SELF-AVOIDING POLYGONS: SHARP ASYMPTOTICS OF CANONICAL PARTITION FUNCTIONS UNDER THE FIXED AREA CONSTRAINT

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Abstract. We study the self-avoiding polygons (SAP) connecting the vertical and the horizontal semi-axes of the positive quadrant of $\mathbb{Z}^2$. For a fixed $\beta > 0$, assign to each such polygon ω the weight $\exp\{-\beta|\omega|\}$, $|\omega|$ denoting the length of ω, and consider the sum $\mathcal{Z}_{Q,+}$ of these weights for all SAP enclosing area $Q > 0$. We study the statistical properties of such SAP and, in particular, derive the exact asymptotics for the partition function $\mathcal{Z}_{Q,+}$ as $Q \to \infty$. The results are valid for any $\beta > \beta_c$, β_c being the critical value for the 2D self-avoiding walks.

1. Introduction

There are several ways to describe the probabilistic picture behind the phenomenon of phase segregation. On the macroscopic level, the co-existing phases can be labelled by local averages on various intermediate spatial scales and, accordingly, phase segregation can be formulated in terms of concentration properties of renormalised piece-wise constant random phase labels. This is essentially the approach of the $\mathbb{L}_1$ -theory, which has been initiated in the works [1, 5, 11, 15, 44] and, more recently, has been accomplished in [16, 6, 17, 7] (see also [8, 18, 9] for adjustments and generalisations).

The $\mathbb{L}_1$-theory is robust in the sense that it makes a clear distinction between (hard) model oriented renormalisation estimates needed to construct phase labels which would possess appropriate decoupling properties and (relatively soft) general tools which are used to derive probabilistic estimates on these phase labels once they are constructed. It has been successfully applied to a wide class of finite range ferromagnetic models on $\mathbb{Z}^d$, super-critical independent and FK percolation models, and ferromagnetic models with Kac potentials. It can be readily employed to describe phase segregation (in the above $\mathbb{L}_1$ sense) in the Widom-Rowlison model of continuum particles and, apparently, it is possible [10] to adjust the theory to some models without the underlying FKG structure, such as low temperature Pirogov-Sinai models or continuum models with Kac potentials [36].

The shortcoming of the $\mathbb{L}_1$-theory is the intrinsic loss of the information on the microscopic level. In particular, in this framework the interfaces between different co-existing phases are defined only in the $\mathbb{L}_1$ sense, and the theory provides only a very little insight into their fluctuation structure.

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A very different point of view on the problems of phase segregation has been suggested in the ground-breaking monograph [24] (see also [39, 40] for earlier works and [2, 23, 35, 41, 42, 45, 46] for important later developments). The DKS theory focuses on the local limit type description of observables directly on the microscopic level. The prominent role in this approach is played by an accurate analysis of the statistical properties of random interfaces. In fact, such an analysis is pivotal for the DKS theory in the sense that the sharpness of all other estimates (e.g., local limit description of the magnetisation in large finite volumes) directly hinges on it.

Random phase boundary is, arguably, the central object in the probabilistic theory of phase segregation. When apply, the results and methods of the DKS approach certainly give much more than just a refinement of limit theorems as provided by the $\mathbb{L}_1$ theory. Apart from giving a scaled description of the interface *per se* (see, e.g. [22, 23]) and apart from providing explicit estimates in terms of sizes of finite systems under consideration, they suggest a probabilistic approach to a variety of interesting and delicate issues such as metastability and fine properties of stochastic dynamics in the phase co-existence regime [38, 46], wetting transition in the vicinity of the wetting point [32] and analytic properties of the free energy and the surface tension [45, 13].

At least for finite range models in two dimensions, random interfaces are expected to exhibit the intrinsic Brownian statistical behaviour based upon a rapid decay of correlations. In particular, the local position of the interface is believed to have the Gaussian scaling, whereas the global description of the fluctuating random phase boundaries is supposed to rely upon the appropriate versions of the conditional invariance principle (e.g., under the Dobrushin type boundary conditions or under the fixed magnetisation constraint).

So far rigorous results in this direction have been obtained mainly in the context of the nearest neighbour very low temperature Ising model [27, 28, 21, 23, 30, 31]. Technically all these works are based upon the low temperature cluster expansions and employ non-probabilistic computations, such as various estimates on generating functions of distributions with small negative weights. Furthermore, the low temperature assumption was frequently used to discount the entropy beyond the intrinsic probabilistic structure of the model.

The main objective of our work here is to develop a "model" purely probabilistic analysis of microscopic random interfaces in two dimensions. We consider the simplest non-trivial ensemble of microscopic phase boundaries, namely the ensemble of self-avoiding $\mathbb{Z}^2$-lattice polygons γ with the associated weights $e^{-\beta|\gamma|}$, $\beta > \beta_c$, where β_c is the critical point for the self-avoiding planar walks (see (1.3) below). This collection of polygons corresponds to the ensemble of phase boundaries of the Ising model in the corner $\mathbb{Z}^2_+$ with free boundary conditions [19]. We obtain the expansion up to zero order terms of the canonical (with fixed area inside the polygons) partition function. As a byproduct of these considerations (and a very weak version of our main result) one can easily deduce the asymptotic concentration property of the appropriately rescaled polygons on the quarter of the related Wulff shape. Of course, enclosed area of self-avoiding polygons is an interesting topic in its own right, see e.g., [26] or [14] for a more physical-style treatment.

The model is described in Sect. 1.1 and the result is stated in Sect. 1.2. As we mention in Remark 1.1.1 below the techniques we develop lead to the sharp local limit description of fluctuations of random interfaces (polygons) around the limiting quarter Wulff shape.

1.1. **The model.** Consider the family $\mathcal{Z}_+$ of all self-avoiding $\mathbb{Z}^2$-lattice paths $\omega = (\omega(0), ..., \omega(n))$ which cross the positive quadrant

$$\mathsf{Quad}_+ = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x > 0 \text{ and } y > 0 \right\}.$$

More precisely, let us define positive lattice semi-axes $\mathcal{P}_0^+$ and $\mathcal{L}_0^+$ as

$$\mathcal{P}_0^+ = \left\{ (0,k) \in \mathbb{Z}^2 \,\middle|\, k > 0 \right\} \quad \text{and} \quad \mathcal{L}_0^+ = \left\{ (k,0) \in \mathbb{Z}^2 \,\middle|\, k > 0 \right\}.$$

Then we say that $\omega \in \mathcal{Z}_+$ iff

$$\omega(0) \in \mathcal{P}_0^+, \ \omega(n) \in \mathcal{L}_0^+ \quad \text{and} \quad \omega(l) \in \mathsf{Quad}_+ \cap \mathbb{Z}^2 \quad \forall l = 1, .., n-1. \tag{1.1}$$

Notice that in the above definition the length $|\omega| = n$ of a path ω was not assumed to be fixed. Each path $\omega \in \mathcal{Z}_+$ splits Quad_+ into two components — the bounded one (containing the origin) and the unbounded one. Let $\mathbf{A}_+(\omega)$ denote the area of the bounded component.

Definition 1.1. Given $Q \in \mathbb{N}$ define the canonical set of paths $\mathcal{Z}_{Q,+}$ via

$$\mathcal{Z}_{Q,+} = \left\{ \omega \in \mathcal{Z}_+ \,\middle|\, \mathbf{A}_+(\omega) = Q \right\}. \tag{1.2}$$

Given $\beta > 0$ we associate with any self-avoiding path ω the weight $e^{-\beta|\omega|}$. It is known [37] that there exists the critical value $\beta_c \in (0, \infty)$, such that

$$\beta > \beta_c \iff \sum_{x \in \mathbb{Z}^2} g_\beta(x) \overset{\mathsf{def}}{=} \sum_{x \in \mathbb{Z}^2} \sum_{\omega : 0 \to x} e^{-\beta|\omega|} < \infty, \tag{1.3}$$

where the last sum above is over all lattice paths with endpoints at 0 and x.

For every $\beta > \beta_c$ our basic canonical partition function is given by

$$\mathbb{Z}_{\beta,+}(Q) = \sum_{\omega \in \mathcal{Z}_{Q,+}} e^{-\beta|\omega|}. \tag{1.4}$$

1.2. **The result.** The $Q \to \infty$ asymptotic properties of $\mathbb{Z}_{\beta,+}(Q)$ and, accordingly, the limit properties of the induced probability distribution

$$\mathbb{P}^\beta_{Q,+}(\omega) = \frac{e^{-\beta|\omega|}}{\mathbb{Z}_{\beta,+}} \mathbb{I}_{\left\{ \mathcal{Z}_{Q,+} \right\}}(\omega) \tag{1.5}$$

on $\mathcal{Z}^\beta_{Q,+}$ are closely related to the asymptotic properties of the connectivity function g_β, which have been studied in detail in [33]: for every $\beta > \beta_c$ the following Ornstein-Zernike type formula holds:

$$g_\beta(x) = \frac{\Psi_\beta(\vec{\mathfrak{n}}(x))}{\sqrt{\|x\|}} e^{-\tau_\beta(x)} \left(1 + o(1)\right), \tag{1.6}$$

where Ψ_β is a positive analytic function on the circle $\mathbb{S}^1$, the vector $\vec{\mathfrak{n}}(x) = x/\|x\|$, and τ_β is an analytic equivalent norm (called the inverse correlation length in the physical literature), which is uniformly strictly convex in the sense that

$$\min_{\xi \in \mathbb{S}^1} \mathsf{det}\left[\mathsf{Hess}\, \tau_\beta(\xi)\right] > 0. \tag{1.7}$$

In two dimensions τ_β can be viewed as the surface tension of the model, with the corresponding Wulff shape given by

$$\mathbf{K}_\beta = \bigcap_{n \in \mathbb{S}^1} \left\{ x \in \mathbb{R}^2 \,\middle|\, (x,n) \leq \tau_\beta(n) \right\}.$$

The positive stiffness condition (1.7) implies that the curvature of $\partial\mathbf{K}_\beta$ is uniformly strictly positive. It will be convenient to reformulate (1.7) in the form of the following strict triangle inequality ([34], [43]): For every $\beta > \beta_c$ there exists a positive constant $\delta = \delta(\beta)$, such that for every two points $x, y \in \mathbb{R}^2$,

$$\tau_\beta(x) + \tau_\beta(y) - \tau_\beta(x+y) \geq \delta\left(\|x\| + \|y\| - \|x+y\|\right). \tag{1.8}$$

In fact δ could be chosen as the minimal radius of curvature of the Wulff shape $\mathbf{K}_\beta$. Of course, the Wulff shape $\mathbf{K}_\beta$ inherits all the lattice symmetries of $\mathbb{Z}^2$. Define now the quarter shape $\mathbf{K}_{\beta,+}$ as

$$\mathbf{K}_{\beta,+} = \mathbf{K}_\beta \bigcap \overline{\mathsf{Quad}_+}.$$

It is relatively easy to show that $\mathbb{P}^\beta_{Q,+}$ concentrates, as $Q \to \infty$, near the curve

$$j^Q_{\beta,+} \overset{\mathsf{def}}{=} \sqrt{\frac{Q}{q_{\beta,+}}}\, j_{\beta,+},$$

where $j_{\beta,+}$ is the boundary of the quarter Wulff shape, $j_{\beta,+} = \partial\mathbf{K}_{\beta,+} \bigcap \mathsf{Quad}_+$, and $q_{\beta,+} \overset{\mathsf{def}}{=} \mathrm{Area}(\mathbf{K}_{\beta,+})$. Define the total surface energy of $j^Q_{\beta,+}$ as

$$\mathcal{W}_\beta\left(j^Q_{\beta,+}\right) = \int_{j^Q_{\beta,+}} \tau_\beta\left(n(s)\right) ds = \sqrt{\frac{Q}{q_{\beta,+}}}\, w_{\beta,+},$$

where s is the natural parameter along $j^Q_{\beta,+}$, $n(s)$ is the normal to $j^Q_{\beta,+}$ at s, and $w_{\beta,+} \overset{\mathsf{def}}{=} \mathcal{W}_\beta(j_{\beta,+})$.

Theorem 1. *For every* $\beta > \beta_c$

$$\mathbb{Z}_{\beta,+}(Q) = \frac{c_\beta}{\sqrt{2Q}} \exp\left\{-\sqrt{\frac{Q}{q_{\beta,+}}} w_{\beta,+}\right\}\left(1 + o\left(1\right)\right). \tag{1.9}$$

At the end of Sect. 2.4 we shall give an explicit formula for the absolute constant c_β.

Remark 1.1.1. When combined with the ideas from [22, 23], our approach makes possible to obtain a Gaussian limit description for the scaled by $1/\sqrt[4]{Q}$ (i.e., by the square root of the typical linear size of the related droplet) fluctuations of the self-avoiding polygons around the limiting (quarter of the Wulff) shape. The rigorous derivation of such a theorem would lead however to further technical complications and we shall not do this here.

1.3. The notation.

Let us start with the

Lattice path notation. The lines and semi-lines are defined as

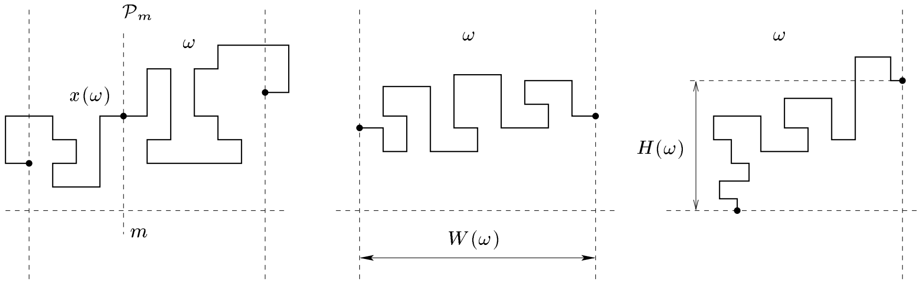
$$\mathcal{P}_n = \left\{(n,k) \in \mathbb{Z}^2 \;\middle|\; k \in \mathbb{Z}\right\} \quad \text{and} \quad \mathcal{L}_n = \left\{(k,n) \in \mathbb{Z}^2 \;\middle|\; k \in \mathbb{Z}\right\},$$

and, respectively,

$$\mathcal{P}^+_n = \left\{(n,k) \in \mathcal{P}_n \;\middle|\; k > 0\right\} \quad \text{and} \quad \mathcal{L}^+_n = \left\{(k,n) \in \mathcal{L}_n \;\middle|\; k > 0\right\};$$

the semi-lines $\mathcal{P}^-_n$ and $\mathcal{L}^-_n$ are defined analogously. The vertical strip $S_{[m,n]}$ is defined as

$$S_{[m,n]} = \left\{(k,l) \in \mathbb{Z}^2 \;\middle|\; m < k \leq n\right\}. \tag{1.10}$$



a) b) c)

FIGURE 1.1. a) $\mathcal{P}_m$ is a break line, $x(\omega)$ is a break point b) Bridge: $W(\omega)$-width of ω c) Corner path: $H(\omega)$-height of ω

We shall always orient $\mathbb{Z}^2$-lattice paths $\omega = \big(\omega(0), \omega(1), ..., \omega(n)\big)$, $\omega(j) \in \mathbb{Z}^2$, in such a way that the horizontal coordinates $\omega_1(0)$ and $\omega_2(n)$ of $\omega(0)$ and $\omega(n)$ satisfy $\omega_1(0) \leq \omega_1(n)$. A line $\mathcal{P}_m$ is called a *break line* of ω if $\omega_1(0) < m < \omega_1(n)$ and ω intersects $\mathcal{P}_m$ at exactly one point, $\# \{\omega \cap \mathcal{P}_m\} = 1$. In the latter case the point $x = \omega \cap \mathcal{P}_m$ is called a *break point* of ω (Fig. 1.1 a)).

We say that a path ω is a *bridge* if the inequality $\omega_1(0) < \omega_1(l) < \omega_1(n)$ holds for any $l = 1, ..., n-1$ (Fig. 1.1 b)). Given a bridge ω we define its span $\mathsf{Span}(\omega)$, its width $W(\omega)$, and its height $H(\omega)$ via

$$\mathsf{Span}(\omega) = \big(\omega_1(0), \omega_1(n)\big), \quad W(\omega) = \omega_1(n) - \omega_1(0), \quad H(\omega) = \omega_2(n) - \omega_2(0)$$

respectively.

Notice that given a bridge ω every break line $\mathcal{P}_m$ induces the splitting of ω into the union of two bridges with disjoint spans and the common vertex at the corresponding break point (Fig. 1.2 a)). In general, given two bridges ω and ω', we shall say that ω' is an *embedded* bridge of ω if $\omega' \subset \omega$ and each of the two end-points of ω' is either a break point of ω or else coincides with the corresponding end-point of ω. We shall write $\omega' \subseteq^e \omega$ if ω' is an embedded bridge of ω.

Any bridge ω which has no break points is said to be *irreducible*. In the sequel we shall use the letter γ for the irreducible bridges.

In a similar fashion we say that ω is a corner path if $\omega_2(l) > \omega_2(0)$ and $\omega_1(l) < \omega_1(n)$ for every $l = 1, ..., n-1$. The span, width, and height of corner paths are defined exactly as in the bridge case. Any break line of a corner path splits it into the union of a corner path and a bridge (Fig. 1.2 b)).

Ensembles of paths and curves. We use the following basic microscopic ensembles of $\mathbb{Z}^2$-lattice paths:

$\mathcal{Z}_+$	is the ensemble of all crossing paths of Quad_+ already defined in (1.1).
$\mathcal{Z}_{Q,+}$	contains all the paths from $\mathcal{Z}_+^0$ which chop out area Q from Quad_+, see (1.2).
$\mathcal{B}_n$	is the set of all bridges ω of the width $W(\omega) = n$ starting at the point $\omega_1(0) = (0,0)$.
$\mathcal{F}_n$	is the set of all the irreducible bridges γ from $\mathcal{B}_n$.
$\mathcal{A}_n$	is the ensemble of all corner paths with the end-points at $(-n, 0)$ and on the positive semi-line $\mathcal{P}_0^+$.

The macroscopic counterparts of the paths from $\mathcal{Z}_+$ and $\mathcal{Z}_{Q,+}$ are:

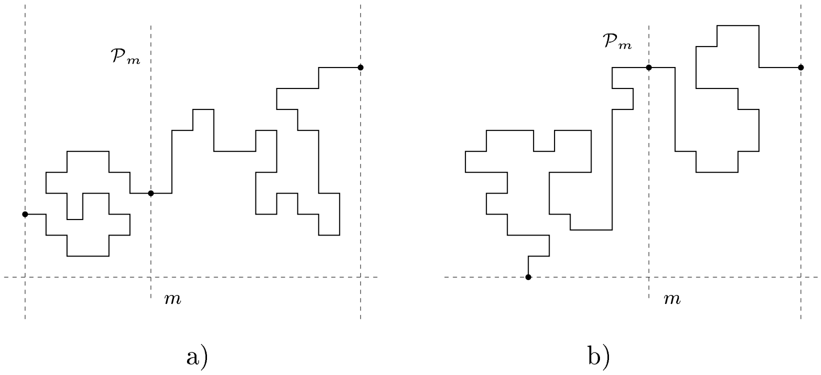


FIGURE 1.2. Splitting of a) Bridge b) Corner path by a break line $\mathcal{P}_m$

$\mathcal{C}_+$	the family of all Jordan curves lying inside Quad_+ with the end-points on the positive vertical and horizontal coordinate semi-axes.
$\mathcal{C}_{q,+}$	the sub-family of those curves from $\mathcal{C}_+$, which chop out area q from Quad_+.

Partition functions. The partition functions associated with the ensembles $\mathcal{Z}_+$, $\mathcal{B}_n$, $\mathcal{F}_n$, and $\mathcal{A}_n$ are denoted as $\mathbb{Z}_{\beta,+}$, $\mathbb{B}_n$, $\mathbb{F}_n$ and $\mathbb{A}_n$ respectively. Given a subset $\mathcal{E}$ of $\mathcal{Z}_+$ (respectively of $\mathcal{B}_n$, $\mathcal{F}_n$, $\mathcal{A}_n$) and a function f on $\mathcal{E}$ we use $\mathbb{Z}_{\beta,+}(f;\mathcal{E})$ (respectively $\mathbb{B}_n(f;\mathcal{E})$, $\mathbb{F}_n(f;\mathcal{E})$, $\mathbb{A}_n(f;\mathcal{E})$) to denote the following sum:

$$\sum_{\omega\in\mathcal{E}} f(\omega)e^{-\beta|\omega|}.$$

In the case of the bridge and corner partition functions a special role will be played by the tilts of the form $f(\omega)=e^{tH(\omega)}$. We define

$$\mathbb{B}_n(t) \overset{\mathsf{def}}{=} \mathbb{B}_n\left(e^{tH(\omega)}\right), \quad \mathbb{F}_n(t) \overset{\mathsf{def}}{=} \mathbb{F}_n\left(e^{tH(\omega)}\right), \quad \mathbb{A}_n(t) \overset{\mathsf{def}}{=} \mathbb{A}_n\left(e^{tH(\omega)}\right). \tag{1.11}$$

For the special collection of bridges of the form $\mathcal{B}^{\mathrm{u}_1,\mathrm{u}_2,\dots}_{n,\mathrm{l}_1,\mathrm{l}_2,\dots}$, where the lower (respectively upper) symbolic indices $\mathrm{l}_1,\mathrm{l}_2,\dots$ (respectively $\mathrm{u}_1,\mathrm{u}_2,\dots$) are intended to label the nature of the corresponding ensembles (e.g the set $\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ of tube bridges defined in Sect. 3.1.2) we shall use the abbreviations of the form $\mathbb{B}^{\mathrm{u}_1,\mathrm{u}_2,\dots}_{n,\mathrm{l}_1,\mathrm{l}_2,\dots}$ to denote the associated partition functions, e.g.,

$$\mathbb{B}^{\mathrm{u}_1,\mathrm{u}_2,\dots}_{n,\mathrm{l}_1,\mathrm{l}_2,\dots}(f) \overset{\mathsf{def}}{=} \mathbb{B}_n\left(f;\mathcal{B}^{\mathrm{u}_1,\mathrm{u}_2,\dots}_{n,\mathrm{l}_1,\mathrm{l}_2,\dots}\right).$$

Finally, for various ensembles of paths we shall define the appropriate notion of area. Accordingly, we shall use the abbreviations

$$\mathbb{B}^{\mathrm{u}_1,\mathrm{u}_2,\dots}_{n,\mathrm{l}_1,\mathrm{l}_2,\dots}(f;Q)$$

to denote the partition function restricted to the set of paths of area Q. There is a slight notational overlap with the tilted partition functions introduced in (1.11), but since the area symbols will always contain either Q or q, there should be no confusion.

1.4. **Skeleton calculus and the tube condition.** With every path ω from $\mathcal{Z}_+$ (respectively from $\mathcal{B}_n$ or from $\mathcal{A}_n$) and with every number $s > 0$ we associate a skeleton $\mathfrak{S}$ in the following way:

Definition 1.2. Let us say that a collection $\mathfrak{S} = \{y_0, ..., y_r\} \subset \mathbb{Z}^2$ is a skeleton of a path $\omega = (\omega(0), ..., \omega(m))$ iff

(i) $y_l \in \omega$ for $l = 0, ..., r$, and $y_0 = \omega(0)$, $y_r = \omega(m)$.

(ii) $s \leq \|y_{l+1} - y_l\| \leq 3s$ for $l = 0, ..., r-1$.

(iii) $\mathsf{d}_{\mathsf{Hausd}}\left(\mathbf{P}\left(\mathfrak{S}\right), \omega\right) \leq s$, where $\mathbf{P}\left(\mathfrak{S}\right)$ is the polygonal line through the vertices of $\mathfrak{S}$, and $\mathsf{d}_{\mathsf{Hausd}}\left(\cdot\right)$ is the Hausdorff distance.

If $\mathfrak{S}$ is an s-skeleton of ω, we write $\omega \overset{s}{\sim} \mathfrak{S}$, and say that ω is s-compatible with $\mathfrak{S}$. Given an s-skeleton $\mathfrak{S}$ let $\mathcal{Z}_{\mathfrak{S},+}$ be the family of all paths $\omega \in \mathcal{Z}_+$, which are s-compatible with $\mathfrak{S}$. Set

$$\mathbb{Z}_{\beta,+}\left(\mathfrak{S}\right) = \mathbb{Z}_{\beta,+}\left(\mathcal{Z}_{\mathfrak{S},+}\right).$$

It is easy to see that

$$\log \mathbb{Z}_{\beta,+}(\mathfrak{S}) \leq \sum_{l=0}^{r-1} \log g_\beta(y_{l+1} - y_l) \leq -\sum_{l=0}^{r-1} \tau_\beta(y_{l+1} - y_l) \overset{\mathsf{def}}{=} -\mathcal{W}_\beta\left(\mathbf{P}(\mathfrak{S})\right). \tag{1.12}$$

As in [24] or in [35] the energy estimate (1.12), stability properties of the Wulff functional $\mathcal{W}_\beta$, and the positivity of the stiffness (1.8) of τ_β imply that for any $\nu \in (1/2, 1]$ there exists a constant $c_\beta(\nu) > 0$, such that

$$\begin{aligned} &\mathbb{Z}_{\beta,+}\left(\mathcal{Z}_{Q,+}; \mathsf{d}_{\mathsf{Hausd}}\left(\omega, j^Q_{\beta,+}\right) > Q^{\nu/2}\right) \\ &\qquad \leq \exp\left\{-\sqrt{\frac{Q}{q_{\beta,+}}} w_\beta - c_\beta(\nu) Q^{\frac{2\nu-1}{2}}\right\} \end{aligned} \tag{1.13}$$

In other words, only the paths ω staying inside the $Q^{\nu/2}$-tube around $\mathcal{Z}_{Q,+}$ could contribute to $\mathbb{Z}_{\beta,+}(Q)$ on the level asserted in (1.9).

Definition 1.3. The set of paths satisfying the tube condition (see Fig. 1.3 a))

$$\mathcal{Z}^{\mathsf{tube}}_{Q,+} = \left\{\omega \in \mathcal{Z}_{Q,+} \;\middle|\; \mathsf{d}_{\mathsf{Hausd}}\left(\omega, j^Q_{\beta,+}\right) \leq Q^{\nu/2}\right\}. \tag{1.14}$$

Remark. The corresponding sets of tube paths will be also defined in the bridge and corner ensembles in the corresponding sections. The scaling between Q and n one should keep in mind is, of course, $n \sim \sqrt{Q}$.

1.5. **Strategy of the proof.** We use the following basic splitting of the path $\omega \in \mathcal{Z}^{\mathsf{tube}}_{Q,+}$ (see Fig. 1.3 b)):

Given $Q \in \mathbb{N}$ define $M = M(Q)$ to be the integer value of the mid-point of the horizontal projection of $j^Q_{\beta,+}$, $M(Q) \sim \sqrt{Q}$. For every path $\omega \in \mathcal{Z}^{\mathsf{tube}}_{Q,+}$ let us define $l = l(\omega)$ via

$$M + l = \min\left\{m > M \;\middle|\; \mathcal{P}_m \text{ is a break line of } \omega\right\}. \tag{1.15}$$

Our considerations in Sect. 3 (see, in particular, Remark 3.4.1) show that the contribution of the paths with long irreducible components sitting on the pole $\mathcal{P}_M$ is asymtotically negligible:

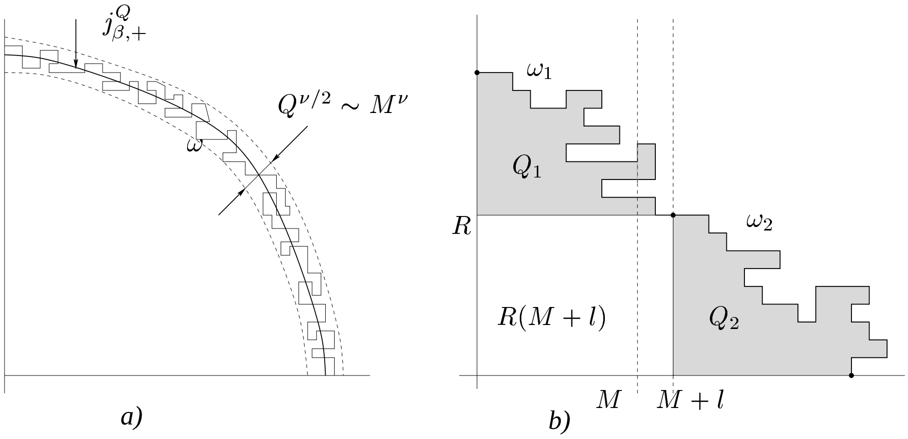


FIGURE 1.3. a) Tube path ω b) Basic splitting $\omega = \omega_1 \cup \omega_2$ of a path from $\mathcal{Z}_{Q,+}$

Lemma 1.4. *For every $\varepsilon > 0$ there exists $c_1 > 0$, such that*

$$\mathbb{Z}_{\beta,+}\big(\mathcal{Z}^{\mathsf{tube}}_{Q,+}\ ;\ l(\omega) \geq M^\varepsilon\big) \leq \exp\left\{-\sqrt{\frac{Q}{q_{\beta,+}}}w_{\beta,+} - c_1 M^\varepsilon\right\}. \tag{1.16}$$

Thus, we can restrict attention only to bridges $\omega \in \mathcal{Z}^{\mathsf{tube}}_{Q,+}$ satisfying the condition $l(\omega) \leq M^\varepsilon$. In the latter case the corresponding break point $x(\omega) = \big(M + l(\omega), R\big)$, $R = R(\omega)$, splits the bridge ω into two pieces, $\omega = \omega_1 \cup \omega_2$, as shown in Fig. 1.3 b). Notice that ω_2 is a corner path, $\omega_2 \in \mathcal{A}_R$, whereas ω_1 is a bridge from $\mathcal{B}_{M+l}$ which, in addition, has no break lines in the interval $[M, M+l]$. Let us use $\mathcal{B}_{M,l}$ to denote the set of all such bridges and $\mathbb{B}_{M,l}$ to denote the corresponding partition function. We obtain:

$$\begin{aligned}\mathbb{Z}_{\beta,+}(Q) = &\sum_{|R-M|\leq M^\nu}\ \sum_{l \leq M^\varepsilon}\ \sum_{Q_1+Q_2+R(M+l)=Q} \mathbb{B}_{M,l}(Q_1)\,\mathbb{A}_R(Q_2)\\ &+ O\Big(\exp\Big\{-\sqrt{\frac{Q}{q_{\beta,+}}}w_{\beta,+} - c_2 M^\nu - c_3 M^\varepsilon\Big\}\Big).\end{aligned} \tag{1.17}$$

This is our basic decomposition of the canonical partition function $\mathbb{Z}_{\beta,+}(Q)$. The target asymptotic formula (1.9) follows from the local limit results on the bridge and corner canonical partition functions (see (1.32) and (1.34) below), which enable an asymptotically sharp re-summation in (1.17), as it is carried in Sect. 2.4.

1.6. **Conjugate quantities and the parametrised representation.** Define $m_\beta(\cdot)$ to be the convex conjugate of $\tau_\beta(1,\cdot)$; thus, for any $|t| < \tau_\beta(1,0)$ or, equivalently, for any $t \in \mathcal{D}_\beta$,

$$\tau_\beta\big(1, m'_\beta(t)\big) + m_\beta(t) = t m'_\beta(t).$$

The function $-m_\beta$ can be thought as the mass of the tilted bridge grand-canonical partition function. In the sequel we shall frequently rely on the results from [33], which we, for the sake of convenience, collect in the properties **P1**-**P5** below:

P1: The interior of the effective domain of m_β is

$$\mathcal{D}_\beta \overset{\mathsf{def}}{=} \operatorname{int}\left\{t \mid m_\beta(t) < \infty\right\} = \left(-\tau_\beta(1,0), \tau_\beta(1,0)\right). \tag{1.18}$$

P2: m_β is analytic on $\mathcal{D}_\beta$.

P3: m_β is strictly convex; moreover, for every $[a,b] \subset \mathcal{D}_\beta$

$$\min_{t\in[a,b]} m''_\beta(t) > 0. \tag{1.19}$$

P4: For every closed interval $[a,b] \subset \mathcal{D}_\beta$,

$$\mathbb{B}_n(t) = \mu(t) e^{n m_\beta(t)} \left(1 + o\left(1\right)\right),$$

uniformly in $t \in [a,b]$. The coefficient μ above is given by:

$$\mu(t) = \Big\{\sum_k k\mathbb{F}_k(t) e^{-k m_\beta(t)}\Big\}^{-1} \tag{1.20}$$

is also an analytic function on $\mathcal{D}_\beta$. Furthermore, there exists a continuous positive function $d_\beta(\cdot)$ on $\mathcal{D}_\beta$, such that, uniformly in n and $t \in [a,b]$,

$$\mathbb{F}_n(t) \le e^{-n d_\beta(t)} \mathbb{B}_n(t). \tag{1.21}$$

P5: The map

$$t \mapsto (t, -m_\beta(t)), \qquad t \in \mathcal{D}_\beta,$$

gives a parametrised representation the boundary $\partial\mathbf{K}_\beta \cap \left\{(x,y) \mid y > 0\right\}$.

1.7. **Separation of masses.** The formula (1.21) is an instance of the separation of the decay rates type phenomenon, which lies in the heart of the Ornstein-Zernike theory. In particular, the corresponding decay estimates are ubiquitous in the local limit approach we pursue in this work. Let us formulate a general point-wise statement of this sort, which will be repeatedly referred to in the sequel:

Proposition 1.5. *Fix two numbers $r, \delta > 0$, and consider the lattice cone*

$$\mathcal{C}_r = \left\{(x_1, x_2) \mid |x_2| \le r x_1\right\}. \tag{1.22}$$

Then there exist positive constants $c_1 = c_1(r,\delta)$ and $c_2 = c_2(r,\delta)$, such that uniformly in $x \in \mathcal{C}_r$ and in all sub-intervals $[a,b] \subset [0, x_1]$ of the length $|b-a| \ge \delta x_1$ one has the following inequality

$$g_\beta\left(x;\ \omega \text{ has no break lines on } [a,b]\right) \le c_1 e^{-c_2 x_1} g_\beta(x); \tag{1.23}$$

here, given a family $\mathcal{E}$ of self-avoiding paths ω we define:

$$g_\beta\left(x; \mathcal{E}\right) = \sum_{\substack{\omega : 0 \to x \\ \omega \in \mathcal{E}}} e^{-\beta|\omega|}.$$

In the case of the self-avoiding walks the estimate (1.23) has been established in [33] following the earlier work of [20]. The intrinsic renormalisation proof of the mass-gap type statements has been recently developed in a more complicated context of the nearest neighbour Bernoulli bond percolation in [12] and in the context of high temperature Ising models in [13].

The inequalities (1.21) and (1.23) imply the following relation between the bridge and full connectivity functions (c.f. [20] and [33]): Given $x \in \mathbb{Z}^2$ (with $x_1 \geq 0$) define $h_\beta(x)$ and $f_\beta(x)$ as

$$h_\beta(x) = \sum_{\substack{\omega:0\to x \\ \omega-\text{ bridge}}} e^{-\beta|\omega|} \quad \text{and} \quad f_\beta(x) = \sum_{\substack{\gamma:0\to x \\ \gamma-\text{ irreducible bridge}}} e^{-\beta|\gamma|}.$$

Then for every number $r > 0$ fixed, there exist positive constants $c_3 = c_3(x)$ and $c_4 = c_4(x)$, such that uniformly in $x \in \mathcal{C}_r$

$$f_\beta(x) \leq e^{-c_3 x_1} h_\beta(x) \qquad \text{and} \qquad g_\beta(x) \leq c_4 h_\beta(x). \tag{1.24}$$

1.8. **Bridge and corner partition functions.** The leading asymptotics of the bridge and corner partition functions $\mathbb{B}_{M,l}$ and $\mathbb{A}_R$, which show up in the basic decomposition formula (1.17), are related to the following variational problem: Find

$$\psi(q) \overset{\mathsf{def}}{=} \min\Big\{ \int_0^1 \tau_\beta(1, u'(\xi))\, d\xi \mid u(0) = 0, \int_0^1 u(\xi)\, d\xi = q \Big\}. \tag{1.25}$$

As it will be explained in Sect. 2.1, the unique minimiser u_q of (1.25) is, in the range of areas we are working with, given by

$$u_q'(\xi) = m_\beta'\big((1-\xi)t\big), \tag{1.26}$$

where $t = t(q)$ can be recovered from the relation

$$\int_0^1 u_q(\xi)\, d\xi = \int_0^1 (1-\xi) m_\beta'\big((1-\xi)t\big)\, d\xi \overset{\mathsf{def}}{=} q(t) = q. \tag{1.27}$$

By properties **P2** and **P3** the function $q(t)$ is analytic on $\mathcal{D}_\beta$ and, moreover, has the analytic inverse $t = t(q)$ on the latter domain.

Notice that by the convex duality between $m_\beta(\cdot)$ and $\tau_\beta(1, \cdot)$ the minimal value $\psi(q)$ can be expressed in terms of $t = t(q)$ as

$$\begin{aligned} \psi(q) &= \int_0^1 \tau_\beta\left(1, m_\beta'[(1-\xi)t]\right) d\xi \\ &= \int_0^1 (1-\xi) t m_\beta'[(1-\xi)t]\, d\xi \; - \; \int_0^1 m_\beta[(1-\xi)t]\, d\xi \\ &= tq \; - \; \int_0^1 m_\beta[(1-\xi)t]\, d\xi. \end{aligned} \tag{1.28}$$

For future references we introduce

$$\sigma(t) = \frac{d}{dt} q(t) = \int_0^1 (1-\xi)^2 m_\beta''\big((1-\xi)t\big)\, d\xi \tag{1.29}$$

and observe that, due to Property **P3**, this derivative is non-degenerate:

$$^\forall[a,b] \subset \mathcal{D}_\beta, \qquad \min_{t\in[a,b]} \sigma(t) > 0.$$

As a result, differentiating $\psi\big(q(t)\big)$ w.r.t. t, we get

$$\psi'(q) = t \tag{1.30}$$

and, integrating by parts, rewrite (1.28) as

$$\psi(q) = 2tq - m_\beta(t) = 2q\psi'(q) - m_\beta(t) \tag{1.31}$$

provided t and q are related via (1.27).

We are in a position now to formulate two crucial propositions, which set up he stage for the asymptotic re-summation in (1.17):

Proposition 1.6. *As $M \to \infty$, the asymptotics of the bridge partition function $\mathbb{B}_{M,l}(Q_1)$ is given by*

$$\mathbb{B}_{M,l}(Q_1) = \sqrt{\frac{\mu(t)\mu(0)}{2\pi M^3 \sigma(t)}}\, e^{-(M+l)\psi(q_1)} \mu_l(t)\big(1 + o(1)\big), \tag{1.32}$$

where $q_1 = Q_1/(M+l)^2$, $t = t(q_1)$, and

$$\mu_l(t) = \sum_{l' \geq l} e^{-l' m_\beta(t)} \mathbb{F}_{l'}(t). \tag{1.33}$$

Proposition 1.7. *As $R \to \infty$, the asymptotics of the corner partition function $\mathbb{A}_R(Q_2)$ is given by*

$$\mathbb{A}_R(Q_2) = \sqrt{\frac{\kappa(s)\mu(0)}{2\pi R^3 \sigma(s)}}\, e^{-R\psi(q_2)}\big(1 + o(1)\big), \tag{1.34}$$

where $q_2 = Q_2/R^2$, $s = s(q_2)$, and the positive real analytic function $\kappa(\cdot)$ is given by

$$\kappa(t) = \lim_{n \to \infty} e^{-n m_\beta(t)} \mathbb{A}_n(t). \tag{1.35}$$

We shall prove the above two statements in Sect. 3 and Sect. 4 respectively.

2. Variational Problems

In this section, assuming validity of Propositions 1.6 and 1.7, we derive sharp estimates on the variational sum $(M+l)\psi(q_1) + R\psi(q_2)$ that enters the basic decomposition formula (1.17) through the relations (1.32) and (1.34). The eventual computation leads to the main assertion (1.9) of Theorem 1, and it will be performed in the concluding Sect. 2.4. The techniques employed in the preparatory Sect. 2.1 and Sect. 2.2 are rather standard, and we shall merely sketch the corresponding arguments without going into the (otherwise routine) complete details.

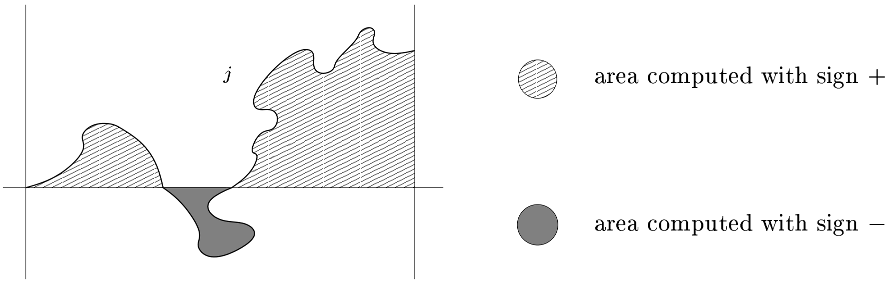


FIGURE 2.4. A curve $j \in \mathcal{J}_a$ and its signed area

2.1. **The parametrised problem.** We use $\mathcal{J}_a$ to denote the class of all Jordan curves j lying inside the strip $\{(x,y) \mid 0 \le x \le a\}$ having the endpoints at the origin and on the vertical axis passing through $(a,0)$. With every curve $j \in \mathcal{J}_a$ we associate the energy

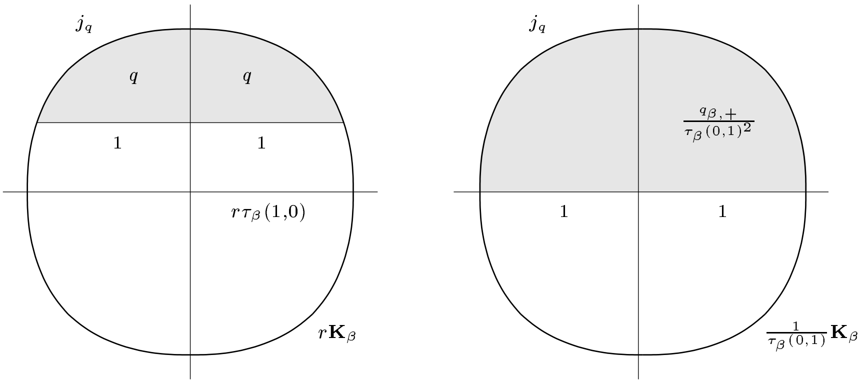
$$\mathcal{W}_\beta\,(j) = \int_j \tau_\beta\,(n(s))\;ds,$$

and the signed area $\mathsf{Area}\,(j)$ (see Fig. 2.4).

Consider the following (parametrised) variational problem:

$$\min\Big\{\mathcal{W}_\beta\,(j) \;\Big|\; j \in \mathcal{J}_1;\ \mathsf{Area}\,(j) = q \ge 0\Big\} \tag{2.1}$$

The first observation is that for the values of q satisfying $0 < q \le q_{\beta,+}/\tau_\beta(1,0)^2$ the solution to (2.1) coincides with the portion of the appropriately dilated Wulff shape $r\partial\mathbf{K}_\beta$. Indeed, just choose r to ensure that the area above the width-2 upper cross section of $r\mathbf{K}_\beta$ equals to $2q$ (Fig. 2.5 a)).



a) b)

FIGURE 2.5. The optimal curve j_q

Then the symmetry of $\mathbf{K}_\beta$ and the elementary surgery considerations imply that the quarter curve j_q (Fig. 2.5 a)) solves (2.1). By the monotonicity of areas this argument can be pushed all the way up to $q = q_{\beta,+}/\tau_\beta(1,0)^2$ which, of course, corresponds to $r = 1/\tau_\beta(1,0)$ (Fig. 2.5 b)). Thus, for the values of q satisfying $0 < q \le q_{\beta,+}/\tau_\beta(1,0)^2$ the solutions to the problems (2.1) and (1.25) coincide; in other words, j_q can be represented as the graph of u_q. Now, as we have already mentioned before, by the results of [33], the map

$$t \mapsto \big(t, -m_\beta(t)\big), \qquad t \in [-\tau_\beta(1,0), \tau_\beta(1,0)]$$

gives the parametrisation of the part of the boundary $\partial\mathbf{K}_\beta$ in the upper half plane. Consequently, for $q \le q_{\beta,+}/\tau_\beta(1,0)^2$, the solution u_q of (1.25) satisfies (1.26). Thus, for $q \le q_{\beta,+}/\tau_\beta(1,0)^2$, the minimal value both in (1.25) and (2.1) is given by

$$\mathcal{W}_\beta\,(j_q) = \int_0^1 \tau_\beta(1, u_q'(\xi))\,d\xi = \int_0^1 \tau_\beta\big(1, m_\beta'\big((1-\xi)t\big)\,d\xi,$$

where the function $t = t(q)$ has been defined in (1.27). Putting things other way around, we conclude that for every $t \in [0, \tau_\beta(1,0))$ the minimal value in the problem (2.1) at $q = q(t)$ equals to

$$e(t) \overset{\mathsf{def}}{=} \int_0^1 \tau_\beta\big(1, m'_\beta\big((1-\xi)t\big)\big)\, d\xi. \tag{2.2}$$

Elementary scaling considerations reveal:

Lemma 2.1. *For every $a \in \mathbb{R}_+$ and $t \in [0, \tau_\beta(1,0))$*

$$\min\Big\{\mathcal{W}_\beta\,(j) \mid j \in \mathcal{J}_a;\ \mathsf{Area}\,(j) = a^2 q(t)\Big\} = a e(t). \tag{2.3}$$

2.2. **The point constraint.** The variational problem we discuss in this section corresponds to the basic path decomposition (1.17).

Let (x_0, x_0) be the mid-point of the boundary $j_{\beta,+}$ of the quarter Wulff shape:

$$(x_0, x_0) = j_{\beta,+} \cap \big\{(x,y) \mid x = y\big\}.$$

We fix a small $\delta > 0$, and, for every point (x,y) satisfying

$$(x - x_0)^2 + (y - y_0)^2 \ \le\ \delta, \tag{2.4}$$

consider the sub-class

$$\mathcal{C}_{x,y} \overset{\mathsf{def}}{=} \Big\{ j \in \mathcal{C}_+ \mid j \cap \mathcal{P}_x = (x,y) \Big\}$$

of curves from $\mathcal{C}_+$, where $\mathcal{P}_x$ is the vertical axis passing through (x,y).

The normalised variational problem with the (x,y)-point constraint is

$$\min\Big\{\mathcal{W}_\beta\,(j) \mid j \in \mathcal{C}_{x,y} \cap \mathcal{C}_{q_\beta,+}\Big\}. \tag{2.5}$$

By the construction, each curve $j \in \mathcal{C}_{x,y} \cap \mathcal{C}_{q_\beta,+}$ splits into the union $j = j_1 \cup j_2$, where $j_1 \in \mathcal{J}_x$, $j_2 \in \mathcal{J}_y$, and $q_1 + q_2 + xy = q_{\beta,+}$, where $q_i \overset{\mathsf{def}}{=} \mathsf{Area}\,(j_i)$; $i = 1,2$ (Fig. 2.6).

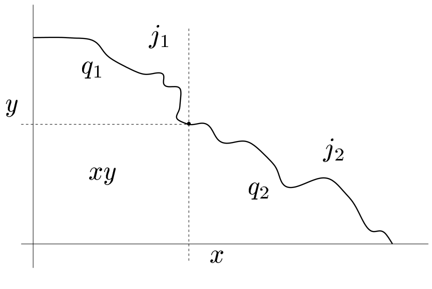


FIGURE 2.6. Splitting of a path $j \in \mathcal{C}_{x,y}$

Consequently, the minimisation problem (2.5) can be rewritten as

$$\min_{q_1+q_2=q_{\beta,+}-xy} \min\Big\{\mathcal{W}_\beta\,(j_1) + \mathcal{W}_\beta\,(j_2) \mid j_1 \in \mathcal{J}_x,\ j_2 \in \mathcal{J}_y,\ \mathsf{Area}\,(j_i) = q_i\Big\}. \tag{2.6}$$

If δ in (2.4) is small enough, we can, without loss of generality, focus only on the areas (q_1, q_2) satisfying the conditions

$$q_1 \le x^2 q_{\beta,+}/\tau_\beta(1,0)^2, \qquad q_2 \le y^2 q_{\beta,+}/\tau_\beta(1,0)^2$$

In the latter case, in view of the relations (1.27) and (2.2), the variational problem (2.6) is equivalent to the following two-dimensional constraint minimisation problem:

$$\min\Bigl\{xe(t)+ye(s)\bigm| x^2q(t)+y^2q(s)=q_{\beta,+}-xy\Bigr\}. \tag{2.7}$$

Let $\hat{\psi}(x,y)$ denote the value of the above minimum, $\hat{t}(x,y)$ and $\hat{s}(x,y)$ — the corresponding minimisers.

Lemma 2.2. *There exists $\delta>0$, such that $\hat{\psi}$, $\hat{t}$ and $\hat{s}$ are analytic on the δ-neighbourhood (2.4) of (x_0,y_0). Moreover, for every (x,y) satisfying (2.4), the functions $\hat{t}(x,y)$ and $\hat{s}(x,y)$ verify the following transversality condition:*

$$\frac{\hat{t}(x,y)}{x}=\frac{\hat{s}(x,y)}{y}. \tag{2.8}$$

Proof of Lemma 2.2. Applying the method of Lagrange multipliers and using the duality relation (recall that $m_\beta(\cdot)$ is the convex conjugate of $\tau_\beta(1,\cdot)$), $\tau'_\beta(1,m'_\beta(t))=t$, we obtain:

$$\begin{cases}\bigl(\hat{t}x-\lambda x^2\bigr)\displaystyle\int_0^1(1-\xi)^2m''_\beta\bigl((1-\xi)\hat{t}\bigr)\,d\xi=0,\\ \bigl(\hat{s}y-\lambda y^2\bigr)\displaystyle\int_0^1(1-\xi)^2m''_\beta\bigl((1-\xi)\hat{s}\bigr)\,d\xi=0,\\ x^2q(\hat{t})+y^2q(\hat{s})+xy-q_{\beta,+}=0.\end{cases} \tag{2.9}$$

Due to the strict convexity of m_β (1.19), the integrals above do not vanish; as a result, the system simplifies to

$$\begin{bmatrix}\hat{t}-\lambda x\\ \hat{s}-\lambda y\\ x^2q(\hat{t})+y^2q(\hat{s})+xy-q_{\beta,+}\end{bmatrix}\stackrel{\mathsf{def}}{=}F(\hat{t},\hat{s},\lambda,x,y)=\mathbf{0}. \tag{2.10}$$

Recalling now the definitions (1.27) and (1.29), we see that the matrix

$$\frac{\partial F(\hat{t},\hat{s},\lambda,x,y)}{\partial(\hat{t},\hat{s},\lambda)}=\begin{bmatrix}1&0&-x\\0&1&-y\\x^2\sigma(\hat{t})&y^2\sigma(\hat{s})&0\end{bmatrix}$$

is non-degenerate. By analyticity of m_β this implies analyticity of the vector $(\hat{t},\hat{s},\lambda)(x,y)$ in a neighbourhood of (x_0,x_0). The analyticity of $\hat{\psi}$ follows then from the relation $\hat{\psi}(x,y)=xe(\hat{t})+ye(\hat{s})$. Finally, notice that $t_0\stackrel{\mathsf{def}}{=}\hat{t}(x_0,x_0)=\hat{s}(x_0,x_0)$ satisfies $m'_\beta(t_0)=1$. Indeed, $-m'_\beta(t)$ is just the slope of the boundary $\partial\mathbf{K}_{\beta,+}$ at the point $\bigl(t,-m_\beta(t)\bigr)$ and $\partial\mathbf{K}_{\beta,+}$ is symmetric with respect to the diagonal. In particular, $t_0\neq 0$ and, consequently $\hat{t},\hat{s}\neq 0$ in a neighbourhood of (x_0,x_0). Thereby, the transversality condition (2.8) follows from the first two equations in (2.10). □

2.3. **Quadratic expansion of $\hat{\psi}$ around** (x_0,x_0)**.** At every point (x,y) lying on the optimal curve $j_{\beta,+}=\partial\mathbf{K}_{\beta,+}\cap\mathsf{Quad}_+$, at (x_0,x_0) in particular, we have

$$\hat{\psi}(x,y)=\min_{(u,v)}\hat{\psi}(u,v).$$

Consequently $\nabla\hat{\psi} \equiv 0$ on $j_{\beta,+}$. Furthermore, using the symmetry of $j_{\beta,+}$ with respect to the diagonal $\{(x,y) \mid x = y\}$, we infer that

$$\frac{\partial^2\hat{\psi}}{\partial x^2}\Big|_{(x_0,x_0)} = \frac{\partial^2\hat{\psi}}{\partial y^2}\Big|_{(x_0,x_0)}, \quad \text{and} \quad \big(\mathsf{Hess}\,(\hat{\psi})v, v\big)\Big|_{(x_0,x_0)} = 0.$$

with $\mathbf{v} = (1,-1)$. As a result,

$$\mathsf{Hess}\,(\hat{\psi})\Big|_{(x_0,x_0)} = \begin{bmatrix} \theta & \theta \\ \theta & \theta \end{bmatrix}.$$

To compute the number θ above, set $\hat{\psi}(x) = \hat{\psi}(x,x)$ and $\hat{t} \equiv \hat{t}_x = \hat{t}(x,x)$. Of course,

$$\frac{d^2\hat{\psi}(x)}{dx^2}\Big|_{x_0} = 4\theta. \tag{2.11}$$

Also, by symmetry,

$$\hat{\psi}(x) = 2xe(\hat{t}_x) = 2x\int_0^1 \tau_\beta\left(1, m'_\beta((1-\xi)\hat{t}_x)\right)\, d\xi, \tag{2.12}$$

whereas $\hat{t}_x$ complies with the following area constraint:

$$x^2\big(2q(\hat{t}_x)+1\big) = 2x^2\int_0^1 (1-\xi)m'_\beta\left((1-\xi)\hat{t}_x\right)\, d\xi + x^2 = q_{\beta,+}. \tag{2.13}$$

Differentiating (2.12) and (2.13) w.r.t. x and using the notations from (1.27)–(1.29), we obtain

$$\frac{d\hat{t}_x}{dx} = -\frac{2q(\hat{t}_x)+1}{x\sigma(\hat{t}_x)}, \qquad \frac{d\hat{\psi}(x)}{dx} = 2\Big(e(\hat{t}_x) - \big(2q(\hat{t}_x)+1\big)\,\hat{t}_x\Big). \tag{2.14}$$

Next, using (1.31), we rewrite

$$e(\hat{t}_x) = \int_0^1 \tau_\beta\left(1, m'_\beta((1-\xi)\hat{t}_x)\right)\, d\xi = 2\hat{t}_x q(\hat{t}_x) - m_\beta\left(\hat{t}_x\right) \tag{2.15}$$

thus simplifying the last equality to

$$\frac{d\hat{\psi}(x)}{dx} = -2\left(m_\beta\left(\hat{t}_x\right) + \hat{t}_x\right).$$

As a result,

$$\frac{d^2\hat{\psi}(x)}{dx^2}\Big|_{x_0} = -2\frac{d\hat{t}_x}{dx}\Big|_{x_0}\left(1 + m'_\beta(t_0)\right),$$

where, as before, $t_0 = \hat{t}_{x_0} = \hat{t}(x_0,x_0)$. As we have already seen above, $m'_\beta(t_0) = 1$. Finally, the relations (2.11), (2.13), and (2.14) together imply

$$\theta = \frac{1}{4}\frac{d^2\hat{\psi}(x)}{dx^2}\Big|_{x_0} = \frac{q_{\beta,+}}{x_0^3\sigma(t_0)}. \tag{2.16}$$

We record the fruits of the above calculation as

Lemma 2.3. *The second order expansion of the analytic function* $\hat{\psi}$ *around* (x_0, x_0) *is given by*

$$\hat{\psi}(x,y) = \hat{\psi}(x_0,x_0) + \frac{1}{2}\frac{q_{\beta,+}}{x_0^3\sigma(t_0)}\big((x-x_0)+(y-y_0)\big)^2 + o\big((x-x_0)^2+(y-y_0)^2\big)\,. \tag{2.17}$$

2.4. **Proof of Theorem 1.** In the notation of Sect. 2.2 the scaling between the microscopic quantities associated with a path $\omega \in \mathcal{Z}_{Q,+}^{\mathsf{tube}}$ (Fig. 1.3) and the macroscopic quantities associated with the optimal quarter shape curve $j_{\beta,+} = \partial\mathbf{K}_{\beta,+} \cap \mathsf{Quad}_+$ is given by (recall that M is the *integer* part of the middle point of the horizontal projection of $j_{\beta,+}^{Q}$)

$$\left|\sqrt{\frac{Q}{q_{\beta,+}}} - \frac{M}{x_0}\right| \le \frac{1}{x_0}, \tag{2.18}$$

where, as in Sect. 2.2, (x_0, x_0) is the middle point of $j_{\beta,+}$.

The tube condition (1.14) and the estimate (1.16) enable to restrict the range of summation in the right hand side of (1.17) to,

$$l \le c_1 M^{\varepsilon}, \quad |R - M| \le M^{\nu}, \quad \text{and} \quad \big|Q_i - M^2 q(t_0)\big| \le c_3 Q^{\frac{1+\nu}{2}}, \tag{2.19}$$

with any fixed values of $\varepsilon > 0$ and $\nu \in (1/2, 1)$. In what follows we shall choose ε and ν in such a way that

$$\nu + \varepsilon < 1 \quad \text{and} \quad \nu < \frac{2}{3}. \tag{2.20}$$

In the asymptotic expressions (1.32) and (1.34) the areas q_1 and q_2 scale as

$$q_1 = \frac{Q_1}{(M+l)^2} \quad \text{and} \quad q_2 = \frac{Q_2}{R^2}.$$

Therefore, by the last of the relations (2.19), $|q_i - q(t_0)| \le c_1 Q^{(\nu-1)/2} \le c_2 M^{\nu-1}$, and, consequently,

$$t(q_1) = t_0 + O\left(M^{\nu-1}\right) \quad \text{and} \quad s(q_2) = t_0 + O\left(M^{\nu-1}\right), \tag{2.21}$$

where $t = t(q_1)$ and $s = s(q_2)$ are recovered from $q(t) = q_1$ and $q(s) = q_2$ (see (1.27)) respectively.

Let us fix l and R in a compliance with the (2.19) and sum over all admissible Q_1 and Q_2 (the third sum in (1.17)). By the scaling relation (2.18), the microscopic break point with coordinates $(M + l, R)$ corresponds, in terms of the variational problem (2.5), to the macroscopic point (x, y) with the coordinates (see Fig. 1.3)

$$x = \frac{M+l}{M}x_0 \quad \text{and} \quad y = \frac{R}{M}x_0,$$

optimal slopes $\hat{t} = \hat{t}(x,y)$, $\hat{s} = \hat{s}(x,y)$, and the areas

$$\hat{Q}_1 = (M+l)^2 q(\hat{t}) \quad \text{and} \quad \hat{Q}_2 = R^2 q(\hat{s}).$$

Notice, that the transversality condition (2.8) reads in the above notation as

$$\frac{\hat{t}}{M+l} = \frac{\hat{s}}{R}. \tag{2.22}$$

By (2.19), the range of admissible areas Q_1, Q_2 (with R, l fixed) is included in

$$\{Q_1, Q_2 \mid |Q_i - \hat{Q}_i| < c_4 M^{1+\nu}\}.$$

Thus, using the relations (1.32), (1.34), and (2.21) as well as the analytic properties of the functions μ, κ, σ and μ_l, we rewrite the third sum in (1.17) as:

$$\frac{\mu(0)\mu_l(t_0)\sqrt{\mu(t_0)\kappa(t_0)}}{2\pi\sigma(t_0)\sqrt{M^3R^3}} \sum_{\substack{|Q_i-\hat{Q}_i|\le c_4 M^{1+\nu} \\ Q_1+Q_2=\hat{Q}_1+\hat{Q}_2}} e^{-(M+l)\psi(q_1)-R\psi(q_2)}\left(1+o\left(1\right)\right). \tag{2.23}$$

Now (recall the definition of $e(t)$ in (2.2)),

$$\begin{aligned}(M+l)\psi(q_1)+R\psi(q_2)-\frac{M}{x_0}\hat{\psi}(x,y) &= (M+l)e(t)+Re(s)-\frac{M}{x_0}\hat{\psi}(x,y) \\ &= (M+l)\left(e(t)-e(\hat{t})\right)+R\left(e(\hat{s})-e(s)\right).\end{aligned} \tag{2.24}$$

Next, using the convex duality between $\tau_\beta(1,\cdot)$ and $m_\beta(\cdot)$, relations (2.21)–(2.20), (1.27)–(1.29), and positivity of $\sigma(\cdot)$, we obtain

$$\begin{aligned}e(t)-e(\hat{t}) =& \hat{t}\int_0^1 (1-\xi)\left(m'_\beta((1-\xi)t)-m'_\beta((1-\xi)\hat{t})\right)d\xi \\ &+\frac{1}{2}\int_0^1 \frac{\left(m'_\beta((1-\xi)t)-m'_\beta((1-\xi)\hat{t})\right)^2}{m''_\beta((1-\xi)\hat{t})}d\xi+o\left((t-\hat{t})^2\right) \\ =& \hat{t}\left(q(t)-q(\hat{t})\right)+\frac{(q(t)-q(\hat{t}))^2}{2\sigma(\hat{t})}+o\left((t-\hat{t})^2\right).\end{aligned}$$

Now, the scaling relation for the area $q(\cdot)$ together with the transversality condition (2.22) and the identity $Q_1+Q_2=\hat{Q}_1+\hat{Q}_2$ imply

$$(M+l)\hat{t}\left(q(t)-q(\hat{t})\right)+R\hat{s}\left(q(s)-q(\hat{s})\right)=0.$$

As a result, the RHS of (2.24) reads, due to relations (2.19), (2.21) and the choice of ν in (2.20),

$$\begin{aligned}&(M+l)\frac{(q(t)-q(\hat{t}))^2}{2\sigma(\hat{t})}+R\frac{(q(s)-q(\hat{s}))^2}{2\sigma(\hat{s})}+o\left(1\right) \\ &\quad=\frac{(Q_1-\hat{Q}_1)^2}{2\sigma(\hat{t})(M+l)^3}+\frac{(Q_2-\hat{Q}_2)^2}{2\sigma(\hat{s})R^3}+o\left(1\right)=\frac{(Q_2-\hat{Q}_2)^2}{\sigma(t_0)R^3}+o\left(1\right).\end{aligned}$$

Thus, by the Gaussian summation formula, (2.23) reduces to

$$\frac{\mu(0)\mu_l(t_0)\sqrt{\mu(t_0)\kappa(t_0)}}{2\sqrt{\pi M^3\sigma(t_0)}}\exp\left\{-\frac{M}{x_0}\hat{\psi}\left(\frac{M+l}{M}x_0,\frac{R}{M}x_0\right)\right\}\left(1+o\left(1\right)\right). \tag{2.25}$$

It remains to sum up the expressions in (2.25) over all admissible (that is, satisfying (2.19)) values of R and l. At this stage we shall rely on the quadratic expansion of $\hat{\psi}$ around $\hat{\psi}(x_0,x_0)$, which has been developed in Lemma 2.3. We rewrite the formula (2.17) in terms of the $(M+l,R)$ coordinates as:

$$\begin{aligned}\frac{M}{x_0}\hat{\psi}\left(\frac{M+l}{M}x_0,\frac{R}{M}x_0\right)-\frac{M}{x_0}\hat{\psi}(x_0,x_0) &= \frac{\theta}{2}Mx_0\left(\frac{l}{M}+\frac{R-M}{M}\right)^2+o\left(1\right) \\ &= \frac{\theta x_0}{2M}\left(R-M\right)^2+o\left(1\right),\end{aligned} \tag{2.26}$$

where the term $o\left(1\right)$ above appears, in both cases, due to the choice of ε and ν in (2.20).

Substituting (2.25) and (2.26) into the basic decomposition formula (1.17), and using the relations (to get rid of the second sum in (1.17))

$$\mu_l(t) = \mu_l(t_0)(1 + o\,(1)), \qquad \sum_l \mu_l(t_0) = \frac{1}{\mu(t_0)},$$

we compute:
(2.27)

$$\mu(0)\sqrt{\frac{\kappa(t_0)}{\mu(t_0)}}\frac{\exp\left\{-\sqrt{\frac{Q}{q_{\beta,+}}}w_{\beta,+}\right\}}{2\sqrt{\pi M^3\sigma(t_0)}}\sum_{|R-M|\le c_5 M^\nu}\exp\left\{-\frac{\theta x_0}{2M}(R-M)^2\right\}(1+o\,(1))$$
$$= \frac{\mu(0)}{\sqrt{2\theta x_0 M^2 \sigma(t_0)}}\,(1+o\,(1))\sqrt{\frac{\kappa(t_0)}{\mu(t_0)}}\exp\left\{-\sqrt{\frac{Q}{q_{\beta,+}}}w_{\beta,+}\right\}.$$

Here, by (2.16) and the scaling relation (2.18),

$$\theta x_0 M^2\sigma(t_0) = \frac{q_{\beta,+}}{x_0^3\sigma(t_0)}x_0 M^2\sigma(t_0) = q_{\beta,+}\left(\frac{M}{x_0}\right)^2 = Q.$$

Finally, using the relation $\kappa(t_0) = \mu(t_0)$, verified in Theorem 4.2 of Sect. 4, we deduce the claim of Theorem 1 with (cf. (1.20))

$$c_\beta = \mu(0) \equiv \Big\{\sum_k k\mathbb{F}_k(0)e^{-km_\beta(0)}\Big\}^{-1},$$

the inverse of mean length of "horizontal" irreducible bridges (i.e., with vanishing tilt t). □

3. ASYMPTOTICS OF BRIDGE PARTITION FUNCTIONS

In this section we develop our basic local limit techniques and use them to prove Proposition 1.6.

3.1. **The notation and the results.**

3.1.1. *Asymptotics of* $\mathbb{B}_n\left(n^2q_n\right)$. Given a bridge $\omega = \big(\omega(0), ..., \omega(m)\big)$, the centred signed area $a(\omega)$ is defined as the algebraic sum of the components bounded by the trajectory of ω and the segment connecting the end-points $\omega(0)$ and $\omega(m)$ (Fig. 3.7).

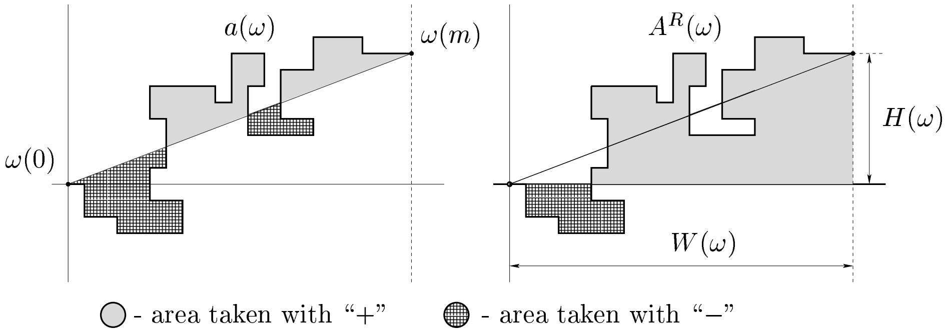


FIGURE 3.7. Signed areas $a(\omega)$ and $A^R(\omega)$ associated with a bridge ω

The signed area $A^R(\omega)$ under the path ω is called the *real area*, see Fig. 3.7. Of course,

$$A^R(\omega) = \frac{W(\omega)H(\omega)}{2} + a(\omega).$$

Notice that if the origin is placed at the end-point $\omega(0)$ and the bridge ω has no overhangs, then $A^R(\omega)$ coincides with the value of the integral of the piecewise constant function corresponding to the path ω.

Given $n \in \mathbb{N}$ and an area Q, it will be convenient to introduce the corresponding rescaled area $q_n = Q/n^2$. The main local limit result of this section asserts:

Theorem 3.1. *Let $\delta > 0$ be fixed. Then, uniformly in the rescaled areas q_n satisfying (see Sect. 2.1)*

$$q_n \le (1-\delta)\,\frac{q_{\beta,+}}{\tau_\beta(0,1)^2}, \tag{3.1}$$

the asymptotic behaviour of the canonical bridge partition function is given by:

$$\mathbb{B}_n\left(n^2 q_n\right) = \sqrt{\frac{\mu(0)\mu(t_n)}{2\pi n^3 \sigma(t_n)}}\, e^{-n\psi(q_n)}\left(1 + o\left(1\right)\right), \tag{3.2}$$

where $t_n = t(q_n)$ and the functions ψ and σ have been defined in Sect. 1.8.

3.1.2. *Tube trajectories.* Recall that we use j_q to denote the unique solution to the variational problem (2.1). For every $n \in \mathbb{N}$ and $\nu \in (1/2, 1]$, we define the following set of the tube bridges

$$\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n} \overset{\mathsf{def}}{=} \left\{\omega \in \mathcal{B}_n \;\middle|\; \mathsf{d}_{\mathsf{Hausd}}\left(\omega, nj_{q_n}\right) \le n^\nu\right\}. \tag{3.3}$$

As in (1.13), the skeleton calculus readily implies that, uniformly in q_n satisfying (3.1),

$$\mathbb{B}_n\left(A^R(\omega) = n^2 q_n;\ \omega \notin \mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}\right) \le \exp\left\{-n\psi(q_n) - c_1 n^{2\nu-1}\right\}, \tag{3.4}$$

where $c_1 = c_1(\nu) > 0$. Comparing (3.4) with (3.2) we see that only the tube trajectories can contribute to $\mathbb{B}_n(n^2 q_n)$ on the level asserted in the latter expression. In fact, the principal contribution to $\mathbb{B}_n(n^2 q_n)$ comes from an even more refined set of bridges, the so called regular bridges, which we shall eventually describe at the end of this subsection.

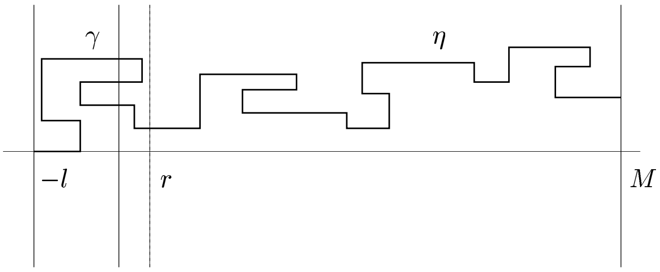


FIGURE 3.8. Bridge $\gamma \vee \eta$ from $\mathcal{B}_{M,l}$: $\gamma \in \mathcal{F}_{l+r}$ and $\eta \in \mathcal{B}_{M-r}$

3.1.3. *Consequence: Asymptotics of* $\mathbb{B}_{M,l}(Q)$. Assuming the claim of Theorem 3.1, it is a short step to deduce Proposition 1.6. Recall that by its very definition (see Sect. 1.5), the partition function $\mathbb{B}_{M,l}(\cdot)$ corresponds to those bridges from $\mathcal{B}_{M+l}$ whose first irreducible sub-bridge γ has width $W(\gamma) > l$ (see Fig. 3.8). We start by rewriting $\mathbb{B}_{M,l}(Q)$ as

$$\frac{\mathbb{B}_{M,l}(Q)}{\mathbb{B}_{M+l}(Q)} = \sum_r \sum_{H,\Delta} \mathbb{F}_{r+l}\left(H,\Delta\right) \frac{\mathbb{B}_{M-r}\big(Q-(M-r)H-\Delta\big)}{\mathbb{B}_{M+l}\left(Q\right)}, \tag{3.5}$$

where $\mathbb{F}_{r+l}\left(H,\Delta\right)$ denotes the restriction of the irreducible partition function to the paths of height H and area $A^R(\gamma) = \Delta$.

By the tube estimate (3.4) we can restrict the summation in (3.5) to the range $|H| \le M^\nu$ and $|\Delta| \le (l+r)M^\nu$. In addition, we shall consider non-negative l and r satisfying $l + r \le M^\varepsilon$ with some small $\varepsilon > 0$, the assumption to be justified in Sect. 3.6 below (see also Proposition 3.4). Set

$$q = Q/(M+l)^2 \qquad \text{and} \qquad \tilde q = \frac{Q-(M-r)H-\Delta}{(M-r)^2}.$$

Applying Theorem 3.1 in the region of values of $r+l$, H, and Δ described above, we obtain

$$\log \frac{\mathbb{B}_{M-r}\big(Q-(M-r)H-\Delta\big)}{\mathbb{B}_{M+l}\left(Q\right)} = -(M-r)\psi\left(\tilde q\right) + (M+l)\psi(q) + o\left(1\right).$$

On the other hand, a direct computation in the same region gives

$$\begin{aligned}(M+l)\psi\left(q\right) - (M-r)\psi\left(\tilde q\right) &= (r+l)\psi(q) - (M-r)\left(\psi\left(\tilde q\right) - \psi\left(q\right)\right)\\ &= H\psi'(q) + (r+l)\big(\psi(q) - 2q\psi'(q)\big) + o\left(1\right).\end{aligned}$$

In view of the duality relations (1.30)–(1.31) the latter reduces to

$$tH - (r+l)m_\beta(t) + o\left(1\right),$$

with $t = t(q)$ being the convex conjugate quantity to q. As a result, uniformly in non-negative l and r satisfying $l + r \le M^\varepsilon$, we get

$$\sum_{H,\Delta} \mathbb{F}_{r+l}\left(H,\Delta\right) \frac{\mathbb{B}_{M-r}\big(Q-(M-r)H-\Delta\big)}{\mathbb{B}_{M+l}\left(Q\right)} = e^{-(r+l)m_\beta(t)}\mathbb{F}_{r+l}(t)\big(1+o\left(1\right)\big),$$

and (1.32) follows immediately from (3.2) and (3.5).

3.1.4. *Regular bridges.* Fix some $\alpha \in (0, 1/2)$ and, without loss of the generality, assume that both n^α and $K = K_n \equiv n^{1-\alpha}$ are integers. n^α is our basic mesoscopic scale. The points from

$$\big\{n^\alpha, 2n^\alpha, \dots, (K-1)n^\alpha\big\} \tag{3.6}$$

generate the splitting of the strip $S_{[0,n]}$ into the mesoscopic components $S_j \equiv S_{\Delta_j}$,

$$\Delta_j \equiv \big((j-1)n^\alpha, jn^\alpha\big], \qquad j \in \big\{1, 2, \dots, K\big\}. \tag{3.7}$$

Given a bridge $\omega \in \mathcal{B}_n$ and an index $j = 1, \dots, K-1$, let us define γ_j to be the minimal embedded bridge of ω whose projection on the horizontal axis contains jn^α. If $\mathcal{P}_{jn^\alpha}$ happens to be a break line of ω, we set $\gamma_j = \varnothing$. Of course, by the very definition, any $\gamma_j \ne \varnothing$ is irreducible. In any case we, somewhat emotionally, shall refer to γ_j as to the irreducible animal sitting on the pole $\mathcal{P}_{jn^\alpha}$. Note that in general some γ_j can be arbitrarily large, spreading over several mesoscopic strips S_{Δ_l}, though such an event will be rendered asymptotically improbable in Proposition 3.4 below.

To avoid such atypical cases, we fix $\varepsilon > 0$ and introduce a collection of "nice" bridges by requiring for each irreducible animal γ_j, $j = 1, \dots, K-1$, to satisfy $W(\gamma_j) < n^{\varepsilon}$. In the sequel we shall always pick ε sufficiently small such that, in particular, $\varepsilon < \alpha$ (see (3.10) below for the precise range of parameters α, ε we are working with). Notice that in the latter case the condition $W(\gamma_j) < n^{\varepsilon}$ for all j implies that the spans of different $\gamma_\bullet$ don't intersect and as a result we obtain the following mesoscopic decomposition (Fig. 3.9):

$$\omega = \omega_1 \vee \gamma_1 \vee \omega_2 \vee \dots \vee \gamma_{K-1} \vee \omega_K, \tag{3.8}$$

where $K = n^{1-\alpha}$, the components $\gamma_1, \dots, \gamma_{K-1}$ are the disjoint irreducible animals sitting on the poles $\mathcal{P}_{n^\alpha}, \dots, \mathcal{P}_{(K-1)n^\alpha}$ respectively, and $\omega_1, \dots, \omega_K$ are the (mesoscopic) embedded bridges. It is useful to rewrite (see Fig. 3.9) the span of ω_j as

$$\mathsf{Span}(\omega_j) = \left((j-1)n^\alpha + R_{j-1}, jn^\alpha - L_j\right). \tag{3.9}$$

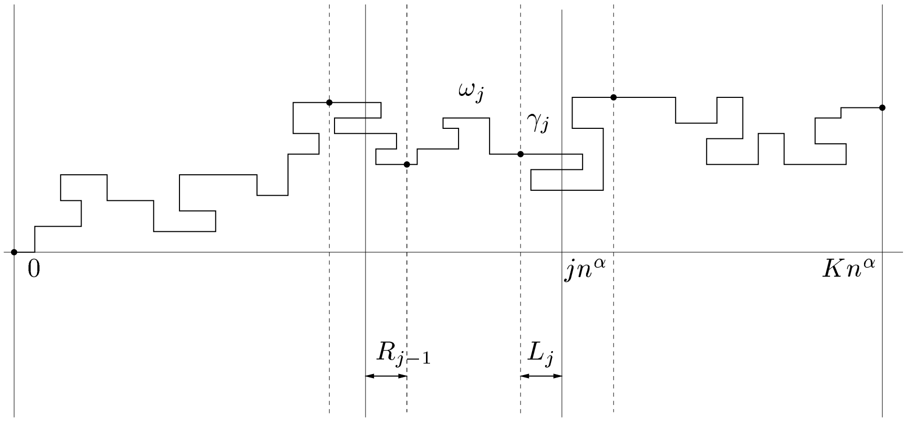


FIGURE 3.9. Basic mesoscopic splitting (3.8) of a bridge $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$

Proceeding to the formal definition, we fix some large enough positive constants r_2, r_3 and positive constants $\alpha > 0$ and $\varepsilon > 0$ satisfying the condition

$$12\varepsilon < 1, \quad 4\varepsilon < \alpha, \quad \text{and} \quad 2\alpha + 2\varepsilon < 1. \tag{3.10}$$

Definition 3.2. *We say that a bridge $\omega \in \mathcal{B}_n$ is regular, if it complies with the conditions (R1)–(R3) below. Namely,*

for any irreducible embedded sub-bridge $\gamma \overset{\mathrm{e}}{\subset} \omega$,

$$W(\gamma) < n^{\varepsilon}, \tag{R1}$$

$$|H(\gamma)| \leq r_2 n^{\varepsilon} \quad \text{and} \quad |a(\gamma)| \leq r_2 n^{2\varepsilon}; \tag{R2}$$

in addition, the mesoscopic components of ω from the decomposition (3.8) satisfy the condition

$$\forall j = 1, \dots, K; \quad |H(\omega_j)| \leq r_3 n^{\alpha} \quad \text{and} \quad |a(\omega_j)| \leq r_3 n^{3\alpha/2+\varepsilon}. \tag{R3}$$

The ensemble of regular bridges is denoted $\mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$.

Clearly, the claim of Theorem 3.1 is an immediate consequence of the following two statements:

Proposition 3.3. *Let $\delta > 0$ be fixed. Then, uniformly in the rescaled areas q_n satisfying (3.1), the asymptotic behaviour of the regular canonical bridge partition function is given by:*

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(n^2 q_n\right) = \sqrt{\frac{\mu(0)\mu(t_n)}{2\pi n^3 \sigma(t_n)}}\, e^{-n\psi(q_n)}\left(1 + o\left(1\right)\right), \tag{3.11}$$

with the conjugate tilt $t_n = t(q_n)$ and the functions ψ, σ defined as in Sect. 1.8.

Proposition 3.4. *Fix any $\delta > 0$ and choose any $\alpha > 0$, $\varepsilon > 0$ satisfying (3.10). Furthermore, suppose that the constants r_2, r_3 in* (R2), (R3) *are chosen to be large enough. Then there exist $c_1, c_2 > 0$ such that uniformly in the rescaled areas $\{q_n\}$ satisfying (3.1), the inequality*

$$\mathbb{B}_n\left(A^R(\omega) = n^2 q_n; \omega \notin \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\right) \leq c_1 \exp\left\{-n\psi(q_n) - c_2 n^{\varepsilon}\right\} \tag{3.12}$$

holds for all n large enough.

Remark 3.4.1. We shall prove Proposition 3.4 in Sect. 3.6 below. The estimate (1.16) of Lemma 1.4 now follows by a straightforward adjustment of the proof of Lemma 3.16 and of the arguments in STEP 1 from the proof of Lemma 3.17.

3.2. **Regular bridges.**

Our proof of the main local limit result—Proposition 3.3 consists of several steps presented in Lemmas 3.5–3.7 below. For reader's convenience we start by sketching the main ideas behind our argument.

3.2.1. *Tilting of the measure.* The first step towards (3.11) is quite natural: we change the measure,

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(n^2 q_n\right) = e^{-ntq_n}\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(e^{tA^R/n}\,;\, A^R(\omega) = n^2 q_n\right), \tag{3.13}$$

and evaluate the RHS at the conjugate value $t = t_n = t(q_n)$ of the tilt (recall (1.27)). However, since A^R is a complicated functional of the trajectory ω, this evaluation is quite a delicate task. A possible way out is suggested by the representation (3.8).

3.2.2. *The ideal area A^I.* Using the elements from the mesoscopic decomposition of $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$, we rewrite $A^R(\omega)$ as

$$A^R(\omega) = \sum_{j=1}^{K} N_j H(\omega_j) + \sum_{j=1}^{K-1} M_j H(\gamma_j) + \sum_{j=1}^{K} a(\omega_j) + \sum_{j=1}^{K-1} a(\gamma_j)$$

with N_j and M_j given by

$$N_j = n - \frac{2j-1}{2} n^{\alpha} + \frac{L_j - R_{j-1}}{2}, \qquad M_j = n - jn^{\alpha} + \frac{L_j - R_j}{2}.$$

Let $\mathcal{E} = \mathcal{E}(\omega)$ denote the collection of the endpoints of the mesoscopic components from (3.8). It is easy to see that for any fixed collection $\mathcal{E}$, the variables $a(\gamma_\bullet)$ and $a(\omega_\bullet)$ are symmetrically distributed (in ω such that $\mathcal{E}(\omega) \equiv \mathcal{E}$, see Fig. 3.10); in particular, the same is true once $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ and the collection $\mathcal{E}$ is compatible with the definition of $\mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$. Thus, it is natural to expect that for $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ the real area $A^R(\omega)$ is well approximated by its averaged analogue

$$A^I(\omega) \overset{\mathsf{def}}{=} \sum_{j=1}^{K} N_j H(\omega_j) + \sum_{j=1}^{K-1} M_j H(\gamma_j), \tag{3.14}$$

to be called the *ideal area.* As we shall see below, the mesoscopic coarsening A^I of A^R is an adequate object to work with.

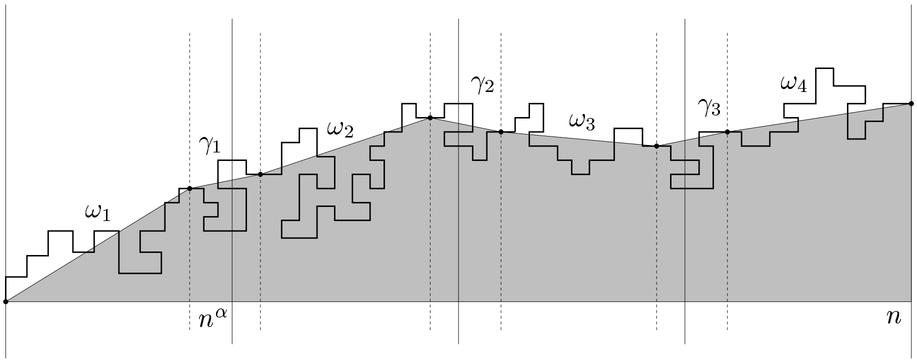


FIGURE 3.10. Ideal area $A^I(\omega)$ related to the mesoscopic decomposition $\omega = \omega_1 \vee \gamma_1 \vee \omega_2 \vee \gamma_2 \vee \omega_3 \vee \gamma_3 \vee \omega_4$

3.2.3. *Regular collections $\mathcal{M}$ of breaking points.* The ensemble $\mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ splits into the disjoint union of ensembles labeled by various different choices of collections of breaking points $\mathcal{M} = (\{L_j\}, \{R_j\})$; the latter clearly satisfy

$$0 \le \min_j (L_j + R_j) \le \max_j (L_j + R_j) < n^\varepsilon. \tag{3.15}$$

We shall call such collections $\mathcal{M}$ regular and use $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\,\cdot\,;\mathcal{M})$ to denote the restriction of the partition function $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\,\cdot\,)$ to the set of $\mathcal{M}$-compatible paths. Thus,

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\,\cdot\,) = \sum_{\mathcal{M}} \mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\,\cdot\,;\mathcal{M}),$$

where the summation is over all regular collections $\mathcal{M}$.

3.2.4. *Reduction to the tilts by A^I.* Let a segment $[a,b] \subset \mathcal{D}_\beta$, $a < 0 < b$, be fixed. For every $t \in [a,b]$ and each regular collection $\mathcal{M}$ define the (tilted) probability distribution

$$\mathbb{P}^{\mathsf{reg}}_{t,n}\bigl(\,\cdot \mid \mathcal{M}\bigr) \overset{\mathsf{def}}{=} \frac{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(\,\cdot\,;e^{tA^I/n};\mathcal{M}\bigr)}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n};\mathcal{M}\bigr)}$$

The following important area concentration property will be proved in Sect. 3.3 below.

Lemma 3.5. *Let $\Delta = \Delta(\omega)$ denote the area difference*

$$\Delta = \Delta(\omega) \overset{\mathsf{def}}{=} \sum_j a(\omega_j) + \sum_j a(\gamma_j).$$

Fix a number $S > 0$. Then there exist positive constants η, ζ, C, and c such that the inequality

$$\bigl|\mathbb{E}^{\mathsf{reg}}_{t,n}\bigl(e^{s\Delta/n};\,|\Delta| > n^{1-\eta}\mid \mathcal{M}\bigr)\bigr| \le Ce^{-cn^{\zeta}} \tag{3.16}$$

holds uniformly in $t \in [a,b]$, in $|s| \le S$, in regular collections $\mathcal{M}$ and in all n large enough.

Now, using the definitions above, we rewrite

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^R/n}\,;\,n^2q_n\bigr) = \sum_{\mathcal{M}}\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\,;\mathcal{M}\bigr)\,\mathbb{E}^{\mathsf{reg}}_{t,n}\bigl(e^{t\Delta/n}\,;\,n^2q_n\mid\mathcal{M}\bigr),$$

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\,;\,n^2q_n\bigr) = \sum_{\mathcal{M}}\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\,;\mathcal{M}\bigr)\,\mathbb{P}^{\mathsf{reg}}_{t,n}\bigl(A^R = n^2q_n\mid\mathcal{M}\bigr).$$

On the other hand, by Lemma 3.5,

$$\begin{aligned}\bigl|\mathbb{E}^{\mathsf{reg}}_{t,n}\bigl(e^{t\Delta/n}\,;\,n^2q_n\mid\mathcal{M}\bigr) &- \mathbb{P}^{\mathsf{reg}}_{t,n}\bigl(A^R = n^2q_n\mid\mathcal{M}\bigr)\bigr|\\ &\le 2Ce^{-cn^{\zeta}} + \bigl|\mathbb{E}^{\mathsf{reg}}_{t,n}\bigl(e^{t\Delta/n}-1\,;|\Delta|\le n^{1-\eta};\,n^2q_n\mid\mathcal{M}\bigr)\bigr|\\ &\le 2Ce^{-cn^{\zeta}} + O\bigl(n^{-\eta}\bigr)\mathbb{P}^{\mathsf{reg}}_{t,n}\bigl(A^R = n^2q_n\mid\mathcal{M}\bigr)\end{aligned}$$

uniformly in $\mathcal{M}$ under consideration. As a result,

$$\begin{aligned}\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^R/n}\,;\,n^2q_n\bigr) &= \mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\bigr)O\bigl(e^{-cn^{\zeta}}\bigr)\\ &\quad+\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\,;\,n^2q_n\bigr)\bigl(1+O(n^{-\eta})\bigr)\end{aligned} \tag{3.17}$$

and it remains to study the asymptotics of the partition functions in the RHS above.

3.2.5. *Asymptotics of the ideal partition function.* Our analysis of the first term in the RHS of (3.17) is based on the analytic properties of the two-point functions collected in Sect. 1.6. The corresponding result reads:

Lemma 3.6. *Let $[a,b] \subset \mathcal{D}_\beta$ be fixed and suppose that α and ε satisfy (3.10) with ε small enough, $0 < 4\varepsilon < \alpha$, and fix any constant $A > 0$. Then the asymptotics*

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{zA^I/n}\bigr) = \sqrt{\mu(0)\mu(z)}\exp\Bigl\{n\int_0^1 m_\beta\bigl((1-\xi)z\bigr)\,d\xi\Bigr\}\bigl(1+o(1)\bigr) \tag{3.18}$$

holds, as $n \to \infty$, uniformly in complex z satisfying the condition

$$\Re z \in [a,b], \qquad \bigl|\Im z\bigr| \le A/\sqrt{n}. \tag{3.19}$$

The proof of the lemma is given in Sect. 3.4 below.

3.2.6. *Local limit asymptotics.* Let the segment $[a,b] \subset \mathcal{D}_\beta$ be as fixed above. For any $t \in [a,b]$, introduce the tilted probability distribution

$$\mathbb{P}^{\mathsf{reg}}_{t,n}(\,\cdot\,) = \frac{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(\,\cdot\,;e^{tA^I/n}\bigr)}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\bigl(e^{tA^I/n}\bigr)}. \tag{3.20}$$

The following local limit result will be proven in Sect. 3.5.

Lemma 3.7. *Let t_n be the conjugate tilt to q_n. Then, as $n \to \infty$, we have*

$$\mathbb{P}^{\mathsf{reg}}_{t_n,n}\bigl(A^R = n^2q_n\bigr) = \frac{1}{\sqrt{2\pi n^3\sigma(t_n)}}\bigl(1+o(1)\bigr) \tag{3.21}$$

uniformly in the rescaled areas q_n satisfying (3.1).

Consequently, in view of the change of measure formula (3.13), it remains to insert the asymptotics (3.18) and (3.21) into the decomposition (3.17) and to apply the duality relation (1.28). The conclusion (3.11) of Proposition 3.3 follows. □

We turn now to the proof of Lemmas 3.5–3.7.

3.3. **Proof of Lemma 3.5.** Given a collection of integer heights

$$\mathcal{H} = \{h_1, g_1, h_2, ..., g_{K-1}, h_K\}$$

we say that a regular bridge $\omega \in \mathcal{B}_n$ is $\mathcal{H}$-compatible if its mesoscopic decomposition (3.8) satisfies:

$$\forall j = 1, ..., K, \quad H(\omega_j) = h_j \qquad \text{and} \qquad \forall j = 1, ..., K-1, \quad H(\gamma_j) = g_j.$$

A collection $\mathcal{H}$ is called *regular* iff

$$\forall j = 1, ..., K, \quad |h_j| \le r_3 n^{\alpha} \qquad \text{and} \qquad \forall j = 1, ..., K-1, \quad |g_j| \le r_2 n^{\varepsilon},$$

Thanks to the conditions **(R1)**–**(R3)** of Definition 3.2, a collection $\mathcal{H}$ is regular if and only if there exists an $\mathcal{H}$-compatible bridge $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$.

For a fixed regular collection $\mathcal{M}$ of breaking points, we use $\mathbb{E}^{\mathsf{reg}}_{t,n}(\,\cdot \mid \mathcal{M}\,)$ to denote the corresponding conditional expectation. We have

$$\bigl|\mathbb{E}^{\mathsf{reg}}_{t,n}\bigl(e^{t\Delta/n};\, |\Delta| > n^{1-\eta} \bigm| \mathcal{M}\bigr)\bigr| \le \max_{\mathcal{H}} \bigl|\mathbb{E}^{\mathsf{reg}}_{n}\bigl(e^{t\Delta/n};\, |\Delta| > n^{1-\eta} \bigm| \mathcal{M}, \mathcal{H}\bigr)\bigr| \tag{3.22}$$

where the maximum runs over all regular collections $\mathcal{H}$. Clearly, fixing $\mathcal{M}$ and $\mathcal{H}$ is equivalent to fixing the collection $\mathcal{E}$ of the endpoints in the mesoscopic decomposition of a bridge (see Fig. 3.10). Recall that for any choice of $\mathcal{M}$ and $\mathcal{H}$, the random variables

$$a(\omega_1), \quad a(\gamma_1), \quad a(\omega_2), \quad \dots, \quad a(\gamma_{K-1}), \quad a(\omega_K)$$

are conditionally independent, symmetric, bounded functions of $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$. Consequently, for any $s \in \mathbb{R}$ and $j = 1, \dots, K$ we, using the second of the conditions (R3), obtain:

$$\begin{aligned}
\mathbb{E}^{\mathsf{reg}}_{n}\bigl(e^{sa(\omega_j)/n} \bigm| \mathcal{M}, \mathcal{H}\bigr) &\le \sum_{k=0}^{\infty} \frac{|s|^{2k}}{(2k)!n^{2k}} \mathbb{E}^{\mathsf{reg}}_{n}\bigl(a(\omega_j)^{2k} \bigm| \mathcal{M}, \mathcal{H}\bigr) \\
&\le \cosh\bigl(r_3 n^{3\alpha/2+\varepsilon-1}|s|\bigr) \le \exp\Bigl\{\frac{{r_3}^2}{2} n^{3\alpha+2\varepsilon-2}|s|^2\Bigr\}, \\
\mathbb{E}^{\mathsf{reg}}_{n}\bigl(e^{sa(\gamma_l)/n} \bigm| \mathcal{M}, \mathcal{H}\bigr) &\le \cosh\bigl(r_2 n^{2\varepsilon-1}|s|\bigr) \le \exp\Bigl\{\frac{{r_2}^2}{2} n^{4\varepsilon-2}|s|^2\Bigr\}
\end{aligned}$$

and thus

$$\mathbb{E}^{\mathsf{reg}}_{n}\bigl(e^{s\Delta/n} \bigm| \mathcal{M}, \mathcal{H}\bigr) \le \exp\bigl\{c_1 n^{2(\alpha+\varepsilon)-1}|s|^2\bigr\}.$$

On the other hand, the Hoeffding-Azuma inequality [29, 3] implies

$$\begin{aligned}
\mathbb{P}^{\mathsf{reg}}_{n}\bigl(\bigl|\Delta(\omega)\bigr| > n^{1-\eta} \bigm| \mathcal{M}, \mathcal{H}\bigr) &\le 2\exp\Bigl\{-\frac{n^{2(1-\eta)}}{2K\bigl({r_3}^2 n^{3\alpha+2\varepsilon} + {r_2}^2 n^{4\varepsilon}\bigr)}\Bigr\} \\
&\le 2\exp\bigl\{-c_2 n^{1-2(\alpha+\varepsilon+\eta)}\bigr\}
\end{aligned}$$

and therefore, using the Cauchy-Schwartz inequality and the last two bounds, we deduce the validity of the estimate

$$\mathbb{E}^{\mathsf{reg}}_{n}\bigl(e^{t\Delta/n};\, |\Delta| > n^{1-\eta} \bigm| \mathcal{M}, \mathcal{H}\bigr) \le c_3 \exp\bigl\{-c_4 n^{1-2(\alpha+\varepsilon+\eta)}\bigr\}$$

uniformly in $t \in [a, b]$, fixed collections $\mathcal{M}$ and $\mathcal{H}$, and all n large enough.

Finally, (3.16) follows directly from (3.22) and the last bound. ☐

3.4. **Proof of Lemma 3.6.** Fix any $t \in [a,b]$ and a regular collection $\mathcal{M}$ of breaking points. On the event $\{\mathcal{M}\}$, the partition function $\mathbb{B}_n(e^{tA^I/n};\mathcal{M})$ (cf. (1.3)) has the following factorisation property:

$$\mathbb{B}_n(e^{tA^I/n};\mathcal{M}) = \prod_{j=1}^{K} \mathbb{B}_{W(\omega_j)}\big(t(\omega_j)\big) \prod_{l=1}^{K-1} \mathbb{F}_{W(\gamma_l)}\big(t(\gamma_l)\big), \tag{3.23}$$

with (cf. (3.14))

$$\begin{aligned} t(\omega_j) &\overset{\mathsf{def}}{=} \frac{N_j}{n}t = \Big(1 - \frac{2j-1}{2K} + \frac{L_j - R_{j-1}}{2n}\Big)t, \\ t(\gamma_l) &\overset{\mathsf{def}}{=} \frac{M_l}{n}t = \Big(1 - \frac{l}{K} + \frac{L_l - R_l}{2n}\Big)t. \end{aligned} \tag{3.24}$$

Next, we define

$$\mathbb{B}_n^{\alpha,\varepsilon}(t) \overset{\mathsf{def}}{=} \sum_{\mathcal{M}} \mathbb{B}_n(e^{tA^I/n};\mathcal{M}), \tag{3.25}$$

where the sum runs over all regular collections $\mathcal{M}$. Let $\mathcal{B}_{n,\alpha,\varepsilon}$ denote the ensemble of all bridges ω whose mesoscopic collections $\mathcal{M}(\omega)$ of breaking points are regular.

As we shall see below, in some complex neighbourhood $\mathcal{U}$ of the segment $[a,b]$ the partition function $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{zA^I/n}\big)$ is well approximated by $\mathbb{B}_n^{\alpha,\varepsilon}(z)$; the latter, however, possesses the following property.

Lemma 3.8. *Assume that α and ε satisfy (3.10) with ε small enough, $0 < 4\varepsilon < \alpha$. Then there exists a complex neighbourhood $\mathcal{U}$ of the segment $[a,b]$ such that the relation*

$$\mathbb{B}_n^{\alpha,\varepsilon}(z) = \sqrt{\mu(0)\mu(z)}\exp\Big\{n\int_0^1 m_\beta\big((1-x)z\big)\,dx\Big\}\big(1 + O(n^{-\varepsilon})\big) \tag{3.26}$$

holds true, as $n \to \infty$, uniformly in $z \in \mathcal{U}$.

Proof. Let t_j denote the statistically averaged value of $t(\omega_j)$,

$$t_j \equiv \Big(1 - \frac{2j-1}{2K}\Big)t, \qquad j = 1,\dots,K. \tag{3.27}$$

Our argument below is based on the following properties [33]: there exist a complex neighbourhood $\mathcal{U}$ of $[a,b]$ and a positive constant $\tilde\alpha = \tilde\alpha(\mathcal{U})$ such that the relations

$$\begin{gathered} \sum_{k\ge 2}^{\infty} e^{\tilde\alpha k}\big|\mathbb{F}_k(z)\exp\{-km_\beta(z)\}\big| < \infty, \\ 1 + \sum_{k=2}^{n^\varepsilon}(k-1)\mathbb{F}_k(z)\exp\{-km_\beta(z)\} = \frac{1}{\mu(z)} + o(n^\varepsilon e^{-\tilde\alpha n^\varepsilon}) \\ \log\mathbb{B}_n(z) - nm_\beta(z) - \log\mu(z) = o(e^{-\tilde\alpha n}), \end{gathered} \tag{3.28}$$

hold uniformly in $z \in \mathcal{U}$ and all n sufficiently large. Here we have used the identity [33]

$$\sum_{k=1}^{\infty} \mathbb{F}_k(\cdot)\mathrm{e}^{-km_\beta(\cdot)} \equiv 1$$

valid everywhere on $\mathcal{U}$. Note that the first two properties in (3.28) imply the following useful estimate:

$$
\begin{aligned}
1+\sum_{l\geq1,r\geq1}^{n^{\varepsilon}} \mathbb{F}_{l+r}\Bigl(z_0+(r-l)\frac{z}{n}\Bigr)\exp\Bigl\{-(l+r)m_\beta\Bigl(z_0+(r-l)\frac{z}{n}\Bigr)\Bigr\}& \\
=\bigl(\mu(z_0)\bigr)^{-1}+o(n^{3\varepsilon-2})&
\end{aligned}
\tag{3.29}
$$

valid uniformly in z_0 and z belonging to $\mathcal{U}$ and all n large enough (thanks to the term-by-term cancellation in the linear part

$$
\sum_{k=2}^{2n^{\varepsilon}}\Bigl[\mathbb{F}'_k(z_0)+\mathbb{F}_k(z_0)m'_\beta(z_0)\Bigr]\exp\Bigl\{-km_\beta(z_0)\Bigr\}\sum_{\substack{1\leq l,r\leq n^{\varepsilon}\\ l+r=k}}(r-l)\frac{z}{n}
$$

of the Taylor expansion of the LHS in (3.29) around z_0).

In addition, we shall use the following simple observation to be checked below.

Let $z(\omega_j)$ and z_j be the complex counterparts of $t(\omega_j)$ and t_j defined in (3.24) and (3.27) correspondingly. Then, uniformly in $\omega\in\mathcal{B}_{n,\alpha,\varepsilon}$ and z under consideration, we have

$$
\begin{aligned}
\log\prod_{j=1}^{K}\mathbb{B}_{W(\omega_j)}\bigl(z(\omega_j)\bigr) &= \sum_{j=1}^{K}\log\mu(z_j)+n^{\alpha}\sum_{j=1}^{K}m_\beta(z_j)\\
&\quad-\sum_{j=1}^{K-1}\bigl(L_j+R_j\bigr)m_\beta\Bigl(\frac{z_j+z_{j+1}}{2}\Bigr)+O(n^{\varepsilon-\alpha}).
\end{aligned}
\tag{3.30}
$$

Let us turn to the proof of (3.26). In view of (3.30), the rescaled partition function

$$
\widetilde{\mathbb{B}}_n^{\alpha,\varepsilon}(z)\overset{\mathsf{def}}{=}\mathbb{B}_n^{\alpha,\varepsilon}(z)\exp\Bigl\{-n^{\alpha}\sum_{j=1}^{K}m_\beta(z_j)\Bigr\}
$$

can be rewritten as (recall (3.25) and (3.23))

$$
\widetilde{\mathbb{B}}_n^{\alpha,\varepsilon}(z)=\prod_{j=1}^{K}\mu(z_j)\sum_{\mathcal{M}}\Bigl[\prod_{l=1}^{K-1}\mathbb{F}_{W(\gamma_l)}\bigl(z(\gamma_l)\bigr)e^{-W(\gamma_l)m_\beta(\bar z_l)}\Bigr]\bigl(1+O(n^{\varepsilon-\alpha})\bigr),
$$

where $\bar z_l$ stands for $(z_l+z_{l+1})/2$. Now, different $\gamma_\bullet$-bridges in the last parentheses are independent; interchanging the order of summation and multiplication, we thus obtain

$$
\widetilde{\mathbb{B}}_n^{\alpha,\varepsilon}(z)=\prod_{j=1}^{K}\mu(z_j)\prod_{l=1}^{K-1}\Bigl[\sum_{\gamma_l:W(\gamma_l)\leq n^{\varepsilon}}\mathbb{F}_k\bigl(z(\gamma_l)\bigr)e^{-W(\gamma_l)m_\beta(\bar z_l)}\Bigr]\bigl(1+O(n^{\varepsilon-\alpha})\bigr).
$$

Now, using the simple fact

$$
W(\gamma_l)\bigl(m_\beta\bigl(z(\gamma_l)\bigr)-m_\beta(\bar z_l)\bigr)=O\Bigl(\frac{W(\gamma_l)^2}{n}\Bigr)
$$

valid uniformly in z and $\gamma_\bullet$ under consideration as well as the self-averaging property (3.29), we rewrite the expression in the brackets above as

$$
\sum_{\gamma_l:W(\gamma_l)\leq n^{\varepsilon}}\mathbb{F}_k\bigl(z(\gamma_l)\bigr)\exp\Bigl\{-W(\gamma_l)m_\beta\bigl(z(\gamma_l)\bigr)\Bigr\}e^{O(W(\gamma_l)^2/n)}=\bigl(\mu(\bar z_l)\bigr)^{-1}+O(n^{3\varepsilon-1});
$$

therefore, using analyticity of $\mu(\cdot)$, we obtain

$$\begin{aligned}\widetilde{\mathbb{B}}_n^{\alpha,\varepsilon}(z) &= \exp\Bigl\{\sum_{j=1}^{K}\log\mu(z_j) - \sum_{l=1}^{K-1}\log\mu(\bar{z}_l)\Bigr\}\exp\Bigl\{O(Kn^{3\varepsilon-1}) + O(n^{\varepsilon-\alpha})\Bigr\}\\ &= \sqrt{\mu(0)\mu(z)}\bigl(1+O(n^{3\varepsilon-\alpha})\bigr).\end{aligned}$$

It remains to observe that (thanks to smoothness of $m_\beta(\cdot)$)

$$n^\alpha\sum_{j=1}^{K} m_\beta(z_j) - n\int_0^1 m_\beta\bigl(z(x)\bigr)\,dx = O(n/K^2) = O(n^{2\alpha-1}).$$

Finally, we justify (3.30). To this end, fix any $\omega\in\mathcal{B}_{n,\alpha,\varepsilon}$. According to the definitions (3.24) and (3.27), we have $z(\omega_j)-z_j = O(n^\varepsilon/n)$ and therefore the last relation in (3.28) implies (recall that $K=n^{1-\alpha}$ and μ is an analytic function)

$$\sum_{j=1}^{K}\log\mathbb{B}_{W(\omega_j)}\bigl(z(\omega_j)\bigr) = \sum_{j=1}^{K}\log\mu(z_j) + \sum_{j=1}^{K}W(\omega_j)m_\beta(z(\omega_j)) + O(n^{\varepsilon-\alpha}).$$

To estimate the last sum, we note that definition (3.9), the complex analogue of (3.27), and the Taylor formula imply the equality

$$\begin{aligned}W(\omega_j)m_\beta\bigl(z(\omega_j)\bigr) &= \bigl(n^\alpha-(L_j+R_{j-1})\bigr)m_\beta(z_j)\\ &\qquad + \frac{z}{2K}(L_j-R_{j-1})m'_\beta(z_j) + O(n^{2\varepsilon-1}).\end{aligned}\tag{3.31}$$

On the other hand, $R_0=L_K=0$; therefore, using again the Taylor formula, we get the relations

$$\begin{aligned}\sum_{j=1}^{K}(L_j+R_{j-1})m_\beta(z_j) &= \sum_{j=1}^{K-1}(L_j+R_j)m_\beta(\bar{z}_j)\\ &\qquad+\frac{z}{2K}\sum_{j=1}^{K-1}(L_j-R_j)m'_\beta(\bar{z}_j) + O(n^\varepsilon/K)\\ \sum_{j=1}^{K}(L_j-R_{j-1})m'_\beta(z_j) &= \sum_{j=1}^{K-1}(L_j-R_j)m'_\beta(\bar{z}_j) + O(n^\varepsilon),\end{aligned}$$

where, as before, $\bar{z}_j$ stands for $(z_j+z_{j+1})/2$. Finally, summing in (3.31) from $j=1$ to K and using the last two identities, we finish the proof. □

A simple use of the Cauchy formula shows that uniformly in z under consideration, one obtains the convergence

$$\frac{1}{n}\frac{d^2}{dz^2}\log\mathbb{B}_n^{\alpha,\varepsilon}(z)\to\frac{d^2}{dz^2}\int_0^1 m_\beta\bigl((1-x)z\bigr)\,dx\qquad\text{as } n\to\infty.$$

Recalling the non-degeneracy of the function $\sigma(t)$ from (1.29) and using the Taylor formula, we immediately deduce the following fact.

Corollary 3.9. *There exist $\delta>0$ and $\eta>0$ such that the estimate*

$$\frac{|\mathbb{B}_n^{\alpha,\varepsilon}(t+i\tau)|}{\mathbb{B}_n^{\alpha,\varepsilon}(t)}\le\exp\bigl\{-\eta n\tau^2\bigr\}$$

holds uniformly in $t\in[a,b]$ and $|\tau|\le\delta$.

We turn now to the proof of (3.18). In a way, similar to (3.25)+(3.23), we rewrite

$$\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(z) \equiv \mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{zA^I/n}\big) = \sum_{\mathcal{M}} \prod_{j=1}^{K} \mathbb{B}^{\mathsf{reg}}_{W(\omega_j)}\big(z(\omega_j)\big) \prod_{l=1}^{K-1} \mathbb{F}^{\mathsf{reg}}_{W(\gamma_l)}\big(z(\gamma_l)\big), \tag{3.32}$$

where the functions $\mathbb{B}^{\mathsf{reg}}_m(\cdot)$ and $\mathbb{F}^{\mathsf{reg}}_m(\cdot)$ are defined in an obvious way to satisfy the conditions **(R1)**–**(R3)** of Definition 3.2. As we shall see below, these functions are well approximated by their non-restricted analogues:

Let the positive constants α, ε and the segment $[a,b]$ be as fixed above. Then there exist large enough r_2, r_3 (recall Definition 3.2) and positive c, ζ such that the relations

$$\begin{aligned} \big|\mathbb{B}_m(z) - \mathbb{B}^{\mathsf{reg}}_m(z)\big| &\le \mathbb{B}_m(t) - \mathbb{B}^{\mathsf{reg}}_m(t) = O(e^{-cm^\zeta}\mathbb{B}_m(t)) \\ \big|\mathbb{F}_m(z) - \mathbb{F}^{\mathsf{reg}}_m(z)\big| &\le \mathbb{F}_m(t) - \mathbb{F}^{\mathsf{reg}}_m(t) = O(e^{-cm^\zeta}\mathbb{F}_m(t)) \end{aligned} \tag{3.33}$$

hold uniformly in $m \in \mathbb{N}$ and $z \in \mathbb{C}$ satisfying $\Re z \in [a,b]$.

We postpone the proof of the bounds (3.33) till the end of the section and deduce validity of the asymptotics (3.18) first.

Our key observation is the next estimate: if $z \in \mathbb{C}$ is such that $\Re z \in [a,b]$, then

$$\big|\mathbb{B}^{\alpha,\varepsilon}_n(t+i\tau) - \mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(t+i\tau)\big| \le \mathbb{B}^{\alpha,\varepsilon}_n(t) - \mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(t) = O(e^{-cn^\zeta})\mathbb{B}^{\alpha,\varepsilon}_n(t), \tag{3.34}$$

with possibly different constant $c > 0$. Indeed, the inequality above follows directly from the mesoscopic representations of the partition functions $\mathbb{B}^{\alpha,\varepsilon}_n(\cdot)$ and $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\cdot)$ (recall (3.32) and its non-restricted analogue (3.25)+(3.23)) combined with the apriori bounds

$$\begin{aligned} \big|\mathbb{B}_m(t+is)\big| &\le \mathbb{B}_m(t), & \big|\mathbb{F}_m(t+is)\big| &\le \mathbb{F}_m(t), \\ \big|\mathbb{B}^{\mathsf{reg}}_m(t+is)\big| &\le \mathbb{B}^{\mathsf{reg}}_m(t), & \big|\mathbb{F}^{\mathsf{reg}}_m(t+is)\big| &\le \mathbb{F}^{\mathsf{reg}}_m(t) \end{aligned} \tag{3.35}$$

valid uniformly in $t \in [a,b]$, real s, and integer $m \ge 2$. On the other hand, the equality is a simple corollary of the estimate

$$\Big|\prod_{i=1}^{l} a_i - \prod_{i=1}^{l} b_i\Big| \le l \max_j \Big|\frac{a_j - b_j}{b_j}\Big| \prod_{i=1}^{l} c_i \qquad \text{with } c_i = \max(|a_i|, |b_i|),$$

the uniform bounds (3.33) and the observation that $KO(e^{-c(n^\varepsilon)^\zeta}) = O(e^{-c'n^{\zeta'}})$ as $n \to \infty$.

Now the target bound (3.18) follows directly from (3.34) and a simple observation that the ratio $\mathbb{B}^{\alpha,\varepsilon}_n(z)/\mathbb{B}^{\alpha,\varepsilon}_n(t)$ is uniformly separated from zero for any z satisfying (3.19).

Finally, we prove the bounds (3.33). Since the inequalities in (3.33) are simple term-by-term estimates combined with the apriori bounds of the type (3.35), we shall concentrate ourselves on the proof of the equalities only. The latter, however, have a natural probabilistic interpretation. Namely, in terms of the (tilted) bridge and irreducible bridge distributions

$$\mathbb{P}^t_m(\cdot) \overset{\mathsf{def}}{=} \frac{\mathbb{B}_m\big(\cdot\,; e^{tH(\omega)}\big)}{\mathbb{B}_m\big(e^{tH(\omega)}\big)}, \qquad \mathbb{Q}^t_m(\cdot) \overset{\mathsf{def}}{=} \frac{\mathbb{F}_m\big(\cdot\,; e^{tH(\gamma)}\big)}{\mathbb{F}_m\big(e^{tH(\gamma)}\big)} \tag{3.36}$$

the equalities in (3.33) follow directly from the properties:

Fix $\varepsilon > 0$ *small enough; then, for some* $c > 0$, $\eta > 0$, *the estimates*

$$\begin{aligned}\mathbb{P}^t_m \begin{pmatrix} |H(\omega)| \le r_3 m, \quad |a(\omega)| \le r_3 m^{3/2+\varepsilon} \\ {}^{\forall}\gamma \overset{\mathrm{e}}{\subset} \omega,\ \gamma\text{-}\textit{irreducible} \implies W(\gamma) < m^{\varepsilon} \end{pmatrix} &\ge 1 - e^{-cm^{\zeta}}, \\ \mathbb{Q}^t_m \Big(|H(\gamma)| \le r_2 m, |a(\gamma)| \le r_2 m^2 \Big) &\ge 1 - e^{-cm^{\zeta}} \end{aligned} \tag{3.33'}$$

hold, as $m \to \infty$, *uniformly in* $t \in [a, b]$.

To verify the first bound above, it is enough to check that for any $\varepsilon > 0$ there are $c > 0$ and $\zeta > 0$ such that the inequalities

$$\mathbb{P}^t_m \big({}^{\exists}\gamma \overset{\mathrm{e}}{\subset} \omega,\ \gamma\text{-irreducible } \&\ W(\gamma) \ge m^{\varepsilon} \big) \le e^{-cm^{\zeta}},$$

$$\mathbb{P}^t_m \big(|H(\omega)| > r_3 m \big) \le e^{-cm^{\zeta}},$$

$$\mathbb{P}^t_m \big(|a(\omega)| > r_3 m^{3/2+\varepsilon} \bigm| |H(\omega)| \le r_3 m \big) \le e^{-cm^{\zeta}}$$

hold, uniformly in $t \in [a, b]$, for all m large enough. However, the first one is a direct corollary of (1.24), the second one is obvious provided $r_3 > 0$ is large enough, and the last bound follows directly from the strict triangle inequality (1.8) for m large enough.

To verify the second bound in (3.33′), we observe first that the irreducible two point function f_β evidently satisfies:

$$f_\beta(m, H) < g_\beta(m, H) = g_\beta(H, m) < \mathrm{e}^{-H\tau_\beta(1, m/H)}.$$

Since, $\tau_\beta(1, \cdot)$ is continuous at zero, the large deviation estimate

$$\mathbb{Q}^t_m \big(|H(\gamma)| > r_2 m \big) \le e^{-cm}$$

holds uniformly in $t \in [a, b] \subset \mathcal{D}_\beta$ for any r_2 satisfying the following two conditions:

$$\delta \overset{\mathsf{def}}{=} \tau_\beta(1, 1/r_2) - \max\{|a|, |b|\} > 0 \quad \text{and} \quad \delta \cdot r_2 > -\liminf_{m} \min_{t \in [a,b]} \frac{1}{m} \log \mathbb{F}_m(t).$$

Similarly, for any fixed r, the strict triangle inequality (1.8) implies that the upper bound

$$\mathbb{Q}^t_m \big(|a(\gamma)| > r_2 m^2 \bigm| |H(\gamma)| \le rm \big) \le e^{-cm},$$

also holds uniformly in $t \in [a, b] \subset \mathcal{D}_\beta$ and for any r_2 large enough. Thus, the second of the inequalities in (3.33′) follows once r_2 has been chosen to be sufficiently large. $\square$

3.5. **Proof of Lemma 3.7.** From the point of view of the tilted probability distribution $\mathbb{P}^{\mathsf{reg}}_{t,n}(\,\cdot\,)$ defined in (3.20), Lemma 3.6 implies the (uniform) central limit asymptotics for the properly rescaled ideal area A^I. In particular, the mean value satisfies

$$\mathbb{E}^{\mathsf{reg}}_{t,n}\big(A^I/n\big) = n \frac{d}{ds} \int_0^1 m_\beta\big((1-\xi)s\big)\, d\xi \,\Big|_{s=t}.$$

Taking now $t = t_n$ to be the conjugate value to q_n in (1.27) and recalling the self-averaging property (3.16) of the area correction $\Delta(\omega) \equiv A^R(\omega) - A^I(\omega)$, we obtain

$$\mathbb{E}^{\mathsf{reg}}_{t_n,n}\big(A^R/n\big) = n \int_0^1 (1-\xi) m'_\beta\big((1-\xi)t_n\big)\, d\xi = n q_n.$$

In other words, t_n is the proper tilt in the study the asymptotics of the probability $\mathbb{P}_n^{\mathsf{reg}}\big(A^R = n^2 q_n\big)$.

Let $\chi_n(\tau)$ be the t_n-tilted characteristic function of the centred normalised real area,

$$\begin{aligned}\chi_n(\tau) &\overset{\mathsf{def}}{=} \mathbb{E}^{\mathsf{reg}}_{t_n,n} \exp\Big\{ \frac{i\tau}{n^{3/2}} \big(A^R - n^2 q_n\big)\Big\} \\ &= \frac{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{i\tau\Delta/n^{3/2}}; e^{(t_n+i\tau/\sqrt{n})A^I/n}\big)}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{t_n A^I/n}\big)} \exp\big\{-i\tau\sqrt{n} q_n\big\}.\end{aligned} \tag{3.37}$$

The function $\chi_n(\tau)$ being $2\pi n^{3/2}$-periodic, the inversion formula for the Fourier transform implies

$$\mathbb{P}^{\mathsf{reg}}_{t_n,n}\big(A^R = n^2 q_n\big) = \frac{1}{2\pi n^{3/2}} \int_{-\pi n^{3/2}}^{\pi n^{3/2}} \chi_n(\tau)\, d\tau.$$

Thus, the local limit result (3.20) is an immediate corollary of the following three facts.

Claim 3.10. *Fix any $A > 0$. Then the convergence*

$$\log \chi_n(\tau) + \frac{1}{2}\sigma(t_n)\tau^2 \longrightarrow 0 \tag{3.38}$$

holds, as $n \to \infty$, uniformly in $|\tau| \le A$.

Claim 3.11. *There exist positive constants δ, η, C, c, and ζ such that the bound*

$$\big|\chi_n(\tau)\big| \le 2\exp\big\{-\eta\tau^2\big\} + Ce^{-cn^\zeta} \tag{3.39}$$

is true uniformly in $|\tau| \le \delta\sqrt{n}$ and all n large enough.

Claim 3.12. *Let $\delta > 0$ be as fixed above. Then there exist positive constants c and ζ such that the inequality*

$$\big|\chi_n(\tau)\big| \le \exp\big\{-cn^\zeta\big\} \tag{3.40}$$

is satisfied uniformly in $\delta\sqrt{n} \le |\tau| \le \pi n^{3/2}$ and all n large enough.

The rest of this section is devoted to the proof of these facts. The central limit theorem convergence (3.38) and the Gaussian upper bound (3.39) in the region of small τ, $|\tau| \le \delta\sqrt{n}$, will be deduced by comparison with the characteristic function of the ideal area A^I in Sect. 3.5.1. The estimate (3.40), valid in the region $\delta\sqrt{n} \le |\tau| \le \pi n^{3/2}$, will be established in Sect. 3.5.2 as an outcome of certain renormalisation procedure.

3.5.1. *Comparison with the ideal area.* Let $\chi_n^I(\tau)$ be the characteristic function of the centred normalised ideal area,

$$\chi_n^I(\tau) \overset{\mathsf{def}}{=} \mathbb{E}^{\mathsf{reg}}_{t_n,n} \exp\Big\{ \frac{i\tau}{n^{3/2}} \big(A^I - n^2 q_n\big)\Big\}.$$

Using the representation (3.37) and the bound (3.16), we obtain

$$\begin{aligned}\big|\chi_n(\tau) - \chi_n^I(\tau)\big| &= \frac{\big|\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{i\tau\Delta/n^{3/2}} - 1; e^{(t_n+i\tau/\sqrt{n})A^I/n}\big)\big|}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{t_n A^I/n}\big)} \\ &\le 2\mathbb{P}^{\mathsf{reg}}_{t_n,n}\big(|\Delta| > n^{1-\eta}\big) + \big|\chi_n^I(\tau)\big| O\big(n^{-\eta}\big) \\ &= O\big(e^{-cn^\zeta}\big) + \big|\chi_n^I(\tau)\big| O\big(n^{-\eta}\big)\end{aligned}$$

uniformly in $|\tau| < \delta\sqrt{n}$.

Now, let τ satisfy $|\tau| \le A$ with A fixed above. Then a direct calculation based on the asymptotics (3.18) shows that

$$\chi_n^I(\tau) \equiv \frac{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(e^{(t_n+i\tau/\sqrt{n})A^I/n}\right)}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(e^{t_nA^I/n}\right)}\exp\bigl\{-i\tau\sqrt{n}q_n\bigr\} = \exp\Bigl\{-\frac{\sigma(t_n)}{2}\tau^2\Bigr\} + o(1),$$

as $n \to \infty$, uniformly in such τ. The convergence (3.38) follows.

Next, combining together Corollary 3.9 and the uniform bound (3.34) we get

$$\frac{\bigl|\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(t+is)\bigr|}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(t)} \le \exp\bigl\{-n\eta s^2\bigr\} + O\bigl(e^{-cn^\zeta}\bigr), \qquad |s| \le \delta,$$

and thus, uniformly in $|\tau| \le \delta\sqrt{n}$,

$$\bigl|\chi_n^I(\tau)\bigr| \le \exp\bigl\{-\eta\tau^2\bigr\} + O\bigl(e^{-cn^\zeta}\bigr).$$

Finally, the inequality

$$\bigl|\chi_n(\tau)\bigr| \le 2\exp\bigl\{-\eta\tau^2\bigr\} + O\bigl(e^{-cn^\zeta}\bigr)$$

holds uniformly in $|\tau| \le \delta\sqrt{n}$ and all n large enough. This coincides with (3.39).

3.5.2. *Large τ region.* According to Definition 3.2, regular bridges $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ admit the decomposition into irreducible components

$$\omega = \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_N, \tag{3.41}$$

each verifying condition **(R1)**, $W(\gamma_j) < n^\varepsilon$; in particular, the number $N = N(\omega)$ of components satisfies the uniform lower bound $N \ge n^{1-\varepsilon}$. The decomposition (3.41) generates the partition $\mathcal{D} = \mathcal{D}(\omega)$ of the integer number n according to the following rule:

$$\omega \mapsto \mathcal{D}(\omega) = \bigl\{d_1, d_2, \ldots, d_N\bigr\}, \qquad \text{where } \forall j = 1, 2, \ldots, N, \quad W(\gamma_j) = d_j.$$

Given a (positive integer) partition $\overline{\mathcal{D}} = \bigl\{d_1, d_2, \ldots, d_N\bigr\}$ of n,

$$d_1 + d_2 + \cdots + d_N = n,$$

and a bridge $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$, we say that ω is *$\overline{\mathcal{D}}$-compatible* if and only if $\mathcal{D}(\omega) = \overline{\mathcal{D}}$.

To avoid the degenerated case $a(\gamma_\bullet) \equiv 0$ (occurring for $W(\gamma_\bullet) = 1$ or 2), we shall consider the partitions $\mathcal{D}$ of an integer number n having reach enough structure:

Definition 3.13. *A (positive integer) partition $\mathcal{D} = \bigl\{d_1, d_2, \ldots, d_N\bigr\}$ of a natural number n is called regular, if*

1) $\forall j = 1, 2, \ldots, N, \qquad d_j < n^\varepsilon,$

2) $\forall k = 1, 2, \ldots, n^{1-\varepsilon}$, $\exists j$ *such that* $d_j \ge 3$ *and* $\sum_{l=1}^{j-1} d_l \in \bigl[(k-1)n^\varepsilon, kn^\varepsilon\bigr).$

Note that for any regular partition $\mathcal{D} = \mathcal{D}_n$ of an integer number n and any $\mathcal{D}$-compatible bridge $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$, each segment $\bigl[(k-1)n^\varepsilon, kn^\varepsilon\bigr)$ intersects a span of an irreducible component $\gamma_\bullet$ of width $W(\gamma_\bullet) \ge 3$.

Due to the massgap condition (1.21), the function $\mathcal{D}(\omega)$ thought as a random variable in $\omega \in \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ has the following important property:

Fix any $\varepsilon > 0$ *small enough. Then there exist positive constants* c *and* ζ *such that the estimate*

$$\mathbb{P}^{\mathsf{reg}}_{t,n}\big(\,\mathcal{D}(\omega) \text{ is not regular }\big) < e^{-cn^{\zeta}} \tag{3.42}$$

holds, as $n \to \infty$*, uniformly in* $t \in [a,b]$.

Let $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(\cdot \mid \mathcal{D}\big)$ denote the restriction of the bridge partition function $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}(\cdot)$ to the ensemble of $\mathcal{D}$-compatible bridges. Thanks to the uniform bound (3.42), the target inequality (3.40) follows directly from the following fact.

Lemma 3.14. *Suppose that* ε *satisfies* $0 < 12\varepsilon < 1$ *and let* $[a,b] \subset \mathcal{D}_\beta$ *be as fixed above. Then there exist positive constants* c *and* ζ *such that the estimate*

$$\frac{\Big|\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\Big(\exp\Big\{\sum_1^N t_k H(\gamma_k) + \frac{i\tau}{n^{3/2}}A^R(\omega)\Big\} \mid \mathcal{D}\Big)\Big|}{\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\Big(\exp\Big\{\sum_1^N t_k H(\gamma_k)\Big\} \mid \mathcal{D}\Big)} \le e^{-c_1 n^{\varepsilon}} \tag{3.43}$$

is true, for all n *large enough, uniformly in* $n^{1/2-\varepsilon} \le |\tau| \le \pi n^{3/2}$*, in regular partitions* $\mathcal{D} \equiv \mathcal{D}_n$ *of* n*, and in all collections of tilts* $t_1, \ldots, t_N \in [a,b]$.

The rest of this section is devoted to the proof of this lemma. It will be performed in several steps, the key idea being as follows.

Let ω be any $\mathcal{D}$-compatible bridge. Using the microscopic decomposition of the real area,

$$A^R(\omega) = \sum_1^N n_j H(\gamma_j) + \sum_1^N a(\gamma_j), \qquad \text{with } n_j = d_j/2 + \sum_{k=j}^N d_k,$$

we rewrite the LHS of (3.43) as

$$\prod_1^N \mathbb{E}^{t_j}_{d_j}\Big(\exp\Big\{\frac{i\tau}{n^{3/2}}\big(n_j H(\gamma_j) + a(\gamma_j)\big)\Big\}\Big), \tag{3.44}$$

where $\mathbb{E}^t_d(\cdot)$ denotes the expectation corresponding to the tilted irreducible distribution $\mathbb{Q}^t_d$, cf. (3.36).

Let $\varepsilon > 0$ be as fixed above. We claim that the following three properties hold true uniformly in $3 \le d \le n^{\varepsilon}$.

a) There exists a positive constant c_1 *such that uniformly in* $|s| \le n^{-\varepsilon}$ *and* $t \in [a,b]$,

$$\big|\mathbb{E}^t_d\big(e^{isH(\gamma)}\big)\big| \le 1 - c_1 s^2; \tag{3.45}$$

b) There exists a positive constant c_2 *such that uniformly in* $|s| \le n^{-2\varepsilon}$ *and* $|h| \le r_2 n^{\varepsilon}$,

$$\mathbb{E}_d\big(e^{isa(\gamma)} \mid H(\gamma) = h\big) \ge 1 - c_2 s^2 n^{4\varepsilon}; \tag{3.46}$$

c) There exists a positive constant c_3 *such that uniformly in* $|s| \le \pi$ *and* $|h| \le r_2 n^{\varepsilon}$,

$$\mathbb{E}_d\big(e^{isa(\gamma)} \mid H(\gamma) = h\big) \le e^{-c_3 s^2}. \tag{3.47}$$

Postponing the proof of these properties for a while, we deduce first the claim (3.43) of the lemma.

To begin with, suppose that $n^{1+\varepsilon} < |\tau| \le \pi n^{3/2}$. Then the absolute value of the product (3.44) is bounded above, thanks to estimate (3.47), by

$$\begin{aligned}
&\bigotimes_1^N \mathbb{Q}^{t_j}_{d_j}\bigl(\max_j |H(\gamma_j)| > r_2 n^{\varepsilon}\bigr) \\
&\qquad\qquad + \Bigl(\max_{3\le d<n^{\varepsilon}}\ \max_{|h|\le r_2 n^{\varepsilon}} \mathbb{E}_d\bigl(e^{i\tau n^{-3/2}a(\gamma)} \mid H(\gamma)=h\bigr)\Bigr)^{n^{1-\varepsilon}} \\
&\le \exp\bigl\{-c_4 n^{\varepsilon}\bigr\} + \exp\bigl\{-c_3\tau^2 n^{-3} n^{1-\varepsilon}\bigr\} \le \exp\bigl\{-c_5 n^{\varepsilon}\bigr\}.
\end{aligned}$$

On the other hand, for $n^{1/2-\varepsilon} \le |\tau| \le n^{3/2-5\varepsilon}$ we decompose

$$\begin{aligned}
&\Bigl|\mathbb{E}^{t_j}_{d_j}\Bigl(\exp\Bigl\{\frac{i\tau}{n^{3/2}}\bigl(n_j H(\gamma_j) + a(\gamma_j)\bigr)\Bigr\}\Bigr)\Bigr| \\
&\qquad \le \Bigl|\mathbb{E}^{t_j}_{d_j}\Bigl(\exp\Bigl\{\frac{i\tau}{n^{3/2}} n_j H(\gamma_j)\Bigr\}\Bigr)\Bigr| + \mathbb{Q}^{t_j}_{d_j}\bigl(|H(\gamma_j)| > r_2 n^{\varepsilon}\bigr) \qquad (3.48)\\
&\qquad + \max_{|h|\le r_2 n^{\varepsilon}} \mathbb{E}_{d_j}\Bigl(1 - \exp\Bigl\{\frac{i\tau}{n^{3/2}} a(\gamma_j)\Bigr\} \mid H(\gamma_j)=h\Bigr).
\end{aligned}$$

Now, in each interval

$$\frac{n^{3/2}}{(k+1)n^{2\varepsilon}} < |\tau| \le \frac{n^{3/2}}{k n^{2\varepsilon}}, \qquad k = n^{3\varepsilon}, \dots, n^{1-\varepsilon}, \tag{3.49}$$

we take only those factors in (3.44), which verify the condition $n_j \le k n^{\varepsilon}$; as a result, $\bigl|\tau n_j n^{-3/2}\bigr| \le n^{-\varepsilon}$, and, applying the estimates (3.45)–(3.46), we majorate the RHS of (3.48) by

$$1 - c_1\frac{{n_j}^2}{n^3}\tau^2 + c_2\frac{n^{4\varepsilon}}{n^3}\tau^2 \le \exp\Bigl\{-c_6\frac{{n_j}^2}{n^3}\tau^2\Bigr\}$$

if only $n_j \ge c_7 n^{2\varepsilon}$. In this way we bound above the absolute value of the product (3.44) by $\exp\bigl\{-c_6\tau^2 n^{-3}\sum {n_j}^2\bigr\}$ with the sum running over all j for which the following conditions hold:

$$c_7 n^{2\varepsilon} \le n_j \le k n^{\varepsilon}, \qquad 3 \le d_j.$$

However, due to the very definition of regular partitions $\mathcal{D}$ of n, this sum is uniformly bounded below by

$$n^{2\varepsilon} \sum_{l=[c_7 n^{\varepsilon}]+1}^{k} l^2 \ge c_8 n^{2\varepsilon} k^3,$$

and as a result, the absolute value of the product (3.44) is majorated, uniformly in τ from (3.49), by

$$\exp\Bigl\{-c_9\frac{\tau^2}{n^3} n^{2\varepsilon} k^3\Bigr\} \le \exp\Bigl\{-c_9\frac{k^3}{n^{2\varepsilon}(k+1)^2}\Bigr\} \le \exp\bigl\{-c_{10} n^{\varepsilon}\bigr\}.$$

It remains to observe that $0 < 12\varepsilon < 1$ implies $3/2 - 5\varepsilon > 1 + \varepsilon$ and thus the two considered intervals cover the whole region of the values of τ mentioned in the lemma. The target bound (3.43) follows.

We turn now to the proof of the properties (3.45)–(3.47) used above.

Step 1: proof of (3.45). The key observation behind the bound (3.45) is the uniform non-degeneracy of the variance $\mathsf{Var}_d^t\big(H(\gamma)\big)$ of $H(\gamma)$ under the tilted distribution $\mathbb{Q}_{t,d}^{\mathsf{reg}}(\,\cdot\,)$.

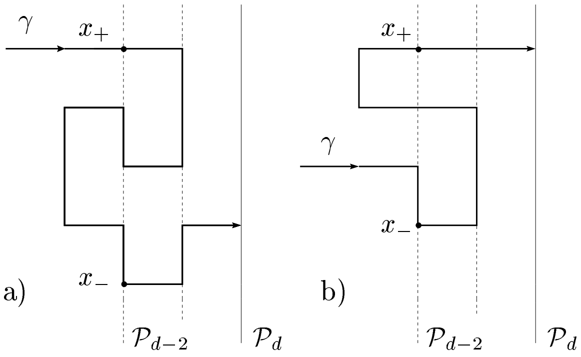


Figure 3.11. The points $x_+(\gamma)$ and $x_+(\gamma)$: a) the case $\gamma \in \mathcal{F}_{d,-}$; b) the case $\gamma \in \mathcal{F}_{d,+}$

Namely, given d, $3 \le d \le n^{\varepsilon}$, and an irreducible bridge γ of width d,

$$\gamma = \big(\gamma(0),\dots,\gamma(m)\big) \in \mathcal{F}_d,$$

define $x_+(\gamma)$ and $x_-(\gamma)$ as the highest and the lowest points of the intersection $\gamma \cap \mathcal{P}_{d-2}$ respectively (Fig. 3.11). Defining further $m_+(\gamma)$ and $m_-(\gamma)$ via

$$x_+(\gamma) = \gamma(m_+) \qquad \text{and} \qquad x_-(\gamma) = \gamma(m_-),$$

we decompose $\mathcal{F}_d = \mathcal{F}_{d,-} \vee \mathcal{F}_{d,+}$ according to which relation, $m_+ < m_-$ or $m_- < m_+$, is satisfied. By the conditional variance inequality it is enough to show that for some $c_{11} > 0$, the estimate

$$\min\Big(\mathsf{Var}_d^t\big(H(\gamma) \mid \mathcal{F}_{d,+}\big), \mathsf{Var}_d^t\big(H(\gamma) \mid \mathcal{F}_{d,-}\big)\Big) \ge c_{11} > 0 \tag{3.50}$$

holds uniformly in $t \in [a,b]$ and $3 \le d \le n^{\varepsilon}$.

To this end, take any path $\gamma = \big(\gamma(0),...,\gamma(m)\big) \in \mathcal{F}_{d,-}$ and split it as $\gamma_{\mathsf{bulk}} \vee \gamma_{\mathsf{tail}}$, where

$$\gamma_{\mathsf{bulk}} \overset{\text{def}}{=} \big(\gamma(0),...,\gamma(m_-)\big), \qquad \gamma_{\mathsf{tail}} \overset{\text{def}}{=} \big(\gamma(m_-),...,\gamma(m)\big).$$

The bulk part γ_{bulk} evidently satisfies:

(a) γ_{bulk} has no break points in the strip $\mathcal{S}_{[0,d-2]}$

(b) The set $\gamma_{\mathsf{bulk}} \cap \mathcal{P}_{d-1}$ lies above the point $x_- + \vec{e}_1$,

and, consequently, the tail part γ_{tail} is under no constraint to go down along the line $\mathcal{P}_{d-1}$. Since, by (1.24),

$$\sum_{\gamma_{\mathsf{tail}}} \exp\big\{th(\gamma_{\mathsf{tail}}) - \beta|\gamma_{\mathsf{tail}}|\big\} \le \sum_{k\in\mathbb{Z}} e^{tk} g_\beta\big((2,k)\big) \le c_{12}\mathbb{B}_2(t) < \infty$$

uniformly in $t \in [a,b]$, we infer from the above that (one-path estimate)

$$^{\forall}h = 1,2,... \qquad \mathbb{Q}_d^t\big(h(\gamma_{\mathsf{tail}}) = -h \mid \mathcal{F}_{d,-}\big) \ge c_{13}e^{-h(\beta+t)},$$

also uniformly in $3 \le d \le n^{\varepsilon}$ and $t \in [a,b]$. The set $\mathcal{F}_{d,+}$ can be treated similarly, and thus (3.50) follows.

Finally, the bound (3.45),

$$\left|\mathbb{E}_d^t\left(e^{isH(\gamma)}\right)\right| \leq 1 - c_1 s^2,$$

holds uniformly in $|s| \leq n^{-\varepsilon}$, $t \in [a,b]$, and $d \leq n^{\varepsilon}$.

Step 2: proof of (3.46). According to the definition of the irregular bridges, the upper bound

$$\mathsf{Var}_d\big(a(\gamma) \bigm| H(\gamma) = h\big) \leq c_{14} n^{4\varepsilon}$$

holds uniformly in $|h| \leq r_2 n^{\varepsilon}$ and $d \leq n^{\varepsilon}$. Consequently, the inequality (3.46),

$$\mathbb{E}_d^{\mathsf{reg}}\left(e^{isa(\gamma)} \bigm| H(\gamma) = h\right) \geq 1 - c_2 s^2 n^{4\varepsilon}$$

is valid uniformly in $|s| \leq n^{-2\varepsilon}$, $d \leq n^{\varepsilon}$, and $|h| \leq r_2 n^{\varepsilon}$.

Step 3: proof of (3.47). Fix a natural d, $3 \leq d \leq n^{\varepsilon}$, and $|h| \leq r_2 n^{\varepsilon}$. Recall that by definition, $a(\gamma)$ is either integer or half-integer (i.e., $a(\gamma) - 1/2 \in \mathbb{Z}$). For definiteness, we shall consider the first case, the second being its obvious modification.

For any $k \in \mathbb{Z}$, we put

$$p_k \overset{\mathsf{def}}{=} \mathbb{Q}_d\big(a(\gamma) = k \bigm| H(\gamma) = h\big).$$

Clearly, $p_k = p_{-k}$, and thus the corresponding characteristic function is real-valued. We claim that the bound (3.47) follows directly from the property

$$\forall k \in \mathbb{Z}, \quad \frac{1}{2} e^{-4\beta} \leq \frac{p_k}{p_{k+1}} \leq 2e^{4\beta}. \tag{3.51}$$

Indeed, once this property is established, we use the simple formula

$$\begin{aligned}\mathbb{E}_d\left(e^{isa(\gamma)} \bigm| H(\gamma) = h\right) &= \sum_{k=-\infty}^{\infty} p_k e^{isk} \\ &= \sum_{k=-\infty}^{\infty} (p_{2k} + p_{2k+1}) e^{i\tau 2k} \Big(\frac{p_{2k}}{p_{2k} + p_{2k+1}} + \frac{p_{2k+1}}{p_{2k} + p_{2k+1}} e^{i\tau}\Big)\end{aligned}$$

and observe that due to (3.51),

$$\mathbb{E}_d\left(e^{i\tau a(\gamma)} \bigm| H(\gamma) = h\right) \leq \max_{1 + e^{-4\beta}/2 \leq 1/\lambda \leq 1 + 2e^{4\beta}} \left|\lambda + (1-\lambda) e^{is}\right|.$$

The target bound (3.47) follows.

It remains to verify the uniform bound (3.51). By symmetry, it is enough to consider the case $h \leq 0$ and the ratios p_{k+1}/p_k. Define

$$\mathcal{F}_{d,k} = \big\{\gamma \in \mathcal{F}_d \bigm| a(\gamma) = k\big\}.$$

Then one can construct a bounded degree map $\Phi_{d,k} : \mathcal{F}_{d,k} \to \mathcal{F}_{d,k+1}$, such that for every $k \in \mathbb{Z}$ the length $|\Phi_{d,k}(\gamma)| \leq |\gamma| + const.$

For example, consider the map $\Phi_{d,k}$ described on Fig. 3.12: given a path $\omega = (\omega(0), ..., \omega(N)) \in \mathcal{F}_{d,k}$ let us say that a horizontal segment $[\omega(l), \omega(l+1)]$ is an inner maximum, if $l > 0$, $l + 1 < N$ and the vertical coordinate of $\omega(l)$ is larger or equal than the vertical coordinate of any other point $\omega(j)$; $j = 0, ..., N$, along the path ω. Then for any $\omega \in \mathcal{F}_{d,k}$ whose set of inner maximums is not empty, the map $\Phi_{d,k}$ puts the unit square on the top of the left-most inner maximum $[\omega(l), \omega(l+1)]$ (Fig. 3.12 a) and right-to-left arrows on Fig. 3.12 e) and f)). All the

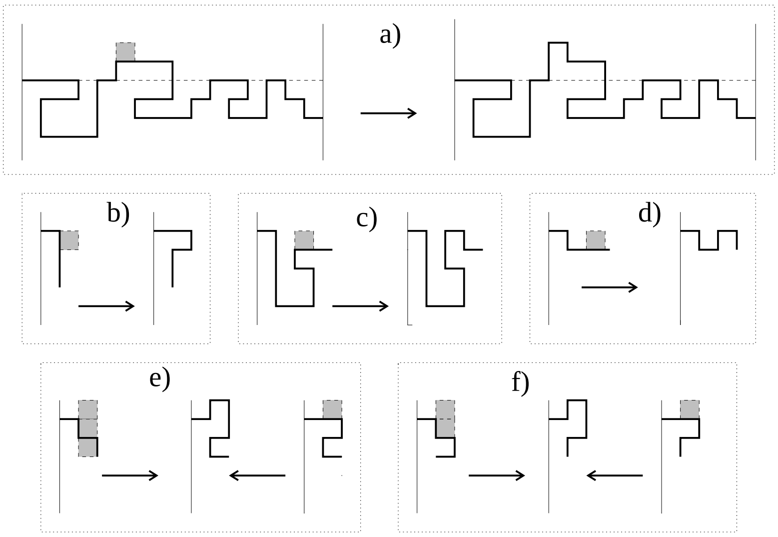


FIGURE 3.12. A map $\Phi_{d,k} : \mathcal{F}_{d,k}$ into $\mathcal{F}_{d,k+1}$ in the case $h \geq 0$:
a) and right-to-left arrows in e), f): Placing a square on the top of the left-most inner maximum of the path
b)-d) and left-to-write arrows on e)-f): Attaching squares in the cases without inner maximums.

possible cases of $\omega \in \mathcal{F}_{d,k}$ without inner maximums[1] correspond to Fig. 3.12 b)-d) and left-to-right arrows on Fig. 3.12 e),f) along with the corresponding algorithms for constructing the map $\Phi_{d,k}$. As a result, $\Phi_{d,k}$ is at most two-to-one (it is one-to-one except the cases e),f) on Fig. 3.12), and the length of the image $\Phi_{d,k}(\gamma)$ always satisfies $|\Phi_{d,k}(\gamma)| \leq |\gamma| + 4$.

The inequality (3.51) follows and thus the proof of Lemma 3.7 is finished. □

3.6. **Residual Bridges.** Our proof of Proposition 3.4 involves several reduction steps. The first such step is, actually, already accomplished — it is the tube estimate (3.4), which enables us to restrict our attention to the set of bridges $\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ sitting in the n^ν-tube around the rescaled minimiser nj_{q_n}. The main import of the tube constraint is the possibility to control areas under the conjugate tilts. However, the membership in $\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ is a global condition and as such is not convenient to work with. Thus our second reduction step is to introduce a somewhat localised version of $\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$.

Definition 3.15. *Given $s, k \in \mathbb{R}_+$, we define $\mathcal{B}^{\mathsf{loc}}_{n,s,k}$ as the set of bridges ω such that every embedded irreducible sub-bridge $\gamma \overset{\mathsf{e}}{\subset} \omega$ satisfies the conditions*

$$W(\gamma) \leq 2ks \qquad \text{and} \qquad |a(\gamma)| \leq 4sW(\gamma).$$

---

[1] A straightforward combinatorial consideration shows that indeed these are all typical cases (all the paths without inner maximum can be classified according to whether they pass or not through the point $(x_0 + 2, y_0 - 1)$, where (x_0, y_0) is the left-hand side endpoint of a path).

The locality property of bridges from $\mathcal{B}^{\mathsf{loc}}_{n,s,k}$ is reflected in the following important relation:

$$\omega_1 \in \mathcal{B}^{\mathsf{loc}}_{n_1,s,k} \quad \text{and} \quad \omega_2 \in \mathcal{B}^{\mathsf{loc}}_{n_2,s,k} \qquad \Longleftrightarrow \qquad \omega_1 \vee \omega_2 \in \mathcal{B}^{\mathsf{loc}}_{n_1+n_2,s,k}. \tag{3.52}$$

We shall eventually work with (varying) $s \sim n^\nu$ and (fixed) k large enough.

3.6.1. *Localised tube condition.* All the reduction estimates we shall develop in the sequel rely on the property (3.52). The first of these estimates is a rather coarse bound, which, in view of (3.4), suggests that $\mathcal{B}^{\mathsf{loc}}_{n,n^\nu,k}$ is, indeed, a reasonable ensemble to consider:

Lemma 3.16. *Let* $\frac{1}{2} < \nu < 1$. *There exist* $c_1, c_2, c_3 > 0$, *such that for every fixed* $k > 0$ *the inequality*

$$\mathbb{B}^{\mathsf{tube}}_{n,\nu,q_n}\left(\omega \notin \mathcal{B}^{\mathsf{loc}}_{n,n^\nu,k}\right) \leq c_1 \exp\left\{-n\psi(q_n) + c_2 n^\nu - c_3 k n^\nu\right\} \tag{3.53}$$

holds, for all n *large enough, uniformly in* $\{q_n\}$ *satisfying the condition (3.1).*

Proof. Since the curvature of the Wulff shape j_{q_n} is bounded above and $\nu < 1$, any embedded irreducible sub-bridge γ of a bridge $\omega \in \mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ such that $W(\gamma) \leq 2kn^\nu$ satisfies, for fixed $k > 0$ and n large enough, the inequality $|a(\gamma)| \leq 4n^\nu W(\gamma)$. Thus, the claim of the lemma follows directly from the bound

$$\mathbb{B}^{\mathsf{tube}}_{n,\nu,q_n}\left(\exists \gamma \overset{e}{\subset} \omega \text{ irreducible with } W(\gamma) \geq 2kn^\nu\right) \leq c_4 e^{-n\psi(q_n) + c_5 n^\nu - c_6 k n^\nu}$$

with (any) fixed $k > 0$ and all n large enough.

To check this inequality, we fix $k > 0$ and assume, without loss of generality, that $m \overset{\mathsf{def}}{=} kn^\nu$ divides n. Set $M = n/m$, and, given a bridge $\omega \in \mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ define x_0, x_1, ..., x_M to be the first hitting points by ω of the vertical lines $\mathcal{P}_0, \mathcal{P}_m, \mathcal{P}_{2m}, ..., \mathcal{P}_n$. Clearly, the points x_0, ..., x_M generate the (mesoscopic) splitting of ω into disjoint components, $\omega = \eta_1 \vee \cdots \vee \eta_M$ (Fig. 3.13). Given a vector $\underline{x} = (x_0, x_1, ..., x_M)$ with $x_j \in \mathcal{P}_{jm}$; $j = 0, ..., M$, consider the family $\mathcal{B}^{\mathsf{tube},\underline{x}}_{n,\nu,q_n}$ of the tube bridges having $\underline{x}$ as

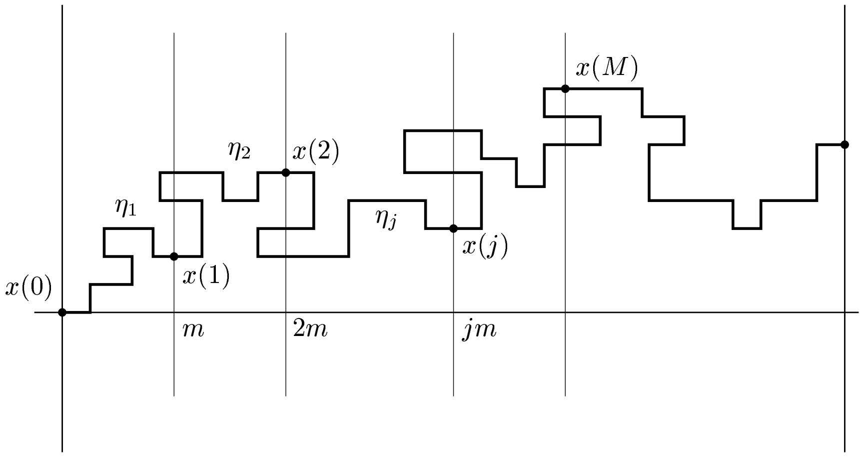


FIGURE 3.13. The mesoscopic splitting $\omega = \eta_1 \vee \cdots \vee \eta_M$ of a bridge ω

the collection of their hitting points. Of course, $\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n}$ can be represented as the disjoint union

$$\mathcal{B}^{\mathsf{tube}}_{n,\nu,q_n} = \bigvee_{\underline{x}} \mathcal{B}^{\mathsf{tube},\underline{x}}_{n,\nu,q_n},$$

over at most $c_7(2n^\nu)^M$ vectors $\underline{x}$, which we shall call ν-tube compatible. Thus, it is enough to show that the following estimate holds uniformly in $j = 1, ..., M$, sequences $\{q_n\}$ satisfying (3.1), and in ν-tube compatible vectors $\underline{x}$:

$$\mathbb{B}_n \left(\mathcal{B}^{\mathsf{tube},\underline{x}}_{n,\nu,q_n};\ \begin{matrix} \omega \text{ has no break points} \\ \text{in the strip } S_{[(j-2/3)m,(j-1/3)m]} \end{matrix} \right) \leq e^{-c_8 m} e^{-n\psi(q_n)+c_9 n^\nu}. \tag{3.54}$$

In order to verify (3.54), note that for every ν-compatible vector $\underline{x} = (x_0, ..., x_M)$ all the increments satisfy

$$\max_{j=1,\dots,M-1} \frac{|x_{j+1} - x_j|}{m} \leq c_{10}. \tag{3.55}$$

Since the latter estimate only improves with the growth of k in $m = kn^\nu$, we can assume that there exists $r > 0$ such that the differences $x_{j+1} - x_j$ belong to the cone $\mathcal{C}_r$ uniformly in $\underline{x}$ and j.

Recalling the natural splitting $\omega = \eta_1 \vee \cdots \vee \eta_M$ of any path $\omega \in \mathcal{B}^{\mathsf{tube},\underline{x}}_{n,\nu,q_n}$ into the disjoint union of paths η_j with the end-points at $x_0, x_1, \dots, x_M$ (see Fig. 3.14 b)), we observe that a bridge ω has no break-points in the strip $S_{[(j-2/3)m,(j-1/3)m]}$ either if the path η_j has no break points there, or if one of η_i-s has a long backtrack hitting $S_{[(j-2/3)m,(j-1/3)m]}$. The former event will be rendered asymptotically improbable thanks to the mass-separation bound (1.23), whereas the latter one will be ruled out by the strict triangle inequality (1.8).

More precisely, let us define the following sub-collections of bridges from $\mathcal{B}^{\mathsf{tube},\underline{x}}_{n,\nu,q_n}$: for every $j = 1, ..., M$ set

$$\mathcal{A}^j_{\underline{x}} = \Big\{ \omega \ \Big| \ \eta_j \text{ has no break points in } S_{[(j-2/3)m,(j-1/3)m]} \Big\},$$

and, for every pair $i > j$,

$$\mathcal{A}^{i,j}_{\underline{x}} = \Big\{ \omega \ \big| \ \eta_i \cap S_{[(j-2/3)m,(j-1/3)m]} \neq \varnothing \Big\}.$$

Then, for every $j = 1, ..., M$,

$$\left\{ \omega \in \mathcal{B}^{\mathsf{tube}}_{n,\underline{x}};\ \begin{matrix} \omega \text{ has no break points} \\ \text{in the strip } S_{[(j-2/3)m,(j-1/3)m]} \end{matrix} \right\} \subseteq \mathcal{A}^j_{\underline{x}} \bigcup \Big\{ \bigcup_{i>j} \mathcal{A}^{i,j}_{\underline{x}} \Big\}.$$

By (3.55) we are entitled to use the separation of decay rates result (1.23) uniformly in ν-tube compatible $\underline{x}$ and $j = 1, ..., M$. Consequently,

$$\mathbb{B}_n \left(\mathcal{A}^j_{\underline{x}} \right) \leq e^{-c_{11} m} \prod_{k=1}^M g_\beta \left(x_k - x_{k-1} \right).$$

Similarly, the strict triangle inequality (1.8) implies that

$$\mathbb{B}_n \left(\mathcal{A}^{i,j}_{\underline{x}} \right) \leq e^{-c_{12}(i-j)m} \prod_{k=1}^M g_\beta \left(x_k - x_{k-1} \right),$$

uniformly in the ν-tube compatible vectors $\underline{x}$ and in the couples of indexes $j < i$. Since by (1.6),

$$\max_{\underline{x}} \prod_{k=1}^{M} g_\beta\left(x_k - x_{k-1}\right) \le e^{-n\psi(q_n)+c_9 n^\nu},$$

we arrive at (3.54) and thus finish the proof. □

3.6.2. *Reduction to conjugate tilts.* By the preceding step,

$$\mathbb{B}_n\left(A^R = n^2 q_n\right) = \mathbb{B}_n\left(A^R = n^2 q_n; \mathcal{B}^{\mathsf{loc}}_{n,n^\nu,k}\right) + o\left(e^{-n\psi(q_n)-c_{13}n^\nu}\right),$$

as $n \to \infty$, if only k is sufficiently large. Let $t_n = t(q_n)$ be the conjugate tilt as specified in (1.27). By the usual "change of measure" argument,

$$\begin{aligned}\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}&\left(A^R = n^2 q_n; \omega \notin \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\right)\\ &= e^{-nt_n q_n}\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\left(e^{t_n A^R/n}, A^R = n^2 q_n; \omega \notin \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\right).\end{aligned}$$

Thus, the target estimate (3.12) of Proposition 3.4 is a direct consequence of the asymptotic behaviour (3.17) of the regular partition functions $\mathbb{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\left(e^{t_n A^R/n}; n^2 q_n\right)$, Lemmas 3.6 and 3.7, combined with the following claim:

Lemma 3.17. *Fix $[a,b] \subset \mathcal{D}_\beta$ and let positive constants ε, α, and ν be as above. There exists a positive constant c_{14} such that uniformly in n and in $t \in [a,b]$ we have*

$$\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\left(e^{tA^R/n}; \omega \notin \mathcal{B}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\right) \le e^{-c_{14}n^\varepsilon}\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\left(e^{tA^R/n}\right). \tag{3.56}$$

Proof. We consider separately each of the three conditions in Definition 3.2 of regular bridges.

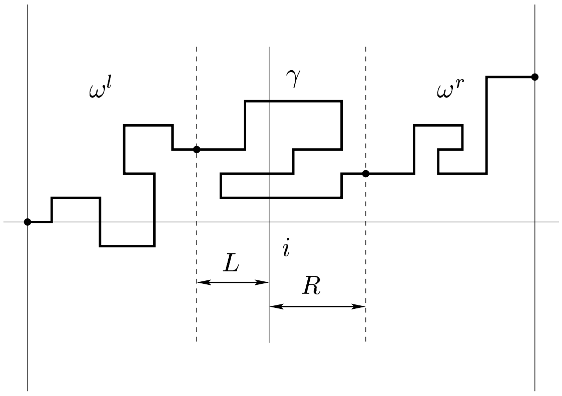


FIGURE 3.14. Decomposition (3.57)

STEP 1: Condition **(R1)**. With $i \in \{1,\dots,n-1\}$ fixed, any bridge $\omega \in \mathcal{B}^{\mathsf{loc}}_{n,n^\nu,k}$ can be split as (Fig. 3.14)

$$\omega = \omega^l \vee \gamma \vee \omega^r, \tag{3.57}$$

where γ is the irreducible bridge sitting on $\mathcal{P}_i$ and ω^l and ω^r are the corresponding left and right embedded sub-bridges of ω. Of course, if $i < 2kn^\nu$ (respectively $k > n - 2kn^\nu$), the part ω^l (respectively ω^r) can be empty. Since the span of γ is $(i-L, i+R)$, the signed real area $A^R(\omega)$ under the path of ω equals

$$\begin{aligned} A^R(\omega) = A^R(\omega^l) + H(\omega^l)(n-i+L) \\ + a(\gamma) + H(\gamma)\big(n - i + (L-R)/2\big) + A^R(\omega^r). \end{aligned}$$

By the locality property (3.52), we have the inclusions

$$\omega^l \in \mathcal{B}^{\mathsf{loc}}_{i-L,n^\nu,k}, \qquad \gamma \in \mathcal{B}^{\mathsf{loc}}_{L+R,n^\nu,k}, \qquad \omega^r \in \mathcal{B}^{\mathsf{loc}}_{n-i-R,n^\nu,k}.$$

In particular, the irreducible part γ satisfies the following apriori bound:

$$|a(\gamma)| n^{-1} \leq 4(L+R)n^{\nu-1} \ll (L+R). \tag{3.58}$$

As a result,

$$\begin{aligned} &\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\Big(e^{\frac{t}{n}A^R}; W(\gamma) > n^\varepsilon\Big) \\ &\leq \sum_{L+R=n^\varepsilon+1}^{2kn^\nu} \mathbb{B}^{\mathsf{loc}}_{i-L,n^\nu,k}\Big(\exp\Big\{\frac{t}{n}A^R + t\Big(1 - \frac{i-L}{n}\Big)H\Big\}\Big) \\ &\quad \times e^{4|t|(L+R)n^{\nu-1}} \mathbb{F}_{L+R}\Big(t\Big(1 - \frac{2i+(R-L)}{2n}\Big)\Big) \mathbb{B}^{\mathsf{loc}}_{n-i-R,n^\nu,k}\Big(e^{\frac{t}{n}A^R}\Big). \end{aligned} \tag{3.59}$$

By the separation of decay rates bound (1.24),

$$\mathbb{B}_{L+R}(t) = \mathbb{B}^{\mathsf{loc}}_{L+R,n^\nu,k}(t)\,(1 + o(1))\,.$$

Together with the strict triangle inequality (1.8) and the apriori bound (3.58) on the area of a local path $\gamma \in \mathbb{B}^{\mathsf{loc}}_{L+R,n^\nu,k}$, this implies the bound

$$\mathbb{F}_{L+R}(t_1) \leq e^{-c_{15}(L+R)}\mathbb{B}_{L+R}(t_1) \leq e^{-c_{16}(L+R)}\mathbb{B}^{\mathsf{loc}}_{L+R,n^\nu,k}\Big(e^{t_1 H + \frac{t_2}{n}a(\gamma)}\Big)\,(1 + o(1))\,,$$

which holds uniformly in n, $n^\varepsilon \leq L+R \leq 2kn^\nu$ and $t_1, t_2 \in [a,b] \subset \mathcal{D}_\beta$. Substituting the last two inequalities into (3.59) we, thus, obtain:

$$\begin{aligned} \mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\Big(e^{\frac{t_n}{n}A^R}; W(\gamma) > n^\varepsilon\Big) &\leq C\big(2kn^\nu\big)^2 e^{-c_{16}n^\varepsilon}\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\Big(e^{\frac{t_n}{n}A^R}\Big), \\ &\leq Ce^{-c_{17}n^\varepsilon}\mathbb{B}^{\mathsf{loc}}_{n,n^\nu,k}\Big(e^{\frac{t_n}{n}A^R}\Big) \end{aligned} \tag{3.60}$$

for every $i \in \{1, ..., n\}$.

Step 2: Condition **(R2)**. By the preceding step we may restrict attention only to those bridges $\omega \in \mathcal{B}^{\mathsf{loc}}_{n,n^\nu,k}$ whose irreducible sub-bridges have spans shorter than n^ε. We claim that uniformly in $m = 1, ..., n^\varepsilon$ and $t_1, t_2 \in [a,b]$,

$$\begin{aligned} \mathbb{F}^{\mathsf{loc}}_{m,n^\nu,k}\Big(e^{\frac{t_1}{n}A^R + t_2 H}; |H(\gamma)| > r_2 n^\varepsilon \text{ or } |a(\gamma)| > r_2 n^{2\varepsilon}\Big) \\ \leq e^{-c_{18}n^\varepsilon}\mathbb{F}^{\mathsf{loc}}_{m,n^\nu,k}\Big(e^{\frac{t_1}{n}A^R + t_2 H}\Big), \end{aligned}$$

as soon as r_2 is chosen to be sufficiently large. Indeed, assuming that ε is small enough to satisfy $\varepsilon + \nu < 1$, we obtain $A^R(\gamma)/n = o(1)$ uniformly in $\gamma \in \mathcal{F}^{\mathsf{loc}}_{m,n^\nu,k}$. Since $\frac{1}{m}\log \mathbb{F}^{\mathsf{loc}}_{m,n^\nu,k}(t)$ is bounded above and below uniformly in m and in $t \in [a,b] \subset \mathcal{D}_\beta$, the last estimate follows from the Hölder inequality and the strict positivity of the connectivity decay rates τ_β, once the constant r_2 has been chosen to be sufficiently large.

Going back to the splitting (3.57) and, accordingly, adjusting the analysis of the decomposition formula (3.59) with respect to the event

$$\{|H(\gamma)| > r_2 n^{\varepsilon} \text{ or } |a(\gamma)| > r_2 n^{2\varepsilon}\},$$

we conclude that

$$\begin{aligned}\mathbb{B}^{\mathsf{loc}}_{n,n^{\nu},k}\Big(e^{\frac{t}{n}A^R};\ \exists\gamma \overset{e}{\subset} \omega \text{ irreducible with } |H(\gamma)| > r_2 n^{\varepsilon} \text{ or } |a(\gamma)| > r_2 n^{2\varepsilon}\Big)\\ \le\ e^{-c_{19}n^{\varepsilon}}\mathbb{B}^{\mathsf{loc}}_{n,n^{\nu},k}\Big(e^{\frac{t}{n}A^R}\Big).\end{aligned} \tag{3.61}$$

Step 3: Condition **(R3)**. With properties (3.60) and (3.61) established, we shall restrict our attention only to those $\omega \in \mathcal{B}^{\mathsf{loc}}_{n,n^{\nu},k}$ which admit the decomposition (3.8) with $|H(\omega_j)| \le r_2 n^{\alpha+\varepsilon}$, and, consequently, $|A^R(\omega_j)| \le r_2 n^{2\alpha+\varepsilon}$, for every $j = 1, ..., n^{1-\alpha}$ (as all the remaining bridges are rendered asymptotically improbable by the strict triangle inequality (1.8)). Therefore, thanks to the scaling assumption (3.10) we can assume that

$$\frac{|A^R(\omega_j)|}{n} = o\,(1) \tag{3.62}$$

uniformly in $j = 1, ..., n^{1-\alpha}$ and the bridges ω_j in the decomposition (3.8). The length of the span of ω_j is at most n^{α}. Because of (3.62) and the locality property (3.52) it remains to show that uniformly in $m \le n^{\alpha}$ and $t \in [a, b] \subset \mathcal{D}_{\beta}$,

$$\begin{aligned}\mathbb{B}^{\mathsf{loc}}_{m,n^{\nu}k}\left(e^{tH(\omega)}; |H(\omega)| > r_3 n^{\alpha} \text{ or } |a(\omega)| > r_3 n^{3/2\alpha+\varepsilon}\right)\\ \le e^{-c_{19}n^{\varepsilon}}\mathbb{B}^{\mathsf{loc}}_{m,n^{\nu}k}\left(e^{tH(\omega)}\right).\end{aligned}$$

However, as in the preceding step, this already follows in a standard way from the Hölder inequality and the strict triangle inequality (1.8), as soon as the constant r_3 is large enough. The proof of Lemma 3.17 is completed. □

4. Asymptotics of corner partition functions

The proof of Proposition 1.7 closely follows the line of reasoning developed in Sect. 3. As in the proof of Theorem 3.1 we employ the $K = n^{1-\alpha}$ mesoscopic splitting of a path $\omega \in \mathcal{A}_n$ with respect to the irreducible animals $\gamma_1, \dots, \gamma_{K-1}$ which happen to sit on the poles $\mathcal{P}_{jn^{\alpha}}$. Exactly as it has been done in the proof of (3.60) it is possible to rule out the case of $\max_j W(\gamma_j) > n^{\varepsilon}$, and consider only those paths $\omega \in \mathcal{A}_n$ which admit the disjoint mesoscopic decomposition

$$\omega\ = \omega_1 \vee \gamma_1 \vee \omega_2 \vee \cdots \vee \gamma_{K-1} \vee \omega_K. \tag{4.1}$$

The decomposition (4.1) differs from (3.8) in two respects: first of all, ω_1 is a corner path; in addition, all the bridges $\gamma_1, \dots, \omega_K$ are subject to the global constraint to stay above the horizontal semi-axis $\mathcal{L}_0^-$. The latter constraint, however, has little impact on the asymptotic properties of $\mathbb{A}_n\left(n^2 q_n\right)$ whenever the sequence of rescaled areas $\{q_n\}$ is bounded away from zero.

Indeed, using $\mathcal{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ to denote the ensemble of regular corner paths satisfying (R1)-(R3) of Definition 3.2 and $\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$ to denote the corresponding partition function, we follow literally the proof of Proposition 3.4 to deduce:

$$\mathbb{A}_n\big(n^2 q_n\big)\ =\ \mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(n^2 q_n\big)\ +\ o\big(e^{-n\psi(q_n)-c_1 n^{\varepsilon}}\big).$$

Next, tilting the area A^R by the conjugate value $t = t(q_n)$, we obtain (cf. (3.13)):

$$\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(n^2 q_n\big) \;=\; e^{-ntq_n}\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^R/n};\, n^2 q_n\big).$$

Further, using an argument similar to the one employed in the proof of Lemma 3.5 we get the relation

$$\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^R/n};\, n^2 q_n\big) \;=\; \mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n};\, n^2 q_n\big)\big(1 + o\,(1)\big)$$

and factorise the partition function in the RHS as

$$\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n};\, n^2 q_n\big) \;=\; \mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big)\mathbb{P}^{\mathsf{reg}}_{t,n}\big(n^2 q_n\big). \tag{4.2}$$

Thus, as in the case of bridge partition functions, we have split the problem into two: finding the sharp asymptotics of $\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big)$ and deriving a local limit result for the tilted probability measure $\mathbb{P}^{\mathsf{reg}}_{t,n}(\cdot)$ in the ensemble of corner paths.

Remark 4.0.1. If the sequence of rescaled areas $\{q_n\}$ is bounded away from zero, then so is the sequence of conjugate tilts $\{t_n = t(q_n)\}$. Thus, since the averaged value of the height $H(\omega_1)$ is of the order n^α, we infer from the strict triangle inequality (1.8) that those remaining parts of the trajectories $\gamma_1 \vee \cdots \vee \omega_K$ in (4.1) which violate the global constraint to stay above the semi-axis $\mathcal{L}_0^-$ give a negligible contribution to either of the terms on the right hand side of (4.2).

As far as the $\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big)$ term is considered, let us remove the aforementioned constraint and denote the modified partition function $\tilde{\mathbb{A}}^{\mathsf{reg}}_{n,\alpha,\varepsilon}$. By (1.8),

$$\mathbb{A}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big) \;=\; \tilde{\mathbb{A}}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big)\big(1 + o\big(e^{-c_2(t)n^\alpha}\big)\big),$$

for every $t > 0$ fixed. In its turn, the partition function $\tilde{\mathbb{A}}^{\mathsf{reg}}_{n,\alpha,\varepsilon}\big(e^{tA^I/n}\big)$ can be studied along the lines of the proof of Lemma 3.6, except that in order to control the contribution of ω_1 one needs to develop a sharp asymptotic expression for the tilted corner generation function $\mathbb{A}_n(t)$.

As in the case of bridges, the local limit result for $\mathbb{P}^{\mathsf{reg}}_{t,n}\big(n^2 q_n\big)$ requires some care, but an obvious and straightforward modification of the arguments employed in the proof of Lemma 3.7 yields:

Claim 4.1. *Fix any $\delta > 0$. As $n \to \infty$,*

$$\mathbb{P}^{\mathsf{reg}}_{t,n}\big(A^R = n^2 q_n\big) \;=\; \frac{1}{\sqrt{2\pi n^3 \sigma(t_n)}}\big(1 + o\,(1)\big),$$

uniformly in the rescaled areas $\{q_n\}$ satisfying

$$\delta \;\le\; q_n \;\le\; \frac{q_{\beta,+}}{\tau_\beta(0,1)^2}(1-\delta). \tag{4.3}$$

Remark 4.1.1. The second inequality above plays the same role as (3.1) of Theorem 3.1, whereas the first constraint in (4.3) suppresses the effect of entropic repulsion of the components $\gamma_1, \ldots, \omega_K$ from $\mathcal{L}_0^-$ (this repulsion becomes non-negligible for $q_n \sim 0$).

To summarise the above discussion: essentially the only new ingredient required for the proof of Proposition 1.7 is an asymptotically sharp computation of the corner generating functions $\mathbb{A}_n(t) = \mathbb{A}_n\big(e^{tH(\omega)}\big)$. More precisely, we should prove the following corner counterpart of (3.28):

Theorem 4.2. *Let* $[a,b] \subset \mathcal{D}_\beta^+ \overset{\mathsf{def}}{=} \mathcal{D}_\beta \cap (0, \tau_\beta(0,1))$. *Then there exists a complex neighbourhood* $\mathcal{U}$ *of* $[a,b]$ *in* $\mathbb{C}$, *a non-vanishing analytic function* $\kappa(\cdot)$ *on* $\mathcal{U}$ *and a positive constant* $\tilde{\alpha} > 0$, *such that*

$$\left| e^{-nm_\beta(z)} \mathbb{A}_n(z) - \kappa(z) \right| \le \frac{1}{\tilde{\alpha}} e^{-\tilde{\alpha} n}, \tag{4.4}$$

uniformly in $n \in \mathbb{N}$ *and in* $z \in \mathcal{U}$.
Furthermore, the function κ *above is related to the bridge prefactor function* μ *(see (1.20)) as follows: For each* $t \in \mathcal{D}_\beta^+$ *define* $t^* \in \mathcal{D}_\beta^+$ *via*

$$m'_\beta(t) m'_\beta(t^*) \;=\; 1. \tag{4.5}$$

Then,

$$\kappa(t)\kappa(t^*) \;=\; \mu(t)\mu(t^*). \tag{4.6}$$

In particular, if t_0 *satisfies* $m'_\beta(t_0) = 1$, *then* $\kappa(t_0) = \mu(t_0)$.

The rest of the section is devoted to the proof of Theorem 4.2.

4.1. **Asymptotic behaviour of** $\mathbb{A}_n(z)$**.** Due to Remark 4.0.1, for every $t \in \mathcal{D}_\beta^+$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{A}_n(t) \;=\; m_\beta(t). \tag{4.7}$$

In order to derive sharper asymptotics as asserted in (4.4), it is convenient to investigate corner generating functions of the form $\mathbb{A}_{n+k}(z)$ enabling, thus, some degree of freedom in choosing n and k. Namely, we shall consider n and m to be of the same order and, moreover, request that they satisfy

$$\frac{1}{4} < \frac{k}{n} < \frac{k+1}{n} < 4. \tag{4.8}$$

Recall that the starting point of corner paths from $\mathcal{A}_{n+k}$ is $(-n-k, 0)$. By (1.23) most of these paths should have break lines in the interval $(-n-\varepsilon n, -n+\varepsilon n)$. To be precise, given $\omega \in \mathcal{A}_{n+k}$ let us use $\gamma = \gamma(\omega)$ to denote the irreducible animal of ω which sits on the pole $\mathcal{P}_{-n}$. Then (1.23) implies that for every fixed $[a,b] \subset \mathcal{D}_\beta^+$ there exists $c_1 > 0$, such that

$$\mathbb{A}_{n+k}\left(e^{tH(\omega)}\,;\, \mathsf{Span}(\gamma) \not\subset (-n-\varepsilon n, -n+\varepsilon n)\right) \;\le\; e^{-c_1 n + (n+k) m_\beta(t)} \tag{4.9}$$

uniformly in $t \in [a,b]$ and n sufficiently large.

Let us remove the condition to stay above the semi-axis $\mathcal{L}_0^-$ on the interval $(-n-\varepsilon n, 0)$ and denote the modified partition function $\mathbb{A}_{n+k,\varepsilon}(t)$. Since for each $t \in [a,b] \subset \mathcal{D}_\beta^+$ the average slope of the paths contributing to $\mathbb{A}_{n+k,\varepsilon}(t)$ is $m'_\beta(t) > 0$, we infer from the strict triangle inequality (1.8) that there exists $c_2 > 0$, such that

$$\mathbb{A}_{n+k,\varepsilon}(t) - \mathbb{A}_{n+k}(t) \;\le\; e^{-c_2 n + (n+k) m_\beta(t)}, \tag{4.10}$$

uniformly in $t \in [a,b]$ and n large enough.

Since m_β is analytic in a complex neighbourhood of $[a,b]$ and since, evidently,

$$|\mathbb{A}_{n+k,\varepsilon}(z) - \mathbb{A}_{n+k}(z)| \;\le\; \mathbb{A}_{n+k,\varepsilon}(\Re z) - \mathbb{A}_{n+k}(\Re z), \tag{4.11}$$

the inequalities (4.9) and (4.10) actually hold in some complex neighbourhood $\mathcal{U}$ of $[a,b]$: There exists $c_3 > 0$, such that the relations

$$\begin{aligned} \mathbb{A}_{n+k}\left(e^{zH(\omega)}\,;\mathsf{Span}(\gamma)\not\subset(-n-\varepsilon n,-n+\varepsilon n)\right) &= o\left(e^{-c_3 n+(n+k)m_\beta(z)}\right) \\ \mathbb{A}_{n+k,\varepsilon}(z)-\mathbb{A}_{n+k}(z) &= o\left(e^{-c_3 n+(n+k)m_\beta(z)}\right) \end{aligned} \tag{4.12}$$

hold uniformly in $z\in\mathcal{U}$. Therefore,

$$\mathbb{A}_{n+k}(z) \;=\; \sum_{l=0}^{\varepsilon n}\sum_{r=0}^{\varepsilon n}\mathbb{A}_{n-l}(z)\mathbb{F}_{l+r}(z)\mathbb{B}_{k-r}(z) \;+\; o\left(e^{-c_3 n+(n+k)m_\beta(z)}\right), \tag{4.13}$$

also uniformly in $z\in\mathcal{U}$.

Define now

$$\kappa_n(z) = e^{-nm_\beta(z)}\mathbb{A}_n(z).$$

Multiplying both sides of (4.13) by $e^{-(n+k)m_\beta(z)}$ and using the relation

$$e^{-nm_\beta(z)}\mathbb{B}_n(z) = \mu(z)\left(1+o(e^{-c_4 n})\right)$$

valid uniformly in n large enough and in z from (possibly further shrinked) complex neighbourhood $\mathcal{U}$ of $[a,b]$, we arrive to the following recursion type relation for the functions $\{\kappa_{n+k}\}$:

$$\kappa_{n+k}(z) = \left(\mu(z)+o(e^{-c_5 n})\right)\sum_{l=0}^{\varepsilon n}\sum_{r=0}^{\varepsilon n}\kappa_{n-l}(z)e^{-(l+r)m_\beta(z)}\mathbb{F}_{l+r}(z) + o\left(e^{-c_3 n}\right), \tag{4.14}$$

which holds uniformly in $z\in\mathcal{U}$ and in n,k satisfying $n/4 < k < 4n$. Thus, given a couple (n,k) as in (4.8),

$$\kappa_{n+k+1}(z)-\kappa_{n+k}(z) = o(e^{-c_6 n})\sum_{l=0}^{\varepsilon n}\sum_{r=0}^{\varepsilon n}\kappa_{n-l}(z)e^{-(l+r)m_\beta(z)}\mathbb{F}_{l+r}(z) + \left(e^{-c_3 n}\right).$$

By the separation of masses (1.21), the neighbourhood $\mathcal{U}$ can be chosen in such a way that the estimate

$$|e^{-nm_\beta(z)}\mathbb{F}_n(z)| \le c_8 e^{-c_7 n}$$

is valid uniformly in $n\in\mathbb{N}$ and in $z\in\mathcal{U}$. It follows that the limit

$$\kappa(z) \overset{\mathsf{def}}{=} \lim_{n\to\infty}\kappa_n(z)$$

exists on $\mathcal{U}$ and, furthermore, the uniform exponential rate of convergence (4.4) holds.

In order to check that $\kappa(\cdot)$ does not vanish in a (possibly smaller than $\mathcal{U}$ neighbourhood of $[a,b]$ assume, to the contrary, that this is not the case. Then there exists $t\in[a,b]$, such that $\kappa(t)=0$. By (4.4) this would imply that

$$\lim_{n\to\infty}\frac{1}{n}\log\kappa_n(t) < -\tilde{c} < 0,$$

in a clear contradiction to the principal rate of decay formula (4.7).

4.2. **Local limit behaviour of the corner connectivities $a_\beta(x)$.** Exactly as in the case of bridge partition functions [33] the analytic control (4.4) over moment generating functions sets up the stage for a local limit description of corner connectivities $a_\beta(x)$,

$$a_\beta(x) \overset{\mathsf{def}}{=} \sum_{\substack{\omega:(-x_1,0)\mapsto(0,x_2)\\ \omega\in\mathcal{A}_{x_1}}} e^{-\beta|\omega|}.$$

In view of the conditions of Theorem 4.2 these estimates should hold in the language of conjugate tilts uniformly in $t\in[a,b]\subset\mathcal{D}_\beta^+$, which translates to uniform results for $a_\beta(x)$ for directions

$$x\in\tilde{\mathcal{C}}_r \overset{\mathsf{def}}{=} \{(y_1,y_2)\,:\,y_2/r<y_1<y_2r\}.$$

As in [33], the only remaining property to be checked is an exponential decay of $\mathbb{A}_n(t+is)$ for large values of s, that is in the case when $t+is$ does not necessarily belongs to the complex neighbourhood $\mathcal{U}$ described in Theorem 4.2.

Lemma 4.3. *For any $t\in\mathcal{D}_\beta^+$ and for any complex neighbourhood $\mathcal{U}$ of t (such that $\Re\mathcal{U}\subset\mathcal{D}_\beta^+$) there exists a positive constant $\delta>0$, such that uniformly in n sufficiently large one has the following inequality*

$$\sup_{t+is\notin\mathcal{U},|s|<\pi}\left|\frac{\mathbb{A}_n(t+is)}{\mathbb{A}_n(t)}\right| \;<\; e^{-\delta n}. \tag{4.15}$$

In view of the corresponding result for the bridge partition functions [33] the proof of the lemma is essentially contained in (4.13). Indeed, by (4.11), a weaker version of (4.13),

$$\mathbb{A}_{n+k}(z) \;=\; \sum_{l=0}^{\varepsilon n}\sum_{r=0}^{\varepsilon n}\mathbb{A}_{n-l}(z)\mathbb{F}_{l+r}(z)\mathbb{B}_{k-r}(z)+o\left(e^{-c_3n+(n+k)m_\beta(t)}\right),$$

holds for any value of s in $z=t+is$. Since $\kappa(\cdot)$ doesn't vanish in $\mathcal{U}$, we divide both sides of the latter expression by $\mathbb{A}_{n+k}(t)$ and use the asymptotic expressions (4.4) together with the corresponding property **P4** in Sect. 1.6 for the bridge moment generating functions to obtain:

$$\sup_{t+i\notin\mathcal{U},|s|<\pi}\left|\frac{\mathbb{A}_n(t+is)}{\mathbb{A}_n(t)}\right| \;\le\; c_{10}\varepsilon n\sup_{t+is\notin\mathcal{U},|s|<\pi}\left|\frac{\mathbb{B}_n(t+i\tau)}{\mathbb{B}_n(t)}\right|+o\left(e^{-c_3n}\right);$$

the target estimate (4.15) follows. □

As a consequence we deduce the following local limit formula for a_β:

Lemma 4.4. *For every $r\in(1,\infty)$,*

$$a_\beta(x) \;=\; \frac{\kappa(t)}{\sqrt{2\pi n m_\beta''(t)}}\,e^{-\tau_\beta(x)}\big(1+o\,(1)\big), \tag{4.16}$$

uniformly in $x=(n,k)\in\tilde{\mathcal{C}}_r$. The above tilt t is related to the point x by the duality relation $m_\beta'(t)=k/n$.

Remark 4.4.1. Recall [33] that the bridge two-point functions satisfy a similar asymptotic formula: For every $r < \infty$,

$$h_\beta(x) = \frac{\mu(t)}{\sqrt{2\pi n m''_\beta(t)}} e^{-\tau_\beta(x)}\big(1 + o\,(1)\big), \tag{4.17}$$

uniformly in $x = (n,k) \in \mathcal{C}_r = \{(n,k) : |k| \le rn\}$.

4.3. **Relation between the prefactors κ and μ.** Since $a_\beta(n,k) = a_\beta(k,n)$, we deduce from (4.16) the following relation between the values of the corrector function κ at a couple of points (t,t^*) satisfying (4.5) with $m'_\beta(t) = k/n$ and $m'_\beta(t^*) = n/k$:

$$\frac{\kappa(t)}{\sqrt{n m''_\beta(t)}} = \frac{\kappa(t^*)}{\sqrt{k m''_\beta(t^*)}}. \tag{4.18}$$

Consider now the bridge two-point function $h_\beta(2n,2k)$. For every $\omega \in \mathbb{B}_{2n}$; $\omega : 0 \mapsto (2n,2k)$ and each $j = 1,\dots,k-1$ let us say that $\mathcal{L}_j$ is a *horizontal* break line of ω, if ω intersects $\mathcal{L}_j$ at exactly one point; $\#\{\omega \cap \mathcal{L}_j\} = 1$. Given $\varepsilon > 0$ we, by (1.23), can ignore those ω which do not have at least one horizontal break line in each of the horizontal strips $\mathbb{Z} \times [k - n^\varepsilon, \dots k]$ and $\mathbb{Z} \times [k, k + n^\varepsilon]$. For the remaining paths ω let us use γ to denote the irreducible horizontal component which intersects $\mathcal{L}_k$ (if $\mathcal{L}_k$ is a break line, we set $\gamma = \varnothing$). This induces the splitting,

$$\omega = \omega_1 \vee \gamma \vee \omega_2, \tag{4.19}$$

where ω_1, ω_2 are corner paths and γ is an irreducible bridge. Precisely, if we denote the end points of γ as $(n-r, k-q)$ and $(n+l, k+p)$, then $\omega_1 \in \mathcal{A}_{k-q}$, $\gamma \in \mathcal{F}_{q+p}$ and $\omega_2 \in \mathcal{A}_{k-p}$, see Fig. 4.15.

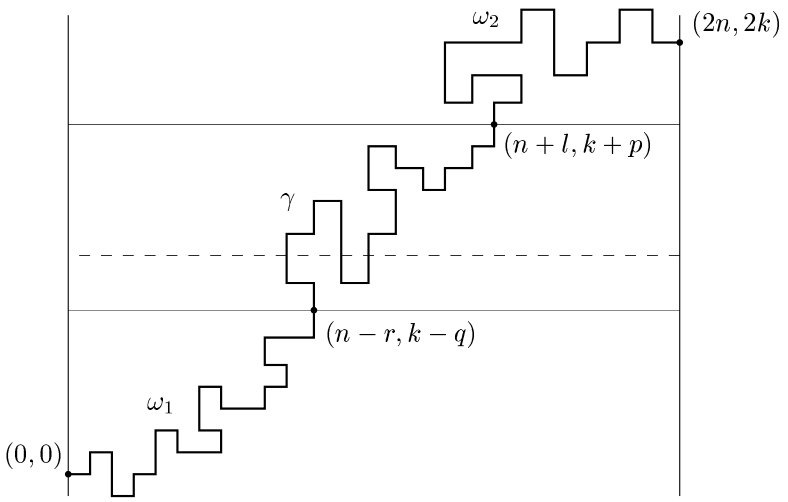


FIGURE 4.15. Decomposition (4.19)

Recall that we can restrict attention to $|p|, |q| \leq n^{\varepsilon}$. By (1.8) there is no loss to assume that $|r|, |l| \leq n^{\nu}$ for some $\nu \in (1/2, 1)$. Therefore,

$$\begin{aligned} &h_{\beta}(2n, 2k)\,(1 + o\,(1)) \\ &\qquad = \sum_{q=0}^{n^{\varepsilon}} \sum_{p=0}^{n^{\varepsilon}} \sum_{r=-n^{\nu}}^{n^{\nu}} \sum_{l=-n^{\nu}}^{n^{\nu}} a_{\beta}(k-q, n-r) f_{\beta}(p+q, l+r) a_{\beta}(k-p, n-l). \end{aligned} \tag{4.20}$$

Substituting the asymptotic expressions (4.17) and (4.16) into both hand sides of (4.20) and using (4.18):

$$\begin{aligned} &\mu(t) e^{-2\tau_{\beta}(n,k)} \big(1 + o\,(1)\big) \\ &\quad = \frac{\chi(t)\chi(t^*)}{\sqrt{\pi k m_{\beta}''(t^*)}} \sum_{q=0}^{n^{\varepsilon}} \sum_{p=0}^{n^{\varepsilon}} \sum_{r=-n^{\nu}}^{n^{\nu}} \sum_{l=-n^{\nu}}^{n^{\nu}} e^{-\tau_{\beta}(k-q,n-r)-\tau_{\beta}(k-p,n-l)} f_{\beta}(p+q, l+r). \end{aligned} \tag{4.21}$$

Under an appropriate choice of ε and $\nu \in (1,2)$ in (4.21) (for example $\nu < 2/3$ and $\varepsilon < 1/4$ qualifies) it is easy, using the duality relations $\tau_{\beta}'(1, n/k) = t^*$ and $\tau_{\beta}''(t^*) = 1/m_{\beta}''(t^*)$ that uniformly in the range of summation in (4.21),

$$\begin{aligned} &\tau_{\beta}(k-q, n-r) + \tau_{\beta}(k-p, n-l) - 2\tau_{\beta}(k, n) \\ &\qquad = (p+q) m_{\beta}(t^*) - t^*(l+r) + \frac{r^2 + l^2}{2k m_{\beta}''(t^*)} + o\,(1)\,. \end{aligned} \tag{4.22}$$

Evidently, up to smaller order terms, it is possible to restrict summation only to the range $|l + r| \leq C n^{\nu}$, where $C = C(n/k)$ is a large enough constant. However, uniformly in $|u| \leq C k^{\varepsilon}$,

$$\sum_{r=-n^{\nu}}^{n^{\nu}} e^{-\frac{r^2 + (u-r)^2}{2k m_{\beta}''(t^*)}} = \sqrt{\pi k m_{\beta}''(t^*)}\big(1 + o\,(1)\big).$$

Thus, the target relation (4.6) follows from (4.21), the Gaussian summation formula and (1.20). ☐

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