A LOCAL TRACE FORMULA FOR RESONANCES OF PERTURBED PERIODIC SCHRÖDINGER OPERATORS

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Abstract : We give a local trace formula for the pair $(P_1(h){=}P_0{+}W(hy),\ P_0)$, where P_0 is a periodic Schrödinger operator, W is a decreasing perturbation and h is a small positive parameter. We apply this result to establish the existence of $\sim h^{-n}$ resonances near some energy λ of $\sigma(P_0)$.

Introduction

Formulas relating the scattering resonances and the trace of some functions of the perturbed and unperturbed operators play an important role in the scattering theory and in the analysis of the distributions of resonances. The classical one, called the Poisson formula in Lax-Phillips theory, links the trace of the wave group to resonances for compactly supported perturbation of the Laplacian in odd dimension. It was proved by Lax-Phillips [18] and then with successive extension by Bardos-Guillot-Ralston [2], Melrose [19], Sjöstrand-Zworski [29].

Recently Sjöstrand [26,27] obtained a general local trace formula with remainder, valid in the semi-classical cases and in many situations where the resonances can only be defined in some neighborhood of the real axis.

The originality of Sjöstrand's results is that he does not require the scattering theory and this permitted him to cover the even dimension cases and a very large class of perturbations including the long range perturbations. His approach was an effective tool in the study of the distribution of resonances and the spectral shift function. See [3,4,5,6,20,21,28,29,30].

This paper concern a local trace formula for resonances for the pair of self-adjoint operators $\Big(P_1(h) = P_0 + W(hy),\ P_0 = -\Delta + V(y)\Big)$. Here V is real-valued and periodic with respect to the lattice Γ in $\mathbb{R}^n$, W is a decreasing perturbation and h is a small positive parameter. Since the potential V is periodic, our model is excluded in [26,27].

The Hamiltonian $P_1(h)$ describes the quantum motion of an electron in a crystal placed in an externel field. Many works exist on such a model where various spectral properties are described such as the dynamics, abscence of embedded eigenvalues and scattering properties [7,9,12,13,14,17].

In the study of resonances of $P_1(h)$, one may introduce the Fermi surface $F(\lambda)$ for

$\lambda \in \mathbb{R}$, where $F(\lambda)$ is the set of Bloch numbers ξ such that λ is an eigenvalue of the reduced Hamiltonian P_ξ obtained by the Floquet Bloch theory, (see definition 1.1). If $F(\lambda)$ is the union of smooth manifolds we identify the resonances of $P_1(h)$ with the complex eigenvalues of a closed operator $P_1(t,h)$, which is obtained by the method of spectral deformation, t is the distortion parameter, see [10]. This method is a standard technique in resonance theory since the work of Aguilar-Combes [1], see also [8].

Following [10], the invertibility of $(z - P_1(t,h))$, for z in small neighborhood Ω of λ is controlled by the invertibility of an $h-$pseudodifferential operator $E^t_{1,-+}(z,h)$ defined on $L^2\big(\mathbb{R}^n/\Gamma^*\big)$, where Γ^* is the dual lattice of Γ.

Using the reduced Hamiltonian $E^t_{-+}(z,h)$, we establish a trace formula for the pair $\Big(P_1(h),\ P_0\Big)$ similar to the one obtained by Sjöstrand [26,27].

An application of this trace formula yields a lower bound for the number of resonances of $P_1(h)$ near any point of the analytic singular support of $w_R(s) = \displaystyle\int_{|x|<R} w\big(s - W(x)\big)\,dx$. Here R is a large constant and $w(s)$ is the density of states of P_0.

The paper is organized as follows : In the next section, we introduce some notations and state the assumptions and the results precisely. Some technical results are given in section 2. The asymptotic expansion of $\mathrm{tr}\big[f(P_1(h)) - f(P_0)\big]$ (resp. the density of resonances) is proved in section 3 (resp. section 5). Section 4 is devoted to the proof of the local trace formula.

1. Preliminaries and main results

Let $\Gamma = \oplus_{i=1}^n \mathbb{Z} e_i$ be a lattice generated by some basis $(e_1, e_2, \cdots, e_n)$ of $\mathbb{R}^n$. The dual lattice Γ^* is given by $\Gamma^* := \{\gamma^* \in \mathbb{R}^n;\ \langle \gamma | \gamma^* \rangle \in 2\pi\mathbb{Z},\ \forall \gamma \in \Gamma\}$. A fundamental domain of Γ (resp. Γ^*) is denoted by E (resp. E^*). If we identify opposite edges of E (resp. E^*) then it becomes a flat torus denoted by $\mathbb{T} = \mathbb{R}^n/\Gamma$ (resp. $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$).

Let V be a real-valued potential, C^∞ and $\Gamma-$periodic. For $\xi \in \mathbb{R}^n$, we define the operator P_ξ on $L^2(\mathbb{T})$ by :

$$P_\xi := (D_y + \xi)^2 + V(y).$$

The operator P_ξ is a semi-bounded self-adjoint with ξ-independent domain $H^2(\mathbb{T})$. Since the resolvent of P_ξ is compact, P_ξ has a complete set of (normalized) eigenfunctions $\Phi_n(\cdot, \xi) \in H^2(\mathbb{T})$, $n \in \mathbb{N}$, called Bloch functions. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities, $\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots$. The operator P_ξ satisfies the identity $e^{-iy\cdot\gamma^*} P_\xi e^{iy\cdot\gamma^*} = P_{\xi+\gamma^*}$, then for every $p \geq 1$, the function $\xi \mapsto \lambda_p(\xi)$ is Γ^*-periodic.

Ordinary perturbation theory shows that $\lambda_p(\xi)$ are continuous functions of ξ for any fixed p, and $\lambda_p(\xi)$ is even an analytic function of ξ near any point $\xi_0 \in \mathbb{T}^*$ where $\lambda_p(\xi_0)$ is a simple eigenvalue of P_{ξ_0}. The function $\lambda_p(\xi)$ is called the band function and the closed intervals $\Lambda_p := \lambda_p(\mathbb{T}^*)$ are called bands. (See [22] and [25]).

Consider the self-adjoint operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$:

$$P_0 = -\triangle + V(y). \tag{1.1}$$

By Bloch-Floquet theory, it is well known that $\sigma(P_0) = \sigma_{\mathrm{ac}}(P_0) = \cup_{p\geq 1}\Lambda_p$. (See [22]).

Definition 1.1. Let $\lambda \in \mathbb{R}$ and $F(\lambda) = \{\xi \in \mathbb{T}^*;\ \lambda \in \sigma(P_\xi)\}$ the corresponding Fermi-surface.
a) We will say that $\lambda \in \sigma(P_0)$ is a simple energy level if and only if λ is a simple eigenvalue of P_ξ, for every $\xi \in F(\lambda)$.
b) Suppose that λ is a simple energy level of P_0 and let $\lambda(\xi)$ be the unique eigenvalue defined on a neighborhood of $F(\lambda)$ such that $\lambda(\xi) = \lambda,\ \forall \xi \in F(\lambda)$. We say that λ is a non-critical energy if $d_\xi \lambda(\xi) \neq 0$ for all $\xi \in F(\lambda)$.

Now, let us recall some well-known facts about the density of states associated with P_0, see [23]. The density of states measure ρ is defined as follows :

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{p\geq 1} \int_{\{\xi \in E^*;\ \lambda_p(\xi)\leq \lambda\}} d\xi, \tag{1.2}$$

where $\mathrm{vol}(E^*)$ is the Lebesgue measure of E^*.

Since the spectrum of P_0 is absolutely continuous, the measure ρ is absolutely continuous with respect to the Lebesgue measure $d\lambda$. Therefore the density of states of P_0, denoted by $w(E) = \frac{d\rho}{dE}(E)$, is locally integrable.

We now consider the perturbed periodic Schrödinger operator :

$$P_1(h) := P_0 + W(hy), \quad h \searrow 0, \tag{1.3}$$

where $W \in C^\infty(\mathbb{R}^n; \mathbb{R})$. We assume that there exist positive constants a and C such that W extends analytically to $D(a) := \{z \in \mathbb{C}^n;\ |\Im(z)| \leq a < \Re(z) >\}$ and

$$|W(z)| \leq C < z >^{-n-\epsilon}, \text{ uniformly on } z \in D(a), \tag{\textbf{H1}}$$

where ϵ is a positive constant and $< z >= (1 + |z|^2)^{\frac{1}{2}}$. Here $\Re(z)$, $\Im(z)$ designe respectively the real and imaginary part of z.

Remark 1.2. Assumption **(H1)** and Cauchy formula imply that $\forall \alpha \in \mathbb{N}^n$, $\exists C_\alpha > 0$ such that

$$|\partial_z^\alpha W(z)| \leq C_\alpha < z >^{-n-\epsilon-|\alpha|}, \textit{ uniformly on } z \in D(a). \tag{\textbf{H2}}$$

In the next section, we prove :

Theorem 1.3.
Assume **(H1)**. *For* $f \in C_0^\infty(\mathbb{R})$, *the operator* $\big[f(P_1(h)) - f(P_0)\big]$ *is of trace class and*

$$\operatorname{tr}\big[f(P_1(h)) - f(P_0)\big] \sim \sum_{j=0}^{+\infty} a_j h^{j-n}, \quad h \searrow 0, \tag{1.4}$$

with

$$a_0 = \int_{\mathbb{R}^n}\int_{\mathbb{R}} f'(t)\Big[\rho\big(t\big) - \rho\big(t - W(x)\big)\Big]\, dtdx = \int_{\mathbb{R}} f(s)\widetilde{w}(s)\, ds, \tag{1.5}$$

where

$$\widetilde{w}(s) := \int_{\mathbb{R}^n} \Big[w\big(s - W(x)\big) - w(s)\Big]\, dx. \tag{1.6}$$

Notice that for the proof of this theorem, we need only the assumption (**H**.**2**) for z real.

The second result involves the resonances of $P_1(h)$. Let us recall the definition. We follow essentially the presentation of [10].

Let $v \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$, be Γ^*-periodic. For real t, we introduce the spectral deformation family $\mathcal{U}_t$ defined by : $\forall u \in \mathcal{S}$,

$$\big[\mathcal{U}_t u\big](x) := \mathcal{F}_h^{-1}\big[\big(J_t^{\frac{1}{2}}\mathcal{F}_h(u\circ v_t)\big)(\cdot)\big](x), \quad \forall x \in \mathbb{R}^n, \tag{1.7}$$

where $v_t(x) = x - tv(x)$ and $J_t(x)$ its Jacobian. Here $\mathcal{F}_h$ is the semi-classical Fourier transform :

$$\big[\mathcal{F}_h u\big](\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}x\xi}u(x)\, dx, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{1.8}$$

Let $\widehat{P}_1(h) := -h^2\Delta + V(\dfrac{x}{h}) + W(x)$, the operator obtained from $P_1(h)$ by the change of variables $x = hy$.

Lemma 1.4. [10, Proposition 2.8] *Assume* **(H.1)**. *The self-adjoint operator* $P_1(t,h) := \mathcal{U}_t\widehat{P}_1(h)\mathcal{U}_t^{-1}$, *defined for* $t \in]-t_0, t_0[$, *extends to an analytic type-A family of operators on* $D(t_0) := \{t \in \mathbb{C};\ |t| < t_0\}$, *with domain* $H^2(\mathbb{R}^n)$.

From now on, we assume that λ_0 is a simple non-degenerate energy of P_0. By [10], there exist a complex neighborhood Ω of λ_0 such that for $t \in i]0, t_0[,\ t_0 > 0$ small enough, the spectrum of $P_1(t,h)$ in Ω consist of discrete eigenvalues of finite multiplicities. Moreover, these eigenvalues are $t-$independent and so they can be identified with resonances of $P_1(h)$.

Let $\widetilde{\Omega} \subset \Omega$ be an open relatively compact subset of Ω. We assume that $I = \widetilde{\Omega} \cap \mathbb{R}$ and $J = \Omega \cap \mathbb{R}$ are intervals. See the following **Fig.1.**

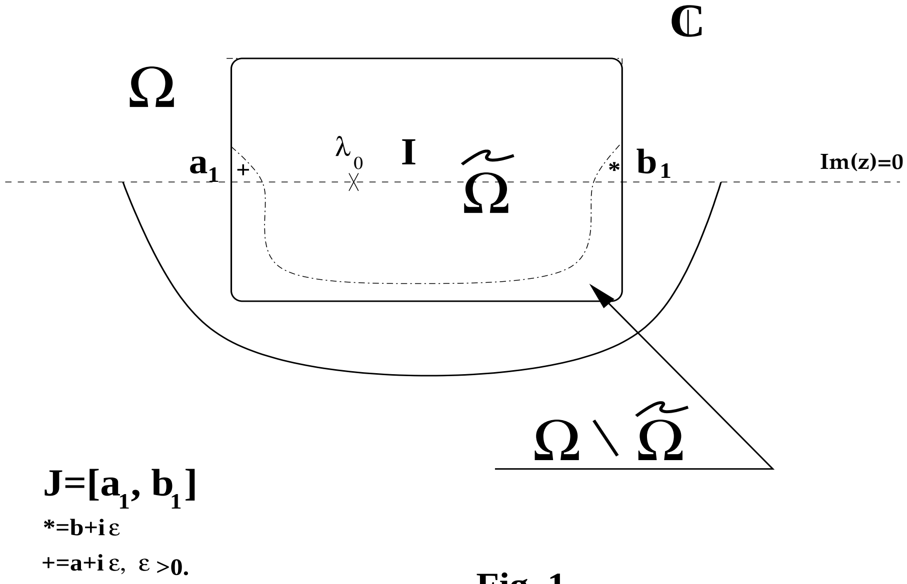

Fig. 1.

Our main result is :

Theorem 1.5.
Let $\lambda_0, \Omega, \widetilde{\Omega}, I, J$ *be as above and* W *satisfies* **(H1)**. *Then for* h *sufficiently small the following holds : Let* $g = g(z,h)$ *be holomorphic for* $z \in \Omega$ *and satisfies* $|g(z,h)| \leq 1$ *for* $z \in \Omega \setminus \widetilde{\Omega}$. *Let* $\chi \in C_0^\infty(J)$, *be independent of* h *with* $\chi \equiv 1$ *near* $\overline{I}$. *Then, uniformly with respect to* g,

$$\operatorname{tr}\big[(\chi g)(P_1(h)) - (\chi g)(P_0)\big] = \sum_{z \in \operatorname{Res}(P_1(h)) \cap \widetilde{\Omega}_-} g(z) + \mathcal{O}(h^{-n}), \tag{1.9}$$

where $\widetilde{\Omega}_- = \widetilde{\Omega} \cap \{w \in \mathbb{C};\ \Im(w) \leq 0\}$.

In the following Theorem, we give a lower bound for the number of resonances of $P_1(h)$ near any point of the analytic singular support of $\widetilde{w}$.

Theorem 1.6.
Let Ω *be as above and let* $E_0 \in \Omega \cap \operatorname{singsupp}_a(\widetilde{w})$. *Then for every* $h-$*independent complex neighborhood* $\widetilde{\Omega}$ *of* E_0, *there exist* $h_0 = h(\widetilde{\Omega}) > 0$ *sufficiently small and* $C = C(\widetilde{\Omega}) > 0$ *large enough such that for* $h \in]0, h_0[$,

$$\#\big[\operatorname{Res}(P_1(h)) \cap \widetilde{\Omega}\big] \geq \frac{1}{C(\widetilde{\Omega})} \cdot h^{-n}. \tag{1.10}$$

Here $\operatorname{singsupp}_a(\widetilde{w})$ *denote the analytic singular support of the distribution* $\widetilde{w}$.

Remark 1.7. Note that if λ_0 is a simple non-critical energy of P_0 then the function $\lambda \mapsto w(\lambda)$ is analytic near λ_0. See [12, Theorem 4.4]. Consequently, for R large enough the function $\displaystyle\int_{|x|>R} \Big[w(s - W(x)) - w(s)\Big]\, dx$ is analytic near λ_0, since $W(x)$ tends to zero when

$|x|$ tends to infinity. Therefore, for $E \in \mathbb{R}$ and $R > 0$ sufficiently large, we have

$$E \in \operatorname{singsupp}_a(\widetilde{w}) \Leftrightarrow E \in \operatorname{singsupp}_a\Big(\int_{|x|<R} w(s - W(x))\, dx\Big).$$

2. Effective resonant Hamiltonian

Denote by T_Γ the distribution in $\mathcal{S}'(\mathbb{R}^{2n})$ defined by

$$T_\Gamma(x,y) = \frac{1}{\operatorname{vol}(E) h^n} \sum_{\beta^* \in \Gamma^*} e^{i(x-hy)\frac{\beta^*}{h}}. \tag{2.1}$$

We recall that E is a fundamental domain of Γ.

For $m \in \mathbb{N}$, put $\mathbb{L}^m := \{u(x) T_\Gamma(x,y);\ \partial_x^\alpha u \in L^2(\mathbb{R}^n),\ \forall \alpha,\ |\alpha| \le m\}$.
It was shown in [10] that the operator $P_1(t,h)$, $t \in D(t_0)$, acting on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ is unitary equivalent to

$$\mathbb{P}_1(t,h) := \big(D_y + v_t(D_x)\big)^2 + V(y) + W_t^w(x, hD_x; h),\ W_t^w := \mathcal{U}_t W \mathcal{U}_t^{-1}, \tag{2.2}$$

acting on $\mathbb{L}^0$ with domain $\mathbb{L}^2$, and that

Proposition 2.1. *Assume* **(H2)** *and let $t \in D(t_0)$, t_0 small. There exist $N \in \mathbb{N}$, a complex neighborhood Ω of $J =]a_1, b_1[$, and a bounded operator R_+ in $\mathcal{L}\big(\mathbb{L}^0; L^2(\mathbb{T}^*; \mathbb{C}^N)\big)$ such that for all $z \in \Omega$ and $0 < h < h_0$ small enough, the operator*

$$\mathcal{P}_1^t(z,h) := \begin{pmatrix} \mathbb{P}_1(t,h) - z & R_+^* \\ R_+ & 0 \end{pmatrix} : \mathbb{L}^2 \times L^2(\mathbb{T}^*; \mathbb{C}^N)) \to \mathbb{L}^0 \times L^2(\mathbb{T}^*; \mathbb{C}^N) \tag{2.3}$$

is bijective with bounded two-sided inverse

$$\mathcal{E}_1^t(z,h) := \begin{pmatrix} E_1^t(z,h) & E_{1,+}^t(z,h) \\ E_{1,-}^t(z,h) & E_{1,-+}^t(z,h) \end{pmatrix}. \tag{2.4}$$

Here $E_{1,-+}^t := E_{1,-+}^{t^w}(k, -hD_k; z, h)$ is an $h-$pseudodifferential operator with symbol

$$E_{1,-+}^t(k,x;z,h) \sim \sum_{j\ge 0} E_{1,-+}^{t,j}(k,x;z)\, h^j, \quad \forall\, 0 < h < h_0, \tag{2.5}$$

in $S^0\Big(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)\Big)$.

Remark 2.2.
1) We denote by $\mathcal{P}_0^t(z,h)$ and $\mathcal{E}_0^t(z,h) := \begin{pmatrix} E_0^t(z,h) & E_{0,+}^t(z,h) \\ E_{0,-}^t(z,h) & E_{0,-+}^t(z,h) \end{pmatrix}$ the operators given by Proposition 2.1 when $W = 0$.
2) Note that, R_+ depends only on the non-perturbed periodic Schrödinger operator P_0. See [13, Proposition 2.1] and [11, Chapter 13]. Therefore, we may take the same R_+ for $\mathcal{P}_1^t(z,h)$ and $\mathcal{P}_0^t(z,h)$.

When $t = 0$ we drop the exponent 0 in the $\mathcal{P}_i^0$, $\mathcal{E}_i^0$ and $E_{ij}^0(z,h)$, with $i = 0,1$ and $j = -,+,-+$. Recall that $\mathbb{P}_1(0,h) = \widehat{P}_1(h)$. The following well-known formulas are a consequence of Proposition 2.1 (see also [16]), for $j = 0,1$.

$$\big(\widehat{P}_j(h) - z\big)^{-1} = E_j(z,h) - E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h), \tag{2.6}$$

$$E_{j,-+}(z,h)^{-1} = -R_+\big(\widehat{P}_j(h) - z\big)^{-1}R_+^* \tag{2.7}$$

and

$$\partial_z E_{j,-+}(z,h) = E_{j,-}(z,h)E_{j,+}(z,h). \tag{2.8}$$

Here $\widehat{P}_0(h) := -h^2\Delta + V(\frac{y}{h})$.

In the following, we write $[a_j]_{j=0}^1 = a_1 - a_0$.

Lemma 2.3. *We have*

$$\Big[E_{j,+}(z,h)\Big]_{j=0}^1 = E_1(z,h)W(x)E_{0,+}(z,h), \tag{2.9}$$

$$\Big[E_{j,-}(z,h)\Big]_{j=0}^1 = E_{0,-}(z,h)W(x)E_1(z,h) \tag{2.10}$$

and

$$\Big[E_{j,-+}(z,h)\Big]_{j=0}^1 = E_{1,-}(z,h)W(x)E_{0,+}(z,h). \tag{2.11}$$

In particular, if **(H.1)** *is satisfied then*

$$\Big[E_{j,-+}\big(k,x;z,h\big)\Big]_{j=0}^1 \in S^0\Big(\mathbb{T}_k^* \times \mathbb{R}_x^n, <x>^{-n-\epsilon}; \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)\Big). \tag{2.12}$$

Proof. Identities (2.9), (2.10) and (2.11) follow from the first resolvent equation

$$\begin{aligned}\Big[\mathcal{E}_j(z,h)\Big]_{j=0}^1 &= \mathcal{E}_1(z,h)\big[\mathcal{P}_0(z,h) - \mathcal{P}_1(z,h)\big]\mathcal{E}_0(z,h)\\ &= -\mathcal{E}_0(z,h)\big[\mathcal{P}_1(z,h) - \mathcal{P}_0(z,h)\big]\mathcal{E}_1(z,h)\end{aligned}$$

and the fact that $\Big[\mathcal{P}_j(z,h)\Big]_{j=0}^1 = \begin{pmatrix} W(x) & 0 \\ 0 & 0\end{pmatrix}$.

Formula (2.12) is a simple consequence of (2.11) and standard $h-$pseudodifferential calculus.

□

Proposition 2.4. *Under assumption* **(H.1)**, *the operator*

$$W(x)E_{0,+}(z,h) : L^2(\mathbb{T}^*;\mathbb{C}^N) \to L^2(\mathbb{R}^n) \tag{2.13}$$

and

$$E_{0,-}(z,h)W(x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^*;\mathbb{C}^N) \tag{2.14}$$

are of trace class.

Proof. Since $\Big(E_{0,-}(z,h)W(x)\Big)^{*} = W(x)E_{0,+}(\overline{z},h)$ it suffice to prove (2.13). Without any loss of generality, we may assume that $N=1$.
Consider the operator $H = \big(\mathrm{Id} - h^2\Delta_{\mathbb{T}^*}\big)^{\frac{-n-\epsilon}{2}}$ on $L^2(\mathbb{T}^*;\mathbb{C})$. Set $A = W(x)E_{0,+}(z,h)$, $B = A^*A$ and $C = H^{-1}BH^{-1}$.
Since $W \in S^0\big(\mathbb{R}^{2n}, < x >^{-n-\epsilon}\big)$ and $E_{0,+}(k,x;z,h) \in S^0\big(\mathbb{T}^*_k \times \mathbb{R}^n_x\big)$, a standard result of h-pseudo-differential calculus shows that $C \in S^0(\mathbb{T}^*_k \times \mathbb{R}^n_x)$. Therefore, C extends to a bounded operator from $L^2(\mathbb{T}^*;\mathbb{C})$ into $L^2(\mathbb{T}^*;\mathbb{C})$ (see [11, Chapter 13]). Combining this with the fact that B is positif, we get :

$$0 \le B = HCH \le \|C\|\, H^2$$

which implies

$$0 \le B^{\frac{1}{2}} \le \sqrt{\|C\|}\, H.$$

Since $H : L^2(\mathbb{T}^*;\mathbb{C}) \to L^2(\mathbb{T}^*;\mathbb{C})$ is of trace class, the lemma follows from the above inequality. ☐

Proposition 2.5. *For $z \in \mathbb{C}$ such that $\Im(z) \neq 0$, the operator*

$$\Big[E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h)\Big]_{j=0}^{1}$$

is of trace class from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and

$$\mathrm{tr}\Big(\Big[E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h)\Big]_{j=0}^{1}\Big) = \mathrm{tr}\Big(\Big[E_{j,-+}(z,h)^{-1}\partial_z E_{j,-+}(z,h)\Big]_{j=0}^{1}\Big). \tag{2.15}$$

Here the operator in the right member of (2.15) *is defined on $L^2(\mathbb{T}^*;\mathbb{C}^N)$.*

Proof. Let $z \in \mathbb{C}$ s.t. $\Im(z) \neq 0$, we have the following identity :

$$\Big[E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h)\Big]_{j=0}^{1} = \tag{2.16}$$

$$\Big[\big([E_{j,+}(z,h)]_{j=0}^{1}\big)E_{1,-+}(z,h)^{-1}E_{1,-}(z,h)\Big] +$$

$$\Big[E_{0,+}(z,h)E_{0,-+}(z,h)^{-1}\big([E_{j,-}(z,h)]_{j=0}^{1}\big)\Big] -$$

$$\Big[E_{0,+}(z,h)E_{1,-+}(z,h)^{-1}\big([E_{j,-+}(z,h)]_{j=0}^{1}\big)E_{0,-+}(z,h)^{-1}E_{1,-}(z,h)\Big].$$

According to Lemma 2.3 and Proposition 2.4, all the term of the right member in the last equality are of trace class. Using the cyclicity of the trace and identity (2.8), we obtain the proposition.

☐

The main result in this section is

Proposition 2.6. *Let* $\psi \in C_0^\infty(\mathbb{R})$ *and let* $\widetilde{\psi}$ *be an almost analytic extension of* ψ. *Then the operator* $[\psi(P_1(h)) - \psi(P_0)]$ *is of trace class as an operator from* $L^2(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$ *and*

$$\operatorname{tr}[\psi(P_1(h)) - \psi(P_0)] = \operatorname{tr}[\psi(\widehat{P}_1(h)) - \psi(\widehat{P}_0(h))] = \tag{2.17}$$

$$-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\operatorname{tr}\Big(\big[E_{j,-+}(z)^{-1}\partial_z E_{j,-+}\big]_{j=0}^{1}\Big)\, L(dz).$$

Here $\overline{\partial} = \dfrac{\partial}{\partial \overline{z}}$.

Proof. By Helffer-Sjöstrand formula (see [15]), we have

$$\psi(\widehat{P}_1(h)) - \psi(\widehat{P}_0(h)) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\Big[(z-\widehat{P}_1(h))^{-1} - (z-\widehat{P}_0(h))^{-1}\Big]\, L(dz).$$

Combining this with (2.6), we obtain

$$\psi(\widehat{P}_1(h)) - \psi(\widehat{P}_0(h)) = \frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\big[E_j(z,h)\big]_{j=0}^{1}\, L(dz) \tag{2.18}$$

$$-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\Big[E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h)\Big]_{j=0}^{1}\, L(dz).$$

Since $E_j(z,h)$, $j=0,1$ is holomorphic in a neighborhood of $\operatorname{supp}(\widetilde{\psi})$, the first term in the right member of (2.18) vanishes. Consequently,

$$\psi(\widehat{P}_1(h)) - \psi(\widehat{P}_0(h)) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\Big[E_{j,+}(z,h)E_{j,-+}(z,h)^{-1}E_{j,-}(z,h)\Big]_{j=0}^{1}\, L(dz).$$

Using Proposition 2.5, we conclude that $[\psi(\widehat{P}_1(h)) - \psi(\widehat{P}_0(h))]$ is of trace class and applying (2.15), we obtain the second equality of (2.17). The first equality follows from the fact that $\widehat{P}_j(h)$ is unitary equivalent to $P_j(h)$, $j=0,1$ $\square$

3. Proof of Theorem 1.3

The proof of Theorem 1.3 is a simple consequence of Proposition 2.6 (with $\psi = f$) and symbolic calculus. Here, we only give an outline of the proof. For the details, we refer to [9]. Fix δ in $]0,\frac{1}{2}[$. The integral (2.17) over $\{z \in \mathbb{C};\ |\Im z| \le h^\delta\}$ is $\mathcal{O}(h^\infty)$, since $\overline{\partial}\widetilde{f}(z) = \mathcal{O}(|\Im z|^\infty)$ and

$$\Big\|\Big[E_{j,-+}(z)^{-1}\partial_z E_{j,-+}(z)\Big]_{j=0}^{1}\Big\|_{\operatorname{tr}} = \mathcal{O}\big(|\Im z|^{-1}\big).$$

On the other hand, $\Big[E_{j,-+}(z,h)^{-1}\partial_z E_{j,-+}(z,h)\Big]_{j=0}^{1}$ has an asymptotic expansion in powers of h uniformly on z in $\{z \in \operatorname{supp}\widetilde{f};\ |\Im(z)| \ge h^\delta\}$ (see [9]). Therefore, as in [11, Theorem 13.28], we have

$$-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)\operatorname{tr}\Big(\big[E_{j,-+}(z)^{-1}\partial_z E_{j,-+}(z)\big]_{j=0}^{1}\Big)\, L(dz) \sim \sum_{j\ge 0} a_j h^{j-n},\ (h \searrow 0)$$

with

$$a_0 = (2\pi)^{-n} \sum_k \int_{\mathbb{R}^n_x} \Big(\int_{E^*} \big[f\big(W(x) + \lambda_k(\xi)\big) - f\big(\lambda_k(\xi)\big)\big] \, d\xi \Big) \, dx.$$

Note that, the sum in the last equality is finite, since $\lim\limits_{k\to\infty} \lambda_k(\xi) = +\infty$ and W is bounded. It remains only to show (1.5). Clearly

$$a_0 = (2\pi)^{-n} \sum_k \int_{\mathbb{R}^n_x} \Big(\int_{E^*} \int_{\mathbb{R}_t} \big[1_{[\lambda_k(\xi), +\infty[}(t) - 1_{[\lambda_k(\xi)+W(x), +\infty[}(t) \big] f'(t) \, dt d\xi \Big) \, dx.$$

Remembering the definition of $\rho(t)$, we get

$$a_0 = \int_{\mathbb{R}^n_x} \int_{\mathbb{R}_t} \Big(\rho(t) - \rho\big(t - W(x)\big) \Big) f'(t) \, dt dx.$$

Integrating by parts, we obtain $a_0 = \displaystyle\int_{\mathbb{R}} f(s) \widetilde{w}(s) \, ds.$ □

4. Proof of the local trace formula

Let $\lambda_0, \Omega, \widetilde{\Omega}, I$ and J be as in Theorem 1.5. Applying Proposition 2.6, (with $\psi := \chi g$), we obtain

$$\begin{aligned} \mathcal{I} = \operatorname{tr}[(\chi g)(P(h)) - (\chi g)(P_0(h))] = \qquad\qquad (4.1) \\ -\frac{1}{\pi} \int_{\mathbb{C}} \big(\overline{\partial} \widetilde{\chi}\big)(z) g(z) \operatorname{tr}\Big(\big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z,h) \big]_{j=0}^1 \Big) \, L(dz), \end{aligned}$$

where $\chi \in C_0^\infty(J)$, $\equiv 1$ near $\overline{I}$, and $h-$independent. Here $g = g(z,h)$ is an holomorphic function in Ω and satisfies $|g(z,h)| \leq 1$ for $z \in \Omega \setminus \widetilde{\Omega}$.

Since $\operatorname{supp}(\overline{\partial}\widetilde{\chi}) \subset \{w \in \mathbb{C};\ |g(w)| \leq 1\} \subset \Omega \backslash \widetilde{\Omega}$, the integral (4.1) over $\{z \in \mathbb{C};\ \Im(z) < 0\}$ is $\mathcal{O}(h^{-n})$ uniformly with respect to g. Consequently,

$$\mathcal{I} = -\frac{1}{\pi} \int_{\mathbb{C}_+} \big(\overline{\partial} \widetilde{\chi}\big)(z) g(z) \operatorname{tr}\Big(\big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z) \big]_{j=0}^1 \Big) \, L(dz) + \mathcal{O}(h^{-n}), \qquad (4.2)$$

where $\mathbb{C}_+ = \{w \in \mathbb{C};\ \Im(w) \geq 0\}$.

Let $\tilde{\gamma} : [0,1] \to \mathbb{C}_+$, $t \mapsto [ta + (1-t)b] + i\epsilon$. In the following, we fix $\epsilon > 0$ small enough (independent of h) such that $\{a + i\epsilon, b + i\epsilon\} \in \Omega \setminus \overline{\widetilde{\Omega}}$. See **Fig.1**. Green's formula gives :

$$\begin{aligned} -\frac{1}{\pi} \int_{\mathbb{C}_+} \big(\overline{\partial} \widetilde{\chi}\big)(z) g(z) \big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z,h) \big]_{j=0}^1 \, L(dz) = \qquad\qquad (4.3) \\ -\frac{1}{\pi} \int_{0 \leq \Im(z) < \epsilon} \big(\overline{\partial} \widetilde{\chi}\big)(z) g(z) \big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z,h) \big]_{j=0}^1 \, L(dz) \\ + \frac{1}{2\pi i} \int_{\tilde{\gamma}} \widetilde{\chi}(z) g(z) \big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z,h) \big]_{j=0}^1 \, dz. \end{aligned}$$

The integrand in the first integral in the right member of (4.3) has its support in the region where $g = O(1)$, so we get (uniformly with respect to g)

$$\mathcal{I} = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \widetilde{\chi}(z) g(z) \operatorname{tr}\Big(\big[E_{j,-+}(z,h)^{-1} \partial_z E_{j,-+}(z,h) \big]_{j=0}^1 \Big) \, dz + \mathcal{O}(h^{-n}).$$

Since $\operatorname{supp}(1-\widetilde{\chi}) \subset \{w \in \mathbb{C};\ |g(w)| \leq 1\}$, we may replace $\mathcal{I}$ by :

$$\mathcal{I} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} g(z)\mathrm{tr}\Big(\big[E_{j,-+}(z,h)^{-1}\partial_z E_{j,-+}(z,h)\big]_{j=0}^{1}\Big)\,dz + \mathcal{O}(h^{-n}). \tag{4.4}$$

In $\tilde{\gamma}$, $\Im(z) = \epsilon > 0$ fixed. Since $E^s_{j,-+}(z,h)$ is an analytic type-A family on $D(t_0) = \{t \in \mathbb{C}; |t| \leq t_0\}$ and $E_{j,-+}(z,h)$ is invertible, there exist $t_1 > 0$ sufficiently small such that for all $s \in D(t_1)$, $E^s_{j,-+}(z,h)$ is invertible and $E^s_{j,-+}(z,h)^{-1}$ is an analytic type-A family on $D(t_1)$. Applying this with the fact that $\mathcal{U}_s$ is unitary when s is real, we get

$$\mathrm{tr}\Big(\Big[E_{j,-+}(z,h)^{-1}\partial_z E_{j,-+}(z,h)\Big]_{j=0}^{1}\Big) = \mathrm{tr}\Big(\Big[E^s_{j,-+}(z,h)^{-1}\partial_z E^s_{j,-+}(z,h)\Big]_{j=0}^{1}\Big),\ \forall s \in D(t_1). \tag{4.5}$$

On the other hand, since the right hand side of (4.5) is analytic for $s \in D(t_0)\cap\{s \in \mathbb{C};\ \Im(s) \geq 0\}$ and $z \in \tilde{\gamma}$, then the above equality remains true for all $s \in D(t_0)\cap\{s \in \mathbb{C};\ \Im(s) \geq 0\}$ and $z \in \tilde{\gamma}$.

From now on, we fix $s \in i]0,t_0[$. The identity (4.4) becomes (uniformly with respect to g)

$$\mathcal{I} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} g(z)\mathrm{tr}\Big(\big[E^s_{j,-+}(z,h)^{-1}\partial_z E^s_{j,-+}(z,h)\big]_{j=0}^{1}\Big)\,dz + \mathcal{O}(h^{-n}). \tag{4.6}$$

Since the operator $\big(E^s_{1,-+}(z,h) - E^s_{0,-+}(z,h)\big)E^s_{0,-+}(z,h)^{-1}$ is compact, we decompose it in sum of two operators, one of them is of finite rank and the other has a small norm.

Lemma 4.1. *There exist two operators $K_1,\ K_2$ from $L^2(\mathbb{T}^*;\mathbb{C}^N)$ into its self, such that*

$$\mathrm{Id} + K_1 \text{ is invertible},\ K_2 \text{ is of finite rank } (= \mathcal{O}(h^{-n})) \tag{4.7}$$

and $K_1 + K_2 = \big(E^s_{1,-+}(z,h) - E^s_{0,-+}(z,h)\big)E^s_{0,-+}(z,h)^{-1}$. Here Id *designe the identity operator on $L^2(\mathbb{T}^*;\mathbb{C}^N)$.*

Proof. Without any loss of generality, we may assume that $N = 1$.

Consider the operator on $L^2(\mathbb{T}^*)$, $A_0 = -h^2\Delta_{\mathbb{T}^*}$. The spectrum of A_0 is discrete. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ in $[-C,C]$ and $\chi \equiv 0$ in $\mathbb{R}\setminus[-2C,2C]$. By functional calculus, we define the operator $A = \chi(-h^2\Delta_{\mathbb{T}^*})$. The operator A is of finite rank, $\big(= \mathcal{O}(h^{-n})\big)$. Standard $h-$pseudodifferential calculus on the torus and the assumption **(H.1)** on W give

$$\Big\|(\mathrm{Id}-A)\Big[E^s_{1,-+}(z,h) - E^s_{0,-+}(z,h)\Big]E^s_{0,-+}(z,h)^{-1}\Big\| \to 0 \quad \text{when} \quad C \to +\infty.$$

To finish the proof of the lemma, it suffices to take :

$$K_1 = (\mathrm{Id}-A)\Big[E^s_{1,-+}(z,h) - E^s_{0,-+}(z,h)\Big]E^s_{0,-+}(z,h)^{-1} \tag{4.8}$$

and

$$K_2 = A\Big[E^s_{1,-+}(z,h) - E^s_{0,-+}(z,h)\Big]E^s_{0,-+}(z,h)^{-1}. \tag{4.9}$$

The remainder of the proof of the theorem 1.5, is close to the one in [27, Section 4], for that we omit the details.

Set $K(z,h) = K_2(z,h)\big(\mathrm{Id} + K_1(z,h)\big)^{-1}$. A simple calculus gives :

$$E^s_{1,-+}(z,h) = \big(\mathrm{Id} + K(z,h)\big)(\mathrm{Id} + K_1)E^s_{0,-+}(z,h). \tag{4.10}$$

Differentiating the last identity with respect to z and inserting it in the integrand of the right member of (4.6), we get

$$\Big[E^s_{j,-+}(z,h)^{-1}\partial_z E^s_{j,-+}(z,h)\Big]_{j=0}^{1} = \tag{4.11}$$

$$E^s_{0,-+}(z,h)^{-1}\big(\mathrm{Id} + K_1\big)^{-1}(\mathrm{Id} + K)^{-1}\partial_z K(z,h)\big(\mathrm{Id} + K_1(z,h)\big)E^s_{0,-+}(z,h) +$$
$$E^s_{0,-+}(z,h)^{-1}\big(\mathrm{Id} + K_1(z,h)\big)^{-1}\partial_z K_1(z,h)E^s_{0,-+}(z,h).$$

Clearly, all the terms in (4.11) are of trace class. Using the cyclicity of the trace we obtain :

$$\mathrm{tr}\Big(\big[E^s_{j,-+}(z,h)^{-1}\partial_z E^s_{j,-+}(z,h)\big]_{j=0}^{1}\Big) = \tag{4.12}$$

$$\mathrm{tr}\Big(\big[\big(\mathrm{Id} + K(z,h)\big)^{-1}\partial_z K(z,h)\big]\Big) + \mathrm{tr}\Big(\big[(\mathrm{Id} + K_1)^{-1}\partial_z K_1\big]\Big).$$

Here the last term is holomorphic in Ω, so committing another error $\mathcal{O}(h^{-n})$, we obtain (uniformly with respect to g)

$$\mathcal{I} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} g(z)\mathrm{tr}\big[(\mathrm{Id} + K(z,h))^{-1}\partial_z K(z,h)\big]\,dz + \mathcal{O}(h^{-n}). \tag{4.13}$$

It follows from Proposition 2.1 (see also [10]) that, z is a resonance of $P_1(h)$ if and only if $0 \in \sigma\big(E^s_{1,-+}(z,h)\big)$ with the same multiplicity. Combining this with the fact that $(\mathrm{Id} + K_1)E^s_{0,-+}$ is holomorphic in Ω and have a bounded inverse everywhere, we deduce that the zeros of $D(z,h) := \det\big(\mathrm{Id} + K(z,h)\big)$ are the resonances of $P_1(h)$ and that the multiplicities agree. Therefore, for all $\Omega_1 \subset\subset \Omega$ such that $\partial\Omega_1 = \tilde{\gamma} \cup \tilde{\gamma}_1$, where $\tilde{\gamma}_1 \subset \Omega \setminus \overline{\tilde{\Omega}}$ and avoiding the resonances of $P_1(h)$, we have

$$\frac{1}{2\pi i}\int_{\tilde{\gamma}\cup\tilde{\gamma}_1} g(z)\mathrm{tr}\big[(\mathrm{Id} + K(z,h))^{-1}\partial_z K(z,h)\big]\,dz = \tag{4.14}$$

$$\mathrm{tr}\Big[\frac{1}{2\pi i}\int_{\tilde{\gamma}\cup\tilde{\gamma}_1} g(z)\frac{d}{dz}\big(\log D(z,h)\big)\,dz\Big] = \sum_{z\in\mathrm{Res}(P_1(h))\cap\Omega_1} g(z).$$

Since K_2 is of finite rank $\big(= \mathcal{O}(h^{-n})\big)$, we have

$$D(z,h) = O(1)e^{\mathcal{O}(1)h^{-n}}, \quad \forall z \in \Omega. \tag{4.15}$$

In $\Omega_{+,\frac{\epsilon}{2}} := \{z \in \Omega;\ \Im(z) \geq \frac{\epsilon}{2}\}$, the operator $E^s_{1,-+}(z,h)$ is uniformly invertible, hence

$$\big(\mathrm{Id} + K(z,h)\big)^{-1} = \big(\mathrm{Id} + K_1(z,h)\big)E^s_{0,-+}(z,h)E^s_{1,-+}(z,h)^{-1} = \mathcal{O}(1) \tag{4.16}$$

as an operator on $L^2(\mathbb{T}^*, \mathbb{C}^n)$ and

$$|D(z,h)| \geq e^{-Ch^{-n}}, \quad z \in \Omega_{+,\frac{\epsilon}{2}}. \tag{4.17}$$

Let $N(P, \Omega; h)$ be the number of resonances in Ω counted with their multiplicity. Combining (4.15), (4.17) and Jensen's inequality, we get

$$N(P, \Omega; h) \leq Ch^{-n}. \tag{4.18}$$

Let $z_j,\ j = 1, \cdots, M = N(P, \Omega; h)$ repeated according to their multiplicity and put

$$D_w(z,h) = \prod_{j=1}^{M} (z - z_j). \tag{4.19}$$

The identity (4.18) implies

$$|D_w(z,h)| \leq e^{Ch^{-n}} \text{ in } \Omega, \quad |D_w(z,h)| \geq e^{-Ch^{-n}} \text{ in } \Omega_{+,\frac{\epsilon}{2}}. \tag{4.20}$$

Using [27, Lemma 4.3], we may choose $\tilde{\gamma}_1$ such that

$$|D_w(z,h)| \geq e^{-Ch^{-n}}, \quad z \in \tilde{\gamma}_1. \tag{4.21}$$

Here the idea is to avoid small fixed neighborhoods of resonances of $P_1(h)$ in $\Omega \setminus \overline{\tilde{\Omega}}$.

We factorize D :

$$D(z,h) = D_w(z,h) G(z,h), \quad z \in \Omega, \tag{4.22}$$

where G and $\dfrac{1}{G}$ are holomorphic in Ω. Combining (4.20), (4.21) and (4.22), we get

$$|G(z,h)| \leq e^{Ch^{-n}}, \quad z \in \Omega_{+,\frac{\epsilon}{2}} \cup \tilde{\gamma}_1. \tag{4.23}$$

For $\Omega_1 \subset\subset \Omega$ a simply connected relatively open $h-$independent set with $\overline{\tilde{\Omega}}$ in its (relative) interior, by maximum principle, we have

$$|G(z,h)| \leq e^{Ch^{-n}}, \quad z \in \Omega_1. \tag{4.24}$$

Thanks to (4.17), (4.20) and (4.22), we have

$$|G(z,h)| \geq e^{-Ch^{-n}}, \quad z \in \Omega_{+,\frac{\epsilon}{2}}. \tag{4.25}$$

For $C_0 > 0$ large enough, put

$$F(z,h) := C_0 h^{-n} - \log |G(z,h)| \geq 0. \tag{4.26}$$

(4.25) imply that

$$F(z,h) \leq Ch^{-n}, \quad \forall z \in \Omega_{+,\frac{\epsilon}{2}}. \tag{4.27}$$

Notice that F is harmonic. Harnack's inequality for non-negative harmonic functions and (4.27) give (after an arbitrarily small decrease of Ω_1) :

$$F(z,h) \leq Ch^{-n}, \quad \forall z \in \Omega_1. \tag{4.28}$$

Hence, $\log|G| \geq -Ch^{-n}$ on Ω_1. Using this with (4.24), we get

$$\big|\log|G|\big| \leq Ch^{-n} \quad \text{on } \Omega_1. \tag{4.29}$$

Since $\log|G(z,h)| = \Re(\log G)$ is harmonic, we get after an arbitrarily small decrease of Ω_1 :

$$\nabla\big(\Re(\log G)\big) = \mathcal{O}(h^{-n}) \quad \text{on } \Omega_1. \tag{4.30}$$

The Cauchy Riemann equations for $\log G$ imply

$$\frac{d}{dz}\log G = \mathcal{O}(h^{-n}) \quad \text{on } \Omega_1. \tag{4.31}$$

For the curve $\tilde{\gamma}_1 \in \Omega \setminus \overline{\tilde{\tilde{\Omega}}}$ fixed as above and from (4.18), (4.19), we get

$$\int_{\tilde{\gamma}_1} g(z)\frac{d}{dz}\log D_w(z,h)\,dz = \mathcal{O}(h^{-n}). \tag{4.32}$$

(4.22), (4.31) and (4.32) imply that

$$\int_{\tilde{\gamma}_1} g(z)\frac{d}{dz}\log D(z,h)\,dz = \mathcal{O}(h^{-n}). \tag{4.33}$$

Combining (4.13), (4.14) and (4.33), we obtain (uniformly with respect to g)

$$\mathcal{I} = \sum_{z\in \mathrm{Res}(P_1(h))\cap\Omega_1} g(z) + \mathcal{O}(h^{-n}). \tag{4.34}$$

Since $|g| \leq 1$ on $\Omega_1 \setminus \widetilde{\Omega}$ and the fact that the number of the resonances of $P_1(h)$ in Ω_1 is $\mathcal{O}(h^{-n})$, we may replace the sum in (4.34) by

$$\sum_{z\in \mathrm{Res}(P_1(h))\cap\widetilde{\Omega}_-} g(z),$$

which proves Theorem 1.5. □

5. Proof of the lower bound

As indicated in the introduction, the proof is a simple consequence of Theorem 1.3 and Theorem 1.5. Let Ω be as in Theorem 1.5 and let $E_0 \in \Omega$. Suppose that $E_0 \in \mathrm{singsupp}_a(\widetilde{w})$.

Since $\widetilde{w}$ is real, it is clear that $(E_0, 1)$ and $(E_0, -1)$ are in $\mathrm{WF}_a(\widetilde{w})$. It follows from the definition of the analytic wave front set by the F.B.I transformation that : there exist sequences $(\alpha_j, \beta_j) \to (E_0, 1)$, $\lambda_j \to +\infty$ and $\epsilon_j \to 0$ such that

$$\Big| \int (f_j \chi_0)(E) \widetilde{\omega}(E)\, dE \Big| \geq e^{-\epsilon_j \lambda_j}, \tag{5.1}$$

where $f_j(E) = e^{i\lambda_j(\alpha_j - E)\beta_j} e^{-\frac{\lambda_j}{2}(\alpha_j - E)^2}$ and χ_0 is a cut-off function defined in a small real neighborhood of E_0, $\equiv 1$ near E_0. See [24].

Let a be a small positive constant. Set $\Omega_0 :=]E_0 - 2a, E_0 + 2a[+ i] - 2a^2, a^2]$ and $\widetilde{\Omega} :=]E_0 - a, E_0 + a[+ i] - a^2, a^2]$. By construction, there exist $C_0 > 0$ such that

$$\big| f_j(E) \big| \leq e^{-\frac{1}{C_0}\lambda_j} \quad \text{uniformly on} \quad E \in \Omega_0 \setminus \widetilde{\Omega}. \tag{5.2}$$

Let $\chi \in C_0^\infty\big( ]E_0 - 2a, E_0 + 2a[; [0, 1] \big)$, $\equiv 1$ on $]E_0 - a, E_0 + a[$. Then the inequality (5.1) remains true when we replace χ_0 by χ.

Theorem 1.3 gives

$$\mathrm{tr}\big[f_j(P_1(h)) - f_j(P_0) \big] = h^{-n} \int \big(f_j \chi \big)(E) \widetilde{\omega}(E)\, dE + \mathcal{O}_j(h^{1-n}). \tag{5.3}$$

On the other hand, applying Theorem 1.5 with $g = f_j$ and Ω replaced by Ω_0, we have

$$\mathrm{tr}\big[f_j(P_1(h)) - f_j(P_0) \big] = \sum_{z \in \mathrm{Res}(P_1(h)) \cap \widetilde{\Omega}_-} f_j(z) + \mathcal{O}(1) h^{-n} e^{-\frac{1}{C_0}\lambda_j}, \tag{5.4}$$

where $\widetilde{\Omega}_- = \{ w \in \widetilde{\Omega};\ \Im(w) \leq 0 \}$. Combining (5.3) and (5.4) we get

$$\sum_{z \in \mathrm{Res}(P_1(h)) \cap \widetilde{\Omega}_-} f_j(z) = h^{-n} \int \big(f_j \chi \big)(E) \widetilde{\omega}(E)\, dE + \mathcal{O}(h^{-n}) e^{-\frac{1}{C_0}\lambda_j} + \mathcal{O}_j(h^{1-n}),$$

which together with (5.1) imply

$$\Big| \sum_{z \in \mathrm{Res}(P_1(h)) \cap \widetilde{\Omega}_-} f_j(z) \Big| \geq h^{-n} \Big[e^{-\epsilon_j \lambda_j} - \mathcal{O}(1) e^{-\frac{1}{C_0}\lambda_j} \Big] + \mathcal{O}_j(h^{1-n}).$$

Fix j large enough and then let h be sufficiently small, we conclude that

$$\Big| \sum_{z \in \mathrm{Res}(P_1(h)) \cap \widetilde{\Omega}_-} f(z) \Big| \geq \frac{1}{C} h^{-n}, \tag{5.5}$$

where $f = f_j$ (with j fixed) is independent of h. The inequality (5.5) gives the required lower bound on the number of resonances in $\widetilde{\Omega}_-$ and since we can choose $\widetilde{\Omega}$ as small as we like, we get the theorem 1.6. □

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