, the function of complex variable *w* is discontinuous too, because at least one of equalities of (6) is violated.

The same simply we can prove the relation between the one-valuedness of mapping *z* onto *w* and one-valuedness of functions of a real argument *u* and *v*.

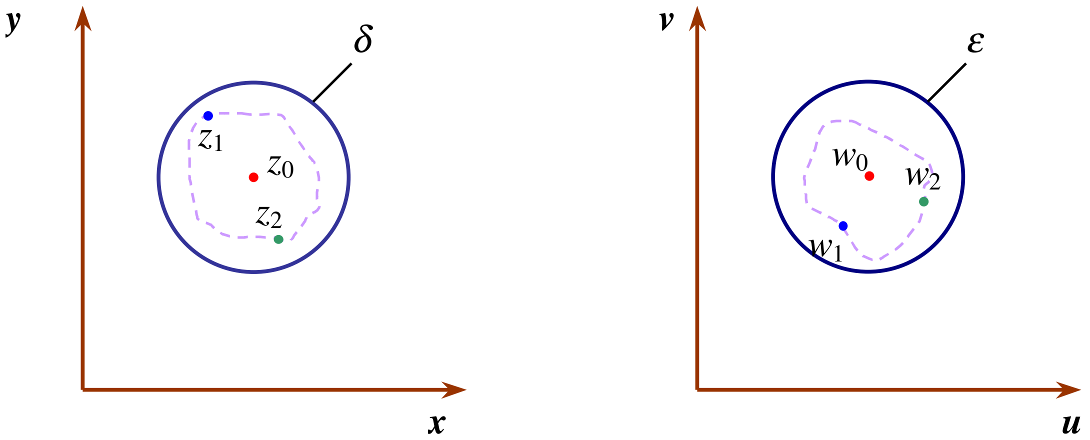

Fig. 1. Possible mapping of points z_1 and z_2 belonging to δ-vicinity of the complex plane Z into the points w_1 and w_2 belonging to ε-vicinity of the complex plane W

80

Noting the above definitions, consider some complex function $f(z)$ proceeding the one-valued mapping of δ-vicinity of the point z_0 of the complex plane Z into ε-vicinity of the point w_0 of the complex plane W (see Fig. 1). Choose in the δ-vicinity of z_0 two points $z_1(x_1, y_1)$ and $z_2(x_2, y_2)$. In accord with the complex function definition, some points $w_1(u_1, v_1)$ and $w_2(u_2, v_2)$ of mapping on the complex plane W correspond them. And according to the condition of one-valued mapping, if

$$z_1 \neq z_2 \ , \quad \text{then} \quad w_1 \neq w_2 \,. \tag{7}$$

Form the differences between the picked points z_1, w_1, z_2, w_2, z_0 and w_0 correspondingly:

81

$$\begin{cases} z_1 - z_0 = (x_1 - x_0) + i(y_1 - y_0) = \Delta x_1 + i\Delta y_1 \,; \\ w_1 - w_0 = (u_1 - u_0) + i(v_1 - v_0) = \Delta u_1 + i\Delta v_1 \,; \end{cases}$$
$$\begin{cases} z_2 - z_0 = (x_2 - x_0) + i(y_2 - y_0) = \Delta x_2 + i\Delta y_2 \,; \\ w_2 - w_0 = (u_2 - u_0) + i(v_2 - v_0) = \Delta u_2 + i\Delta v_2 \,. \end{cases} \tag{8}$$

Noting (7), in general case

$$\Delta x_1 \neq \Delta x_2 \,;\ \Delta y_1 \neq \Delta y_2 \,;\ \Delta u_1 \neq \Delta u_2 \,;\ \Delta v_1 \neq \Delta v_2 \,.$$

At the same time

$$|\Delta x_1| < \delta \,;\ |\Delta y_1| < \delta \,;\ |\Delta u_1| < \varepsilon \,;\ |\Delta v_1| < \varepsilon \,;$$
$$|\Delta x_2| < \delta \,;\ |\Delta y_2| < \delta \,;\ |\Delta u_2| < \varepsilon \,;\ |\Delta v_2| < \varepsilon \,.$$

Thus, even from the condition

$$|z_1 - z_0| = |z_2 - z_0|$$

generally there does not follow

$$|w_1 - w_0| = |w_2 - w_0| .$$

This last is caused by the fact that (5) cannot note the equality of $w \to w_0$ subtend velocities, because the ε-vicinity is usually chosen with respect to the most distanced point $w(z)$ corresponding to z and falling in the δ-vicinity of z_0. But if

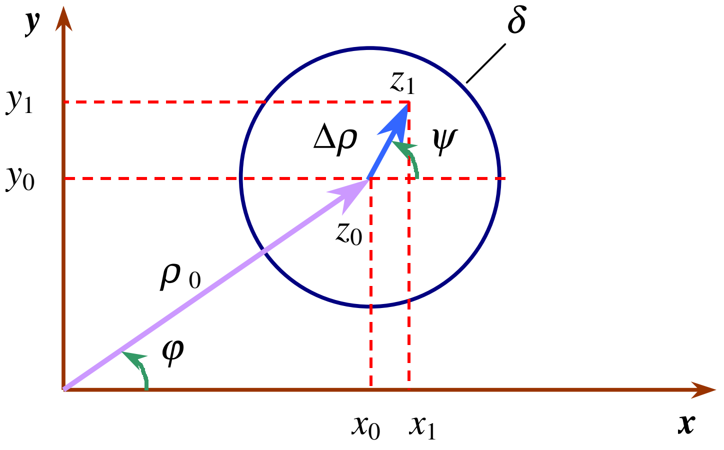

Fig. 2. Diagram for calculation the increment Δz on the complex plane Z

82

we outline the real border of the mapping $z \xrightarrow{f} w$, then dependently on $f(z)$ it can take any complex form (e.g., $\Gamma(\delta)$ in Fig. 1).

But the inconstancy of $w \to w_0$ subtend velocity dependently on the subtend direction causes that the relation

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{9}$$

becomes dependent on the $w \to w_0$ subtend direction.

83

As is known, (9) determines the total derivative of the complex function $f(z)$ with respect to complex argument z. As we can see from this analysis, this function is a complex analogue of the derivative with respect to direction in vector algebra.

In order to reveal the salient features of the total complex derivative, determine the differentials of z and w.

To determine the differential of z, pick on the complex plane Z the δ-vicinity of the point z_0 (see Fig. 2). Pick in this δ-vicinity a point $z_1(x_1, y_1)$, and let

$$|z_1 - z_0| = \Delta\rho < \delta . \tag{10}$$

We see from the construction in Fig. 2 that

$$\begin{aligned} x_1 - x_0 &= \Delta x_1 = \Delta\rho \cos\psi ; \\ y_1 - y_0 &= \Delta y_1 = \Delta\rho \sin\psi ; \end{aligned} \tag{11}$$

$$z_1 - z_0 = \Delta x_1 + i\Delta y_1 = \Delta\rho\left(\cos\psi + i\sin\psi\right). \tag{12}$$

Tending $z_1 \to z_0$ and noting (10), we yield in the limit

$$dz = \lim_{z_1 \to z_0}\left(z_1 - z_0\right) = d\rho\left(\cos\psi + i\sin\psi\right) = |dz|\left(\cos\psi + i\sin\psi\right). \tag{13}$$

84

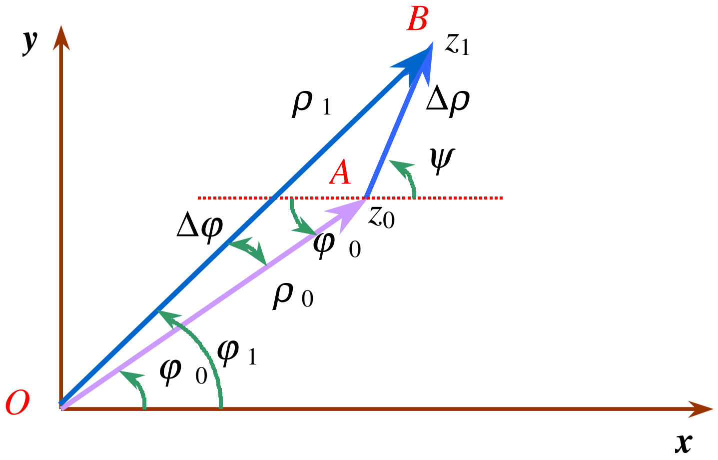

Fig. 3. Geometric construction in addition to Fig. 2 for calculation the increment Δz on the complex plane Z

As we see from (13), *dz* depends only on one real variable ρ, and at the same time it notes all partial differentials in δ-vicinity of z_0. This property of the differential of z reveals the most visually when representing z in polar form.

To show it, consider the triangle OAB (see Fig. 3) formed by the radius-vectors

$$z_0 = \rho_0 e^{i\varphi_0}; \qquad z_1 = \rho_0 e^{i\varphi_1}; \qquad \Delta\rho. \tag{14}$$

In accord with the sine theorem

$$\frac{\rho_1}{\sin(\pi+\varphi_0-\psi)} = \frac{\Delta\rho}{\sin\Delta\varphi},$$

whence

$$\sin\Delta\varphi = \frac{\Delta\rho}{\rho_1}\sin(\psi-\varphi_0); \tag{15}$$

$$\cos\Delta\varphi = \sqrt{1-\left(\frac{\Delta\rho}{\rho_1}\right)^2 \sin^2(\psi-\varphi_0)}. \tag{16}$$

In accord with the cosine theorem,

$$\rho_1 = \sqrt{\rho_0^2+(\Delta\rho)^2+2\Delta\rho\rho_0\cos(\psi-\varphi_0)}. \tag{17}$$

Substituting (17) into (16), we yield

$$\cos\Delta\varphi = \frac{\rho_0+\Delta\rho\cos(\psi-\varphi_0)}{\rho_1}. \tag{18}$$

85

Noting that

$$z_1 = \rho_0 e^{i\varphi_1} = \rho_0 e^{i\varphi_0}(\cos\Delta\varphi + i\sin\Delta\varphi),$$

yield

$$z_1 = \rho_0 e^{i\varphi_0} + \Delta\rho e^{i\varphi_0}\left[\cos(\psi-\varphi_0)+i\sin(\psi-\varphi_0)\right] = \rho_0 e^{i\varphi_0} + \Delta\rho e^{i\psi}. \tag{19}$$

And from (19) we yield

86

$$\begin{aligned}&\Delta z = \Delta\rho e^{i\psi};\\ &dz = \lim_{\Delta\rho\to 0}\Delta z = d\rho e^{i\psi}.\end{aligned} \tag{20}$$

We see from this derivation that the form itself of writing the total differential of dz with the fixed value of the direction of differentiation (the angle ψ) turns one of the independent variables (φ) into a non-differentiable parameter depending only on the location of a point in which this total differential is sought, and the second differential of $d\rho$ does not depend on $z\to z_0$ subtend direction, so this is the differential in a common sense.

This derivation demonstrates also that the form of total differential does not depend on the way, how z is presented.

And finally, at $\psi = 0$

$$dz = d\rho\cos\psi = dx,$$

and at $\psi = \pi/2$

$$dz = id\upsilon\sin\psi = idy.$$

If we abstract from the way, how the partial differentials dx and dy were compared, and substitute into (20) the limits taken from (11), we will obtain

$$dz = dx + idy,$$

i.e. the conventional value of the total differential. This fully corroborates the validity to identify dz with the total differential in the form (20).

Now determine the differential of w at the point w_0 at the condition that there

exists a one-valued mapping of the δ-vicinity of the complex plane z into the ε-vicinity of the complex plane w.

As we proved before, a function of a complex variable can be presented the most generally as

$$w = f(z,x,y) = f\left[\rho(\cos\varphi + i\sin\varphi);\ \rho\cos\varphi;\ \rho\sin\varphi\right].$$

The increment in the function at the point w_0 takes the following form:

$$\begin{aligned}&\Delta w = f\left(\left[\rho_0 e^{i\varphi_0} + \Delta\rho e^{2\psi}\right];\ \left[\rho_0\cos\varphi_0 + \Delta\rho\cos\psi\right];\ \left[\rho_0\sin\varphi_0 + \Delta\rho\sin\psi\right]\right) -\\ &-f\left(\rho_0(\cos\varphi_0 + i\sin\varphi_0);\ \rho_0\cos\varphi_0;\ \rho_0\sin\varphi_0\right).\end{aligned} \tag{21}$$

We see from (21) that Δw depends, as Δz, on one variable ρ. Thus we can write

$$\Delta w = \Delta f(\rho)$$

and

$$dw(z) = \lim_{\Delta\rho\to 0}\Delta f(\rho) = df(\rho). \tag{22}$$

With the help of obtained differentials of an independent variable z and complex function $w(z)$, we can easy obtain an expression for the total derivative of a complex function. For it we will use (9), (20) and (22):

$$\frac{dw(z)}{dz} = \lim_{\Delta z\to 0}\frac{\Delta w(z)}{\Delta z} = \lim_{\Delta\rho\to 0}\frac{\Delta f(\rho)}{\Delta\rho(\cos\psi + i\sin\psi)} =$$

$$=\left[\lim_{\Delta\rho\to 0}\frac{\Delta f(\rho)}{\Delta\rho}\right](\cos\psi - i\sin\psi)=\frac{df(\rho)}{d\rho}(\cos\psi - i\sin\psi). \tag{23}$$

Now find the form of presentation of the total derivative of a complex variable with respect to z presented in the co-ordinate form, using (2). (23) takes the following form:

$$\begin{aligned}\frac{dw(z)}{dz}&=\frac{\partial u}{\partial\rho}(\cos\psi - i\sin\psi)+i\frac{\partial v}{\partial\rho}(\cos\psi - i\sin\psi)=\\&=\left(\frac{\partial u}{\partial\rho}\cos\psi+\frac{\partial v}{\partial\rho}\sin\psi\right)+i\left(\frac{\partial v}{\partial\rho}\cos\psi-\frac{\partial u}{\partial\rho}\sin\psi\right).\end{aligned} \tag{24}$$

To transit in (24) from partial derivative with respect to ρ to those with respect to x and y, we have to make a substitution:

$$\begin{aligned}\frac{\partial u}{\partial\rho}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\rho}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\rho};\\\frac{\partial v}{\partial\rho}&=\frac{\partial v}{\partial x}\frac{\partial x}{\partial\rho}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\rho}.\end{aligned} \tag{25}$$

And it follows from (20) that

$$\frac{\partial x}{\partial\rho}=\cos\psi \quad \text{and} \quad \frac{\partial y}{\partial\rho}=\sin\psi. \tag{26}$$

Substituting sequentially (26) into (25) and (24), we yield

$$\begin{aligned}\frac{dw(z)}{dz}&=\left[\frac{\partial u}{\partial x}\cos^2\psi+\frac{\partial v}{\partial y}\sin^2\psi+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\sin\psi\cos\psi\right]+\\&+i\left[\frac{\partial v}{\partial x}\cos^2\psi-\frac{\partial u}{\partial y}\sin^2\psi+\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\sin\psi\cos\psi\right]=\\&=\left(\frac{\partial w}{\partial x}\cos\psi+\frac{\partial w}{\partial y}\sin\psi\right)(\cos\psi - i\sin\psi)=\\&=\left(\frac{\partial w}{\partial x}\cos^2\psi-i\frac{\partial w}{\partial y}\sin^2\psi\right)-i\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)\sin\psi\cos\psi.\end{aligned} \tag{27}$$

To the point, when Caushy – Riemann conditions

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}; \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

are true, the intermediate expression (27) becomes independent on the angle ψ. This last is one of the proofs that in their essence Caushy – Riemann conditions only define the class of functions of a complex variable having a central symmetry.

89

To write the form of the second total derivative for the function of a complex variable, use the principle of the double sequential mapping

$$z\xrightarrow{f}w\xrightarrow{d/dz}\frac{dw}{dz}. \tag{28}$$

Note that the differentiation directions of the first and second derivatives for the function of a complex variable can be generally not the same. So

$$\frac{d^2w}{dz^2}=\frac{d}{dz}\left(\frac{dw}{dz}\right)=\frac{d}{dz}\left(\frac{dw}{d\rho}e^{-i\psi_1}\right)=\frac{\partial^2 w}{\partial\rho^2}e^{-i(\psi_1-\psi_2)} \tag{29}$$

To present the second derivative in co-ordinate form, substitute to (27) the value of *dw/dz* taken from (27). After transformation we yield

$$\frac{d^2w}{dz^2}=\left[\frac{\partial^2 w}{\partial x^2}\cos\psi_1\cos\psi_2+\frac{\partial^2 w}{\partial y^2}\sin\psi_1\sin\psi_2+\frac{\partial^2 w}{\partial x\partial y}\sin\left(\psi_1+\psi_2\right)\right]e^{-i(\psi_1+\psi_2)}. \tag{30}$$

For functions of complex variable satisfying Caushy – Riemann conditions, we can find the second derivative, noting that according to (25), the following equality is a complex analogue of Caushy – Riemann equations:

90

$$\frac{\partial w}{\partial x}=i\frac{\partial w}{\partial y}. \tag{31}$$

Differentiate (31) with respect to *x*

$$\frac{\partial^2 w}{\partial x^2}=i\frac{\partial^2 w}{\partial x\partial y}, \tag{32}$$

and with respect to *y*

$$i\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial x\partial y}. \tag{33}$$

Multiply (33) by *i* and subtract it from (32):

$$\frac{\partial^2 w}{\partial x^2}=-\frac{\partial^2 w}{\partial y^2}. \tag{34}$$

Substituting in (30) the terms $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial x\partial y}$ in accord with (32) and (34), we yield

$$\frac{d^2w}{dz^2}=\frac{\partial^2 w}{\partial x^2}. \tag{35}$$

Proceeding the same with the rest derivatives, we yield

$$\frac{\partial^2 w}{\partial z^2}=\frac{\partial^2 w}{\partial x^2}=-\frac{\partial^2 w}{\partial y^2}=i\frac{d^2w}{dxdy}. \tag{36}$$

The equality (36) evidences that for the functions analytical after Caushy – Riemann, the total second derivative with respect to *z* retains its independence of the direction of its calculation and depends neither on ψ_1 nor on ψ_2.

Now express $w(x, y)$ in (33) through *u* and *v*. It leads us to the following system:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{37}$$

and

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, \tag{38}$$

which are Laplace equations for two variables. One more property of the centrally symmetric functions of complex variable follows from this:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0. \tag{39}$$

It means that both their real and imaginary parts must satisfy Laplace equations.

Summing up, the carried out investigation shows that the theory of complex variables contains much wider scope. And a part of them can be extended to the vector algebra tool, to the theory of functions of several variables. And some results can be used even to analyse real functions of one variable. In this connection I hope that despite the simplicity of statements, this paper will be evaluated properly and it will gain a proper development, the same as other areas of mathematics.

91

As an example, consider an application to solving the following second-order differential equation:

$$z^2\frac{d^2w(z)}{dz^2}+z\frac{dw(z)}{dz}+\left(z^2-n^2\right)w(z)=0 \tag{40}$$

[3, p.61].

To simplify the solution, present z in polar form:

$$z=\rho e^{-\varphi}. \tag{41}$$

Substituting (23), (29) and (41) into (40), we yield:

$$\rho^2\frac{\partial^2 w(\rho)}{\partial\rho^2}+\rho e^{i(\psi_2-\varphi)}\frac{\partial w(\rho)}{\partial\rho}+\left[\rho^2 e^{i(\psi_1+\psi_2)}-n^2 e^{i(\psi_1+\psi_2-2\varphi)}\right]w(\rho)=0. \tag{42}$$

It is obvious from the above analysis that in (42) only ρ is a variable, all other variables – φ, ψ_1 and ψ_2 – become the parameters. And their quantities are not fixed by the statement of problem; this can be used when finding the solution of (40).

Note also that after writing the total derivative in the forms (23) and (29), the obtained derivatives with respect to ρ become the derivatives in a common sense. In this connection the substitution

$$t=\rho e^{i(\psi_1+\psi_2)}$$

92

is true, and (42) takes the following form:

$$t^2\frac{d^2w(t)}{dt^2}+te^{i(\psi_2-\varphi)}\frac{dw(t)}{dt}+\left[t^2-n^2e^{i(\psi_1+\psi_2-2\varphi)}\right]w(t)=0. \tag{43}$$

We will seek the solution of (43) in the form

$$w(t)=t^\alpha\Theta(t), \tag{44}$$

where α is some parameter independent of ρ. Substituting (44) into (43), we yield

$$\begin{aligned}&t^2\frac{d^2\Theta(t)}{dt^2}+\left[2\alpha+e^{i(\psi_2-\varphi)}\right]t\frac{d\Theta(t)}{dt}+\\&+\left[t^2-n^2e^{i(\psi_1+\psi_2-2\varphi)}+\alpha(\alpha-1)+\alpha e^{i(\psi_2-\varphi)}\right]\Theta(t)=0.\end{aligned} \tag{45}$$

Take

$$\alpha=-\frac{1}{2}e^{i(\psi_2-\varphi)}. \tag{46}$$

Then

$$t^2\frac{d^2\Theta(t)}{dt^2}+\left[t^2-e^{i(\psi_2-2\varphi)}\left(n^2e^{i\psi_1}-\frac{1}{2}e^{i\varphi}+\frac{1}{4}e^{i\psi_2}\right)\right]\Theta(t)=0. \tag{47}$$

If in (47)

$$n^2e^{i\psi_1}-\frac{1}{2}e^{i\varphi}+\frac{1}{4}e^{i\psi_2}=0, \tag{48}$$

93

then this equation becomes a known Helmholtz differential equation having the standard solution $\Theta(t)=c_1e^{it}+c_2e^{-it}$.

The equation (48) can be zero because of free choosing the parameters φ, ψ_1 and ψ_2 given by the statement of problem.

To determine the conditions at which (48) is zero, equalise to zero the real and imaginary parts of the left-hand part of the equality:

$$\begin{aligned}&n^2\cos\psi_1+\frac{1}{4}\cos\psi_2-\frac{1}{2}\cos\varphi=0;\\&n^2\sin\psi_1+\frac{1}{4}\sin\psi_2-\frac{1}{2}\sin\varphi=0.\end{aligned}\tag{49}$$

The solution of this system is

$$\begin{aligned}&\psi_1=\varphi+\arccos\left(n^2+\frac{3}{16n^2}\right);\\&\psi_2=\varphi+\arccos\left(\frac{5}{4}-4n^4\right).\end{aligned}\tag{50}$$

This solution is true in the range

$$\frac{1}{2}\le n\le\frac{\sqrt{3}}{2};\qquad -\frac{\sqrt{3}}{2}\le n\le-\frac{1}{2}.\tag{51}$$

94

Transiting to the initial independent variable, we obtain the solution of differential equation (43) in the range of n, corresponding to (51):

$$\begin{aligned}&w(z)=\rho\exp\left[i(\psi_1+\psi_2)\right]^{-\frac{1}{2}\exp i(\psi_2-\varphi)}\left\{c_1\exp\left[i\rho\exp i(\psi_1+\psi_2)\right]+\right.\\&\left.+c_2\exp\left[-i\rho\exp i(\psi_1+\psi_2)\right]\right\}.\end{aligned}\tag{52}$$

Though the range (51) is so limited (but can be well widen by the recurrence relations), we see that the "field" principle to present the function of complex variable enables us using better the merits of the theory of functions of complex variables to solve the physical and mathematical problems, particularly for seeking the solutions of non-trivial differential equations.

References:

1. Lavrentiev, M.A. and Shabat, B.V. *The methods of theory of complex variable.* Nauka, Moscow, 1973, 736 pp. (Russian)
2. Bitsadze, A.V. *Foundations of the theory of analytical functions of complex variable.* Nauka, Moscow, 1969, 239 pp. (Russian)
3. Gray, A. and Mathews, G.B. *Bessel functions and their applications to physics and mechanics.* Inosstrannaya literatura, Moscow, 1953, 371 pp. (russian; from edition: Gray, A. and Mathews, G.B. *A treatise on Bessel functions and their applications to physics.* English edition of 1931)

1985

```
ERROR: undefined
OFFENDING COMMAND: ---------------0203261754926--

STACK:
```