HOMOGENEOUS MEASURES AND SPATIAL ERGODICITY OF THE NAVIER-STOKES EQUATIONS

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Abstract. We show that the Vishik-Fursikov homogeneous statistical solutions of the Navier-Stokes equations extending homogeneous measures on initial conditions yield measures ergodic with respect to space translations if the initial measure is ergodic. Therefore, with respect to such a measure, space averages of almost any realization of the flow at any given time equal probability averages at any point in space at that time. Ergodic measures on the initial conditions exist.

1. Introduction

In his "Theory of Homogeneous Turbulence", G.K. Batchelor defines the space average of a quantity F depending on the velocity field u of a homogeneous flow as

$$\lim_{|A|\to\infty} \frac{1}{|A|} \int_A F(u(x,t))\; dx. \tag{1}$$

He then writes:

> "...the space average thereby obtained is the same for almost all realizations of the field and is identical with the probability average..."

[B], page 16.

The mathematical setting for homogeneous flows is translation invariant measures on function spaces. There are at least two constructions of such measures, see [FT] and [VF]. Here we follow the Vishik-Fursikov construction of homogeneous measures on the space of realizations of the flow and show that, in this setting, Batchelor's claim is correct.

According to Batchelor, A. Kolmogorov, and G.I. Taylor, a statistical solution of the incompressible Navier-Stokes equations on $\mathbb{R}^3$

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} \tag{2}$$

for $i = 1, 2, 3$, describing the velocity $\vec{u}(t,x)$ of a (3-dimensional) fluid at x at time t, is a family of random variables

$$u(t,x) : (\mathcal{X}, \mathcal{B}, P) \to \mathbb{R}^3 \tag{3}$$

indexed by t and x, where $(\mathcal{X}, \mathcal{B}, P)$ is a fixed probability space and for each ω in $\mathcal{X}$ the field $u(.,.)(\omega)$ solves the equation. Of course, $u(.,.)(\omega)$ must also solve

$$\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \tag{4}$$

at least weakly.

For each $\omega \in \mathcal{X}$ the $u(.,.)(\omega)$ is a **realization** of the velocity field. Given such a realization, one can try to average over space for each fixed time. This is the spatial (snapshot) average at time t. On the other hand, for each fixed x and t, there are several realizations of the velocity at (t, x), depending on the choice of ω. Averaging over all ω's gives the probability (ensemble) average at (x, t).

Asking that such ensemble averages are independent of x leads to a simplified model of fluid flow where all statistical properties of the statistical solution are independent of position. The standard approximation of such a situation is flow behind a grid. Such a solution is called a **homogeneous** solution. Batchelor's comment applies to this situation and is intuitively clear: if the flow is homogeneous, then some value of the velocity should occur over all points in space at a certain time as often as it occurs at a single point, at the same time, over repeated experiments. As Batchelor points out, and as we shall show below, this should follow from ergodic properties of the measure on $\mathcal{X}$.

Here $\mathcal{X}$ will be the space of all realizations of the flow. Vishik and Fursikov have shown that, after measuring how likely the various initial conditions are - i.e. after defining a measure on the function space of initial conditions, a measure can be defined also on the space of all paths of functions describing the flow. Their method, similar to Leray's argument for the existence of weak solutions, is reviewed in section 2. We state the ergodic theorem that needs to be proved in the same section. Section 3 shows that ergodic measures on the initial conditions yield ergodic measures on the realizations. Section 4 shows that things work as expected for translations and balls in Euclidean space: The multi-parameter ergodic theorem by Wiener, the follow up by Pitt, and the Khinchin-Kolmogorov proof of the standard 1-dimensional ergodic theorem plus the ergodicity of measures yield Batchelor's claim.

Quite often in the turbulence literature one encounters "averages", without much explanation on whether they are over space, time, or probabilistic. Some direct numerical simulations (DNS) use space averages when averaging is needed, see [McC], page 115. Batchelor himself states after the remarks quoted above that he does not distinguish between space and ensemble averages for the rest of the book. We hope that this note provides a surer footing for these settings.

Time ergodicity of the Navier-Stokes equations is a topic of current research, see [ES] for a review.

2. A MATHEMATICAL MODEL FOR HOMOGENEOUS TURBULENCE

2.1. **The Vishik-Fursikov construction.** Let T be some finite time. Following Vishik and Fursikov, [VF], set

$$\mathcal{X} = L^2\left([0,T];\mathcal{H}^0(r)\right) \tag{5}$$

where $\mathcal{H}^0(r)$ is the space of (weakly) divergence-free fields on $\mathbb{R}^3$ with finite $(0,r)$-norm

$$\|u\|_{0,r}^2 = \int_{\mathbb{R}^3}\left(1+|x|^2\right)^r |u(x)|^2\,dx, \quad r\in\mathbb{R}. \tag{6}$$

Note that r can be negative. The space of initial conditions $\mathcal{H}^0(r)$ contains the subspaces

$\mathcal{M}_l = \{l$-periodic vector fields with at most the first l Fourier coefficients $\neq 0\}$.

Given a measure μ on $\mathcal{H}^0(r)$, Vishik and Fursikov construct measures μ_l on $\mathcal{H}^0(r)$, supported by $\mathcal{M}_l$, such that

$$\int_{\mathcal{H}^0(r)} e^{i(u,v)}d\mu_l(u) \to \int_{\mathcal{H}^0(r)} e^{i(u,v)}d\mu(u), \quad l\to\infty \tag{7}$$

for any v in the dual of $C_0^\infty(\mathbb{R}^3)^3$.

The Cauchy problem for the Navier-Stokes equations is well-posed on each $\mathcal{M}_l$ and the map

$$S_l : \mathcal{M}_l \to \mathcal{X} \tag{8}$$

that associates to each initial value u_o in $\mathcal{M}_l$ the corresponding solution in $L_2\left([0,T];\mathcal{H}^0(r)\right)$ (in fact, in $\mathcal{C}^1\left([0,T],\mathcal{M}_l\right)$) for $r < -\frac{n}{2}$, is well defined. This defines a family of measures P_l on $\mathcal{X}$ via pull-back,

$$P_l(A) = \mu_l(S_l^{-1}A), \tag{9}$$

the **Galyerkin approximations** of what will be the final measure on $\mathcal{X}$. Vishik and Fursikov then show that the weak limit of P_l as $l\to\infty$ exists and this defines a measure on $\mathcal{X}$:

$$P_l \rightharpoonup P, \quad l\to\infty \tag{10}$$

which means that for any continuous and bounded f on $\mathcal{X}$

$$\int_{\mathcal{X}} f(u)dP_l(u) \to \int_{\mathcal{X}} f(u)dP(u), \quad l\to\infty. \tag{11}$$

They also show that P is supported by a set W consisting of weak solutions of the Navier-Stokes, and such that for $t_0 \geq 0$, the map

$$\gamma_{t_0} : W \to \mathcal{H}^0(r), \quad u \mapsto \lim_{t\to t_0^+} u(t,.) \tag{12}$$

is well defined if the limit is taken with respect to the norm

$$\|u\|_{\Phi^{-s}}^2 = \sum_{N=1}^{\infty} \frac{1}{2^{2N}}\frac{1}{(N^3+1)^2}\|u(t,.)|_{B_N}\|_{-s}^2. \tag{13}$$

B_N is the sphere of radius N in $\mathcal{R}^3$, and

$$\|u(t,.)|_{B_N}\|^2_{-s} = \sup\left\{ \frac{|<u,w>_{L^2}|}{\|w\|_{W^{2,s}}}; w \in C_0^\infty(B_N) \right\}. \tag{14}$$

In particular, the initial condition map γ_0 is well-defined. Then the pull-back measure

$$\mu_0 = P \circ \gamma_0^{-1} \tag{15}$$

is also well defined on $\mathcal{H}^0(r)$. As $t \to 0$ the measure P recaptures the original measure μ on $\mathcal{H}^0(r)$:

$$\mu_0 = \mu \quad \text{on } \mathcal{H}^0(r). \tag{16}$$

It is important in what follows that if $u(.,.)$ is in W then for all but a countable set of t's in the fixed interval $[0,T]$ the map $t \to u(.,t)$ is continuous in the Φ^s-norm. From now on, when we fix t given a u, we always stay away from the points of discontinuity.

2.2. **Homogeneous flows.**

To discuss homogeneous flows, define (weakly) for each fixed vector in $\mathbb{R}^3$ the translation operation on $\mathcal{X}$

$$(T_h u)(x,t) = u(x+h,t). \tag{17}$$

If the initial μ_0 is translation invariant, i.e.

$$\int_{\mathcal{H}^0(r)} f(u)\ d\mu_0(u) = \int_{\mathcal{H}^0(r)} f(T_h u)\ d\mu_0(u), \tag{18}$$

for any integrable f, then the μ_l's, P_l's, and finally P of the previous section are also translation invariant:

$$\int_{\mathcal{X}} F(u(t,x))\ dP(u) = \int_{\mathcal{X}} F(u(t,x+h))\ dP(u). \tag{19}$$

In particular, if

$$ev_{(t_0,x_0)} : u \mapsto u(x_0,t_0) \tag{20}$$

then averages of the form

$$\int_{\mathcal{X}} F(u(t_1,x_1),...,u(t_n,x_n))\ dP(u) = \int_{\mathcal{X}} F \circ \left(ev_{(t_1,x_1)},...,ev_{(t_n,x_n)}\right) u\ dP(u)$$

are translation invariant, which is exactly Batchelor's definition of homogeneity. The problem, of course, is that $ev_{(t_i,x_i)}$ is not well defined on $\mathcal{X}$. However, for any fixed t and any smooth function $w : \mathbb{R}^3 \to \mathbb{R}$ with compact support, the translation invariance of the measure on $\mathcal{X}$ implies that

$$\int_{\mathcal{X}} \int_{\mathbb{R}^3} F(u(t,x))w(x)\ dx\ dP(u) \tag{21}$$

is invariant under translations of w, see [VF] page 206. Since there is only one translation invariant measure on $\mathbb{R}^3$ up to constant, there is constant E_F such that

$$\int_{\mathcal{X}} \int_{\mathbb{R}^3} F(u(t,x))w(x)\ dx\ du = E_F(t) \int_{\mathbb{R}^3} w(x)dx. \tag{22}$$

for each $w : \mathbb{R}^3 \to \mathbb{R}$ smooth with compact support. Following Vishik and Fursikov, define **the ensemble average of F at time t** (and at any point x)

$$\int_{\mathcal{X}} F(u(t,x))\ du \overset{\text{def}}{=} E_F(t). \tag{23}$$

On the other hand, the standard **space average of a realization u of the flow at time t** is

$$\lim_{R^3 \to \infty} \frac{1}{\alpha R^3} \int_{B(0,R)} F(u(t,x))\ d^3x, \tag{24}$$

where $\alpha =$ volume of the unit ball in $\mathbb{R}^3$. A corollary to the ergodic theorem proved here is that this limit exists for P-almost all u for appropriate F's.

2.3. **The ergodic statement.** To understand why the ergodic theorem provides the correct setting for what we are after, assume for a minute that the evaluation $u(t,x)$ is well defined for each x in $\mathbb{R}^3$. After rewriting the space average as

$$\lim_{R^3 \to \infty} \frac{1}{\alpha R^3} \int_{B(0,R)} F(u(t,x))d^3x = \lim_{R^3 \to \infty} \frac{1}{\alpha R^3} \int_{B(0,R)} F(ev_{(t,0)}T_\lambda(u))d^3\lambda,$$

to show that space average = probability average, is to show that for almost any fixed realization u

$$\lim_{R^3 \to \infty} \frac{1}{\alpha R^3} \int_{B(0,R)} F(ev_{(t,0)}T_\lambda(u))\ d^3\lambda = \int_{\mathcal{X}} F(ev_{(t,0)}(u'))\ dP(u').$$

where the integral on the right-hand-side is over all realizations u'. Of course, for homogeneous flows the integral on the right hand side is

$$\int_{\mathcal{X}} F((u'(t,x))\ dP(u') \tag{25}$$

for any x in $\mathbb{R}^3$.

3. Ergodicity of the Vishik-Fursikov measure

A set A of functions is **invariant under translations** if for any fixed vector h and u in A the translation $T_h u$ is also in A.

A measure P is **ergodic with respect to translations** if any set invariant under translations has P-measure 0 or 1.

Lemma 3.1. *For any measurable subset A of $\mathcal{X}$,*

$$P(A) = \mu(\gamma_0(A)) \tag{26}$$

Proof. Clearly, $P(A) \leq \mu(\gamma_0(A))$. Suppose that there is A such that

$$P(A) < \mu(\gamma_0(A)). \tag{27}$$

Then there is A_1 with $\gamma_0(A_1) \subseteq \gamma_0(A)$ and $P(A_1 \backslash A) \neq 0$. Since P is the weak limit of $P_l' s$, $P_l(A_1 \backslash A) \neq 0$ for sufficiently large l. Then the definition of P_l gives that $\mu_l(S_l^{-1}(A_1 \backslash A)) \neq 0$. But $S_l^{-1}(\mathcal{X}_1 \backslash \mathcal{X}) = \emptyset$, a contradiction. □

Proposition 3.2. *If μ is ergodic with respect to translations, then P is also ergodic.*

Proof. Since the measure P is supported by the set of solutions W, we only need to look at invariant subsets of W. On the other hand the initial value map γ_0 is well defined on W. Since by Lemma 3.1 $\mu(\gamma_0 \mathcal{X}) = P(\mathcal{X})$, with μ ergodic, we only need to show that if $\mathcal{X}$ is translation invariant then $\gamma_0 \mathcal{X}$ is also translation invariant. Suppose then that $\mathcal{X}$ is a translation invariant Borel subset of $L^2(0, T, \mathcal{H}^0(r))$. Since $(\gamma_0 u)(x) = \lim_{t \to 0+} u(t,x)$ where the limit is in Φ^{-s}, we have as $t \to 0^+$:

$$\begin{aligned}
&\|u(t,.) - \gamma_0 u\|^2_{\Phi^{-s}} \\
&= \sum_{N=1}^{\infty} \frac{1}{2^{2N}} \frac{1}{(N^3+1)^2} \left(\sup_{w_N} \frac{\mid \int_{\mathbb{R}^3} (u_t(x) - (\gamma_0 u)(x)) w_N(x)\ dx \mid}{\left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} |D^\alpha w_N(x)|^2\ dx \right)^{1/2}} \right)^2 \to 0
\end{aligned}$$

where w_N is smooth with compact support in K_N. Then for each fixed vector h

$$\begin{aligned}
&\|T_h u(t,.) - T_h \gamma_0 u\|^2_{\Phi^{-s}} \\
&= \sum_{N=1}^{\infty} \frac{1}{2^{2N}} \frac{1}{(N^3+1)^2} \left(\sup_{w_N} \frac{\mid \int_{\mathbb{R}^3} (u_t(x+h) - (\gamma_0 u)(x+h)) w_N(x)\ dx \mid}{\left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} |D^\alpha w_N(x)|^2\ dx \right)^{1/2}} \right)^2 \\
&= \sum_{N=1}^{\infty} \frac{1}{2^{2N}} \frac{1}{(N^3+1)^2} \left(\sup_{w_N} \frac{\mid \int_{\mathbb{R}^3} (u_t(x) - (\gamma_0 u)(x)) w_N(x-h)\ dx \mid}{\left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} |D^\alpha w_N(x-h)|^2\ dx \right)^{1/2}} \right)^2 \\
&\leq \sum_{N=1}^{\infty} \frac{1}{2^{2N}} \frac{1}{(N^3+1)^2} \left(\sup_{w_{N'}} \frac{\mid \int_{\mathbb{R}^3} (u_t(x) - (\gamma_0 u)(x)) w_{N'}(x)\ dx \mid}{\left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} |D^\alpha w_{N'}(x)|^2\ dx \right)^{1/2}} \right)^2 \\
&\leq 2^{2|h|+2} \sum_{N=1}^{\infty} \frac{((N')^3+1)^2}{(N^3+1)^2 2^{2N'} ((N')^3+1)^2} \\
&\quad \times \left(\sup_{w_{N'}} \frac{\mid \int_{\mathbb{R}^3} (u_t(x) - (\gamma_0 u)(x)) w_{N'}(x)\ dx \mid}{\left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} |D^\alpha w_{N'}(x)|^2\ dx \right)^{1/2}} \right)^2
\end{aligned}$$

for $N' = N + |h| + 1$.

Therefore

$$
\begin{aligned}
&\|T_h u(t,.) - T_h \gamma_0 u\|^2_{\Phi^{-s}} \\
\leq\ & C_h \sum_{N=[h]+1}^{\infty} \frac{1}{2^{2N}} \frac{1}{(N^3+1)^2} \left(\sup_{w_N} \frac{\left| \int_{\mathbb{R}^3} (u_t(x) - (\gamma_0 u)(x)) w_N(x)\ dx \right|}{\left(\sum\limits_{|\alpha| \leq s} \int\limits_{\mathbb{R}^3} |D^\alpha w_N(x)|^2\ dx \right)^{1/2}} \right)^2 \\
\leq\ & C_h \|u(t,.) - \gamma_0 u\|^2_{\Phi^{-s}} \quad \to 0
\end{aligned}
$$

which shows that $\gamma_0 \mathcal{X}$ is translation invariant. □

4. The ergodic theorem

4.1. **Existence of space averages.** The theory of homogeneous turbulence is mainly interested with correlation functions, i.e. averages of functions on $\mathcal{X}$ of the form

$$G_{x,r_1,\ldots,r_n}(u) = u_{i_1}(t,x) u_{i_2}(t,x+r_2) \ldots u_{i_n}(t,x+r_n) \tag{28}$$

where u_{i_k} are components of u, n some positive integer, x in $\mathbb{R}^3$ is fixed, and so are the r_n's and t. Clearly, G is not well defined on $\mathcal{H}^0(r)$, but

$$\int_{\mathbb{R}^3} u_{i_1}(t,x) u_{i_2}(t,x+r_2) \ldots u_{i_n}(t,x+r_n) w(x)\ dx, \tag{29}$$

is well defined for all w in $C_0^\infty(\mathbb{R}^3)$.

Recall now the multi-parameter version of the ergodic theorem from [W], page 675 [1]:

Theorem 4.1. *For $(\mathcal{E}, \mu)$ a space of finite measure, f a mu-integrable function on $\mathcal{X}$, $T_i^\lambda, i = 1, \ldots, n$ commuting groups of measure-preserving transformations on $\mathcal{E}$ with measure preserving inverses and such that $f \circ (T_1^{\lambda_1} \cdots T_n^{\lambda_n})$ is measurable with respect to the product measure, the limit*

$$\lim_{R \to \infty} \frac{1}{\alpha_n R^n} \int_{B(0,R)} f(T_1^{\lambda_1} \cdots T_n^{\lambda_n} p)\ d\lambda_1 \cdots d\lambda_n \tag{30}$$

exists for almost all p in $\mathcal{E}$.

And its completion in [P]:

Theorem 4.2.

$$\overline{f(p)} = \lim_{R \to \infty} \frac{1}{\alpha_n R^n} \int_{B(0,R)} f(T_1^{\lambda_1} \cdots T_n^{\lambda_n} p)\ d\lambda_1 \cdots d\lambda_n \tag{31}$$

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[1] In page 643 of [W], Wiener remarks that he needed this generalization of the ergodic theorem in the study of turbulence.

is invariant under T^λ, integrable with respect to μ, and

$$\int_{\mathcal{X}} \overline{f(p)}\ d\mu(p) = \int_{\mathcal{X}} f(u)\ d\mu(p). \tag{32}$$

It is standard that ergodicity of the measure implies that since the limit is translation invariant it is constants and therefore equal to its ensemble average. We therefore have:

Theorem 4.3. *Let f be such that for for each w in $C_0^\infty(\mathbb{R}^3,\mathbb{R})$, $F_w(u) = \int_{\mathbb{R}^3} f(u(x))w(x)dx$ defines an element of $L^1(\mathcal{X},\mathbb{R})$. Then for almost all u with respect to any translation invariant measure on $\mathcal{X}$ the spatial average limit*

$$\overline{f(u)}^w = \lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} \int_{\mathbb{R}^3} f(u(t,x+\lambda))w(x)\ dx\ d\lambda \tag{33}$$

exists for all but a countable number of t's in $[0,T]$. $\overline{f(u)}^w$ is always translation invariant.

Whenever the measure on $\mathcal{X}$ is ergodic with respect to translations $\overline{f(u)}^w$ is almost always independent of the realization u and

$$\overline{f(u)}^w = \int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(t,x))w(x)\ dx\ du \tag{34}$$

for almost all u.

Proof. Apply 4.1 and 4.2 on $\mathcal{H}^0(r)$ for the family of measure preserving transformations

$$(T_\lambda u)(t,x) = u(t,x+\lambda),\ \lambda \in \mathbb{R}^3 \tag{35}$$

and for the L^1 function F_w on $\mathcal{X}$. □

Hence a homogeneous measure selects u's for which the limit exists, although the limit does not exist for all u: The Vishik-Fursikov construction requires that $r < -3/2$. Then for $0 < p < -r - 3/2$, let $u(x) = |x|^p$. Then $\|u\|_{H^0(r)}$ is finite, but the spatial average of u itself is not finite.

Recalling the defining equation for the ensemble average E_f,

$$\int_{\mathcal{X}} \int_{\mathbb{R}^3} f(u(t,x))w(x)\ dx\ du = E_f \int_{\mathbb{R}^3} w(x)dx \tag{36}$$

the theorem reads

$$E_f = \frac{1}{\int_{\mathbb{R}^3} w(x)\ dx} \overline{f(u)}^w, \tag{37}$$

for almost all u in $\mathcal{X}$ and all w in $C_0^\infty(\mathbb{R}^3,\mathbb{R})$.

On the other hand, repeating the proof of the ergodic theorem for continuous parameters as in [K], for example, shows that the limit

$$\lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,x+\lambda))\ d\lambda \tag{38}$$

exists for almost all u and that for each x

$$\lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,x+\lambda))\ d\lambda = \lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,\lambda))\ d\lambda,$$

see the Appendix for some of the steps. Therefore for any w in C_0^∞, using the dominated convergnce Theorem,

$$\begin{aligned} &\int_{\mathbb{R}^3} \lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,x+\lambda))w(x)d^3\lambda\ dx \\ &= \lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,x+\lambda))d^3\lambda \int_{\mathbb{R}^3} w(x)\ dx. \end{aligned} \tag{39}$$

Then equation (37) yields precisely Batchelor's claim:

Corollary 4.4.

$$\lim_{R\to\infty} \frac{1}{\alpha R^3} \int_{B(0,R)} f(u(t,x+\lambda))\ d\lambda = \int_{\mathcal{X}} f(u(t,x))\ du \tag{40}$$

for almost all u in $\mathcal{X}$ and x in $\mathbb{R}^3$.

5. Examples of ergodic measures

Vishik and Fursikov give three examples of homogeneous measures on $\mathcal{H}^0(r)$: Dirac, finite measures form circles, and Gausssian measures. We point out here that the fist two examples are ergodic. We shall discuss ergodicity within the class of Gaussian measures elsewhere.

5.1. **The Dirac measure.** For $A \subset \mathcal{X}$, set $\mu(A) = 1$ if $u_0(x) = \vec{a}$ belongs to $\mathcal{X}_0$, $\mu(\mathcal{X}_0) = 0$ otherwise. Clearly, $u_0 \in \mathcal{H}^r(0)$, μ is translation invariant, and μ is ergodic with respect to translations.

5.2. **Circle measure.** For $A \subset \mathcal{X}$, define $\mu(A)$ to be the measure of the the subset of trigonometric polynomials of degree N in $\mathcal{X}_0$.

To define the measure on such polynomials

$$u_0(x) = \sum_{n=1}^{N} a_n \cos(\lambda_n \cdot x) + b_n \sin(\lambda_n \cdot x), \tag{41}$$

Vishik and Fursikov fix radii $r_1, \ldots, r_N$, the standard measure (normalized) on the circles $C_n = B(0,r_n) \cap \{\lambda_n \cdot y = 0\}$, and take the pull back measure under the map

$$u_0 \mapsto \frac{1}{2}(a_1 - ib_1, \ldots, a_N - ib_N), \tag{42}$$

the image equipped with a product measure.

If A is invariant under all translations in x and has non-zero measure, each component map $u_0 \mapsto \frac{1}{2}(a_n - ib_n)$ has image of nonzero measure and invariant under all rotations, not only roots of unity. Therefore each component has image of measure 1, c.f. [Wl].

APPENDIX A. THE Φ^{-s} NORM

The Φ^{-s}-norm in [VF] is given as

$$\|u\|^2_{\Phi^{-s}} = \sum_{N=1}^{\infty} \frac{1}{2^{2N}} \frac{1}{C(N)^2} \|u(t,.)|_{B_N}\|^2_{-s} \tag{43}$$

for $C(N)$ such that

$$\int \|u_t|_{B_N}\|_{-s} P_l(du) \leq C(N). \tag{44}$$

That this is equivalent to the one used here follows from

$$\int \|u_t|_{B_N}\|_{-s} P_l(du) \leq \mathrm{Vol}(B_n) \int |u(t,x)|^2 P_l(du) + 1, \tag{45}$$

[VF], page 232, and

$$\int |u(t,x)|^2 P_l(du) \leq \int |u_0(x)|^2 \mu_l(du_0) \leq \int |u_0(x)|^2 \mu(du_0), \tag{46}$$

see [VF], p. 220 and especially estimate (2.6) in page 217 which shows that if u_0 is the initial condition of u then on the cube T_l of size l

$$\|u(t,.)|T_l\|^2 \leq \|u_0|T_l\|^2. \tag{47}$$

Therefore

$$\int \|u_t|_{B_N}\|_{-s} P_l(du) \leq \mathrm{Vol}(B_N) C(\mu) + 1, \tag{48}$$

with $C(\mu)$ depending only on the initial measure μ.

APPENDIX B. TRANSLATION INVARIANCE.

We now give some details on how to repeat the Kolmogorov-Khinchin proof in [K] for the case $f(u) = u_i(t,0)$, suppressing the index i.

Proposition B.1. *For all u in $H^0(r)$ for which the limit exists and all x in $\mathbb{R}^3$,*

$$\lim_{R\to\infty} \frac{1}{R^3} \int_{B(0,R)} u(t,x+\lambda)\ d\lambda = \lim_{R\to\infty} \frac{1}{R^3} \int_{B(0,R)} u(\lambda)\ d\lambda \tag{49}$$

Proof. Since the proof of existence of the limit can be repeated separately for the positive and negative part of any given u, we can assume that u is positive. Then, c.f. [P],

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R)} u(x+\lambda)d^3\lambda &= \lim_{R\to\infty}\frac{1}{R^3}\int_{B(x,R)} u(\lambda)d^3\lambda \\
&\leq \lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R+x)} u(\lambda)d^3\lambda \\
&= \lim_{R\to\infty}\frac{1}{(R+x)^3}\int_{B(0,R+x)} u(\lambda)d^3\lambda \\
&= \lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R)} u(\lambda)d^3\lambda.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R)} u(\lambda)d^3\lambda &= \lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R)} u(x+\lambda-x)d^3\lambda \\
&= \lim_{R\to\infty}\frac{1}{R^3}\int_{B(-x,R)} u(x+\lambda)d^3\lambda \\
&\leq \lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R+|x|)} u(x+\lambda)d^3\lambda \\
&= \lim_{R\to\infty}\frac{1}{(R+|x|)^3}\int_{B(0,R+|x|)} u(x+\lambda)d^3\lambda \\
&= \lim_{R\to\infty}\frac{1}{R^3}\int_{B(0,R)} u(x+\lambda)d^3\lambda.
\end{aligned}$$

□

Appendix C. Equality of Averages

Assume ergodicity, i.e. only sets invariant under all translations have measure zero or 1. That the limit is constant follows immediately from the translation invariance and ergodicity of the measure. To show that the constant equals the ensemble average, first observe that the average of the (partial space) averages is the (ensemble) average: For w in C_0^∞, set

$$u_R = \frac{1}{R^3}\int_{B(0,R)}\int_{\mathbb{R}^3} u(t,x+\lambda)w(x)\ dx\ d\lambda, \tag{50}$$

to get that

$$\begin{aligned}
\int_{\mathcal{X}} u_R \; du &= \int_{\mathcal{X}} \frac{1}{R^3} \int_{B(0,R)} \int_{\mathbb{R}^3} u(t, x+\lambda) w(x) \; dx \; d\lambda \;\; du \\
&= \frac{1}{R^3} \int_{B(0,R)} \int_{\mathcal{X}} \int_{\mathbb{R}^3} u(t, x+\lambda) w(x) \; dx \;\; du \; d\lambda \\
&= \frac{1}{R^3} \int_{B(0,R)} \int_{\mathcal{X}} \int_{\mathbb{R}^3} u(t, x) w(x) \; dx \;\; du \; d\lambda \\
&= \int_{\mathcal{X}} \int_{\mathbb{R}^3} u(t, x) w(x) \; dx \;\; du.
\end{aligned}$$

On the other hand,

$$\int_{\mathcal{X}} \left(\lim_{R\to\infty} \frac{1}{R^3} \int_{B(0,R)} \int_{\mathbb{R}^3} u(t, x+\lambda) w(x) dx \; d\lambda \right. \\ \left. - \frac{1}{R^3} \int_{B(0,R)} \int_{\mathbb{R}^3} u(t, x+\lambda) w(x) \; dx \; d\lambda \right) du$$

goes to zero exactly as in [K], page 31.

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