Bosonic Central Limit Theorem for the One-Dimensional XY Model

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We prove the central limit theorem for Gibbs states and ground states for quasifree Fermions (bilinear Hamiltonians) and for the off critical XY model on a one-dimensional integer lattice.

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1 Introduction

The central limit theorem for mixing classical spin systems has been established in a number of situations. (See pp.459 of [9] for references.) For mixing quantum spin models, a similar limit theorem was considered in [11] by D.Goderis and P.Vets. We refer to the limit theorem of D.Goderis and P.Vets as *Bosonic Central Limit Theorem*. Historically the limit theorem was expected in a study of quantum mean field models (c.f. [21] [13]). For such exactly soluble models the formal Fourier transform of the local quantum observables obeys the canonical commutation relations with respect to a highly degenerate symplectic form while the Fourier transform of the state gives rise to a quasifree state of Boson. D.Goderis and P.Vets tried to justify the limit for asymptotically abelian quantum systems with states which are not necessarily of product type . We employ the word *Bosonic* to make a clear distinction from other central limit theorems in non-commutative probability theories. In fact, the Central Limit Theorem we consider here is the standard central limit theorem of mixing systems . The difference is that the family of measures is replaced by the spectral measure of (mutually non-commuting) asymptotically commuting operators on a Hilbert space.

Although more than 10 years have passed since the work of D.Goderis and P.Vets appeared in [11], the validity of their uniform mixing condition has never been investigated in concrete examples seriously. In this paper, we consider the ground states of the exactly soluble XY model on a one-dimensional integer lattice $\mathbf{Z}$. We prove the *Bosonic Central Limit Theorem* for any local quantum observable (e.g. the energy density). The limit theorem is valid for several other examples such as quantum Gibbs states of finite range interactions, finite correlated pure states and their infinite range analogue. The condition for our *Bosonic Central Limit Theorem* is not exactly same as that of D.Goderis and P.Vets. The limit theorem we use here is proved in Section 2 . Our proof is based on an idea of E.Bolthauzen in [6]. The reason why we mention only one-dimensional models here is due to lack of non trivial examples in higher dimensional models at our present stage of research. There is no essential difference to prove our Bosonic Central Limit Theorem for the multi-dimensional case . In practice, the crucial part in proof is not the general limit theorem but to show uniform mixing condition for concrete states. In multi-dimensional quantum systems, we still do not have many tools to prove such uniform mixing condition other than cluster expansions.

In what follows, we abbreviate "Bosonic Central Limit Theorem" to CLT. To prove CLT for the XY model, we first show CLT for general local observables in quasi-free states of Fermions on lattices.

When restricted to the abelian algebra of the number operator density , any (gauge invariant) quasifree state of Fermions gives rise to a Fermion point process. The Laplace transform of the stationary measure for this process is written by certain infinite determinants. This enables us to show CLT for the number operator density in various situations. See the articles [17] , [18] and [19].

In fact, via the Jordan Wigner transform the XY model is formally equivalent to quasifree (bilinear) Hamiltonian of Fermions on $\mathbf{Z}$. However, to prove CLT for the XY model , it seems necessary to obtain CLT for general observables in quasifree states. This is due to non local nature of the Jordan Wigner transform. We use detail knowledge of

unitary implementors of Bogoliubov automorphisms developped by H.Araki in [2] and [3].

The one-dimensional XY model is an exactly solved model. The Hamiltonian is

$$H = -\sum_{j\in\mathbf{Z}}\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)} + 2\lambda\sigma_z^{(j)}\}$$

where $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$ are Pauli spin matrices at the site j and γ and λ are real parameters (anisotropy and magnetic field). We already claimed that the model is formally equivalent to a quasifree fermion Hamiltonian on a one-dimensional lattice. However , they are not physical equivalent in that the ergodic behavior of the time evolution is different. In [4] , H.Araki introduced the crossed product formalism for the XY model to handle analytic aspects of the XY model and in [5] we have shown lack of ergodicity for the time evolution in the ground state representations when $\gamma \neq 0$ and $|\lambda| < 1$. We use the idea of H.Araki to show Bosonic central limit theorem for the XY model

To explain our results more precisely, we introduce our notation now. By $\mathcal{A}$ we denote the UHF C^*-algebra d^∞ (the infinite tensor product of d by d matrix algebras) :

$$\mathcal{A} = \overline{\bigotimes_{\mathbf{Z}} M_d(\mathbf{C})}^{C^*}.$$

Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$. By $Q^{(j)}$ we denote the element of $\mathcal{A}$ with Q in the jth component of the tensor product and the identity in any other component. For a subset Λ of $\mathbf{Z}$, $\mathcal{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathcal{A}$ generated by elements supported in Λ. We set

$$\mathcal{A}_{loc} = \cup_{\Lambda\subset\mathbf{Z}:|\mathbf{\Lambda}|<\infty}\ \mathcal{A}_\Lambda \tag{1.1}$$

where the cardinality of Λ is denoted by $|\Lambda|$. We call an element of $\mathcal{A}_\Lambda$ a local observable or a strictly local observable.

When φ is a state of $\mathcal{A}$ the restriction to $\mathcal{A}_\Lambda$ will be denoted by φ_Λ :

$$\varphi_\Lambda = \varphi|_{\mathcal{A}_\Lambda}.$$

For simplicity we set

$$\mathcal{A}_R = \mathcal{A}_{[0,\infty)}\ ,\ \mathcal{A}_L = \mathcal{A}_{(-\infty,-1]}\ .$$

Let τ_j be the lattice translation determined by $\tau_j(Q^{(k)}) = Q^{(j+k)}$ for any j and k in $\mathbf{Z}$. Note that τ_j leaves $\mathcal{A}_R$ (resp. $\mathcal{A}_L$) globally invariant if j is positive (resp. negative) .

Next we present our central limit theorem. Consider a translationally invariant state φ of $\mathcal{A}$. ($\varphi\tau_j = \varphi$ for any j.) Suppose a τ invariant dense *subalgebra $\mathcal{B}$ of $\mathcal{A}$ for which φ has the summability of two point correlation in the following sense:

$$\sum_{j\in\mathbf{Z}} |\varphi(Q_1\tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| < \infty \tag{1.2}$$

for any Q_1 and Q_2 in $\mathcal{B}$.

Let N be a positive integer and let Q be a selfadjoint element of $\mathcal{B}$. We define $Q_{<N>}$ via the following equation:

$$Q_{<N>} = \frac{1}{\sqrt{2N+1}} \left(\sum_{|j| \leq N} (\tau_j(Q) - \varphi(Q)) \right).$$

Due to our assumption (1.2) , the state φ is factor. Consider the GNS representation $\{\pi(\mathcal{A}), \Omega, \mathcal{H}\}$ associated with φ where Ω is the GNS cyclic vector and π is the representation of $\mathcal{A}$ on the GNS Hilbert space $\mathcal{H}$. Then on the subspace $\pi(\mathcal{B})\Omega$ of $\mathcal{H}$ the following weak limit exists.

$$w - \lim_{N \to \infty} [Q_{<N>}, R_{<N>}] = s(Q, R)1$$

where

$$s(Q, R) = \sum_{j \in \mathbf{Z}} \varphi([Q, \tau_j(R)])$$

The convergence of $s(Q, R)$ is guaranteed by (1.2). Thus the formal limit $B(Q) = \lim_{N \to \infty} Q_{<N>}$ gives rise to an algebra of canonical commutation relations with respect to the degenerate symplectic form $s(Q, R)$ on $\mathcal{B}$. Note that the limit $B(Q)$ does not exist as an unbounded selfadjoint operator on $\mathcal{H}$. However we may consider the limit of the spectral measure for $Q_{<N>}$ evaluated on the state φ. In terms of the Fourier transform of the spectral measure we introduce the notion of the central limit. Set

$$t(Q, R) = \lim_{N \to \infty} \varphi(Q_{<N>} R_{<N>}).$$

Definition 1.1 *Let φ be a translationally invariant state of $\mathcal{A}$. We say that the central limit theorem holds for φ and $\mathcal{B}$ if and only if*

$$\lim_{N \to \infty} \varphi(e^{iTQ_{<N>}}) = e^{-\frac{T^2}{2} t(Q,Q)} \tag{1.3}$$

for any selfadjoint Q in $\mathcal{B}$ satisfying $t(Q, Q) \neq 0$.

Theorem 1.2 *Let φ be a translationally invariant state of $\mathcal{A}$. Let $\alpha(j)$ be a positive function on the set of integers such that $\alpha(j) = \alpha(-j)$ $\alpha(j) = O(j^{-2-\delta})$ as $j \to \infty$ with some positive δ . The central limit theorem holds for φ and $\mathcal{A}_{loc}$ if*

$$|\varphi(Q_L \tau_j(Q_R)) - \varphi(Q_L)\varphi(Q_R)| \leq K \, \|Q_L\| \, \|Q_R\| \, \alpha(j) \tag{1.4}$$

for any Q_L in $\mathcal{A}_L$ and Q_R in $\mathcal{A}_R$.

In classical dynamical systems, one way to show CLT is to use the Ruelle transfer operator (c.f. [15]). In [1] H.Araki developed a quantum analogue of the Ruelle transfer operator for quantum spin chains. H.Araki did not investigate the uniformity of decay of correlation for Gibbs states. In [12] ,V.Golodets and S.V.Neshveyev considered Gibbs states of AF algebras with one-dimensional locality. For finite range interactions they derived estimates of correlation functions which imply our CLT.

Corollary 1.3 *The central limit theorem holds for a Gibbs state of any finite range translationally invariant interaction at arbitrary temperature.*

In particular CLT is valid for Gibbs states of the XY model.

Our main result of this paper is as follows. Now the algebra $\mathcal{A}$ is the infinite tensor product of 2 by 2 matrices.

Theorem 1.4 *Suppose* $(\lambda, \gamma) \neq (0, \pm 1)$. *Let* ψ *be a pure ground state of the XY model. The central limit theorem is valid for any selfadjoint local observable* Q *and any pure ground state* φ *if* $|\lambda| > 1$ *or if* $|\lambda| < 1$ *and* $\gamma \neq 0$.

To compare the above result with classical spin models, we make a brief remark on two dimensional Ising model(with zero magnetic field). Two dimensional Ising model can be solved by the same Jordan Wigner transformation. Roughly speaking, the region $|\lambda| > 1$ corresponds to the high temperature regime. When $|\lambda| < 1$ and $\gamma \neq 0$ there are precisely two translationally invariant pure ground states. These pure ground states correspond to extremal Gibbs measures (with a plus or minus boundary condition) at low temperatures. The above result tells us CLT for extremal Gibbs measures projected to one dimensional line. Thus we are dealing non Gibbsian measures when $|\lambda| < 1$ and $\gamma \neq 0$. Of course our theorem is valid for non abelian observables Q in the sense that

$$[Q, \tau_j(Q)] \neq 0$$

for small j and such observables are of our prime interest.

At the critical line $|\lambda| = 1$ or $|\lambda| < 1, \gamma = 0$, we believe that scaling different from $\sqrt{n}$ is required to obtain non-trivial limit. In case of certain massless Fermion , H.Spohn obtained a limit theorem for the number operator density for a more singular case. See [20].

We present our proof of Theorem 1.2 in Section 2. Section 3 is devoted to CLT for quasifree states of Fermions and we show CLT for pure ground states of the XY model in Section 4 .

2 Proof of Theorem 1.2

The proof presented here is due to the argument of E.Bolthauzen in [6]. His idea is based on the observation that a probability measure ν on the real line $\mathbf{R}$ is the standard normal distribution if the characteristic function of ν is of the C^1 class and satisfies

$$\int_{\mathbf{R}} xe^{itx}d\nu(x) = it\int_{\mathbf{R}} e^{itx}d\nu(x).$$

The following is Lemma 2 of [6].

Lemma 2.1 *Let ν_n be a sequence of probability measures on on the real line $\mathbf{R}$ and suppose that*

$$\sup_n \int_{\mathbf{R}} |x|^2 d\nu_n(x) < \infty \tag{2.1}$$

and

$$\lim_{n\to\infty} \int_{\mathbf{R}} (x - it)e^{itx}d\nu_n(x) = 0 \tag{2.2}$$

ν_n converges to the standard normal distribution.

In our context,

$$\int_{\mathbf{R}} e^{itx}d\nu_n(x) = \varphi(e^{itQ_{<N>}})$$

and the condtion (2.1) follows from summability of two point functions.

To control non-commutativity we use the following inequality.

Lemma 2.2 *Suppose $A = A^*$ $B = B^*$ are selfadjoint bounded operators on a Hilbert space. Then*

$$\left\| e^{i(A-B)} - e^{-iB}e^{-iA} \right\| \leq \|[A, B]\| . \tag{2.3}$$

Proof
Consider $e^{is(A-B)}e^{-isA}e^{isB}$. By differentiating this identity, We obtain the following equation:

$$e^{i(A-B)} - e^{-iB}e^{iA} = i\int_0^1 e^{is(A-B)}[e^{-isA}, B]e^{-i(1-s)B}e^{iA}ds.$$

Then combined with

$$[e^{-isA}, B] = -i\int_0^1 e^{-itsA}[A, B]e^{-i(1-t)sA}dt$$

we obtain (2.3) .
End of Proof

Proof of Theorem1.2. We now consider Theorem 1.2. We mention modification of the proof of E.Bolthausen now.

First of all, without loss of generality we may assume that $\varphi(Q) = 0$ and $t(Q, Q) = 1$ Let $\{\pi(\mathcal{A}), \Omega, \mathcal{H}\}$ be the GNS triple associated with the state φ where $\pi(\,)$ is the representation $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ and Ω is the GNS cyclic vector. We denote the spectral decomposition of $Q_{<N>}$ by

$$Q_{<N>} = \int \lambda dE_N(\lambda).$$

Then for this sequence of probability measures $d\nu_N(\lambda) = d(\Omega, E_N((-\infty,\lambda])\Omega)$ we apply Lemma 2.1 . Due to Assumption for $\alpha(j)$ in Theorem1.2, we can find an increasing sequence $m(N)$ of positive integers such that

$$\lim_{N\to\infty} \alpha(m(N))\sqrt{2N+1} = 0\,, \ \lim_{N\to\infty} m(N)^{-1}(2N+1)^{1/4} = \infty. \tag{2.4}$$

Set

$$\begin{aligned} Q_{j,N} = & \sum_{|k|\le N, |k-j|\le m(N)} \tau_k(Q) \quad , \quad \beta_N = \sum_{|k|\le N} 1/2\varphi(\{\tau_k(Q)\,,\, Q_{k,N}\}) \\ \overline{Q}_{<N>} & = \beta_N^{-1/2}\left\{\sum_{|k|\le N}\tau_k(Q)\right\} \quad , \quad \overline{Q}_{j,N} = \beta_N^{-1/2} Q_{j,N}. \end{aligned}$$

Then $\beta_N = (2N+1)(1+o(1))$ due to summability of two point correlation functions and $t(Q,Q) = 1$ and now we show

$$\lim_{T\to\infty} \varphi((iT - \overline{Q}_{<N>})e^{iT\overline{Q}_{<N>}}) = 0. \tag{2.5}$$

Then we consider the following expression:

$$\begin{aligned} (iT - \overline{Q}_{<N>})e^{iT\overline{Q}_{<N>}} = & \quad iT(1 - \beta_N^{-1}\sum_{|k|\le N}\tau_k(Q)Q_{k,N})e^{iT\overline{Q}_{<N>}} \\ & -\beta_N^{-1/2}\sum_{|k|\le N}\tau_k(Q)(1 - e^{-iT\overline{Q}_{k,N}} - iT\overline{Q}_{k,N})e^{iT\overline{Q}_{<N>}} \\ & -\beta_N^{-1/2}\sum_{|k|\le N}\tau_k(Q)\left(e^{-iT\overline{Q}_{k,N}}e^{iT\overline{Q}_{<N>}} - e^{-iT(\overline{Q}_{k,N}-\overline{Q}_{<N>})}\right) \\ & -\beta_N^{-1/2}\sum_{|k|\le N}\tau_k(Q)e^{iT(\overline{Q}_{<N>}-\overline{Q}_{k,N})} \\ = & \quad A_1 e^{iT\overline{Q}_{<N>}} - A_2 e^{iT\overline{Q}_{<N>}} + A_3 - A_4. \end{aligned} \tag{2.6}$$

First consider $\varphi(A_1 e^{iT\overline{Q}_{<N>}})$. Note that

$$\left|\varphi(A_1 e^{iT\overline{Q}_{<N>}})\right| \le \varphi(A_1 A_1^*)^{1/2}.$$

By definition ,

$$\begin{aligned} & \varphi(A_1 A_1^*) \\ & = T^2\{1 - \beta_N^{-1}\varphi(\sum_{|k|\le N}\{\tau_k(Q)\,,\, Q_{k,N}\}) + \beta_N^{-2}\sum_{|k|,|l|\le N}\varphi(\tau_k(Q)Q_{k,N}Q_{l,N}\tau_l(Q))\} \\ & = \frac{T^2}{\beta_N^2}\sum\{\varphi(\tau_a(Q)\tau_b(Q)\tau_d(Q)\tau_c(Q)) - \varphi(\tau_a(Q)\tau_b(Q))\varphi(\tau_d(Q)\tau_c(Q))\} + o(1) \end{aligned} \tag{2.7}$$

where the last summation is taken for a,b,c,d satisfying

$$|a|\,,|b|\,|c|\,|d| \le N \quad , \quad |a-b| \le m(N), |c-d| \le m(N).$$

The rest of estimate is same as [6] and we have $\lim_N \varphi(A_1 A_1^*) = 0$

Next we look at the terms A_2 and A_4 . The estimate of $\varphi(A_4)$ is quite similar to the commutative case while the estimate of $\varphi(A_2)$ is slightly different as we show here. First note that

$$\left|\varphi(A_2 e^{iT\overline{Q}_{<N>}})\right| \leq {\beta_N}^{-1/2} \sum_{|k|\leq N} \varphi(\tau_k(Q)\left|1 - e^{-iT\overline{Q}_{k,N}} - iT\overline{Q}_{k,N}\right|^2 \tau_k(Q))^{1/2}. \tag{2.8}$$

Recall that $\left\|\overline{Q}_{k,N}\right\| \leq \frac{2m(N)}{{\beta_N}^{1/2}} \to 0$ as $N \to \infty$ there exists a constant C such that

$$\left|1 - e^{-iT\overline{Q}_{k,N}} - iT\overline{Q}_{k,N}\right|^2 \leq C\overline{Q}_{k,N}^4 \tag{2.9}$$

Thus

$$\begin{aligned}\left|\varphi(A_2 e^{iT\overline{Q}_{<N>}})\right| \quad &\leq C{\beta_N}^{-1/2}(2N+1)\left\|\overline{Q}_{k,N}^2\right\| \leq C(\sqrt{2N+1} + o(1))(\frac{m(N)}{\sqrt{2N+1}})^2 \\ &\leq 2C\frac{m(N)^2}{\sqrt{2N+1}} \end{aligned} \tag{2.10}$$

Due to (2.4) the right-hand side of (2.10) vanishes as N tends to infinity.

Finally we consider $\varphi(A_3)$ in (2.6). We apply Lemma 2.2 and as a consequence, we obtain

$$\left\|e^{-iT\overline{Q}_{k,N}} e^{iT\overline{Q}_{<N>}} - e^{-iT(\overline{Q}_{k,N} - \overline{Q}_{<N>})}\right\| \leq \left\|[\overline{Q}_{k,N}, \overline{Q}_{<N>}]\right\| \leq T^2 2r^2 \beta_N^{-1} \sum_{k\in\mathbf{Z}} \|[Q, \tau_k(Q)]\|\,.$$

Where r is the range of Q. As Q is strictly local, $\sum_{k\in\mathbf{Z}} \|[Q, \tau_k(Q)]\|$ is finite. Thus

$$|\varphi(A_3)| \leq KT^2{\beta_N}^{-3/2}(2N+1)\sum_{k\in\mathbf{Z}} \|[Q, \tau_k(Q)]\| \tag{2.11}$$

and we have $\lim_N \varphi(A_3) = 0$.
End of Proof.

Next consider CLT for non local observables. Here for simplicity of presentation, we consider the case when two point correlation decays exponentially fast. Certainly this can be proved for many examples.The above proof shows that CLT is valid for a non-local observable $Q = Q^*$ if we obtain the following estimates.
(a) We have a subalgebra $\mathcal{F}$ of $\mathcal{A}$ with a norm $\|\|Q\|\|$ such that there exist positive constants C and M such that

$$|\varphi(Q\tau_k(R)) - \varphi(Q)\varphi(R)| \leq Ce^{-kM}\|\|Q\|\|\|\|R\|\| \tag{2.12}$$

for Q and R in $\mathcal{F}$ and any positive integer k.
(b) Any element Q in $\mathcal{F}$ is well localized so that the following sum is finite:

$$\sup_N \sum_{|k|\leq N} \sum_{|l|\geq N} \|[\tau_l(Q), \tau_k(Q)]\| < \infty. \tag{2.13}$$

(c) $\lim_N \varphi(A_4) = 0$
The algebra $\mathcal{A}_{1,x}$ of [1] and F_θ of [14] satisfy the above condition (2.13) . More generally, we introduce the notion of *exponentially localized observables* here.

Definition 2.3 *An element Q of $\mathcal{A}$ is exponentially localized if there exist positive constants $\|\|Q\|\|$ and $0 < \theta < 1$ such that for each positive integer n there exists Q_n in $\mathcal{A}_{[-n,n]}$ satisfying $\|Q_n\| \leq \|Q\|$ and*

$$\|Q - Q_n\| \leq \|\|Q\|\|\theta^n \tag{2.14}$$

θ will be called the localization rate for Q.

Suppose Q and R are exponentially localized. with a localization rate θ . Then, we have

$$\|[Q\ ,\ \tau_k(R)]\| \leq 2(\|Q\|\ \|\|R\|\| + \|R\|\ \|\|Q\|\|)\theta^{k/3} \tag{2.15}$$

The estimate (2.15) implies (2.13) .

Theorem 2.4 *let φ be a translationally invariant state . Suppose there exist positive constants M and K such that*

$$|\varphi(Q_L\tau_j(Q_R)) - \varphi(Q_L)\varphi(Q_R)| \leq K\,\|Q_L\|\,\|Q_R\|\,e^{-Mj} \tag{2.16}$$

for any positive j , any Q_L in $\mathcal{A}_L$ and Q_R in $\mathcal{A}_R$.

The central limit theorem holds for any selfadjoint exponentially localized observable Q .

When Q is exponentially localized, two point correlation decays exponentially fast for Q due to the localization property (2.14) .

$$|\varphi(Q\tau_k(Q)) - \varphi(Q)\varphi(Q)| \leq K'e^{-M'k}$$

where $M' = \min\{M/2, |\ln\theta|\,/2\}$.

Proof of Theorem 2.4.
Suppose that Q has the localization rate θ . Due to exponential localization and (2.15) , it is also straightforward to show (2.13). Thus we concentrate on the estimate of $|\varphi(A_4)|$. We approximate Q by $Q_{r(N)}$ with $\varphi(Q_{r(N)}) = 0$ for suitable $r(N)$. We set

$$\alpha(N) = Ke^{-MN}\ ,\ m(N) = \frac{a}{M}\ln(2N+1)\ ,\ r(N) = b\ln(2N+1). \tag{2.17}$$

If $b = \frac{4}{|\ln\theta|}$, $a > \max\{(2bM+2), 1/2\}$ we have(2.4) and

$$\lim_{N\to\infty}(2N+1)^2\theta^{r(N)} = 0. \tag{2.18}$$

Then

$$\lim_{N\to\infty}\sqrt{2N+1}\left\|\tau_k(Q)e^{iT(\overline{Q}_{<N>}-\overline{Q}_{k,N})} - \tau_k(Q_{r(N)}e^{iT(\overline{Q}_{<N>}-\overline{Q}_{k,N})}\right\| = 0. \tag{2.19}$$

On the other hand, Lemma 2.2 tells us that

$$\left\|e^{iA} - e^{iB}\right\| \leq \left\|e^{iA} - e^{i(A-B)}e^{iB}\right\| + \left\|(1 - e^{i(A-B)})e^{iB}\right\| \leq \|[A,B]\| + \|A-B\|\,.$$

As a result we obtain

$$\begin{aligned}
&\left\|(e^{iT(\overline{Q}_{<N>}-\overline{Q}_{k,N})}-e^{iT(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})}\right\| \\
&\leq T^2\left\|\left[(\overline{Q}_{<N>}-\overline{Q}_{k,N})\,,\,(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})\right]\right\| \\
&+T\left\|(\overline{Q}_{<N>}-\overline{Q}_{k,N})-(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})\right\|. \qquad (2.20)
\end{aligned}$$

Now we look at the first term in the right-hand side of (2.20) :

$$\begin{aligned}
&\left\|\left[\overline{Q}_{<N>}\,,\,\overline{Q_{r(N)}}_{<N>}\right]\right\| = \beta_N^{-1}\left\|\left[\sum_{|k|\leq N}\tau_k(Q)\,,\,\sum_{|l|\leq N}\tau_l(Q-Q_{r(N)})\right]\right\| \\
&\leq (2N+1)^2\beta_N^{-1}\left\|Q\right\|\left\|Q-Q_{r(N)}\right\| \leq C_1(2N+1)^{3/2}\theta^{r(N)}. \qquad (2.21)
\end{aligned}$$

So

$$T^2\left\|\left[(\overline{Q}_{<N>}-\overline{Q}_{k,N})\,,\,(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})\right]\right\| \leq C_2(2N+1)^{-5/2}.$$

The estimate of the second term of the right-hand side of (2.20) is easier.

$$\begin{aligned}
&T\left\|(\overline{Q}_{<N>}-\overline{Q}_{k,N})-(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})\right\|. \\
\leq\quad & C_3T\frac{(2N+1)}{\beta_N^{1/2}}\theta^{r(N)} \qquad (2.22)
\end{aligned}$$

Thus

$$\begin{aligned}
&\sqrt{2N+1}\left\|(e^{iT(\overline{Q}_{<N>}-\overline{Q}_{k,N})}-e^{iT(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})}\right\| \\
&\leq \sqrt{2N+1}(C_1(2N+1)^{3/2}+C_4\sqrt{2N+1})\theta^{r(N)} \leq C_5(2N+1)^2\theta^{r(N)}.
\end{aligned}$$

By definition we have $\theta^{r(N)} = (2N+1)^{-4}$. As a consequence, we have shown that

$$\lim_{N\to\infty}\varphi(A_4) = \lim_{N\to\infty}{\beta_N}^{-1/2}\sum_{|k|\leq N}\varphi(\tau_k(Q_{r(N)})e^{iT(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})}). \qquad (2.23)$$

However the uniform cluster property of φ leads to

$$\left|\varphi(\tau_k(Q_{r(N)})e^{iT(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})})\right| \leq C_4(2N+1)^{-(a-2bM)}. \qquad (2.24)$$

Thus

$$\begin{aligned}
&\left|{\beta_N}^{-1/2}\sum_{|k|\leq N}\varphi(\tau_k(Q_{r(N)})e^{iT(\overline{Q_{r(N)}}_{<N>}-\overline{Q_{r(N)}}_{k,N})})\right| \\
&\leq C_4{\beta_N}^{-1/2}(2N+1)(2N+1)^{-(a-2bM)} \leq C_5(2N+1)^{-1} \qquad (2.25)
\end{aligned}$$

End of Proof

3 Fermions

In this section, we show uniform clustering for quasifree states of Fermions which , in turn , implies CLT. The results mentioned in this section are used in our proof of CLT for pure ground states of the XY model .

We begin with explanation of quasifree states of Fermions on $\mathbf{Z}$. Let $\mathcal{A}^{CAR}$ be the CAR algebra generated by Fermion creation annihilation operators c_j c_j^* . In this paper, by a CAR algebra we mean a unital C^*-algebra $\mathcal{A}^{CAR}$. Thus , c_j and c_j^* satisfy the standard canonical anticommutation relations:

$$\{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad , \quad \{c_j, c_k^*\} = \delta_{j,k} 1 \tag{3.1}$$

for any integer j and k.

For $f = (f_j) \in l_2(\mathbf{Z})$ we set

$$c^*(f) = \sum_{j \in \mathbf{Z}} c_j^* f_j \quad , \quad c(f) = \sum_{j \in \mathbf{Z}} c_j f_j \tag{3.2}$$

where the sum converges in norm topology. Furthermore, let

$$B(h) = c^*(f_1) + c(f_2) \tag{3.3}$$

where $h = (f_1 \oplus f_2)$ is a vector in the test function space $\mathcal{K} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z})$.

By $\overline{f}$ we denote the complex conjugate $\overline{f} = (\overline{f}_j)$ of $f \in l_2(\mathbf{Z})$ and we introduce an antiunitary involution J on the test function space $\mathcal{K} = l_2(\mathbf{Z}) \oplus l_2(\mathbf{Z})$ determined by

$$J(f_1 \oplus f_2) = (\overline{f}_2 \oplus \overline{f}_1). \tag{3.4}$$

Then,

$$\{B(h_1)^*, B(h_2)\} = (h_1, h_2)_{\mathcal{K}} 1 \quad , \quad B(Jh)^* = B(h). \tag{3.5}$$

For later convenience, we introduce the automorphisms τ_k, Θ , Θ_- and $\Theta_k^{(+)}$ of $\mathcal{A}^{CAR}$. τ_k is the shift on the lattice $\mathbf{Z}$.

$$\tau_k(c_{j+k}) = c_{j+k} \quad , \quad \tau_k(c_{j+k}^*) = c_{j+k}^* \tag{3.6}$$

Θ , Θ_- $\Theta_k^{(+)}$ are involutive *automorphisms of $\mathcal{A}^{CAR}$ determined by the following equations.

$$\Theta(c_j) = -c_j \quad , \quad \Theta(c_j^*) = -c_j^* \quad \text{for any } j \ . \tag{3.7}$$

$$\begin{aligned} \Theta_-(c_j) &= \begin{cases} c_j, & \text{if } j \geq 1 \\ -c_j, & \text{if } j \leq 0. \end{cases} \\ \Theta_-(c_j^*) &= \begin{cases} c_j^*, & \text{if } j \geq 1 \\ -c_j^*, & \text{if } j \leq 0. \end{cases} \end{aligned} \tag{3.8}$$

$$\Theta_k^{(+)}(c_j) = \begin{cases} -c_j, & \text{if } j \geq k \\ c_j, & \text{if } j \leq k-1 \ . \end{cases}$$
$$\Theta_k^{(+)}(c_j^*) = \begin{cases} -c_j^*, & \text{if } j \geq k \\ c_j^*, & \text{if } j \leq k-1. \end{cases} \tag{3.9}$$

The automorphisms τ_k, Θ , Θ_- $\Theta_k^{(+)}$ are induced by unitaries , t_k, θ, θ_-, and $\theta_k^{(+)}$ on the test function space $\mathcal{K}$:

$$\begin{aligned} &\tau_k(B(h)) = B(t_k h) \quad , \quad \Theta(B(h)) = -B(h), \\ &\Theta_-(B(h)) = B(\theta_- h) \quad , \quad \Theta_k^{(+)}(B(h)) = B(\theta_k^{(+)} h). \end{aligned} \tag{3.10}$$

Generally any unitary u on $\mathcal{K}$ satisfying $JuJ = u$ gives rise to the automorphism β_u of $\mathcal{A}^{CAR}$ determined by

$$\beta_u(B(h)) = B(uh). \tag{3.11}$$

β_u is called the Bogoliubov automorphism associated with u.

As the automorphism Θ is involutive , $\Theta^2(Q) = Q$, we introduce the Z_2 grading with respect to Θ.

$$\mathcal{A}^{CAR}{}_{\pm} = \left\{ Q \in \mathcal{A}^{CAR} \mid \Theta(Q) = \pm Q \right\} \quad , \quad \mathcal{A}^{CAR} = \mathcal{A}_+^{CAR} \cup \mathcal{A}_-^{CAR}$$

Definition 3.1 *Let A be a positive operator on the test function space $\mathcal{K}$ satisfying*

$$0 \leq A \leq 1 \quad , \quad JAJ = 1 - A. \tag{3.12}$$

The quasifree state φ_A is the state of $\mathcal{A}^{CAR}$ determined by the following equations:

$$\varphi_A(B(h_1)B(h_2)...B(h_{2n+1})) = 0, \tag{3.13}$$

$$\varphi_A(B(h_1)B(h_2)...B(h_{2n})) = \sum \quad sign(p) \prod_{j=1}^{n} (\Gamma h_{p(2j-1)}, A h_{p(2j)})_{\mathcal{K}} \tag{3.14}$$

where the sum is over all permutations p satisfying

$$p(1) < p(3) < ... < p(2n-1) \quad , \quad p(2j-1) < p(2j)$$

and $sign(p)$ is the signature of the permutation p.

Due to (3.13) , any quasifree state φ_A is Θ invariant. A quasifree state φ_A is pure if and only if the operator A is a projection. We call a projection E a basis projection if $JEJ = 1 - E$. The quasifree state φ_E associated with a basis projection E is referred to as the Fock state associated with E and the associated GNS representation (resp. the GNS Hilbert space, the GNS cyclic vector) as Fock representation (resp. Fock space , Fock vacuum vector).

A quasifree state φ_A is faithful if and only if the operator A is positive $0 < A < 1$.

A quasifree state φ_A is translationally invariant if and only if the operator A commutes with the unitary t_1 introduced in (3.10), $t_1 A = A t_1$. It turns out that for a translationally invariant bquasifree state φ_A , the operator A is the Fourier transform of a (2 by 2 matrix

valued) multiplication operator $\tilde{A}(x)$ on $\tilde{\mathcal{K}} = L_2([0,2\pi]) \oplus L_2([0,2\pi])$. More precisely we use the following normalization for the Fourier transform:

$$F(f)(x) = \sum_{n\in\mathbf{Z}} e^{inx} f_n \quad , \quad f_n = (2\pi)^{-1} \int e^{-inx} F(f)(x) dx \tag{3.15}$$

for $f = (f_n) \in l_2(\mathbf{Z})$ and $F(f)(x) \in L_2([0,2\pi])$. If $t_1 A = A t_1$, the operator $\tilde{A} = FAF^{-1}$ on $\tilde{\mathcal{K}}$ commutes with the multiplication operators e^{inx} (for any integer n) and $\tilde{A}$ is a matrix valued multiplication operator. The decay of two point correlation functions is determined by regularity of the function $\tilde{A}(x)$. For example, if the Fourier coefficients of $\tilde{A}(x)$ vanishes exponentially fast, two point correlation functions for local observables decay exponentially as well. However, to obtain uniformity of decay of correlation we impose more conditions on the operator A.

Now we state our central limit theorem of quasifree states. Theorem 1.2 is valid for the even part ${\mathcal{A}^{CAR}}_+$ of $\mathcal{A}^{CAR}$ and we will see that the assumption of Theorem 1.2 holds in the following situations.

Theorem 3.2 *Let φ_A be a translationally invariant quasifree state of $\mathcal{A}^{CAR}$. The Bosonic Central Limit Theorem is valid for φ_A restricted to the even part ${\mathcal{A}^{CAR}}_+ \cap \mathcal{A}^{CAR}_{loc}$ if one of the following conditions is valid:*
(a) The operator A is a basis projection , and all the matrix elements of $\tilde{A}(x)$ are C^∞ functions.
(b) The operator A is strictly positive , $0 < \epsilon < A < 1 - \epsilon < 1$ and all the matrix elements of $\tilde{A}(x)$ are of C^∞ class

To verify the assumption of Theorem1.2, we use results on unitary implementors of Bogoliubov automorphisms and purification technique. The proof of the following proposition can be found in [16], [2] and [3].

Proposition 3.3 *(i) Let E_1 and E_2 be basis projections . The Fock states φ_{E_1} and φ_{E_2} give rise to mutually unitarily equivalent (irreducible) representations of $\mathcal{A}^{CAR}$ if and only if $E_1 - E_2$ is of Hilbert Schmidt Class.*
(ii) Suppose that S_1 and S_2 are bounded positive operators on $\mathcal{K}$ satisfying (3.12). The quasifree states φ_{S_1} and φ_{S_2} are quasi-equivalent if and only if $\sqrt{S_1} - \sqrt{S_2}$ is of Hilbert Schmidt class.

Let $\{\pi_E(\mathcal{A}^{CAR}), \mathcal{H}_E, \Omega_E\}$ be the Fock representation associated with a basis projection E. Let u be a unitary $\mathcal{K}$ satisfying $JuJ = u$. We say that the Bogoliubov automorphism β_u is unitarily implementable in $\{\pi_E(\mathcal{A}^{CAR}), \mathcal{H}_E, \Omega_E\}$ when there exists a unitary $\Gamma(u)$ on $\mathcal{H}_E$ such that

$$\Gamma(u)\pi_E(Q)\Gamma(u)^{-1} = \pi_E(\beta_u(Q)). \tag{3.16}$$

$\Gamma(u)$ is unique up to phase factor. This is due to irreducibility of Fock representation.

If the unitary u commtes with a basis projection E, the Fock state φ_E is invariant under β_u and there exists $\Gamma(u)$ leaving Ω_E invariant:

$$\Gamma(u)\Omega_E = \Omega_E.$$

Proposition 3.4 *(i) Let $\{\pi_E(\mathcal{A}^{CAR}), \mathcal{H}_E\Omega_E\}$ be the Fock representation associated with a basis projection E. The Bogoliubov automorphism β_u associated with u is unitarily implementable in $\mathcal{H}_E$ if and only if $uEu^{-1} - E$ is of Hilbert Schmidt Class.*
(ii) Let E_1 and E_2 be basis projections . Suppose that $E_1 - E_2$ is of Hilbert Schmidt Class and $\|E_1 - E_2\| < 1$. Let $\{\pi(\mathcal{A}^{CAR}), \mathcal{H}\}$ be the irreducible representation equivalent to the Fock representation associated with E_1 (therefore with E_2) and let Ω_{E_a} $(a = 1, 2)$ be the unit vector in $\mathcal{H}$ giving rise to the Fock state φ_{E_a}. Then

$$|(\Omega_{E_1}, \Omega_{E_2})| = det(1 - (E_1 - E_2)^2)^{1/2} \tag{3.17}$$

Furthermore if $E_1 - E_2$ is of trace Class and $\|E_1 - E_2\| < 1$,

$$|(\Omega_{E_1}, \Omega_{E_2})| = det_{E_1}(E_1 E_2 E_1) \tag{3.18}$$

where $det_{E_1}(A)$ is the determinant of an operator A acting on the range of E_1.

The following result is a consequence of (3.18).

Corollary 3.5 *There exist positive constants C_1 and C_2 such that for basis projections E_1 and E_2 satisfying $\|E_1 - E_2\|_{H.S.} \leq C_1$, we have*

$$\|\varphi_{E_1} - \varphi_{E_2}\| \leq C_2 \|E_1 - E_2\|^2_{H.S.} \tag{3.19}$$

We apply (3.19) to derive uniform clustering property of quasifree states. For this purpose we need the following estimate of trace norm.

Lemma 3.6 *Let A be a bounded operator on a Hilbert space $\mathcal{H}$. Suppose there exists a complete orthonormal basis $\{\xi_j \ (j \in \mathbf{Z})\}$ such that the following sum is finite:*

$$\sum_{l \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |(\xi_k, A\xi_l)| < \infty \tag{3.20}$$

Then the operator A is of trace class and

$$|tr(A)| \leq tr(|A|) \leq \sum_{l \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |(\xi_k, A\xi_l)| \,. \tag{3.21}$$

Proof
Consider the polar decomposition of A :

$$A = w\,|A|\,.$$

Then,

$$\begin{aligned} tr(|A|) &\leq \sum_{k \in \mathbf{Z}} |(\xi_k, w^* A \xi_k)| \leq \sum_{l \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |(\xi_k, A\xi_l)|\, |(\xi_l, w^* \xi_k)| \\ &\leq \sum_{l \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |(\xi_k, A\xi_l)| \,. \end{aligned} \tag{3.22}$$

End of Proof

Lemma 3.7 *Let A and B be non-negative bounded operators on $\mathcal{H}$. For a real number p with $0 < p < 1$ there exists a constant $C(p)$ such that*

$$\|A^p - B^p\| \leq C(p) \|A - B\|^p . \tag{3.23}$$

Proof
Recall that

$$A^p = \frac{\sin t\pi}{\pi} \int_0^\infty \lambda^{p-1} - \lambda^p (\lambda + A)^{-1} d\lambda. \tag{3.24}$$

We introduce cut off in the above integral. Set $\nu = \|A - B\|$. Then,

$$\begin{aligned} A^p - B^p = \quad & \frac{\sin t\pi}{\pi} \int_\nu^\infty \lambda^p \left((\lambda + B)^{-1} - (\lambda + A)^{-1} \right) d\lambda \\ & + \frac{\sin t\pi}{\pi} \int_0^\nu (\lambda^{p-1} - \lambda^p (\lambda + A)^{-1}) d\lambda \\ & - \frac{\sin t\pi}{\pi} \int_0^\nu (\lambda^{p-1} - \lambda^p (\lambda + B)^{-1}) d\lambda. \end{aligned} \tag{3.25}$$

The first term of (3.25) can be estimated as follows:

$$\begin{aligned} & \left\| \int_\nu^\infty \left(\lambda^p ((\lambda + B)^{-1} - (\lambda + A)^{-1}) \right) d\lambda \right\| \\ \leq \quad & \left\| \int_\nu^\infty \lambda^p (\lambda + B)^{-1} (B - A)(\lambda + A)^{-1} \right\| \\ \leq \quad & \|B - A\| \int_\nu^\infty \lambda^p (\lambda)^{-2}) = \|B - A\| \frac{\nu^{p-1}}{p-1} = \frac{\|B - A\|^p}{p-1}. \end{aligned} \tag{3.26}$$

Other terms of (3.25) is bounded from above by a constant times $\|B - A\|^p$.

$$0 \leq \int_0^\nu \lambda^{p-1} - \lambda^p (\lambda + A)^{-1} d\lambda \leq \int_0^\nu \lambda^{p-1} = \frac{\|B - A\|^p}{p}. \tag{3.27}$$

As a consequence,

$$\|A^p - B^p\| \leq \left(\frac{2}{\pi p} + \frac{1}{\pi(p-1)} \right) \|B - A\|^p .$$

End of Proof

Purification of Quasifree States

In our proof of CLT, we also use another CAR algebra $\tilde{\mathcal{A}}^{CAR}$ with a larger test function space.

Set $\tilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$ and $\tilde{J} = J \oplus -J$. $\tilde{\mathcal{A}}^{CAR}$ is a unital C^*-algebra with generators $\tilde{B}(f)$ ($f \in \tilde{\mathcal{K}}$) satisfying

$$\{\tilde{B}(h_1)^*, \tilde{B}(h_2)\} = (h_1, h_2)_{\tilde{\mathcal{K}}} 1 \quad , \quad \tilde{B}(\tilde{J}h)^* = \tilde{B}(h)$$

for $h_1, h_2, h \in \tilde{\mathcal{K}}$. We identify $\mathcal{A}^{CAR}$ with a subalgbera of $\tilde{\mathcal{A}}^{CAR}$ via the following equation:

$$B(h) = \tilde{B}(h \oplus 0). \tag{3.28}$$

Suppose S is a positive operator on $\mathcal{K}$ satisfying $JSJ = 1 - S$. Let P_S be the operator on $\tilde{\mathcal{K}}$ determined by

$$P_S = \begin{bmatrix} S & \sqrt{S(1-S)} \\ \sqrt{S(1-S)} & 1-S \end{bmatrix} \tag{3.29}$$

It is easy to see that P_S is a basis projection on $\tilde{\mathcal{K}}$. Let $\tilde{\varphi}_{P_S}$ be the Fock state of $\tilde{\mathcal{A}}^{CAR}$ associated with P_S. The restriction of $\tilde{\varphi}_{P_S}$ to $\mathcal{A}^{CAR}$ is φ_S.

$$\tilde{\varphi}_{P_S}(Q) = \varphi_S(Q) \qquad \text{for Q in } \mathcal{A}^{CAR}. \tag{3.30}$$

This procedure of passing to pure states of a larger algebra $\tilde{\mathcal{A}}^{CAR}$ is referred to as *purification*.

Proof of Theorem 3.2.

We now return to our proof of CLT for quasifree states. Let E be a basis projection on $\mathcal{K}$ and we show CLT for the Fock state φ_E.

Let $\{e(j) \mid j = 0, \pm 1, \pm 2, ...\}$ be the standard basis of $l_2(\mathbf{Z})$ where

$$e(j) = (e(j)_k) \in l_2(\mathbf{Z}) \quad , \quad e(j)_k = \delta_{j,k}$$

Let p be the projection to the subspace of $l_2(\mathbf{Z})$ spanned by $e(j)\,(j \leq 0)$ and let q_N be the projection to the subspace spanned by $e(j)\,(N \leq j)$. Set $r_N = 1 - p - q_N$. We use the same symbol p , q_N , r_N for the projections on $\mathcal{K}$:

$$p = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$$

and so on.

We show the following estimate , which implies CLT . For any integer M there exists a constant C_M

$$\left\| (\varphi_E)_{(-\infty,0]\cup[N,\infty)} - (\varphi_E)_{(-\infty,0]} \otimes (\varphi_E)_{[N,\infty)} \right\| \leq C_M \frac{1}{N^M} \tag{3.31}$$

where $(\varphi_E)_\Lambda$ is the restriction of φ_E to $\mathcal{A}_\Lambda^{CAR}$.

The state $(\varphi_E)_{(-\infty,0]\cup[N,\infty)}$ should be understood as a quasifree state of the CAR algebra with test function space $(1 - r_N)\mathcal{K}$. By abuse of notation we write

$$(\varphi_E)_{(-\infty,0]\cup[N,\infty)} = \varphi_{(1-r_N)E(1-r_N)}.$$

In the same sprit,

$$(\varphi_E)_{(-\infty,0]} \otimes (\varphi_E)_{[N,\infty)} = \varphi_{pEp+q_NEq_N}.$$

Then, (3.30) leads to

$$\left\| (\varphi_E)_{(-\infty,0]\cup[N,\infty)} - (\varphi_E)_{(-\infty,0]} \otimes (\varphi_E)_{[N,\infty)} \right\| \leq \left\| \varphi_{P_{(1-r_N)E(1-r_N)}} - \varphi_{P_{pEp+q_NEq_N}} \right\| \tag{3.32}$$

where the right-hand side should be understood as the norm of difference of pure states for the CAR algebra over $(1 - r_N)\mathcal{K} \oplus (1 - r_N)\mathcal{K}$.

Due to (3.19) we obtain the following estimates:

$$\left\| P_{(1-r_N)E(1-r_N)} - P_{pEp+q_NEq_N} \right\|_{H.S.}^2 \leq C_M \frac{1}{N^M}.$$

Note that

$$P_{(1-r_N)E(1-r_N)} - P_{pEp+q_NEq_N} = \begin{bmatrix} A_N & B_N \\ B_N & -A_N \end{bmatrix}$$

where

$$A_N = (1-r_N)E(1-r_N) - (pEp + q_NEq_N)$$

and

$$B_N = \sqrt{(1-r_N)Er_NE(1-r_N)} - \sqrt{pE(1-p)Ep + q_NE(1-q_N)Eq_N}.$$

Operators $\sqrt{(1-r_N)Er_NE(1-r_N)}$, $\sqrt{pE(1-p)Ep + q_NE(1-q_N)Eq_N}$ and A are of trace class.

It is easy to see that

$$\left\| P_{(1-r_N)E(1-r_N)} - P_{pEp+q_NEq_N} \right\|_{H.S.}^2 = 2Tr(A_N^2 + B_N^2) \tag{3.33}$$

To obtain estimates of (3.33) we use Lemma 3.6 and 3.7.

Recall that the Fourier transform of a multiplication operator $\tilde{D}(x)$ on $L_2([0, 2\pi])$ gives rise to a convolution operator D :

$$(Df)_j = \sum_k D(j-k) f_k$$

where $f = (f_j) \in l_2(\mathbf{Z})$ and $D(j)$ is the fourier coefficient of $\tilde{D}(x)$.

On the other hand, the integration by parts formula implies that

$$|D(j)| \leq (2\pi)^{-1} \left(\int \left| \frac{d^M}{dx^M} \tilde{D}(x) \right| dx \right) \quad |j|^{-M}. \tag{3.34}$$

Combined with Lemma 3.6 , the estimate (3.34) tells us that $|(1-p)Ep|$ is of trace class.

$$Tr(|(1-p)Ep|) = Tr(|(1-q_N)Eq_N|) \leq C \sum_{k \leq 0, 1 \leq l} |k-l|^{-M} < \infty \tag{3.35}$$

Note that we used the fact that both the trace and the operator E are translationally invariant in the first identity of (3.35) .

Let us consider $Tr(A_N^2)$. As $1 - r_N = p + q_N$, $A_N = pEq_N + q_NEp$, for any integer M we can find constants C_1 , C_2 and C_3 such that

$$\begin{aligned} Tr(A_N^2) &= Tr(pEq_NEp + q_NEpEq_N) \leq C_1 \sum_{k \leq 0} \sum_{N \leq l} |k-l|^{-M} \\ &\leq C_2 \sum_{N \leq l} |l|^{-(M-1)} \leq C_3 |N|^{-(M-2)}. \end{aligned} \tag{3.36}$$

for any integer M.

Next we consider $Tr(B_N^2)$. We have seen $\sqrt{pE(1-p)Ep}$ and $\sqrt{q_N E(1-q_N)Eq_N}$ are of trace class while $\sqrt{(1-r_N)Er_N E(1-r_N)}$ is of finite rank. Then

$$\begin{aligned} Tr(B_N^2) \;\leq\; & \left\| \sqrt{(1-r_N)Er_N E(1-r_N)} - \sqrt{pE(1-p)Ep + q_N E(1-q_N)Eq_N} \right\| \\ & \times Tr(\sqrt{q_N E(1-q_N)Eq_N} + \sqrt{pE(1-p)Ep} + \sqrt{(1-r_N)Er_N E(1-r_N)}). \end{aligned} \tag{3.37}$$

Due to Lemma 3.7 we have

$$\begin{aligned} & \left\| \sqrt{(1-r_N)Er_N E(1-r_N)} - \sqrt{pE(1-p)Ep + q_N E(1-q_N)Eq_N} \right\| \\ \leq \quad & C_2 \left\| (1-r_N)Er_N E(1-r_N) - pE(1-p)Ep - q_N E(1-q_N)Eq_N \right\|^{1/2} \end{aligned} \tag{3.38}$$

$$\begin{aligned} & (1-r_N)Er_N E(1-r_N) - pE(1-p)Ep - q_N E(1-q_N)Eq_N \\ = \quad & -pEq_N Ep - q_N EpEq_N + pE(1-p-q_N)Eq_N + Eq_N E(1-p-q_N)Ep. \end{aligned} \tag{3.39}$$

The norm of each operator in (3.39) decays faster than any polynomial order as N tends to ∞. Thus, for any integer M,

$$\begin{aligned} & \left\| \sqrt{(1-r_N)Er_N E(1-r_N)} - \sqrt{pE(1-p)Ep + q_N E(1-q_N)Eq_N} \right\| \\ & \leq C_M N^{-M}. \end{aligned} \tag{3.40}$$

We claim that there exists a constant C such that

$$Tr(\sqrt{q_N E(1-q_N)Eq_N}) + Tr(\sqrt{(1-r_N)Er_N E(1-r_N)}) \leq C \tag{3.41}$$

for any N . In fact both terms of (3.41) are finite due to Lemma 3.6 and the first term does not depend on N because of translational invariance of the trace. We consider the last term in (3.41) .

$$\begin{aligned} Tr(\sqrt{(1-r_N)Er_N E(1-r_N)}) \leq \quad & \sum_{a,b} |(\eta_a, r_N E(1-r_N)\eta_b)| \\ \leq \quad & \sum_{a,b} (|(\eta_a, (1-p)Ep\eta_b)| + |(\eta_a, (1-q_N)Eq_N\eta_b)| \\ & + 2\,|(\eta_a, pEq_N\eta_b)|) \end{aligned} \tag{3.42}$$

where $\{\eta_a = e(i) \oplus e(j)\}$ is a basis of $\mathcal{K}$ and we used Lemma 3.6 again. The last term in the right-hand side of (3.42) vanished as N tends to ∞ and other terms are independent of N (due to translational invariance again), which implies the inequality (3.41)

By (3.40) and (3.41) , we obtain

$$\left\| P_{(1-r_N)E(1-r_N)} - P_{pEp+q_N Eq_N} \right\|_{H.S.}^2 = 2Tr(A_N^2 + B_N^2) \leq C_M N^{-M}.$$

This inequality shows

$$|\varphi_E(Q\tau_j(R)) - \varphi_E(Q)\varphi_E(R)| \leq \|Q\| \, \|R\| \, C_M N^{-M}$$

for $Q \in \mathcal{A}^{CAR}_{(-\infty,0]}$ and $R \in \mathcal{A}^{CAR}_{[1,\infty)}$. Thus CLT is valid for φ_E.

Now consider quasifree states φ_S when S is strictly positive :

$$0 < \epsilon < S < 1 - \epsilon < 1.$$

We can show CLT in this case with our previous argument for pure Fock states. The only difference is the estimate of $Tr(B_N^2)$ in (3.33). In the present situation,

$$\begin{aligned} B_N \quad &= \sqrt{(1-r_N)S(1-r_N) - (1-r_N)S(1-r_N)S(1-r_N)} \\ &\quad -\sqrt{pSp - pSpSp + q_N S q_N - q_N S q_N S q_N}. \end{aligned} \tag{3.43}$$

Both $(1-r_N)S(1-r_N)$ and $pSp + q_N S q_N$ are strictly positive on the range of $1 - r_N$. When two operators A and B are strictly positive , $0 < \epsilon < A, B$, we have

$$\sqrt{A} - \sqrt{B} = \pi^{-1} \int_\epsilon^\infty \lambda^{1/2} \frac{1}{\lambda + A}(B - A)\frac{1}{\lambda + B} d\lambda \, .$$

Using this identity , we obtain

$$Tr(B_N^2) \leq \pi^{-1}(\int_\epsilon^\infty \lambda^{-3/2}) \|A - B\|_{H.S.} \tag{3.44}$$

where

$$\begin{aligned} A &= (1-r_N)S(1-r_N) - (1-r_N)S(1-r_N)S(1-r_N) \\ B &= pSp - pSpSp + q_N S q_N - q_N S q_N S q_N \, . \end{aligned}$$

The operators A and B are linear combination of pSq_N and $q_N Sp$. As before, pSq_N is of trace class and

$$Tr(|pSq_N|) \leq CN^{-M}. \tag{3.45}$$

Thus

$$Tr(B_N^2) \leq C' Tr(|pSq_N|) \leq C_M N^{-M}. \tag{3.46}$$

This estimate gives rise to CLT.
End of Proof

4 Ground States of the XY Model.

We denote the finite volume Hamiltonian of one-dimensional XY model by

$$H_{[a,b]} = -\sum_{j=a}^{b-1}\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)} + 2\lambda\sigma_z^{(j)}\}. \tag{4.1}$$

The following limit exists in norm topology of $\mathcal{A}$ and defines the time evolution α_t of the infinite system in Heisenberg picture:

$$\alpha_t(Q) = \lim_{N\to\infty} e^{itH_{[-N,N]}} Q e^{-itH_{[-N,N]}}$$

for Q in $\mathcal{A}$. The generator of the one-parameter group α_t of automorphisms is denoted by δ:

$$\delta(Q) = \frac{d}{dt}\alpha_t(Q)|_{t=0} = i[H_{[-\infty,\infty]}, Q]$$

for Q in $\mathcal{A}_{loc}$.

A state φ is a ground state for α_t if and only if

$$-i\varphi(Q^*\delta(Q)) \geq 0$$

for Q in $\mathcal{A}_{loc}$. (See 5.3.18 and subsequent sections in [8] .) The complete set of the ground states of the XY model is known.

Theorem 4.1 *(i) The ground state of the XY model is unique if* $|\lambda| \geq 1$ *or* $\gamma = 0$
(ii) If $|\lambda| < 1$, $\gamma \neq 0$ *and* $(\lambda,\gamma) \neq (0,\pm 1)$ *there exist precisely two pure ground states of the XY model .*
(iii) The spectral gap of the Hamiltonian closes if $|\lambda| = 1$ *or if* $|\lambda| < 1$ *and* $\gamma = 0$ *.*

See Theorem 1 of [5]. Note that in the Ising model , $(\lambda,\gamma) = (0,\pm 1)$, any pure ground state is a product state and CLT is trivial.

At a formal level, the Jordan-Wigner transformation maps the XY model to quasifree Hamiltonian of fermions and the ground states of the XY model is mapped into pure Fock states, however, this formal equivalence is not correct physically. For example, the ground state is always unique at the fermionic couterpart while it is not unique as we mentioned above. Another physical difference is ergodic behavior of time evolution. In [5] we have shown that the time evolution is not asymptotically abelian for the XY model. For Fermion the asymptotic abelian property is obvious.

To obtain CLT we use the method of [4]. We enlarge the algebra $\mathcal{A}$ to another algebra $\tilde{\mathcal{A}}$ adding a new selfadjoint unitary element T having the following property:

$$T^2 = 1 \ , \ T^* = T \ , : \ TQT = \Theta_-(Q) \quad \text{for } Q \text{ in } \mathcal{A} \tag{4.2}$$

where

$$\Theta_-(Q) = \lim_{N\to\infty}\left(\prod_{j=-N}^{0}\sigma_z^{(j)}\right) Q \left(\prod_{j=-N}^{0}\sigma_z^{(j)}\right). \tag{4.3}$$

$\tilde{\mathcal{A}}$ is the crossed product by the Z_2 action via Θ_-. Obviously

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}T.$$

We also introduce the automorphism Θ via the formula,

$$\Theta(Q) = \lim_{N\to\infty} \left(\prod_{j=-N}^{N} \sigma_z^{(j)} \right) Q \left(\prod_{j=-N}^{N} \sigma_z^{(j)} \right). \tag{4.4}$$

Set

$$\mathcal{A}_\pm = \{ Q \in \mathcal{A} | \Theta(Q) = \pm Q \} .$$

Then we can realize the creation and annihilation operators of fermions inside $\tilde{\mathcal{A}}$ as follows.

$$c_j^* = TS_j(\sigma_x^{(j)} + i\sigma_y^{(j)})/2 \quad , \quad c_j = TS_j(\sigma_x^{(j)} - i\sigma_y^{(j)})/2 \tag{4.5}$$

where

$$S_j = \begin{cases} \sigma_z^{(1)} \ldots \sigma_z^{(j-1)} & \text{for } j \geq 2 \ , \\ 1 & \text{for } j = 1 \ , \\ \sigma_z^{(-j)} \ldots \sigma_z^{(0)} & \text{for } j \leq -1 \ . \end{cases} \tag{4.6}$$

This gives rise to the canonical anticommutation relations (3.1) and the C^*-subalgbera generated by c_j^* and c_j is denoted by $\mathcal{A}^{CAR}$ again.

We extend Θ_- and Θ to $\tilde{\mathcal{A}}$ via the following formulae:

$$\Theta_-(T) = T \qquad \Theta(T) = T.$$

Then, it is easy to see

$$\mathcal{A}_+ = \mathcal{A}_+^{CAR} \quad , \quad \mathcal{A}_- = \mathcal{A}_-^{CAR} T.$$

Proof of Theorem 1.4.
In what follows, we assume $(\lambda, \gamma) \neq (0, \pm 1)$. As the Hamiltonian of the XY model is Θ invariant, the ground state is Θ invariant when it is unique. In fact, the Θ invariant ground state of the XY model is unique for any λ and γ and we denote it by ψ . When restricted to $\mathcal{A}_+$, ψ is a Fock state $\varphi_{E(\lambda,\gamma)}$ introduced in the previous section.

$$\psi|_{\mathcal{A}_+} = \varphi_{E(\lambda,\gamma)}|_{\mathcal{A}_+^{CAR}} \tag{4.7}$$

The Fourier transform $\tilde{E}(\lambda, \gamma)$ of the operator $E(\lambda, \gamma)$ is a multiplication operator on $\tilde{\mathcal{K}}$ and the explicit form of $\tilde{E}(\lambda, \gamma)$ can be found in [5]. We do not need the precise form of $\tilde{E}(\lambda, \gamma)$ as we only use the fact that matrix elements of $\tilde{E}(\lambda, \gamma)$ are C^∞ functions.

Consider the case when $|\lambda| > 1$. By unicity of the ground state , ψ is Θ invariant. Due to (4.7) and the results of the previous section, we obtain CLT for any local observable Q in $\mathcal{A}_+$. Thus we concentrate on $\mathcal{A}_-$

For simplicity we set $E(\lambda,\gamma)=E$ from now on. Let $\{\pi_E(\mathcal{A}^{CAR}),\mathcal{H}_E,\Omega_E\}$ be the GNS representation associated with φ_E. If Q_1T is an element of $\mathcal{A}_-$ localized in $(\infty,0]$ and Q_2T is in $\mathcal{A}_-$ localized in $[1,\infty)$($Q_1,Q_2\in\mathcal{A}_-^{CAR}$) the expectation value $\psi(Q_1T\tau_k(Q_2T))$ is written in terms of $\mathcal{A}^{CAR}$ as follows:

$$\psi(Q_1T\tau_k(Q_2T))=(\Omega_E,\pi_E(Q_1\tau_k(Q_2)S_k)\Omega_E)\,. \tag{4.8}$$

By use of Lemma 3.6 , we see that both $\theta_-E\theta_--E$ and $\theta_k^{(+)}E\theta_k^{(+)}-E$ are of trace class. Consider the unitary implementors $\Gamma(\Theta_-)$ and $\Gamma(\Theta_k^{(+)})$ for Θ_- and $\Theta_k^{(+)}$. As $Ad(S_k)$ and $\Theta_-\circ\Theta_k^{(+)}\circ\Theta$ give rise to the same involutive automorphism of $\mathcal{A}^{CAR}$ we conclude that

$$\pi_E(S_k)=\pm\Gamma(\Theta_-)\Gamma(\Theta_k^{(+)})\Gamma(\Theta) \tag{4.9}$$

due to irreducibility of Fock representation. Consider the unitaries $\hat{R}(\theta_-E\theta_-/E)$ and $\hat{R}(\theta_k^{(+)}E\theta_k^{(+)}/E)$ where $\hat{R}(E'/E)$ is the unitary defined in (9.3) in Page 424 of [2]. As $\theta_-E\theta_--E$ and $\theta_k^{(+)}E\theta_k^{(+)}-E$ are of trace class, both $\hat{R}(\theta_-E\theta_-/E)$ and $\hat{R}(\theta_k^{(+)}E\theta_k^{(+)}/E)$ belong to $\mathcal{A}^{CAR}$. Furthermore, they satisfy

$$\begin{aligned}&\hat{R}(\theta_-E\theta_-/E)\Omega_E=\Gamma(\Theta_-)\Omega_E\quad,\quad \hat{R}(\theta_k^{(+)}E\theta_k^{(+)}/E)\Omega_E=\Gamma(\Theta_k^{(+)})\Omega\\ &\Gamma(\Theta)\Omega_E=\Omega.\end{aligned} \tag{4.10}$$

On the other hand , as we know rapid uniform decay of two point correlation functions of observables localized in $(-\infty,0]$ and $[1,\infty)$ for Fock states , the following localization property of $\hat{R}(\theta_-E\theta_-/E)$ and $\hat{R}(\theta_k^{(+)}E\theta_k^{(+)}/E)$ suffices to show CLT. We set

$$R_1=\hat{R}(\theta_-E\theta_-/E)\quad,\quad R_2=\hat{R}(\theta_k^{(+)}E\theta_k^{(+)}/E). \tag{4.11}$$

Lemma 4.2 *For any positive integer M, there exist unitaries $R_1^{(n)}$ in $\mathcal{A}^{CAR}_{(-\infty,n]}$, $R_2^{(n)}$ in $\mathcal{A}^{CAR}_{[k-n,\infty)}$ and a constant C_M such that*

$$\left\|R_a-R_a^{(n)}\right\|\le C_M n^{-M}\ \textit{for}\ a=1,2\ . \tag{4.12}$$

We postpone the proof of Lemma 4.2 till the end of this section.

We sketch our proof of CLT of the XY model for the case $|\lambda|<1$, $\gamma\neq 0$ and $(\lambda,\gamma)\neq(0,\pm1)$. In this case, there exist two translationally invariant pure ground states $\psi_\pm$. As the Θ invariant ground state ψ is unique and $\psi=1/2(\psi_++\psi_-)$. In terms of Fock states of $\mathcal{A}^{CAR}$, $\psi_\pm$ are described as follows:

$$\psi_\pm(Q)=(\Omega_E,\pi_E(Q)1/2(1\pm\Gamma(\Theta_-)T)\Omega_E)\ . \tag{4.13}$$

for Q in $\mathcal{A}$.

This time, $\Gamma(\Theta_-)$ belongs to $\mathcal{A}_-^{CAR}$. Lemma 4.2 is still valid. As a consequence, we obtain CLT as before.

Before we start proof of Lemma 4.2, we prepare a few results on operators on $l_2(\mathbf{Z})$. By X we denote the position operator on $l_2(\mathbf{Z})$ determined by

$$X(f_j) = (jf_j) \qquad \text{for } f = (f_j) \text{ in } l_2(\mathbf{Z}).$$

We say that A vector $f = (f_j)$ is rapidly decreasing if $(|X|+1)^n f$ belongs to $l_2(\mathbf{Z})$ for any positive integer n.

Lemma 4.3 *Let A be a bounded operator $l_2(\mathbf{Z})$ such that on the matrix elements $A_{ij} = (e(i), Ae(j))$ satisfy*

$$\sum_{i\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} |A_{ij}|\,|j|^m < \infty \tag{4.14}$$

for any positive integer m.
(i) For any positive integer m , $(|X|+1)^m\,|A|\,(|X|+1)^m$ is of trace class.
(ii) Any eigenvector of $|A|$ is rapidly decreasing.
(iii) For any positive integer m, there exists a positive finite range operator $A_N(m)$ localized in the interval $[-N, N]$) such that

$$\||A| - A_N(m)\|_{tr} \le CN^{-m}. \tag{4.15}$$

Proof of Lemma 4.3.
Proof of (i) is similar to that of Lemma 3.6 so we do not give here. Let ξ be an eigenvector of $|A|$ with an eigenvalue ϵ, $|A|\,\xi = \epsilon\xi$. Then

$$(|X|+1)^m\,|A|\,\xi = \epsilon(|X|+1)^m\xi$$

The left-hand side is in $l_2(\mathbf{Z})$ for any m . Thus ξ is rapidly decreasing.

To prove (iii), we first recall that for any bounded operator Q

$$(|X|+1)^{-2}Q(|X|+1)^{-2}$$

is of trace class. Let q_N be the projection defined in the previous section. We set

$$B(m) = (|X|+1)^m\,|A|\,(|X|+1)^m$$

and

$$A_N(m) = (1-q_N)\,|A|\,(1-q_N).$$

As $B(m)$ is of trace class by our assumption and we have

$$\begin{aligned} & \|A_N(m) - |A|\|_{Tr} \\ = \;& \left\|(|X|+1)^{-m}(q_N B(m) q_N - q_N B(m) - B(m) q_N)(|X|+1)^{-m}\right\|_{Tr} \\ \le \;& 3\left\|(|X|+1)^{-m} q_N\right\| \|B(m)\|_{Tr}\,. \end{aligned} \tag{4.16}$$

By definition if N is positive,

$$\left\|(|X|+1)^{-m} q_N\right\| \le (N+1)^{-m}.$$

This concludes the proof of (4.15).
End of Proof of Lemma 4.3.

Finally we consider the localization property of R_1 and R_2 in (4.15). We examine R_1 only. R_2 can be treated in the same way. Recall that $R_1 = \hat{R}(\theta_- E \theta_- / E)$ has the following form. Set $\sin\phi = |\theta_- E \theta_- - E|$. The operators $\sin\phi$ and ϕ $(0 \leq \phi\ ,\ \psi \leq \pi/2\ )$ are of trace class due to Lemma 3.6. Note that $\sin^2\phi$ commutes with E and $\theta_- E \theta_-$. Consider the spectral resolution of ϕ

$$\phi = \int_0^{\pi/2} x dF(x).$$

As $\sin\phi$ is compact, the spectrum of ϕ is discrete. In particular the multiplicity of the eigenvalue $\pi/2$ is finite. Moreover the eigenspace for $\pi/2$ is characterized by one of the following conditions :

$$E\xi = \xi, \theta_- E \theta_- \xi = 0$$

or

$$E\xi = 0, \theta_- E \theta_- \xi = \xi.$$

Due to this fact , we conclude that

$$\phi(\cos\phi)^{-1}\left[E, \theta_- E \theta_-\right] = \phi(\cos\phi)^{-1}(1 - F(\{\pi/2\})\left[E, \theta_- E \theta_-\right].$$

Thus we may find a constant C such that

$$\phi(\cos\phi)^{-1}(1 - F(\{\pi/2\}) \leq C \sin\phi.$$

This inequality tells us that the operator $\phi(\cos\phi)^{-1}(1 - F(\{\pi/2\})$ satisfies the assumption of Lemma 4.3 as $\sin\phi$ satisfies the same condition. Taking into the account of these facts, we define the trace class operator $H(\theta_- E \theta_- / E)$ via the following equation:

$$H(\theta_- E \theta_- / E) = 2i\phi(\cos\phi)^{-1}\left[E, \frac{E - \theta_- E \theta_-}{|E - \theta_- E \theta_-|}\right]. \tag{4.17}$$

With this notation, R_1 is written as

$$R_1 = e^{i/2(B, H(\theta_- E \theta_- / E)B)} \prod_a B(\xi_a) \tag{4.18}$$

where ξ_a are eigenvectors $\sin\phi$ with the eigenvalue 1 and $(B, H(\theta_- E \theta_- / E)B)$ is defined in [2]. Due to Lemma 4.3 (ii) , the eigenvector ξ of $\sin\phi$ is rapidly decreasing so the Fermion operator $B(\xi)$ is well localized.

To complete our proof of Lemma 4.2 , it suffices to prove the following inequality:

$$\|(B, H(\theta_- E \theta_- / E)B) - (B, (1 - q_N)H(\theta_- E \theta_- / E)(1 - q_N)B)\| \leq C_M N^{-M}. \tag{4.19}$$

Note that $(B, (1 - q_N)H(\theta_- E \theta_- / E)(1 - q_N)B)$ is in $\mathcal{A}^{CAR}_{(-\infty, N]}$. On the other hand,

$$\begin{aligned} & \|(B, H(\theta_- E \theta_- / E)B) - (B, (1 - q_N)H(\theta_- E \theta_- / E)(1 - q_N)B)\| \\ \leq\ & \|H(\theta_- E \theta_- / E) - (1 - q_N)H(\theta_- E \theta_- / E)(1 - q_N)\|_{Tr}. \end{aligned} \tag{4.20}$$

(See (7.11) of [2].) Now we claim that

$$\|H(\theta_- E\theta_-/E) - (1-q_N)H(\theta_- E\theta_-/E)(1-q_N)\|_{Tr} \leq C_M N^{-M}. \tag{4.21}$$

By Lemma 4.3 $(|X|+1)^m \sin\phi(|X|+1)^m$ and the following operator are of trace class for any positive m:

$$(|X|+1)^m H(\theta_- E\theta_-/E)(|X|+1)^m.$$

We can carry out the computation similar to (4.16) and the left hand side of (4.21) is bounded from above by

$$3\left\|q_N(|X|+1)^{-m}\right\| \|(|X|+1)^m H(\theta_- E\theta_-/E)(|X|+1)^m\|_{Tr}$$

, which shows the inequality (4.21) .
End of Proof.

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