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EXISTENCE AND UNIQUENESS OF THE INTEGRATED DENSITY OF STATES FOR SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS AND UNBOUNDED RANDOM POTENTIALS

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Dedicated to Jean-Michel Combes on the occasion of his 60^{th} *birthday*

The object of the present study is the integrated density of states of a quantum particle in multi-dimensional Euclidean space which is characterized by a Schrödinger operator with a constant magnetic field and a random potential which may be unbounded from above and from below. For an ergodic random potential satisfying a simple moment condition, we give a detailed proof that the infinite-volume limits of spatial eigenvalue concentrations of finite-volume operators with different boundary conditions exist almost surely. Since all these limits are shown to coincide with the expectation of the trace of the spatially localized spectral family of the infinite-volume operator, the integrated density of states is almost surely non-random and independent of the chosen boundary condition. Our proof of the independence of the boundary condition builds on and generalizes certain results obtained by S. Doi, A. Iwatsuka and T. Mine [Math. Z. **237** (2001) 335–371] and S. Nakamura [J. Funct. Anal. **173** (2001) 136–152].

Contents

1 Introduction

The integrated density of states is an important quantity in the theory [31, 13, 46] and application [51, 8, 39, 2, 36] of Schrödinger operators for a particle in d-dimensional Euclidean space $\mathbb{R}^d$ ($d = 1, 2, 3, \dots$) subject to a random potential. It determines the free energy of the corresponding non-interacting many-particle system in the thermodynamic limit and also enters formulas for transport coefficients. In accordance with statistical mechanics, to define the integrated density of states one usually considers first the system confined to a bounded box. For the corresponding finite-volume random Schrödinger operator to be self-adjoint one then has to impose a boundary condition on the (wave) functions in its domain. The infinite-volume limit of the number of eigenvalues per volume of this finite-volume operator below a given energy defines the integrated density of states N. Basic questions are whether this limit exists, is independent of almost all realizations of the random potential and of the chosen boundary condition. These are the questions of existence, non-randomness, and uniqueness.

For vanishing magnetic field these questions were settled several years ago [44, 43, 32, 31, 13, 46], see also [35] for a more recent approach. For non-zero magnetic fields the existence and non-randomness of N are known since [41, 55, 9]. Uniqueness, that is, the independence of the boundary condition follows from recent results in [19] and [42] for bounded below or bounded random potentials, respectively. However, a proof of uniqueness is lacking for random potentials which are unbounded from below.

The main goal of the present paper is to give a detailed proof of the existence, non-randomness, and uniqueness of N for the case of constant magnetic fields and a wide class of ergodic random potentials which may be unbounded from above as well as from below and which satisfy a simple moment condition. In particular, N is shown to coincide with the expectation of the trace of the spatially localized spectral family of the infinite-volume operator. As a consequence, the set of growth points of N is immediately identified with the almost-sure spectrum of this operator. Important examples of random potentials which may yield operators unbounded from below and to which our main result, Theorem 3.1, applies, are alloy-type, Poissonian, and Gaussian random potentials.

Our proof of the existence, non-randomness, and uniqueness of N differs from those outlined in [41, 55, 9] and is patterned on the one of analogous statements for vanishing magnetic fields in the monograph of Pastur and Figotin [46]. Since the infinite-volume operator may be unbounded from below, we have to make sure that the sequence of the underlying finite-volume density-of-states measures is "tight near minus infinity". Our proof of the independence of the boundary condition uses an approximation argument which reduces the problem to that of bounded random potentials and therefore heavily relies on results of Doi, Iwatsuka and Mine [19] or Nakamura [42].

2 Random Schrödinger Operators with Constant Magnetic Fields

2.1 Basic Notation

As usual, let $\mathbb{N} := \{1, 2, 3, \dots\}$ denote the set of natural numbers. Let $\mathbb{R}$, respectively $\mathbb{C}$, denote the algebraic field of real, respectively complex, numbers. An open cube Λ in d-dimensional Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, is a translate of the d-fold Cartesian product $I \times \cdots \times I$ of an open interval $I \subseteq \mathbb{R}$. The open unit cube in $\mathbb{R}^d$ which is centered at site $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ and whose edges are oriented parallel to the co-ordinate axes is the product $\Lambda(y) := \times_{j=1}^{d}]y_j - 1/2, y_j + 1/2[$ of open intervals. We call a bounded open cube Λ *compatible* with the (structure of the simple cubic) lattice $\mathbb{Z}^d$ if it is the interior of the closure of a union of finitely many open unit cubes centered at lattice sites, that is

$$\Lambda = \Big(\overline{\bigcup_{y \in \Lambda \cap \mathbb{Z}^d} \Lambda(y)} \Big)^{\mathrm{int}}. \tag{2.1}$$

The Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $|x| := \big(\sum_{j=1}^{d} x_j^2\big)^{1/2}$. We denote the volume of a Borel subset $\Lambda \subseteq \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure as $|\Lambda| := \int_\Lambda \mathrm{d}^d x = \int_{\mathbb{R}^d} \mathrm{d}^d x\, \chi_\Lambda(x)$ where χ_Λ is the indicator function of Λ. In particular, if Λ is the strictly positive half-line, $\Theta := \chi_{]0,\infty[}$ is the left-continuous Heaviside unit-step function. We use the notation $\alpha \mapsto (\alpha - 1)!$ for Euler's gamma function [24]. The Banach space $\mathrm{L}^p(\Lambda)$ consists of the Borel-measurable complex-valued functions $f : \Lambda \to \mathbb{C}$ which are identified if their values differ only on a set of Lebesgue measure zero and possess a finite L^p-norm

$$\|f\|_p := \begin{cases} \Big(\displaystyle\int_\Lambda \mathrm{d}^d x\, |f(x)|^p \Big)^{1/p} & \text{if } p \in [1, \infty[\,, \\ \operatorname*{ess\,sup}_{x \in \Lambda} |f(x)| & \text{if } p = \infty\,. \end{cases} \tag{2.2}$$

We recall that $\mathrm{L}^2(\Lambda)$ is a separable Hilbert space with scalar product $\langle f, g\rangle := \int_\Lambda \mathrm{d}^d x\, \overline{f(x)}\, g(x)$. The overbar denotes complex conjugation. We write $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$, if $f \in \mathrm{L}^p(\Lambda)$ for any bounded Borel set $\Lambda \subset \mathbb{R}^d$. Moreover, $\mathscr{C}^n_0(\Lambda)$ stands for the vector space of functions $f : \Lambda \to \mathbb{C}$ which are n times continuously differentiable and have compact supports. The vector space of functions which have compact supports and are continuous, respectively arbitrarily often differentiable, is denoted by $\mathscr{C}_0(\Lambda)$, respectively $\mathscr{C}^\infty_0(\Lambda)$. Finally, $W^{1,2}(\Lambda) := \{\phi \in \mathrm{L}^2(\Lambda) : \nabla\phi \in (\mathrm{L}^2(\Lambda))^d\}$ is the first-order Sobolev space of L^2-type where ∇ stands for the gradient in the sense of distributions on $\mathscr{C}^\infty_0(\Lambda)$.

The absolute value of a closed operator $F : \mathscr{D}(F) \to \mathrm{L}^2(\Lambda)$, densely defined with domain $\mathscr{D}(F) \subseteq \mathrm{L}^2(\Lambda)$ and adjoint $F^\dagger$, is the positive operator $|F| := (F^\dagger F)^{1/2}$. The (uniform) norm of a bounded operator $F : \mathrm{L}^2(\Lambda) \to \mathrm{L}^2(\Lambda)$ is defined as $\|F\| := \sup\{\, |Ff|_2 \,:\, f \in \mathrm{L}^2(\Lambda)\,, |f|_2 = 1\}$. Finally, for $p \in [1,\infty[$ we will use the notation

$$\|F\|_p := \Big(\mathrm{Tr}\,|F|^p\Big)^{1/p} \tag{2.3}$$

for the (von Neumann-) Schatten norm of an operator F on $\mathrm{L}^2(\Lambda)$ in the Banach space $\mathscr{J}_p(\mathrm{L}^2(\Lambda))$. For these $\mathscr{J}_p$-spaces of compact operators, see [53, 7]. In particular, $\mathscr{J}_1$ is the space of trace-class and $\mathscr{J}_2$ the space of Hilbert-Schmidt operators.

2.2 Basic Assumptions and Definitions of the Operators

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space and $\mathbb{E}\{\cdot\} := \int_\Omega \mathbb{P}(\mathrm{d}\omega)(\cdot)$ be the expectation induced by the probability measure $\mathbb{P}$. By a *random potential* we mean a (scalar) random field $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(\omega, x) \mapsto V^{(\omega)}(x)$ which is assumed to be jointly measurable with respect to the product of the sigma-algebra $\mathscr{A}$ of event sets in Ω and the sigma-algebra $\mathscr{B}(\mathbb{R}^d)$ of Borel sets in $\mathbb{R}^d$. We will always assume $d \geq 2$, because magnetic fields in one space dimension may be "gauged away" and are therefore of no physical relevance. Furthermore, for $d = 1$ far more is known [13, 46] thanks to methods which only work for one dimension.

We list three properties which V may have or not:

(S) There exists some pair of reals $p_1 > p(d)$ and $p_2 > p_1 d/\,[2(p_1 - p(d))]$ such that

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}\Big\{\Big[\int_{\Lambda(y)} \mathrm{d}^d x\, |V(x)|^{p_1}\Big]^{p_2/p_1}\Big\} < \infty. \tag{2.4}$$

Here $p(d)$ is defined as follows: $p(d) := 2$ if $d \leq 3$, $p(d) := d/2$ if $d \geq 5$ and $p(4) > 2$, otherwise arbitrary.

(E) V is $\mathbb{Z}^d$-ergodic or $\mathbb{R}^d$-ergodic.

(I) V satisfies the finiteness condition

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}\Big[\int_{\Lambda(y)} \mathrm{d}^d x\, |V(x)|^{2\vartheta+1}\Big] < \infty, \tag{2.5}$$

where $\vartheta \in \mathbb{N}$ is the smallest integer with $\vartheta > d/4$.

Remarks 2.1. (i) Property (E) requires the existence of a group $\mathscr{T}_x$, $x \in \mathbb{Z}^d$ or $\mathbb{R}^d$, of probability-preserving and ergodic transformations on Ω such that V is $\mathbb{Z}^d$- or $\mathbb{R}^d$-homogeneous in the sense that $V^{(\mathscr{T}_x\omega)}(y) = V^{(\omega)}(y-x)$ for all $x \in \mathbb{Z}^d$ or $\mathbb{R}^d$, all $y \in \mathbb{R}^d$, and all $\omega \in \Omega$; see [31].

(ii) Property (S) assures that the realization $V^{(\omega)} : x \mapsto V^{(\omega)}(x)$ of V belongs to $\mathrm{L}^{p(d)}_{\mathrm{loc}}(\mathbb{R}^d)$ for each ω in some subset $\Omega_{\mathsf{S}} \in \mathscr{A}$ of Ω with full probability, in symbols, $\mathbb{P}(\Omega_{\mathsf{S}}) = 1$. If $d \neq 4$, property (I) in general does not imply property (S) even if property (E) is supposed. Given (E), a sufficient criterion for both (S) and (I) to hold is the finiteness

$$\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\, |V(x)|^p\Big] < \infty \tag{2.6}$$

for some real $p > d+1$. To prove this claim for property (S) we choose $p_1 = p_2 = p$ in (2.4). For (I) the claim follows from $2\vartheta \leq d$. If the random potential is $\mathbb{R}^d$-homogeneous, Fubini's theorem gives $\mathbb{E}[|V(0)|^p]$ for the l.h.s. of (2.6).

In the present paper we mainly consider the case of a *constant magnetic field* in $\mathbb{R}^d$. This is characterized by a skew-symmetric tensor with real constant components B_{jk}, $j,k \in \{1,\dots,d\}$. On account of gauge equivalence, there is no loss of generality in assuming that the *vector potential* $A : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto A(x)$, generating the magnetic field according to $B_{jk} = \partial_j A_k - \partial_k A_j$, satisfies property

(C) A is the vector potential of a constant magnetic field in the symmetric gauge, that is, its components are given by $A_k(x) = \frac{1}{2}\sum_{j=1}^d x_j B_{jk}$ with $k \in \{1,\dots,d\}$.

We are now prepared to precisely define magnetic Schrödinger operators with random potentials on the Hilbert spaces $\mathrm{L}^2(\Lambda)$ and $\mathrm{L}^2(\mathbb{R}^d)$. The finite-volume case is treated in

Proposition 2.2. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube. Let A be a vector potential with property* (C) *and V be a random potential with property* (S). [*Recall from Remark 2.1(ii) the definition of the set* Ω_{S}.] *Then*

(i) the sesquilinear form $\mathscr{Q} \times \mathscr{Q} \ni (\varphi,\psi) \mapsto \frac{1}{2}\sum_{j=1}^d \langle (\mathrm{i}\nabla_j + A_j)\,\varphi\,,\,(\mathrm{i}\nabla_j + A_j)\,\psi\rangle$ with form domain $\mathscr{Q} = W^{1,2}(\Lambda)$ or $\mathscr{Q} = \mathscr{C}_0^\infty(\Lambda)$ is positive, symmetric and closed, respectively closable. Accordingly, both forms uniquely define positive self-adjoint operators on $\mathrm{L}^2(\Lambda)$ which we denote by $H_{\Lambda,\mathrm{N}}(A,0)$ and $H_{\Lambda,\mathrm{D}}(A,0)$, respectively.

(ii) the two operators

$$H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}) := H_{\Lambda,\mathrm{X}}(A,0) + V^{(\omega)}, \qquad \mathrm{X} = \mathrm{D} \text{ or } \mathrm{X} = \mathrm{N}, \tag{2.7}$$

are well defined on $\mathrm{L}^2(\Lambda)$ as form sums for all $\omega \in \Omega_{\mathsf{S}}$, hence for $\mathbb{P}$-almost all $\omega \in \Omega$. They are self-adjoint and bounded below. Moreover, the mapping $H_{\Lambda,X}(A,V) : \Omega_{\mathsf{S}} \ni \omega \mapsto H_{\Lambda,\mathrm{X}}(A,V^{(\omega)})$ is measurable. We call it the finite-volume magnetic Schrödinger operator with random potential *V and Dirichlet or Neumann boundary condition if* $\mathrm{X} = \mathrm{D}$ *or* $\mathrm{X} = \mathrm{N}$, *respectively.*

(iii) *the spectrum of $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ is purely discrete for all $\omega \in \Omega_{\mathrm{S}}$ such that the (random)* finite-volume density-of-states measure*, defined by*

$$\nu_{\Lambda,\mathrm{X}}^{(\omega)}(I) := \mathrm{Tr}\left[\chi_I(H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}))\right], \tag{2.8}$$

is a positive Borel measure on the real line $\mathbb{R}$ for all $\omega \in \Omega_{\mathrm{S}}$. Here, $\chi_I(H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}))$ is the spectral projection operator of $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ associated with the energy regime $I \in \mathscr{B}(\mathbb{R})$. Moreover, the (unbounded left-continuous) distribution function

$$N_{\Lambda,\mathrm{X}}^{(\omega)}(E) := \nu_{\Lambda,\mathrm{X}}^{(\omega)}(\,]-\infty, E[\,) = \mathrm{Tr}\left[\Theta(E - H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}))\right] < \infty \tag{2.9}$$

of $\nu_{\Lambda,\mathrm{X}}^{(\omega)}$, called the finite-volume integrated density of states*, is finite for all energies $E \in \mathbb{R}$.*

Proof. The assumptions of Proposition 2.2 imply those of [28, Prop. 2.1]. □

Remark 2.3. Counting multiplicity, $\nu_{\Lambda,\mathrm{X}}^{(\omega)}(I)$ is just the number of eigenvalues of the operator $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ in the Borel set $I \subseteq \mathbb{R}$. Since this number is almost surely finite for every bounded I, the mapping $\nu_{\Lambda,\mathrm{X}} : \Omega_{\mathrm{S}} \ni \omega \mapsto \nu_{\Lambda,\mathrm{X}}^{(\omega)}$ is a random Borel measure in the sense that $\nu_{\Lambda,\mathrm{X}}^{(\omega)}$ assigns a finite length to each bounded Borel set.

The infinite-volume case is treated in

Proposition 2.4. *Let A be a vector potential with property* (C) *and V be a random potential with property* (S)*. Then*

(i) *the operator $\mathscr{C}_0^\infty(\mathbb{R}^d) \ni \psi \mapsto \frac{1}{2}\sum_{j=1}^d (\mathrm{i}\partial_j + A_j)^2\psi + V^{(\omega)}\psi$ is essentially self-adjoint for all $\omega \in \Omega_{\mathrm{S}}$. Its self-adjoint closure on $\mathrm{L}^2(\mathbb{R}^d)$ is denoted by $H(A, V^{(\omega)})$.*

(ii) *the mapping $H(A, V) : \Omega_{\mathrm{S}} \ni \omega \mapsto H(A, V^{(\omega)})$ is measurable. We call it the* infinite-volume magnetic Schrödinger operator with random potential V.

Proof. See for example [28, Prop. 2.2]. □

Remarks 2.5. (i) For alternative or weaker criteria instead of (S) guaranteeing the almost-sure self-adjointness of $H(0, V)$, see [46, Thm. 5.8] or [31, Thm. 1 on p. 299].

(ii) The infinite-volume magnetic Schrödinger operator without scalar potential, $H(A, 0)$, is unitarily invariant under so-called *magnetic translations* [60, 37]. The latter form a family of unitary operators $\{T_x\}_{x\in\mathbb{R}^d}$ on $\mathrm{L}^2(\mathbb{R}^d)$ defined by

$$(T_x\psi)(y) := \exp\left[\frac{\mathrm{i}}{2}\sum_{j,k=1}^{d}(x_j - y_j)B_{jk}x_k\right]\psi(y - x), \qquad \psi \in \mathrm{L}^2(\mathbb{R}^d). \tag{2.10}$$

In the situation of Proposition 2.4 and if the random potential V has property (E), we have

$$T_x\, H(A, V^{(\omega)})\, T_x^\dagger = H(A, V^{(\mathscr{T}_x\omega)}) \tag{2.11}$$

for all $\omega \in \Omega_S$ and all $x \in \mathbb{Z}^d$ or $x \in \mathbb{R}^d$, depending on whether V is $\mathbb{Z}^d$- or $\mathbb{R}^d$-ergodic. Hence, following standard arguments, $H(A, V)$ is an *ergodic operator* and its spectral components are non-random, see [55, Thm. 2.1]. Moreover, the discrete spectrum of $H(A, V^{(\omega)})$ is empty for $\mathbb{P}$-almost all $\omega \in \Omega$, see [31, 13, 55], because the family $\{T_x\}_{x\in\mathbb{Z}^d}$ and hence $\{T_x\}_{x\in\mathbb{R}^d}$ is *total*. The latter is true by definition, since the subset $\{T_x\psi\}_{x\in\mathbb{Z}^d} \subset \mathrm{L}^2(\mathbb{R}^d)$ contains an infinite set of pairwise orthogonal functions for each $\psi \in \mathscr{C}_0(\mathbb{R}^d)$ which is dense in $\mathrm{L}^2(\mathbb{R}^d)$.

3 The Integrated Density of States

3.1 Existence and Uniqueness

The quantity of main interest in the present paper is the integrated density of states and its corresponding measure, called the density-of-states measure. The next theorem deals with its definition and its representation as an infinite-volume limit of the suitably scaled finite-volume counterparts (2.9). It is the main result of the present paper.

Theorem 3.1. *Let $\Gamma \subset \mathbb{R}^d$ be a bounded open cube compatible with the lattice $\mathbb{Z}^d$ [recall (2.1)] and let χ_Γ denote the multiplication operator associated with the indicator function of Γ. Assume that the potentials A and V have the properties* (C), (S), (I), *and* (E). *Then the* (infinite-volume) integrated density of states

$$N(E) := \frac{1}{|\Gamma|}\,\mathbb{E}\Big\{\operatorname{Tr}\big[\chi_\Gamma\,\Theta(E - H(A,V))\,\chi_\Gamma\big]\Big\} < \infty \tag{3.1}$$

is well defined for all energies $E \in \mathbb{R}$ in terms of the spatially localized spectral family of the infinite-volume operator $H(A, V)$. It is the (unbounded left-continuous) distribution function of some positive Borel measure on the real line $\mathbb{R}$ and independent of Γ. Moreover, let $\Lambda \subset \mathbb{R}^d$ stand for bounded open cubes centered at the origin. Then there is a set $\Omega_0 \in \mathscr{A}$ of full probability, $\mathbb{P}(\Omega_0) = 1$, such that

$$N(E) = \lim_{\Lambda\uparrow\mathbb{R}^d} \frac{N^{(\omega)}_{\Lambda,\mathrm{X}}(E)}{|\Lambda|} \tag{3.2}$$

holds for both boundary conditions $\mathrm{X} = \mathrm{D}$ *and* $\mathrm{X} = \mathrm{N}$*, all $\omega \in \Omega_0$, and all $E \in \mathbb{R}$ except the (at most countably many) discontinuity points of N.*

Proof. See Section 4. □

Remarks 3.2. (i) As to the limit $\Lambda \uparrow \mathbb{R}^d$, we here and in the following think of a sequence of open cubes centered at the origin whose edge lengths tend to infinity. But there exist more general sequences of expanding regions in $\mathbb{R}^d$ for which the theorem remains true, see for example [46, Rem. 1 on p. 105] and [13, p. 304].

(ii) The homogeneity of the random potential and the magnetic field with respect to $\mathbb{Z}^d$ renders the r.h.s. of (3.1) independent of Γ. In case V is even $\mathbb{R}^d$-ergodic, one may pick an arbitrarily shaped bounded subset $\Gamma \in \mathscr{B}(\mathbb{R}^d)$ with $|\Gamma| > 0$ or even any non-zero square-integrable function instead of the indicator function; for details see the next corollary.

(iii) A proof of the *existence* of the infinite-volume limits in (3.2) under slightly different hypotheses was outlined in [41]. It uses functional-analytic arguments first presented in [32] for the case $A = 0$. A different approach to the existence of these limits for $A \neq 0$, using Feynman-Kac(-Itô) functional-integral representations of Schrödinger semigroups [52, 11], can be found in [55, 9]. It dates back to [44, 43] for the case $A = 0$ and, to our knowledge, works straightforwardly in the case $A \neq 0$ for $\mathrm{X} = \mathrm{D}$ only. For $A \neq 0$ *uniqueness* of the infinite-volume limit in (3.2), that is, its independence of the boundary condition X (previously claimed without proof in [41]) follows from [42] if the random potential V is bounded and from [19] if V is bounded from below. So the main new point about Proposition 3.1 is that it establishes existence and uniqueness for a wide class of V unbounded from below. This class also includes many V yielding operators $H(A, V)$ which are unbounded from below. Even for $A = 0$, Proposition 3.1 is partially new in that the corresponding result [46, Thm. 5.20], only shows vague convergence of the underlying measures, see Lemma 3.5 and Remarks 3.6 below.

(iv) Property (S) is only assumed to guarantee the almost-sure essential self-adjointness of the infinite-volume operator on $\mathscr{C}_0^\infty(\mathbb{R}^d)$. Property (I) is mainly technical. It ensures the existence of a sufficiently high integer moment of V needed for the applicability of standard resolvent techniques. In particular, (I) does not distinguish between the positive part $V_+ := \max\{0, V\}$ and the negative part $V_- := \max\{0, -V\}$ of V. This stands in contrast to proofs based on functional-integral representations, which require much stronger assumptions on V_- but much weaker assumptions on V_+, see [55, Thm. 3.1]. Instead of constant magnetic fields as demanded by property (C), the subsequent proof in Sect. 4 can be extended straightforwardly to cover also ergodic random magnetic fields as in [41, 55].

(v) The convergence (3.2) holds for any other boundary condition X for which the self-adjoint operator $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ obeys the inequalities $H_{\Lambda,\mathrm{N}}(A, V^{(\omega)}) \leq H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}) \leq H_{\Lambda,\mathrm{D}}(A, V^{(\omega)})$ in the sense of forms [49, Def. on p. 269]. This follows from the min-max principle [49, Sec. XIII.1] which implies that the finite-volume integrated density of states $N_{\Lambda,\mathrm{X}}^{(\omega)}$ associated with $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ obeys the sandwiching estimates

$$0 \leq N_{\Lambda,\mathrm{D}}^{(\omega)}(E) \leq N_{\Lambda,\mathrm{X}}^{(\omega)}(E) \leq N_{\Lambda,\mathrm{N}}^{(\omega)}(E) < \infty \tag{3.3}$$

for every bounded open cube $\Lambda \subset \mathbb{R}^d$ and all energies $E \in \mathbb{R}$.

(vi) Under the assumptions of Theorem 3.1 there is some $\Omega_1 \in \mathscr{A}$ with $\mathbb{P}(\Omega_1) = 1$ such that

$$N(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \operatorname{Tr}\left[\chi_\Lambda \, \Theta(E - H(A, V^{(\omega)})) \, \chi_\Lambda\right] \tag{3.4}$$

for all $\omega \in \Omega_1$ and all $E \in \mathbb{R}$ except the (at most countably many) discontinuity points of N. This follows from the fact that $\operatorname{Tr}\left[\chi_\Lambda \, \Theta(E - H(A, V^{(\omega)})) \, \chi_\Lambda\right] = \sum_{j \in \Lambda \cap \mathbb{Z}^d} \operatorname{Tr}\left[\chi_{\Lambda(j)} \, \Theta(E - H(A, V^{(\omega)})) \, \chi_{\Lambda(j)}\right]$ for all bounded open cubes Λ which are compatible with the lattice $\mathbb{Z}^d$, the Birkhoff-Khintchine ergodic theorem in the formulation of [46, Prop. 1.13] and the considerations in [30, p. 80]. Alternative representations of the integrated density of states as in (3.4) seem to date back to [3], see also [17, 31, 13, 56].

(vii) As a by-product, our proof of Theorem 3.1 yields (see (4.20) below) the following rough upper bound on the low-energy fall-off of N,

$$N(E) \leq C\, |E|^{d/2-2\vartheta} \tag{3.5}$$

for all $E \in]-\infty, -1]$ with some constant $C \geq 0$, see also [46, Thm. 5.29] for the case $A = 0$. The true leading behavior of $N(E)$ for $E \to -\infty$ is, of course, consistent with (3.5), but typically much faster. For example, in the case of a Gaussian random potential, in the sense of Subsection 3.3 below, it is known that $\lim_{E\to-\infty} E^{-2} \log N(E) = -(2C(0))^{-1}$, also in the presence of a constant magnetic field [41, 9, 55]. The leading low-energy behavior is less universal in case of a positive Poissonian potential and a constant magnetic field [10, 21, 26, 27, 22, 57], where N vanishes for negative energies anyway. In this context we recall from [41, 55] that the high-energy asymptotics is neither affected by the magnetic field nor by the random potential and given by $\lim_{E\to\infty} E^{-d/2} N(E) = [(d/2)!\,(2\pi)^{d/2}]^{-1}$ in accordance with Weyl's celebrated asymptotics for the free particle [59].

(viii) In case $H(A, V)$ is unbounded from below almost surely and serves as the one-particle Hamiltonian of a macroscopic system of non-interacting (spinless) fermions, the corresponding free energy and resulting basic thermostatic quantities may nevertheless be well defined, provided that $N(E)$ falls off to zero sufficiently fast as $E \to -\infty$. An at least algebraic decay in the sense that $N(E) \leq C\, |E|^{d/2-2\alpha}$ with sufficiently large $\alpha \in \mathbb{N}$, $\alpha > \vartheta$, is assured by simply requiring the ergodic random potential V to satisfy (2.5) with ϑ replaced by α. The proof of this assertion follows the same lines of reasoning leading to (3.5).

In analogy to [46, Prob. II.4] Theorem 3.1 implies

Corollary 3.3. *Assume that the potentials A and V have the properties* (C), (S), *and* (I). *Moreover, let V be $\mathbb{R}^d$-ergodic (and not only $\mathbb{Z}^d$-ergodic). Then*

$$N(E) = \frac{1}{\|f\|_2^2}\, \mathbb{E}\Big\{ \mathrm{Tr}\Big[\overline{f}\, \Theta(E - H(A, V))\, f\Big]\Big\}, \qquad E \in \mathbb{R}, \tag{3.6}$$

for any non-zero $f \in \mathrm{L}^2(\mathbb{R}^d)$ which is to be understood as a multiplication operator inside the trace.

Remark 3.4. Assume the situation of Corollary 3.3 and that the spectral projection $\Theta(E - H(A, V))$ possesses $\mathbb{P}$-almost surely a jointly continuous integral kernel $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Theta(E - H(A, V))(x, y) \in \mathbb{C}$. Then (3.6) with $f \in \mathscr{C}_0(\mathbb{R}^d)$ gives by [48, Lemma on pp. 65–66], Fubini's theorem, and the $\mathbb{R}^d$-homogeneity of V the formula

$$N(E) = \mathbb{E}\Big[\Theta(E - H(A, V))(0, 0)\Big], \qquad E \in \mathbb{R}, \tag{3.7}$$

see also [55, Prop. 3.2]. A sufficient condition for the existence and continuity of the integral kernel [11, Remark 6.1.(ii)] is that V_- and $V_+\chi_\Lambda$ belong for any bounded $\Lambda \in \mathscr{B}(\mathbb{R}^d)$ $\mathbb{P}$-almost surely to the Kato class

$$\mathscr{K}(\mathbb{R}^d) := \Big\{ v : \mathbb{R}^d \to \mathbb{R} :\ v \text{ Borel measurable and } \lim_{t\downarrow 0} \varkappa_t(v) = 0 \Big\}, \tag{3.8}$$

where $\varkappa_t(v) := \sup_{x\in\mathbb{R}^d} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}^d\xi \, \mathrm{e}^{-|\xi|^2} |v(x+\xi\sqrt{s})|$. While property (S) implies $V_+\chi_\Lambda \in \mathscr{K}(\mathbb{R}^d)$, it does not ensure $V_- \in \mathscr{K}(\mathbb{R}^d)$ even when combined with property (I). This is in agreement with the fact that $H(A,V)$ would else be bounded from below, which, for example, is not the case if V is a Gaussian random potential (in the sense of Subsection 3.3 below). For weaker conditions which ensure the validity of (3.7) for rather general random potentials including Gaussian ones, see [12].

Proof of Corollary 3.3. We may assume $f \geq 0$, because the general case $f \in \mathrm{L}^2(\mathbb{R}^d)$ follows therefrom. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(\mathbb{R}^d)$ be a monotone increasing sequence, $f_n \leq f_m$ if $n \leq m$, of positive simple functions approximating f. More precisely, these functions are assumed to be of the form $f_n(x) = \sum_{k=1}^n f_{n,k}\chi_{\Gamma_{n,k}}(x)$ with suitable constants $f_{n,k} \geq 0$ and bounded Borel sets $\Gamma_{n,k} \in \mathscr{B}(\mathbb{R}^d)$ which are pairwise disjoint for each fixed n. Using (3.1) and the $\mathbb{R}^d$-homogeneity of the random potential (see Remark 3.2(ii)) one verifies that (3.6) is valid for all simple functions. Thanks to the convergence $f_n \to f$ as $n \to \infty$ in $\mathrm{L}^2(\mathbb{R}^d)$ this implies

$$\lim_{n,m\to\infty} \int_\Omega \mathbb{P}(\mathrm{d}\omega) \, \|\Theta^{(\omega)}(f_n - f_m)\|_2^2 = N(E) \lim_{n,m\to\infty} |f_n - f_m|_2^2 = 0, \tag{3.9}$$

where we are using the abbreviation $\Theta^{(\omega)} := \Theta(E - H(A, V^{(\omega)}))$. Hence there exists some sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$\mathbb{E}\Big[\|\Theta\, f_{n_{j+1}} - \Theta\, f_{n_j}\|_2\Big] \leq \Big\{\mathbb{E}\Big[\|\Theta\, f_{n_{j+1}} - \Theta\, f_{n_j}\|_2^2\Big]\Big\}^{1/2} \leq 2^{-j} \tag{3.10}$$

for all $j \in \mathbb{N}$ by Jensen's inequality and (3.9). Thanks to monotonicity the r.h.s. of the estimate

$$\|\Theta^{(\omega)} f_{n_i} - \Theta^{(\omega)} f_{n_j}\|_2 \leq \sum_{k=\min\{i,j\}}^{\infty} \|\Theta^{(\omega)} f_{n_{k+1}} - \Theta^{(\omega)} f_{n_k}\|_2, \tag{3.11}$$

converges pointwise for all $\omega \in \Omega_{\mathsf{S}}$ as $i, j \to \infty$. Since $\lim_{i,j\to\infty} \sum_{k=\min\{i,j\}}^\infty \mathbb{E}[\|\Theta\, f_{n_{k+1}} - \Theta\, f_{n_k}\|_2] = 0$ by (3.10), the monotone- and dominated-convergence theorems imply that the r.h.s. (and hence the l.h.s.) of (3.11) converges in fact to zero for $\mathbb{P}$-almost all $\omega \in \Omega$. In other words, the subsequence $(\Theta^{(\omega)} f_{n_j})_j$ is Cauchy in $\mathscr{J}_2(\mathrm{L}^2(\mathbb{R}^d))$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Since the space $\mathscr{J}_2(\mathrm{L}^2(\mathbb{R}^d))$ is complete, this sequence converges with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$ to some $F^{(\omega)} \in \mathscr{J}_2(\mathrm{L}^2(\mathbb{R}^d))$. Let $f : \mathscr{C}_0^\infty(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$ denote a multiplication operator associated with f. The above convergence and $\lim_{n\to\infty} |(f - f_n)\psi|_2 = 0$ for all $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ imply that $F^{(\omega)}$ is the unique continuous extension of $\Theta^{(\omega)} f$ from $\mathscr{C}_0^\infty(\mathbb{R}^d)$ to the whole Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$. Denoting this extension also by $\Theta^{(\omega)} f$, we thus have

$$\lim_{j\to\infty} \|\Theta^{(\omega)} f_{n_j} - \Theta^{(\omega)} f\|_2 = 0 \tag{3.12}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. We therefore get

$$\begin{aligned}\mathbb{E}\Big[\|\Theta(E - H(A,V)) f\|_2^2\Big] &= \mathbb{E}\Big[\lim_{j\to\infty} \|\Theta(E - H(A,V)) f_{n_j}\|_2^2\Big] \\ &= \lim_{j\to\infty} \mathbb{E}\Big[\|\Theta(E - H(A,V)) f_{n_j}\|_2^2\Big] = N(E) \lim_{j\to\infty} |f_{n_j}|_2^2 = N(E)\, |f|_2^2. \end{aligned} \tag{3.13}$$

For the second equality we used the monotone-convergence theorem. Note that $(\|\Theta^{(\omega)} f_n\|_2^2)_n$ is monotone increasing since $\|\Theta^{(\omega)} f_n\|_2^2 = \|\Theta^{(\omega)} f_n^2 \Theta^{(\omega)}\|_1$. □

3.2 Some Properties of the Density-of-States Measure

The proof of Theorem 3.1 will be based on the (almost-sure) vague convergence [6, Def. 30.1] of the two *spatial eigenvalue concentrations* $|\Lambda|^{-1}\,\nu_{\Lambda,\mathrm{X}}^{(\omega)}$, with $\mathrm{X}=\mathrm{D}$ or $\mathrm{X}=\mathrm{N}$, to the same non-random measure ν in the infinite-volume limit $\Lambda\uparrow\mathbb{R}^d$. This measure is called the density-of-states measure and uniquely corresponds to the integrated density of states (3.1) in the sense that $N(E)=\nu(]-\infty,E[)$ for all $E\in\mathbb{R}$.

Lemma 3.5. *Assume the situation of Theorem 3.1. Then the* (infinite-volume) density-of-states measure

$$\nu(I):=\frac{1}{|\Gamma|}\,\mathbb{E}\Big\{\operatorname{Tr}\left[\chi_\Gamma\,\chi_I(H(A,V))\,\chi_\Gamma\right]\Big\},\qquad I\in\mathscr{B}(\mathbb{R}),\tag{3.14}$$

is a positive Borel measure on the real line $\mathbb{R}$, well defined in terms of the spatially localized projection-valued spectral measure of the infinite-volume random Schrödinger operator, and independent of Γ. Moreover, in the sense of vague convergence

$$\nu=\lim_{\Lambda\uparrow\mathbb{R}^d}\frac{\nu_{\Lambda,\mathrm{X}}^{(\omega)}}{|\Lambda|}\tag{3.15}$$

for both $\mathrm{X}=\mathrm{D}$ *and* $\mathrm{X}=\mathrm{N}$ *and $\mathbb{P}$-almost all $\omega\in\Omega$.*

Proof. See Section 4. □

Remarks 3.6. (i) Lemma 3.5 generalizes [46, Thm. 5.20] which deals with the case $A=0$. In fact, our proof in Section 4 closely follows the arguments given there. Concerning the independence of X of the infinite-volume limit in (3.15), we build on a result in [42] for bounded V. (Alternatively, one may use a result in [19].)

(ii) Lemma 3.5 alone does not imply the existence of the integrated density of states N. Moreover, even if the finiteness of $\nu(]-\infty,E[)$ for all $E\in\mathbb{R}$ were known, see (3.5), the vague convergence (3.15) alone would not imply the pointwise convergence (3.2) of the distribution functions in case their supports are not uniformly bounded from below. The latter occurs for random potentials with realizations $V^{(\omega)}$ which yield operators $H(A,V^{(\omega)})$ unbounded from below. On the other hand, (3.2) implies (3.15), see Proposition 4.3 below.

Using (3.14) one may relate properties of the density-of-states measure ν to simple spectral properties of the infinite-volume magnetic Schrödinger operator. Examples are the support of ν and the location of the almost-sure spectrum of $H(A,V^{(\omega)})$ or the absence of a point component in the Lebesgue decomposition of ν and the absence of "immobile eigenvalues" of $H(A,V^{(\omega)})$. This is the content of

Corollary 3.7. *Under the assumptions of Theorem 3.1 and letting $I\in\mathscr{B}(\mathbb{R})$ the following equivalence holds: $\nu(I)=0$ if and only if $\chi_I(H(A,V^{(\omega)}))=0$ for $\mathbb{P}$-almost all $\omega\in\Omega$. This immediately implies:*

(i) $\operatorname{supp}\nu=\operatorname{spec}H(A,V^{(\omega)})$ *for $\mathbb{P}$-almost all $\omega\in\Omega$. [Here* $\operatorname{spec}H(A,V^{(\omega)})$ *denotes the spectrum of $H(A,V^{(\omega)})$ and* $\operatorname{supp}\nu:=\{E\in\mathbb{R}:\nu(]E-\varepsilon,E+\varepsilon[)>0$ for all $\varepsilon>0\}$ *is the topological support of ν.]*

(ii) $0 = \nu(\{E\})$ $\big(= \lim_{\varepsilon\downarrow 0}[N(E+\varepsilon) - N(E)]\big)$ *if and only if* $E \in \mathbb{R}$ *is not an eigenvalue of* $H(A, V^{(\omega)})$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$.

Proof. If $\chi_I(H(A, V^{(\omega)})) = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$, then $\nu(I) = 0$ using (3.14). Conversely, for every $\psi \in \mathscr{C}_0(\mathbb{R}^d) \subset \mathrm{L}^2(\mathbb{R}^d)$, normalized in the sense $\langle\psi, \psi\rangle = 1$, there exists a bounded open cube $\Gamma \subset \mathbb{R}^d$ compatible with $\mathbb{Z}^d$ such that $\operatorname{supp}\psi \subseteq \Gamma$ and therefore

$$\Big\langle \psi\,,\, \chi_I(H(A, V^{(\omega)}))\,\psi \Big\rangle \le \operatorname{Tr}\Big[\chi_\Gamma\, \chi_I(H(A, V^{(\omega)}))\,\chi_\Gamma\Big]. \tag{3.16}$$

Taking the probabilistic expectation on both sides and using (3.14) we arrive at the sandwiching estimate $0 \le \mathbb{E}\left[\langle\psi\,,\, \chi_I(H(A, V))\,\psi\rangle\right] \le |\Gamma|\,\nu(I) = 0$ by the assumption $\nu(I) = 0$. Since the magnetic translations $\{T_x\}$ with $x \in \mathbb{R}^d$ or $\mathbb{Z}^d$ are total, the proof of [13, Lemma V.2.1] shows that $\chi_I(H(A, V^{(\omega)})) = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$. □

Remark 3.8. The equivalence (ii) of the above corollary is a continuum analogue of [16, Prop. 1.1], see also [46, Thm. 3.3]. In the one-dimensional case [45] and the multi-dimensional lattice case [18], the equivalence has been exploited to show in case $A = 0$ the (global) continuity of the integrated density of states N under practically no further assumptions on the random potential beyond those ensuring the existence of N. The proof of such a statement in the multi-dimensional continuum case is considered an important open problem [54]. In case $A \neq 0$ one certainly needs additional assumptions as [20] illustrates. Under certain additional assumptions the integrated density of states is not only continuous but even (locally) Hölder continuous of arbitrary order strictly smaller than one [15, 25] or even equal to one [14, 4, 5, 28]. The latter is equivalent to N being absolutely continuous with locally bounded derivative [50, Chap. 7, Exc. 10].

3.3 Examples

In this subsection we list three examples of (possibly unbounded) random potentials to which the results of the preceding subsections can be applied. While the first one models (crystalline) disordered alloys, the other two model (non-crystalline) amorphous solids. These are typical examples considered in the literature. Each of them is characterized by one of the following properties. We recall from properties (S) and (I) the definitions of the constants $p(d)$ and ϑ.

(A) V is an *alloy-type random field*, that is, a random field with realizations given by

$$V^{(\omega)}(x) = \sum_{j\in\mathbb{Z}^d} \lambda_j^{(\omega)} u(x - j). \tag{3.17}$$

The random variables $\{\lambda_j\}$ are $\mathbb{P}$-independent and identically distributed according to the common probability measure $\mathscr{B}(\mathbb{R}) \ni I \mapsto \mathbb{P}\{\lambda_0 \in I\}$. Moreover, we suppose that the Borel-measurable function $u : \mathbb{R}^d \to \mathbb{R}$ satisfies the Birman-Solomyak condition $\sum_{j\in\mathbb{Z}^d}\big(\int_{\Lambda(j)}\mathrm{d}^d x\,|u(x)|^{p_1}\big)^{1/p_1} < \infty$ with some real $p_1 \ge 2\vartheta + 1$ and that $\mathbb{E}\left(|\lambda_0|^{p_2}\right) < \infty$ for some real p_2 satisfying $p_2 \ge 2\vartheta + 1$ and $p_2 > p_1 d/[2(p_1 - p(d))]$.

(P) V is a *Poissonian field*, that is, a random field with realizations given by

$$V^{(\omega)}(x) = \int_{\mathbb{R}^d} \mu_\varrho^{(\omega)}(\mathrm{d}^d y)\, u(x-y), \tag{3.18}$$

where μ_ϱ denotes the (random) Poissonian measure on $\mathbb{R}^d$ with parameter $\varrho \geq 0$. Moreover, we suppose that the Borel-measurable function $u : \mathbb{R}^d \to \mathbb{R}$ satisfies the Birman-Solomyak condition $\sum_{j\in\mathbb{Z}^d} \big(\int_{\Lambda(j)} \mathrm{d}^d x\, |u(x)|^{2\vartheta+1} \big)^{1/(2\vartheta+1)} < \infty$.

(G) V is a *Gaussian random field* [1, 40] which is $\mathbb{R}^d$-homogeneous. It has zero mean, $\mathbb{E}[\,V(0)] = 0$, and its covariance function $x \mapsto C(x) := \mathbb{E}[\,V(x)V(0)]$ is continuous at the origin where it obeys $0 < C(0) < \infty$.

The following remarks further explain the above three examples.

Remarks 3.9. (i) Consider an *alloy-type random potential*, that is, a random potential with property (A). Such a potential models a (generalized) disordered alloy [34, Ch. 21] which is composed of different atoms occupying, at random, the sites of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Which kind of atom at site $j \in \mathbb{Z}^d$ actually interacts with the quantum particle (classically) located at $x \in \mathbb{R}^d$ through the potential $\lambda_j^{(\omega)} u(x-j)$, is determined by the value $\lambda_j^{(\omega)} \in \mathbb{R}$ of the coupling strength at site j. An alloy-type random potential V is $\mathbb{Z}^d$-ergodic and hence has property (E). Moreover, V is a random field of the form (3.21) below, since one may choose μ there as the random signed pure-point measure given by $\mu^{(\omega)} = \sum_{j\in\mathbb{Z}^d} \lambda_j^{(\omega)} \delta_j$ where δ_y denotes the Dirac measure on $\mathbb{R}^d$ supported at $y \in \mathbb{R}^d$. Lemma 3.10 below with $q = p_1$ and $r = p_2$ shows that V has property (S). Choosing $q = r = 2\vartheta + 1$ it is seen to obey property (I).

(ii) Consider a *Poissonian potential*, that is, a random potential with property (P). Then V is $\mathbb{R}^d$-ergodic and hence has property (E). Using the fact that the Poissonian measure μ_ϱ is a random Borel measure which is pure point and positive-integer valued, each realization of V is informally given by $V^{(\omega)}(x) = \sum_j u(x - x_j^{(\omega)})$. Here the Poissonian points $\{x_j^{(\omega)}\}$ are interpreted as the positions of impurities, each of them generating the same potential u. The random variable $\mu_\varrho(\Lambda)$ then equals the number of impurities in the bounded Borel set $\Lambda \subset \mathbb{R}^d$ and is distributed according to Poisson's law

$$\mathbb{P}(\,\mu_\varrho(\Lambda) = n\,) = \frac{(\varrho\,|\Lambda|)^n}{n!}\,\mathrm{e}^{-\varrho|\Lambda|}, \qquad n \in \mathbb{N} \cup \{0\}, \tag{3.19}$$

so that the parameter ϱ is identified as the mean spatial concentration of impurities. By choosing $\mu = \mu_\varrho$, $q = p_1 = 2\vartheta + 1$, and $r = p_2 > p_1 d/[2p_1 - p(d)]$ in Lemma 3.10 below, one verifies that a Poissonian potential statisfies property (S). Moreover, choosing $q = r = 2\vartheta + 1$ there, it is seen to obey property (I). If $u \geq 0$, (3.7) holds for the Poissonian potential.

(iii) Consider a random field with the Gaussian property (G). Then its covariance function C is bounded and uniformly continuous on $\mathbb{R}^d$. Consequently, [23, Thm. 3.2.2] implies the existence of a separable version V of this field which is jointly measurable. Speaking about a *Gaussian random potential*, it is tacitly assumed that only this version will be dealt with. By the Bochner-Khintchine theorem [47, Thm. IX.9] there is a one-to-one correspondence between Gaussian random potentials and finite positive (and even) Borel measures on $\mathbb{R}^d$. Using the identity

$$\mathbb{E}\left[|V(0)|^p\right] = \frac{1}{[2\pi C(0)]^{1/2}} \int_{\mathbb{R}} \mathrm{d}v\, \mathrm{e}^{-v^2/2C(0)}\, |v|^p = \Big(\frac{p-1}{2}\Big)!\, \frac{[2C(0)]^{p/2}}{\pi^{1/2}}, \tag{3.20}$$

a Gaussian random potential is seen by Fubini's theorem to satisfy (2.6) and hence properties (S) and (I). A simple sufficient criterion ensuring $\mathbb{R}^d$-ergodicity, hence property (E), is the mixing condition $\lim_{|x|\to\infty} C(x) = 0$. We note that the operator $H(A, V)$ is almost surely unbounded from below for any Gaussian random potential V.

The next lemma has already been used to verify properties (S) and (I) for the examples (A) and (P). It is patterned on [33, Prop. 2], see also [13, Cor. V.3.4].

Lemma 3.10. *Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, $\mu : \Omega \ni \omega \mapsto \mu^{(\omega)}$ be a random signed Borel measure on $\mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$ be a Borel-measurable function. Let V be the random field given by the realizations*

$$V^{(\omega)}(x) := \int_{\mathbb{R}^d} \mu^{(\omega)}(\mathrm{d}^d y)\, u(x-y), \qquad x \in \mathbb{R}^d, \quad \omega \in \Omega. \tag{3.21}$$

Then the estimate

$$\begin{aligned} &\Big\{\mathbb{E}\Big[\Big(\int_{\Lambda(j)} \mathrm{d}^d x\, |V(x)|^q\Big)^{r/q}\Big]\Big\}^{1/r} \\ &\quad \leq 3^{d/q} \sup_{l\in\mathbb{Z}^d} \Big\{\mathbb{E}\Big[\big(|\mu|(\Lambda(l))\big)^r\Big]\Big\}^{1/r} \sum_{k\in\mathbb{Z}^d} \Big(\int_{\Lambda(k)} \mathrm{d}^d x\, |u(x)|^q\Big)^{1/q} \end{aligned} \tag{3.22}$$

holds for all $j \in \mathbb{Z}^d$ and all those $q, r \in [1, \infty[$, for which the r.h.s. of (3.22) is finite. [Here $|\mu^{(\omega)}|$ denotes the total-variation measure of $\mu^{(\omega)}$.]

Proof. Minkowski's inequality [38, Thm. 2.4], a subsequent shift in the $\mathrm{d}^d x$-integration, and an enlargement of its domain show that

$$\begin{aligned} \Big(\int_{\Lambda(j)} \mathrm{d}^d x\, |V^{(\omega)}(x)|^q\Big)^{1/q} &\leq \int_{\mathbb{R}^d} |\mu^{(\omega)}|(\mathrm{d}^d y)\, \Big(\int_{\Lambda(j)} \mathrm{d}^d x\, |u(x-y)|^q\Big)^{1/q} \\ &\leq \sum_{k\in\mathbb{Z}^d} |\mu^{(\omega)}|(\Lambda(k))\, \Big(\int_{\Lambda(j)-\Lambda(k)} \mathrm{d}^d x\, |u(x)|^q\Big)^{1/q}, \end{aligned} \tag{3.23}$$

where the cube $\Lambda(j) - \Lambda(k) := \{x - y \in \mathbb{R}^d : x \in \Lambda(j) \text{ and } y \in \Lambda(k)\}$ is the arithmetic difference of the unit cubes $\Lambda(j)$ and $\Lambda(k)$. Using Minkowski's inequality again, we thus arrive at

$$\begin{aligned} \Big\{\mathbb{E}\Big[\Big(\int_{\Lambda(j)} \mathrm{d}^d x\, |V(x)|^q\Big)^{r/q}\Big]\Big\}^{1/r} &\leq \sum_{k\in\mathbb{Z}^d} \Big\{\mathbb{E}\Big[\big(|\mu|(\Lambda(k))\big)^r\Big]\Big\}^{1/r} \Big(\int_{\Lambda(j)-\Lambda(k)} \mathrm{d}^d x\, |u(x)|^q\Big)^{1/q} \\ &\leq \sup_{l\in\mathbb{Z}^d} \Big\{\mathbb{E}\Big[\big(|\mu|(\Lambda(l))\big)^r\Big]\Big\}^{1/r} \sum_{k\in\mathbb{Z}^d} \Big(\int_{\Lambda(0)-\Lambda(0)} \mathrm{d}^d x\, |u(x-k)|^q\Big)^{1/q}. \end{aligned} \tag{3.24}$$

Since $\Lambda(0) - \Lambda(0)$ is contained in the cube centered at the origin and consisting of 3^d unit cubes, the proof is complete. □

4 Proof of the Main Result

The purpose of this section is to prove Lemma 3.5 and Theorem 3.1, which is done in Subsection 4.2. There we first show vague convergence of the density-of-states measures as claimed in Lemma 3.5. Apart from minor modifications, we will thereto adapt the strategy of the proof of [46, Thm. 5.20] which presents an approximation argument for the case $A = 0$. The latter permits us to take advantage of the independence of the infinite-volume limits of the boundary conditions in case V is bounded [42, 19]. Moreover, the argument also allows us to use established results [55, 9] in the case $\mathrm{X} = \mathrm{D}$. This procedure requires auxiliary trace estimates, which are proven in Subsections 4.3 and 4.4. In a second step, we use a criterion which provides conditions under which vague convergence of measures implies pointwise convergence of their distribution functions. This finally proves Theorem 3.1. In Subsection 4.1 we supply such a criterion and, to begin with, a criterion ensuring vague convergence.

4.1 On Vague Convergence of Positive Borel Measures on the Real Line

We recall [6, Def. 25.2(i)] that a positive measure on the real line $\mathbb{R}$ is a Borel measure if it assigns a finite length to each bounded Borel set in $\mathbb{R}$. Note that every Borel measure on $\mathbb{R}$ is regular and hence a Radón measure [6, Thm. 29.12]. Moreover, we recall from [6, § 30, Exc. 3] that vague convergence of a sequence of positive Borel measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ to a measure μ is equivalent to the convergence $\lim_{n\to\infty} \widehat{\mu}_n(f) = \widehat{\mu}(f)$ for all $f \in \mathscr{C}_0^1(\mathbb{R})$. Here and in the following, we occasionally use the abbreviation

$$\widehat{\nu}(f) := \int_{\mathbb{R}} \nu(\mathrm{d}E)\, f(E) \tag{4.1}$$

for the integral of a function f with respect to a measure ν.

Our proof of Lemma 3.5 relies on the following generalization of [46, Lemma 5.22] which provides a criterion for vague convergence.

Proposition 4.1. *Let $p \in]1, \infty[$ and let μ and μ_n, for each $n \in \mathbb{N}$, be positive (not necessarily finite) Borel measures on the real line $\mathbb{R}$ such that the integrals*

$$\widetilde{\mu}_n(z, p) := \int_{\mathbb{R}} \frac{\mu_n(\mathrm{d}E)}{|E - z|^p} \qquad \text{and} \qquad \widetilde{\mu}(z, p) := \int_{\mathbb{R}} \frac{\mu(\mathrm{d}E)}{|E - z|^p} \tag{4.2}$$

are finite for all $z \in \mathbb{C}\backslash\mathbb{R}$ and all $n \in \mathbb{N}$. If $\lim_{n\to\infty} \widetilde{\mu}_n(z, p) = \widetilde{\mu}(z, p)$ for all $z \in \mathbb{C}\backslash\mathbb{R}$, then μ_n converges vaguely to μ as $n \to \infty$.

Remark 4.2. The following implication is immediate. If $\widetilde{\mu}(z, p) = \widetilde{\nu}(z, p) < \infty$ for some $p \in]1, \infty[$ and all $z \in \mathbb{C}\backslash\mathbb{R}$, then the underlying positive Borel measures μ and ν are equal.

Proof of Proposition 4.1. We first define the following one-parameter family

$$E \mapsto \delta_0^{(\varepsilon)}(E) := \Upsilon_p \frac{\varepsilon^{p-1}}{|E - \mathrm{i}\varepsilon|^p}, \qquad (\Upsilon_p)^{-1} := \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{|\xi - \mathrm{i}|^p}, \qquad \varepsilon > 0, \tag{4.3}$$

of smooth Lebesgue probability densities on $\mathbb{R}$ which approximates the Dirac measure δ_0 on $\mathbb{R}$ supported at $E=0$ as $\varepsilon \downarrow 0$. Moreover, let $f_\varepsilon := \delta_0^{(\varepsilon)} * f$ denote the convolution of $\delta_0^{(\varepsilon)}$ and $f \in \mathscr{C}_0^1(\mathbb{R})$. The fundamental theorem of calculus yields $f(E-E') = f(E) - \int_0^{E'} \mathrm{d}\eta\, f'(E-\eta)$ and hence

$$\sup_{E\in\mathbb{R}} |f(E) - f_\varepsilon(E)| \le \int_{\mathbb{R}} \mathrm{d}E'\, \delta_0^{(1)}(E') \sup_{E\in\mathbb{R}} \left| \int_0^{\varepsilon E'} \mathrm{d}\eta\, f'(E-\eta) \right|. \tag{4.4}$$

The supremum on the r.h.s. does not exceed $\varepsilon\, |E'|\, |f'|_\infty$ and hence converges to zero as $\varepsilon \downarrow 0$ for all $E' \in \mathbb{R}$. On the other hand, the supremum may be estimated by $|f'|_1$ such that the dominated-convergence theorem is applicable and one has $\lim_{\varepsilon\downarrow 0} f_\varepsilon = f$ uniformly on $\mathbb{R}$. We now claim that there exists some $C(\varepsilon) > 0$, depending on f, with $\lim_{\varepsilon\downarrow 0} C(\varepsilon) = 0$ such that

$$|f(E) - f_\varepsilon(E)| \le C(\varepsilon)\, \delta_0^{(1)}(E) \tag{4.5}$$

for all $E \in \mathbb{R}$. To prove this, we pick a compact subset G of $\mathbb{R}$ such that $\operatorname{supp} f \subset G$ and $\operatorname{dist}(\mathbb{R}\backslash G, \operatorname{supp} f) > 1$. Since f_ε converges uniformly to f as $\varepsilon \downarrow 0$, the bound (4.5) is valid for all $E \in G$. For any other $E \in \mathbb{R}\backslash G$ the claim (4.5) follows from the estimate $|f_\varepsilon(E)| \le |f|_\infty \int_{\operatorname{supp} f} \mathrm{d}E'\, \delta_0^{(\varepsilon)}(E-E')$ and an explicit computation. Inequality (4.5) may then be employed to show

$$\int_{\mathbb{R}} \mu_n(\mathrm{d}E)\, |f(E) - f_\varepsilon(E)| \le C(\varepsilon)\, \widetilde{\mu}_n(\mathrm{i}, p). \tag{4.6}$$

The same holds true with μ and $\widetilde{\mu}$ taking the place of μ_n and $\widetilde{\mu}_n$, respectively. We then estimate

$$\begin{aligned}
|\widehat{\mu}_n(f) - \widehat{\mu}(f)| &\le |\widehat{\mu}_n(f) - \widehat{\mu}_n(f_\varepsilon)| + |\widehat{\mu}(f) - \widehat{\mu}(f_\varepsilon)| + |\widehat{\mu}_n(f_\varepsilon) - \widehat{\mu}(f_\varepsilon)| \\
&\le C(\varepsilon)\, (\widetilde{\mu}_n(\mathrm{i}, p) + \widetilde{\mu}(\mathrm{i}, p)) \\
&\quad + \Upsilon_p \varepsilon^{p-1} \int_{\mathbb{R}} \mathrm{d}E\, f(E)\, |\widetilde{\mu}_n(E+\mathrm{i}\varepsilon, p) - \widetilde{\mu}(E+\mathrm{i}\varepsilon, p)|.
\end{aligned} \tag{4.7}$$

Here we have used the triangle inequality and Fubini's theorem in the integrals $\widehat{\mu}_n(f_\varepsilon) = \int_{\mathbb{R}} \mu_n(\mathrm{d}E)\, f_\varepsilon(E)$ and $\widehat{\mu}(f_\varepsilon) = \int_{\mathbb{R}} \mu(\mathrm{d}E)\, f_\varepsilon(E)$. The integral on the r.h.s. of (4.7) tends to zero as $n \to \infty$ by the dominated-convergence theorem. It is applicable since the estimate $\widetilde{\mu}_n(E+\mathrm{i}\varepsilon, p) \le (1 + |E|/\varepsilon)^p\, \widetilde{\mu}_n(\mathrm{i}\varepsilon, p)$ shows that the integrand in (4.7) is bounded on $\operatorname{supp} f$. Moreover, since $\widetilde{\mu}_n(\mathrm{i}\varepsilon, p) \to \widetilde{\mu}(\mathrm{i}\varepsilon, p)$ as $n \to \infty$, this bound may be chosen independent of n. To complete the proof, we note that the other terms in (4.7) stay finite as $n \to \infty$ and can be made arbitrarily small as $\varepsilon \downarrow 0$. □

In case each term of a sequence (μ_n) of measures possesses a finite (in general unbounded) distribution function $E \mapsto \mu_n(]-\infty, E[)$, vague convergence of (μ_n) does in general *not* imply pointwise convergence of the sequence of distribution functions. Even worse, if the latter convergence holds true, its limit is in general not equal to the distribution function of the limit of (μ_n). If one desires this equality, one needs a further criterion. This is provided by (4.9) in

Proposition 4.3. *Let μ and μ_n, for each $n \in \mathbb{N}$, be positive (not necessarily finite) Borel measures on the real line $\mathbb{R}$. Then the following two statements are equivalent:*

(i) *The finiteness* $\mu(\,]-\infty, E[\,) < \infty$ *holds for all* $E \in \mathbb{R}$ *and the relation*

$$\lim_{n\to\infty} \mu_n(\,]-\infty, E[\,) = \mu(\,]-\infty, E[\,) \tag{4.8}$$

holds for all $E \in \mathbb{R}$ *except the at most countably many with* $\mu(\{E\}) \neq 0$.

(ii) *The sequence* (μ_n) *converges vaguely to* μ *as* $n \to \infty$ *and the relation*

$$\lim_{E\downarrow-\infty} \limsup_{n\to\infty} \mu_n(\,]-\infty, E[\,) = 0 \tag{4.9}$$

holds.

Remark 4.4. A sequence (μ_n) obeying (4.9) might be called "tight near minus infinity". This naturally extends the usual notion of tightness [6, § 30 Rem. 3] for finite measures to ones having only finite (in general unbounded) distribution functions and ensures that no mass is lost at minus infinity as μ_n tends to μ. More precisely, for each $E \in \mathbb{R}$ the sequence of truncated measures $(\mu_n I_E)_{n\in\mathbb{N}}$, defined below (4.11), is tight in the usual sense. This follows either from the definition of the latter or alternatively from the subsequent proof and [6, Thm. 30.8].

Proof of Proposition 4.3. *(i)* $\Rightarrow$ *(ii):* Equation (4.9) follows from (4.8) and the finiteness $\mu(\,]-\infty, E[\,) < \infty$. Moreover, for every $f \in \mathscr{C}_0^1(\mathbb{R})$ one has

$$\int_{\mathbb{R}} \mu_n(\mathrm{d}E)\, f(E) = -\int_{\mathbb{R}} \mathrm{d}E\, \mu_n(]-\infty, E[)\, f'(E) \tag{4.10}$$

by partial integration. Vague convergence of (μ_n) to μ is now a consequence of the dominated-convergence theorem. It is applicable since (4.8) implies the existence of a locally bounded function dominating all but finitely many of the non-decreasing functions $E \mapsto \mu_n(\,]-\infty, E[\,)$.

(ii) $\Rightarrow$ *(i):* For every $E \in \mathbb{R}$ we define the following continuous "indicator function"

$$\mathbb{R} \ni E' \mapsto I_E(E') := \chi_{]-\infty,E[}(E') + \Big(E + 1 - E'\Big)\, \chi_{[E,E+1[}(E') \tag{4.11}$$

of the half-line $]-\infty, E[\subset \mathbb{R}$. Moreover, we let $\mu\, I_E$ denote the μ-continuous Borel measure with density I_E, that is, $(\mu\, I_E)(B) = \int_B \mu(\mathrm{d}E')\, I_E(E')$ for all $B \in \mathscr{B}(\mathbb{R})$, and the Borel measures $\mu_n\, I_E$ are defined accordingly. From (4.9) it follows that $\limsup_{n\to\infty} \mu_n(\,]-\infty, E_0[\,) < \infty$ for some $E_0 \in \mathbb{R}$. Hence the vague convergence $\mu_n\, I_E \to \mu\, I_E$ as $n \to \infty$ and [6, Lemma 30.3] imply that

$$\mu(\,]-\infty, E[\,) \le \widehat{\mu}(I_E) \le \liminf_{n\to\infty} \widehat{\mu}_n(I_E) \le \liminf_{n\to\infty} \widehat{\mu}_n(]-\infty, E+1[) < \infty \tag{4.12}$$

for all $E \in]-\infty, E_0 - 1]$. Since μ is a Borel measure, this implies the finiteness of $\mu(\,]-\infty, E[\,) = \mu(\,]-\infty, E_0 - 1[\,) + \mu(\,[E_0 - 1, E[\,)$ for all $E \in \mathbb{R}$.

The sequence of total masses of $\mu_n\, I_E$ converges to the total mass of the limiting measure $\mu\, I_E$. More precisely, defining the function $J_{E_1,E} := I_E - I_{E_1} \in \mathscr{C}_0(\mathbb{R})$ for each $E_1 < E$, it follows that

$$\begin{aligned}\lim_{n\to\infty} \widehat{\mu}_n(I_E) &= \lim_{E_1\downarrow-\infty} \lim_{n\to\infty} \widehat{\mu}_n(I_{E_1}) + \lim_{E_1\downarrow-\infty} \lim_{n\to\infty} \widehat{\mu}_n(J_{E_1,E}) \\ &= \lim_{E_1\downarrow-\infty} \widehat{\mu}(J_{E_1,E}) = \widehat{\mu}(I_E).\end{aligned} \tag{4.13}$$

Here the first term on the r.h.s. of the first equality tends to zero using (4.9) and $0 \le \widehat{\mu}_n(I_{E_1}) \le \mu_n(]-\infty, E_1+1[)$. The second equality is a consequence of the vague convergence of μ_n to μ as $n \to \infty$. The third equality follows from the monotone-convergence theorem. Hence [6, Thm. 30.8] implies that $\mu_n I_E$ converges weakly to μI_E as $n \to \infty$, not only vaguely. We recall from [6, Def. 30.7] that weak convergence of the latter sequence requires that $\widehat{\mu}_n(I_E f)$ tends to $\widehat{\mu}(I_E f)$ as $n \to \infty$ for every bounded continuous function f. The claimed convergence (4.8) of the corresponding distribution functions is therefore reduced to the content of [6, Thm. 30.12]. $\square$

4.2 Proofs of Lemma 3.5 and Theorem 3.1

We first give a

Proof of Lemma 3.5. To show that ν is a positive Borel measure on $\mathbb{R}$, it suffices that

$$\begin{aligned} \nu(I)\,|\Gamma| &= \mathbb{E}\Big\{ \operatorname{Tr}\big[\chi_\Gamma\, \chi_I(H(A,V))\, \chi_\Gamma\big]\Big\} \\ &\le \big(\sqrt{2}\varepsilon\big)^{2\vartheta}\, \mathbb{E}\Big\{ \operatorname{Tr}\big[\chi_\Gamma\, |H(A,V) - E_0 - \mathrm{i}\varepsilon|^{-2\vartheta}\, \chi_\Gamma\big]\Big\} < \infty \end{aligned} \tag{4.14}$$

for any compact energy interval $I = [E_0 - \varepsilon, E_0 + \varepsilon]$, $E_0 \in \mathbb{R}$, $\varepsilon > 0$. This follows from the elementary inequality $\chi_I(E) \le (\sqrt{2}\varepsilon)^{2\vartheta}\, |E - E_0 - \mathrm{i}\varepsilon|^{-2\vartheta}$, the spectral theorem applied to $H(A, V^{(\omega)})$ and the functional calculus. Proposition 4.15(i) below and property (I) ensure that the r.h.s. of (4.14) is indeed finite.

To prove (3.15) we employ an approximation argument with bounded truncated random potentials given by

$$V_n^{(\omega)}(x) := V^{(\omega)}(x)\ \Theta\Big(n - |V^{(\omega)}(x)|\Big), \quad n \in \mathbb{N}. \tag{4.15}$$

We denote by $\nu_{\Lambda,\mathrm{X},n}^{(\omega)}$, with $\mathrm{X} = \mathrm{D}$ or $\mathrm{X} = \mathrm{N}$, the approximate finite-volume density-of-states measure associated with V_n, see (2.8). Moreover,

$$\nu_n(I) := \frac{1}{|\Gamma|}\, \mathbb{E}\Big\{ \operatorname{Tr}\big[\chi_\Gamma \chi_I(H(A,V_n)) \chi_\Gamma\big]\Big\}, \quad I \in \mathscr{B}(\mathbb{R}), \tag{4.16}$$

defines the approximate (infinite-volume) density-of-states measure. It is a positive Borel measure on $\mathbb{R}$, see (4.14), and independent of the bounded open cube $\Gamma \subset \mathbb{R}^d$ due to $\mathbb{Z}^d$-homogeneity. In case $\mathrm{X} = \mathrm{D}$ we let $f \in \mathscr{C}_0^1(\mathbb{R})$ and estimate as follows

$$\begin{aligned} \big|\, |\Lambda|^{-1}\, \widehat{\nu}_{\Lambda,\mathrm{D}}^{(\omega)}(f) - \widehat{\nu}(f)\,\big| &\le \big|\,\widehat{\nu}_n(f) - \widehat{\nu}(f)\,\big| \\ &\quad + |\Lambda|^{-1}\, \big|\,\widehat{\nu}_{\Lambda,\mathrm{D},n}^{(\omega)}(f) - \widehat{\nu}_{\Lambda,\mathrm{D}}^{(\omega)}(f)\,\big| + \big|\, |\Lambda|^{-1}\, \widehat{\nu}_{\Lambda,\mathrm{D},n}^{(\omega)}(f) - \widehat{\nu}_n(f)\,\big|. \end{aligned} \tag{4.17}$$

We first consider the limit $\Lambda \uparrow \mathbb{R}^d$. In this limit, the third difference on the r.h.s. of (4.17) vanishes for all $\omega \in \widehat{\Omega} := \bigcap_{n\in\mathbb{N}} \Omega_n$ and all $n \in \mathbb{N}$ by Lemma 4.5 below. Next we consider the limit $n \to \infty$, in which the second difference vanishes for all $\omega \in \widetilde{\Omega}$ by Lemma 4.6. In the latter limit, the first difference vanishes by Lemma 4.7. This proves the claimed vague convergence of $|\Lambda|^{-1}\nu_{\Lambda,\mathrm{D}}^{(\omega)}$ to ν as $\Lambda \uparrow \mathbb{R}^d$ for all $\omega \in \widehat{\Omega} \cap \widetilde{\Omega}$, hence for $\mathbb{P}$-almost all $\omega \in \Omega$. In case $\mathrm{X} = \mathrm{N}$ we estimate

$$\big|\, |\Lambda|^{-1}\, \widehat{\nu}_{\Lambda,\mathrm{N}}^{(\omega)}(f) - \widehat{\nu}(f)\,\big| \le \big|\, |\Lambda|^{-1}\, \widehat{\nu}_{\Lambda,\mathrm{D}}^{(\omega)}(f) - \widehat{\nu}(f)\,\big| + |\Lambda|^{-1}\, \big|\,\widehat{\nu}_{\Lambda,\mathrm{N}}^{(\omega)}(f) - \widehat{\nu}_{\Lambda,\mathrm{D}}^{(\omega)}(f)\,\big|. \tag{4.18}$$

As $\Lambda \uparrow \mathbb{R}^d$ the first term on the r.h.s. converges to zero for $\mathbb{P}$-almost all $\omega \in \Omega$ and the same is true for the second term thanks to Proposition 4.8 below. $\square$

We now prove our main result.

Proof of Theorem 3.1. Since we have already established the vague convergence of the density-of-states measures in Lemma 3.5, it remains to verify relation (4.9) of Proposition 4.3 for the corresponding random distribution functions $|\Lambda|^{-1}\, N^{(\omega)}_{\Lambda,\mathrm{X}}$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

To this end, we employ the elementary inequality $\Theta(E) \leq 2\int_{-\infty}^{E} \mathrm{d}E'\, \delta_0^{(\varepsilon)}(E')$ valid for all $E \in \mathbb{R}$, $\varepsilon > 0$ with $\delta_0^{(\varepsilon)}$ defined in (4.3). Choosing $\varepsilon = 1$ and $p = 2\vartheta + 1$ there, we get

$$
\begin{aligned}
N^{(\omega)}_{\Lambda,\mathrm{X}}(E) &= \mathrm{Tr}\left[\Theta(E - H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}))\right] \\
&\leq 2\,\Upsilon_{2\vartheta+1} \int_{-\infty}^{E} \mathrm{d}E'\ \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(E' - \mathrm{i}, 2\vartheta + 1) \\
&\leq 4\,|E|^{\frac{d}{2}-2\vartheta}\ \frac{\Upsilon_{2\vartheta+1}}{4\vartheta - d}\ C_1(1) \int_{\Lambda} \mathrm{d}^d x\ \left(2 + |V^{(\omega)}(x)|\right)^{2\vartheta+1}
\end{aligned}
\tag{4.19}
$$

for all $E \in]-\infty, -1]$. Here, the second inequality results from (4.31) and Proposition 4.10(i) choosing $E_1 = E'$ there. Dividing (4.19) by the volume $|\Lambda|$ and using the Birkhoff-Khintchine ergodic theorem in the formulation [46, Prop. 1.13] we get

$$
\limsup_{\Lambda\uparrow\mathbb{R}^d} \frac{N^{(\omega)}_{\Lambda,\mathrm{X}}(E)}{|\Lambda|} \leq 4\,|E|^{\frac{d}{2}-2\vartheta}\ \frac{\Upsilon_{2\vartheta+1}}{4\vartheta - d}\ C_1(1)\ \mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\ \left(2 + |V(x)|\right)^{2\vartheta+1}\Big] \tag{4.20}
$$

for $\mathbb{P}$-almost all $\omega \in \Omega$ and all $E \in]-\infty, -1]$. Since the r.h.s. of (4.20) converges to zero as $E \downarrow -\infty$, relation (4.9) of Proposition 4.3 is fulfilled. The existence of the distribution function $N(E) = \nu(]-\infty, E[)$ of the limiting measure ν for all $E \in \mathbb{R}$ as well as the claimed convergence are thus warranted by Proposition 4.3. □

The proof of Lemma 3.5 was based on three lemmas and two propositions. The first lemma basically recalls known facts [55, 9] for $\mathrm{X} = \mathrm{D}$ and V bounded.

Lemma 4.5. *Let $\Lambda \subset \mathbb{R}^d$ stand for bounded open cubes. Suppose A and V have the properties* (C) *and* (E). *Then for every $n \in \mathbb{N}$ there exists $\Omega_n \in \mathscr{A}$ with $\mathbb{P}(\Omega_n) = 1$ such that*

$$
\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{\nu^{(\omega)}_{\Lambda,\mathrm{D},n}}{|\Lambda|} = \nu_n \tag{4.21}
$$

vaguely for all $\omega \in \Omega_n$.

Proof. See [55, Thm. 3.1 and Prop. 3.1(ii)] where the appropriate Feynman-Kac-Itô formula for the infinite-volume and the Dirichlet-finite-volume Schrödinger semigroup is employed; see also [9]. □

Lemma 4.6. *Let $\Lambda \subset \mathbb{R}^d$ stand for bounded open cubes. Let A and V be supplied with the properties* (C), (I), *and* (E). *Then there exists $\widetilde{\Omega} \in \mathscr{A}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that*

$$
\lim_{n\to\infty} \limsup_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|} \left|\, \widehat{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(f) - \widehat{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(f) \,\right| = 0 \tag{4.22}
$$

for all $f \in \mathscr{C}^1_0(\mathbb{R})$, all $\omega \in \widetilde{\Omega}$ and both boundary conditions $\mathrm{X} = \mathrm{D}$ *and* $\mathrm{X} = \mathrm{N}$.

Proof. Thanks to (4.31), Proposition 4.10(i) below and property (I), the integrals

$$\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(z,2\vartheta) = \int_{\mathbb{R}} \frac{\nu^{(\omega)}_{\Lambda,\mathrm{X},n}(\mathrm{d}E)}{|E-z|^{2\vartheta}} = \mathrm{Tr}\left[\left|H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}_n) - z\right|^{-2\vartheta}\right] \tag{4.23}$$

and (analogously) $\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(z,2\vartheta)$ are finite for all $z \in \mathbb{C}\backslash\mathbb{R}$ and $\mathbb{P}$-almost all $\omega \in \Omega$ such that (4.7) yields

$$\begin{aligned} |\,\widehat{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(f) - \widehat{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(f)\,| \le C(\varepsilon)\left[\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(\mathrm{i},2\vartheta) + \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(\mathrm{i},2\vartheta)\right] \\ + \Upsilon_{2\vartheta}\,\varepsilon^{2\vartheta-1}\,|f|_1 \sup_{E\in\operatorname{supp} f} \left|\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(E+\mathrm{i}\varepsilon,2\vartheta) - \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(E+\mathrm{i}\varepsilon,2\vartheta)\right|. \end{aligned} \tag{4.24}$$

Here the quantity $C(\varepsilon)$, which depends on ε and f, was introduced in (4.5) and vanishes for $\varepsilon \downarrow 0$. We further estimate the first term with the help of (4.31) and Proposition 4.10(i) choosing $E_1 = -1$ there. The upper limit $\Lambda \uparrow \mathbb{R}^d$ of the first term after dividing by the volume $|\Lambda|$ is then seen to be finite by the Birkhoff-Khintchine ergodic theorem [46, Prop. 1.13],

$$\limsup_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|}\left[\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(\mathrm{i},2\vartheta) + \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(\mathrm{i},2\vartheta)\right] \le 2\,C_1(1)\,\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\,(3+|V(x)|)^{2\vartheta}\Big] \tag{4.25}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. The second term in (4.24) is bounded with the help of (4.32) and Proposition 4.10(ii) where we again choose $E_1 = -1$. This bound together with the same ergodic theorem yields

$$\begin{aligned} &\limsup_{\Lambda\uparrow\mathbb{R}^d} \sup_{E\in\operatorname{supp} f} \frac{1}{|\Lambda|}\,|\,\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(E+\mathrm{i}\varepsilon,2\vartheta) - \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(E+\mathrm{i}\varepsilon,2\vartheta)| \\ &\quad\le C_2(\varepsilon)\left\{\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\,(2+\sup_{E\in\operatorname{supp} f}|E+\mathrm{i}\varepsilon| + |V(x)|)^{2\vartheta+1}\Big]\right\}^{\frac{2\vartheta}{2\vartheta+1}} \\ &\qquad\times\left\{\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\,|V(x)-V_n(x)|^{2\vartheta+1}\Big]\right\}^{\frac{1}{2\vartheta+1}} \end{aligned} \tag{4.26}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. In the limit $n \to \infty$, the r.h.s. and hence the l.h.s. of (4.26) vanishes for $\mathbb{P}$-almost all $\omega \in \Omega$ thanks to property (I). This completes the proof since the first term on the l.h.s. of (4.24) may be made arbitrarily small as $\varepsilon \downarrow 0$. □

The last lemma shows in which sense the approximate (infinite-volume) density-of-states measures approach the exact one.

Lemma 4.7. *Suppose A and V have the properties* (C), (S), (I), *and* (E). *Then ν_n converges vaguely to ν as $n \to \infty$.*

Proof. Thanks to Proposition 4.15(i) and property (I), the integrals

$$\widetilde{\nu}_n(z,2\vartheta) = \int_{\mathbb{R}} \frac{\nu_n(\mathrm{d}E)}{|E-z|^{2\vartheta}} = \frac{1}{|\Gamma|}\,\mathbb{E}\left\{\mathrm{Tr}\left[\chi_\Gamma\,|H(A,V_n)-z|^{-2\vartheta}\,\chi_\Gamma\right]\right\} \tag{4.27}$$

and (analogously) $\widetilde{\nu}(z,2\vartheta)$ are finite for all $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, $\lim_{n\to\infty}\widetilde{\nu}_n(z,2\vartheta) = \widetilde{\nu}(z,2\vartheta)$ for all $z \in \mathbb{C}\backslash\mathbb{R}$ by (4.56), Proposition 4.15(ii) and property (I) again. This implies vague convergence by Proposition 4.1. □

In the following proposition we exploit recent results of Nakamura [42] or Doi, Iwatsuka and Mine [19] on the independence of the density-of-states measure of the chosen boundary condition for the present setting, thereby heavily relying on either of these results.

Proposition 4.8. *Let $\Lambda \subset \mathbb{R}^d$ stand for bounded open cubes. Assume A is a vector potential with property* (C) *and V is a random potential with properties* (I) *and* (E). *Then*

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|} \left| \int_{\mathbb{R}} \nu_{\Lambda,\mathrm{N}}^{(\omega)}(\mathrm{d}E)\, f(E) - \int_{\mathbb{R}} \nu_{\Lambda,\mathrm{D}}^{(\omega)}(\mathrm{d}E)\, f(E) \right| = 0 \tag{4.28}$$

for all $f \in \mathscr{C}_0^1(\mathbb{R})$ and all $\omega \in \widetilde{\Omega}$. [*The set $\widetilde{\Omega}$ is defined in Lemma 4.6.*]

Remark 4.9. An extension of Nakamura's result [42, Thm. 1] (without his error estimate) to unbounded *non-random* potentials v may be achieved with the subsequent techniques under the uniform local integrability condition $\sup_{y\in\mathbb{Z}^d} \int_{\Lambda(y)} \mathrm{d}^d x\, |v(x)|^{2\vartheta+1} < \infty$, where ϑ is the smallest integer with $\vartheta > d/4$. Properties (I) and (E) of a random potential in general do not imply this condition $\mathbb{P}$-almost surely.

Proof of Proposition 4.8. The proof consists of an approximation argument. To this end, we recall the definition (4.15) of the truncated random potential V_n. Since V_n is bounded and A enjoys property (C), we may apply [42, Thm. 1] (or [19, Thm. 1.2] together with Lemma 4.5) which gives

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|} \left| \mathrm{Tr}\left[f(H_{\Lambda,\mathrm{N}}(A, V_n^{(\omega)})) - f(H_{\Lambda,\mathrm{D}}(A, V_n^{(\omega)})) \right] \right| = 0 \tag{4.29}$$

for all $n \in \mathbb{N}$, all $f \in \mathscr{C}_0^1(\mathbb{R})$ and all $\omega \in \Omega$. Using the triangle inequality we estimate

$$|\widehat{\nu}_{\Lambda,\mathrm{N}}^{(\omega)}(f) - \widehat{\nu}_{\Lambda,\mathrm{D}}^{(\omega)}(f)| \leq |\widehat{\nu}_{\Lambda,\mathrm{N},n}^{(\omega)}(f) - \widehat{\nu}_{\Lambda,\mathrm{D},n}^{(\omega)}(f)| + \sum_{\mathrm{X}=\mathrm{D},\mathrm{N}} |\widehat{\nu}_{\Lambda,\mathrm{X},n}^{(\omega)}(f) - \widehat{\nu}_{\Lambda,\mathrm{X}}^{(\omega)}(f)|. \tag{4.30}$$

The proof is then completed with the help of (4.29) and Lemma 4.6. ☐

Various proofs in the present subsection rely on estimates stated in Proposition 4.10 and Proposition 4.15. These propositions will be proven in the remaining two subsections. In fact, they extend parts of Lemma 5.4 (resp. 5.7) and Lemma 5.12 (resp. 5.14) in [46] to the case of non-zero vector potentials. Basically, the extensions follow from the so-called diamagnetic inequality. For this inequality the reader may find useful the compilation [28, App. A.2] which covers the Neumann-boundary-condition case $\mathrm{X} = \mathrm{N}$.

4.3 Finite-volume trace-ideal estimates

Our first aim is to estimate the trace norm $\|\cdot\|_1$ (recall the notation (2.3)) of a power of the resolvent of the finite-volume magnetic Schrödinger operator $H_{\Lambda,\mathrm{X}}(a, v)$ and of the difference of two such powers.

Proposition 4.10. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube with $|\Lambda| \geq 1$ and $\mathrm{X} = \mathrm{D}$ or $\mathrm{X} = \mathrm{N}$. Let $k \in \mathbb{N}$ with $k > d/4$ and $E_1 \in]-\infty, -1]$. Let a be a vector potential with $|a|^2 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $v, v' \in \mathrm{L}^{2k+1}_{\mathrm{loc}}(\mathbb{R}^d)$ be two scalar potentials with $|v'| \leq |v|$. Then for every $z \in \mathbb{C}\backslash\mathbb{R}$ there exist two constants $C_1(\mathrm{Im}\, z)$, $C_2(\mathrm{Im}\, z) > 0$, which depend on d and p, but are independent of Λ, X, a, v, v' and E_1, such that* *(i)* $\| (H_{\Lambda,\mathrm{X}}(a,v) - z)^{-p} \|_1 \leq C_1(\operatorname{Im} z)\, |E_1|^{\frac{d}{2}-p}\, |1 + |z - E_1| + |v||_p^p$

for all $p \in [2, 2k+1] \cap]d/2, 2k+1]$,

(ii) $\| (H_{\Lambda,\mathrm{X}}(a,v) - z)^{-2k} - \left(H_{\Lambda,\mathrm{X}}(a,v') - z\right)^{-2k} \|_1$

$$\leq C_2(\operatorname{Im} z)\, |E_1|^{\frac{d}{2}-2k}\, |1 + |z - E_1| + |v||_{2k+1}^{2k}\, |v - v'|_{2k+1}.$$

Remarks 4.11. (i) We recall from [28, App.] that the assumptions of Proposition 4.10 guarantee that the operators $H_{\Lambda,\mathrm{X}}(a,v)$ for $\mathrm{X} = \mathrm{D}$ and $\mathrm{X} = \mathrm{N}$ are well defined as self-adjoint operators via forms.

(ii) From (4.23) we conclude that

$$\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(z,p) = \|(H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}) - z)^{-p}\|_1, \tag{4.31}$$

because the resolvent of $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ commutes with its adjoint. Therefore, Proposition 4.10 provides upper bounds on $\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(z, 2\vartheta)$, $\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(z, 2\vartheta)$, and $\widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(z, 2\vartheta + 1)$ as well as on the r.h.s. of the estimate

$$\begin{aligned} &\left| \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X}}(z, 2\vartheta) - \widetilde{\nu}^{(\omega)}_{\Lambda,\mathrm{X},n}(z, 2\vartheta) \right| \\ &\quad \leq \left\| (H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}) - z)^{-2\vartheta} - (H_{\Lambda,\mathrm{X}}(A, V_n^{(\omega)}) - z)^{-2\vartheta} \right\|_1. \end{aligned} \tag{4.32}$$

This estimate is just the triangle inequality for the trace norm.

The proof of Proposition 4.10 uses trace-ideal and resolvent techniques. It is based on two lemmas. The first one gives estimates on the Schatten p-norm of a function of the free Schrödinger operator $H_{\Lambda,\mathrm{X}}(0,0)$ times a multiplication operator and on the trace of a power of the free resolvent.

Lemma 4.12. *Let* $\Lambda \subset \mathbb{R}^d$ *be a bounded open cube and let* $\mathrm{X} = \mathrm{D}$ *or* $\mathrm{X} = \mathrm{N}$.

(i) Let $p \in [2, \infty[$, $Q \in \mathrm{L}^\infty(\Lambda)$ *and* $f : \mathbb{R} \to \mathbb{C}$ *be Borel measurable. Moreover, assume* $f(H_{\Lambda,\mathrm{X}}(0,0)) \in \mathscr{J}_p(\mathrm{L}^2(\Lambda))$. *Then* $f(H_{\Lambda,\mathrm{X}}(0,0))\, Q \in \mathscr{J}_p(\mathrm{L}^2(\Lambda))$ *and*

$$\|f(H_{\Lambda,\mathrm{X}}(0,0))\, Q\|_p \leq \left(\frac{2^d}{|\Lambda|}\right)^{1/p} \|f(H_{\Lambda,\mathrm{X}}(0,0))\|_p\, |Q|_p. \tag{4.33}$$

(ii) Let $\alpha \in]d/2, \infty[$ *and assume* $|\Lambda| \geq 1$. *Then*

$$\begin{aligned} r(E_1, \alpha) &:= \frac{2^d}{|\Lambda|} \| (H_{\Lambda,\mathrm{X}}(0,0) - E_1)^{-\alpha} \|_1 \\ &\leq |E_1|^{\frac{d}{2}-\alpha} \frac{2^d}{(\alpha-1)!} \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-\xi}\, \xi^{\alpha-1} \left(1 + (2\pi\xi)^{-1/2}\right)^d < \infty \end{aligned} \tag{4.34}$$

for all $E_1 \in]-\infty, -1]$.

Remark 4.13. The validity of (4.33) for all $Q \in \mathrm{L}^\infty(\Lambda)$ may be extended to all $Q \in \mathrm{L}^p(\Lambda)$ by an approximation argument. In this regard, Lemma 4.12(i) is a finite-volume analogue of [53, Thm. 4.1]. However, the bound in [53, Thm. 4.1] is sharper than the one obtained by simply taking the limit $\Lambda \uparrow \mathbb{R}^d$ in (4.33). In fact, in contrast to the version of the latter theorem for $p = 2$, equality can never hold in (4.33). Nevertheless, the constant $\left(2^d/|\Lambda|\right)^{1/p}$ in (4.33) is the best possible for all $p \geq 2$.

Proof of Lemma 4.12. (i) Let $(\operatorname{sgn} g)(x) := g(x)/|g(x)|$ if $g(x) \neq 0$ and zero otherwise stand for the signum function of a complex-valued function g. The polar decompositions $Q = |Q| \operatorname{sgn} Q$ and $f(H_{\Lambda,\mathrm{X}}(0,0)) = |f(H_{\Lambda,\mathrm{X}}(0,0))| \operatorname{sgn} f(H_{\Lambda,\mathrm{X}}(0,0))$ together with Hölder's inequality [53, Eq. (2.5b)] and [53, Cor. 8.2] show that

$$\begin{aligned}\| f(H_{\Lambda,\mathrm{X}}(0,0))\, Q\|_p^p &\leq \| \, |f(H_{\Lambda,\mathrm{X}}(0,0))| \; |Q| \, \|_p^p \leq \| \, |f(H_{\Lambda,\mathrm{X}}(0,0))|^{\frac{p}{2}} \; |Q|^{\frac{p}{2}} \, \|_2^2 \\ &= \operatorname{Tr}\big[\, |Q|^{\frac{p}{2}} \, |f(H_{\Lambda,\mathrm{X}}(0,0))|^p \, |Q|^{\frac{p}{2}} \,\big]. \end{aligned} \tag{4.35}$$

Let $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathrm{L}^2(\Lambda)$ denote an orthonormal eigenbasis associated with $H_{\Lambda,\mathrm{X}}(0,0)$ and ε_j the eigenvalue corresponding to φ_j. Then the trace in (4.35) may be calculated in this eigenbasis and estimated as follows

$$\sum_{j=1}^{\infty} |f(\varepsilon_j)|^p \int_\Lambda \mathrm{d}^d x \; |\varphi_j(x)|^2 \; |Q(x)|^p \leq \frac{2^d}{|\Lambda|} \, \| f(H_{\Lambda,\mathrm{X}}(0,0))\|_p^p \; |Q|_p^p. \tag{4.36}$$

The inequality is a consequence of the uniform boundedness $|\varphi_j|^2 \leq 2^d/\,|\Lambda|$ for all $j \in \mathbb{N}$ which follows from the explicitly known expressions for $\{\varphi_j\}$, see [49, p. 266].

(ii) Using the integral represention of powers of resolvents, we get

$$\| \, (H_{\Lambda,\mathrm{X}}(0,0) - E_1)^{-\alpha} \, \|_1 = \frac{1}{(\alpha-1)!} \int_0^\infty \mathrm{d}t \; t^{\alpha-1} \, \mathrm{e}^{tE_1} \, \operatorname{Tr}\Big[\mathrm{e}^{-tH_{\Lambda,\mathrm{X}}(0,0)}\Big]. \tag{4.37}$$

The claimed bound hence follows from the estimates $\operatorname{Tr}\Big[\mathrm{e}^{-t\, H_{\Lambda,\mathrm{X}}(0,0)}\Big] \leq \operatorname{Tr}\Big[\mathrm{e}^{-t\, H_{\Lambda,\mathrm{N}}(0,0)}\Big]$ $\leq |\Lambda| (|\Lambda|^{-1/d} + (2\pi t)^{-1/2})^d$ which are obtained by Dirichlet-Neumann bracketing [49, Prop. 4(b) on p. 270] and the explicitly known [49, p. 266] spectrum of $H_{\Lambda,\mathrm{N}}(0,0)$.

□

The second lemma estimates Schatten norms of certain products involving bounded multiplication operators and the resolvent of the magnetic Schrödinger operator $H_{\Lambda,\mathrm{X}}(a,0)$ without scalar potential.

Lemma 4.14. *Assume the situation of Proposition 4.10 and introduce*

$$R_a := (H_{\Lambda,\mathrm{X}}(a,0) - E_1)^{-1}, \qquad E_1 \in]-\infty, -1]. \tag{4.38}$$

Then the following two assertions hold:

(i) Let $p \in [2,\infty[$ and $\alpha > 0$ such that $\alpha p > d/2$. Moreover, let $Q \in \mathrm{L}^\infty(\Lambda)$. Then

$$\|R_a^\alpha \, Q\|_p^p \leq r(E_1, \alpha p) \; |Q|_p^p. \tag{4.39}$$

(ii) Let $Q_1, \ldots, Q_{k+1} \in \mathrm{L}^\infty(\Lambda)$. Then

$$\| |Q_{k+1}|^{\frac{1}{2}} R_a \, Q_k \cdots R_a \, Q_1 \|_2^2 \leq r(E_1, 2k) \, |Q_{k+1}|_{2k+1} \prod_{j=1}^{k} |Q_j|_{2k+1}^2 \,. \tag{4.40}$$

Proof. (i) The claim follows from the chain of inequalities

$$\begin{aligned} \|R_a^\alpha \, Q\|_p^p &\leq \|R_a^\alpha \; |Q| \, \|_p^p \leq \|R_a^{\frac{\alpha p}{2}} \; |Q|^{\frac{p}{2}} \|_2^2 \\ &\leq \|R_0^{\frac{\alpha p}{2}} \; |Q|^{\frac{p}{2}} \|_2^2 \leq \frac{2^d}{|\Lambda|} \, \|R_0^{\alpha p}\|_1 \, |Q|_p^p. \end{aligned} \tag{4.41}$$

Here the first inequality is a consequence of the polar decomposition $Q = |Q| \operatorname{sgn} Q$. The second one is a special case of [53, Cor. 8.2]. For the third one we used the diamagnetic inequality [28, Eq. (A.23)] in the version

$$\left| R_a^{\frac{\alpha p}{2}} \, |Q|^{\frac{p}{2}} \, \varphi \right| \leq R_0^{\frac{\alpha p}{2}} \, |Q|^{\frac{p}{2}} \, |\varphi| \tag{4.42}$$

for any $\varphi \in \mathrm{L}^2(\Lambda)$, together with [53, Thm. 2.13]. The fourth inequality eventually follows from Lemma 4.12.

(ii) We repeatedly use Hölder's inequality [53, Eq. (2.5b)] for Schatten norms

$$\begin{aligned} \left\| |Q_{k+1}|^{\frac{1}{2}} R_a \, Q_k \cdots R_a \, Q_1 \right\|_2 &\leq \prod_{j=1}^{k} \left\| |Q_{j+1}|^{\frac{j}{2k}} \, R_a \, |Q_j|^{\frac{2k+1-j}{2k}} \right\|_{2k} \\ &\leq \prod_{j=1}^{k} \left\| R_a^{\frac{j}{2k+1}} \, |Q_{j+1}|^{\frac{j}{2k}} \right\|_{\frac{2k(2k+1)}{j}} \left\| R_a^{\frac{2k+1-j}{2k+1}} \, |Q_j|^{\frac{2k+1-j}{2k}} \right\|_{\frac{2k(2k+1)}{2k+1-j}} . \end{aligned} \tag{4.43}$$

The proof is completed using part (i) of the present lemma.

□

We are now ready to present a

Proof of Proposition 4.10. The proof is split into the following three parts:

a) Proof of part (i) for $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$,

b) Proof of part (ii) for $v, v' \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$,

c) Approximation argument for the validity of part (i) and (ii).

Throughout the proof we use the abbreviations

$$R_{a,v}(z) := (H_{\Lambda,\mathrm{X}}(a, v) - z)^{-1} \quad \text{and} \quad R_a = R_{a,0}(E_1), \tag{4.44}$$

in agreement with (4.38).

As to a). Let $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. We may then apply the (second) resolvent equation [58]

$$R_{a,v}(z) = R_a + R_a \, Q \, R_{a,v}(z), \qquad Q := z - E_1 - v. \tag{4.45}$$

By the triangle inequality for the Schatten p-norm $\|\cdot\|_p$, Hölder's inequality and the standard estimate $\|R_{a,v}(z)\| \leq |\operatorname{Im} z|^{-1}$, involving the usual (uniform) operator norm $\|\cdot\|$, the resolvent equation yields

$$\|R_{a,v}(z)\|_p \leq \|R_a\|_p + |\operatorname{Im} z|^{-1} \, \|R_a \, Q\|_p \, . \tag{4.46}$$

Using Lemma 4.14(i) we thus have

$$\begin{aligned} \|(R_{a,v}(z))^p\|_1 = \|R_{a,v}(z)\|_p^p &\leq r(E_1, p) \left(|\Lambda|^{1/p} + |\operatorname{Im} z|^{-1} \, |Q|_p \right)^p \\ &\leq r(E_1, p) \left(1 + |\operatorname{Im} z|^{-1} \right)^p \, |1 + |Q||_p^p, \end{aligned} \tag{4.47}$$

because $\max\{|\Lambda|^{1/d}, |Q|_p\} \leq \big\| 1 + |Q| \big\|_p$. The proof is finished by the upper bound in Lemma 4.12(ii).

As to b). We start from the resolvent equation for powers of resolvents

$$(R_{a,v}(z))^{2k} - (R_{a,v'}(z))^{2k} = \sum_{j=1}^{2k} (R_{a,v'}(z))^{2k+1-j} \left(v' - v\right) (R_{a,v}(z))^{j}, \tag{4.48}$$

see [46, Eq. (5.4)]. Moreover, by the standard iteration of the resolvent equation (4.45) we have

$$(R_{a,v}(z))^{j} = \sum_{r=0}^{j} \sum_{\substack{J\subseteq\{1,\ldots,j\} \\ \#J=r}} R_a M_1(J) \cdots R_a M_j(J)\, (R_{a,v}(z))^{r}, \tag{4.49}$$

where $M_s(J) := 1$ if $s \in J$ and $M_s(J) := Q$ if $s \notin J$, and the second sum extends over all subsets $J \subseteq \{1, \ldots, j\}$ with $\#J = r$ elements. Using (4.49) and a suitable analogue with v replaced by v' in (4.48), the trace norm of the l.h.s. of (4.48) is seen to be bounded from above by a sum of finitely many terms of the form

$$\begin{aligned}\| \,(R_{a,v'}(z))^{s}\, Q_1 R_a \cdots Q_k\, R_a\, Q_{k+1}\, R_a\, Q_{k+2} \cdots R_a Q_{2k+1}\, (R_{a,v}(z))^{r} \|_1 \\ \leq |\mathrm{Im}\, z|^{-s-r}\, \|Q_1\, R_a \cdots Q_k\, R_a\, |Q_{k+1}|^{\frac{1}{2}} \|_2 \\ \times \| \,|Q_{k+1}|^{\frac{1}{2}}\, R_a\, Q_{k+2} \cdots R_a\, Q_{2k+1}\|_2,\end{aligned} \tag{4.50}$$

with $s,\ r \in \{0, 1, \ldots, 2k\}$. Each of these terms involves multiplication operators $Q_1, \ldots, Q_{2k+1}$ suitably chosen from the set $\{1, z - E_1 - v, z - E_1 - v', v' - v\}$ where *exactly one* is equal to $v' - v$. The estimate (4.50) is again Hölder's inequality. The proof is then finished with the help of Lemma 4.14(ii) and Lemma 4.12(ii), because $v' - v$ appears only once and the other three operators in the above set are all bounded by $1 + |z - E_1| + |v|$ since $|v'| \leq |v|$.

As to c). We approximate $v \in \mathrm{L}^{2k+1}_{\mathrm{loc}}(\mathbb{R}^d)$ by v_n^m defined through

$$v_n^m(x) := \max\{-n, \min\{m, v(x)\}\} \tag{4.51}$$

with $x \in \mathbb{R}^d$ and $n, m \in \mathbb{N}$. Consequently, we let $v_\infty^m(x) := \min\{m, v(x)\}$. Monotone (decreasing) convergence for forms [47, Thm. S.16] yields the strong convergence

$$R_{a,v_\infty^m}(z) = \operatorname*{s\text{-}lim}_{n\to\infty} R_{a,v_n^m}(z) \tag{4.52}$$

for all $m \in \mathbb{N}$ and all $z \in \mathbb{C}\backslash\mathbb{R}$. On the other hand, monotone (increasing) convergence for forms [47, Thm. S.14] yields

$$R_{a,v}(z) = \operatorname*{s\text{-}lim}_{m\to\infty} R_{a,v_\infty^m}(z) \tag{4.53}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$. We therefore have

$$\|R_{a,v}(z)\|_p \leq \limsup_{m\to\infty}\, \limsup_{n\to\infty}\, \|R_{a,v_n^m}(z)\|_p \tag{4.54}$$

where we used the non-commutative version of Fatou's lemma [49, Prob. 167 on p. 385] (see also [53, Thm. 2.7.(d)]) twice. Similarly,

$$\|(R_{a,v}(z))^{2k} - (R_{a,v'}(z))^{2k}\|_1 \le \limsup_{m\to\infty} \limsup_{n\to\infty} \|(R_{a,v_n^m}(z))^{2k} - (R_{a,v'^m_n}(z))^{2k}\|_1 \tag{4.55}$$

by the strong resolvent convergences (4.52) and (4.53), its analogue with v replaced by v' and [49, Prob. 167 on p. 385]. Applying part a) and b) of the present proof to the pre-limit expressions in (4.54) and (4.55) completes the proof of Proposition 4.10(i) and 4.10(ii). □

4.4 Infinite-volume trace-ideal estimates

It remains to prove the substitute of Lemma 5.12 (resp. 5.14) in [46]. It is the infinite-volume analogue of Proposition 4.10 above. Accordingly, we will use the notation (2.3) with $\Lambda = \mathbb{R}^d$.

Proposition 4.15. *Assume the situation of Theorem 3.1 and recall the definition (4.15) of the truncated random potential V_n. Let $E_2 \in]-\infty, 0[$. Then for every $z \in \mathbb{C}\setminus\mathbb{R}$ there exist two constants $C_3(\operatorname{Im} z)$, $C_4(\operatorname{Im} z) > 0$, which depend on d and p, but are independent of Γ, n, A, V, and E_2, such that*

(i) $$\mathbb{E}\Big\{\|\chi_\Gamma\,|H(A,V_n)-z|^{-2\vartheta}\,\chi_\Gamma\|_1\Big\}$$
$$\le C_3(\operatorname{Im} z)\,|\Gamma|\;|E_2|^{\frac{d}{2}-2\vartheta}\;\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\;\big(1+|z-E_2|+|V(x)|\big)^{2\vartheta}\Big].$$

The same holds true if V_n is replaced by V.

(ii) $$\mathbb{E}\Big\{\|\chi_\Gamma\big[\,|H(A,V)-z|^{-2\vartheta} - |H(A,V_n)-z|^{-2\vartheta}\,\big]\chi_\Gamma\|_1\Big\}$$
$$\le C_4(\operatorname{Im} z)\,|\Gamma|\;|E_2|^{\frac{d}{2}-2\vartheta}\;\Big\{\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\;\big(1+|z-E_2|+|V(x)|\big)^{2\vartheta+1}\Big]\Big\}^{\frac{2\vartheta}{2\vartheta+1}}$$
$$\times\Big\{\mathbb{E}\Big[\int_{\Lambda(0)} \mathrm{d}^d x\,|V(x)-V_n(x)|^{2\vartheta+1}\Big]\Big\}^{\frac{1}{2\vartheta+1}}.$$

Remark 4.16. We recall from (4.27) that the l.h.s. of Proposition 4.15(i) coincides with $\widetilde{\nu}_n(z,2\vartheta)\,|\Gamma|$. Moreover, by the triangle inequality we have

$$|\,\widetilde{\nu}(z,2\vartheta)-\widetilde{\nu}_n(z,2\vartheta)| \le \frac{1}{|\Gamma|}\,\mathbb{E}\Big\{\|\,\chi_\Gamma\big[\,|H(A,V)-z|^{-2\vartheta} - |H(A,V_n)-z|^{-2\vartheta}\,\big]\chi_\Gamma\|_1\Big\}. \tag{4.56}$$

The proof of Proposition 4.15 is split into two parts. In the first part, the assertion is proven for $\mathbb{R}^d$-ergodic random potentials. We will thereby closely follow [46, Lemma 5.12/5.14]. The $\mathbb{Z}^d$-ergodic case is treated afterwards with the help of the so-called *suspension construction* [29, 30].

Proof of Proposition 4.15 in case V is $\mathbb{R}^d$-ergodic. Throughout, we assume that V is $\mathbb{R}^d$-ergodic. The proof is split into three parts.

a) Proof of part (i),

b) Proof of part (ii) with V replaced by V_m with $m \in \mathbb{N}$ arbitrary,

c) Approximation argument for the validity of part (i) with V_n replaced by V and of part (ii).

We use the abbreviations

$$R_{A,V}(z) := (H(A,V) - z)^{-1} \quad \text{and} \quad R_A := R_{A,0}(E_2) \tag{4.57}$$

for the resolvents of $H(A,V)$ and $H(A,0)$.

As to a). We write $|R_{A,V_n}(z)|^{2\vartheta} = \left(R_{A,V_n}(\overline{z})\right)^{\vartheta} \left(R_{A,V_n}(z)\right)^{\vartheta}$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. Suitably iterating the (second) resolvent equation [58]

$$R_{A,V_n}(z) = R_A + R_A\, Q\, R_{A,V_n}(z), \qquad Q := z - E_2 - V_n, \tag{4.58}$$

we obtain the analogue of (4.49) for $\left(R_{A,V_n}(z)\right)^{\vartheta}$. Using this equation and its adjoint, we are confronted with estimating finitely many terms of the form

$$\begin{aligned}
&\mathbb{E}\Big\{\|\chi_\Gamma\, R^{(s)}\, \widetilde{Q}_1\, R_A \cdots \widetilde{Q}_\vartheta\, R_A\, R_A\, Q_\vartheta \cdots R_A\, Q_1 R^{(r)}\, \chi_\Gamma\|_1\Big\} \\
&\quad \le \Big\{\mathbb{E}\Big[\|\chi_\Gamma\, R^{(s)}\, \widetilde{Q}_1\, R_A \cdots \widetilde{Q}_\vartheta\, R_A\|_2^2\Big]\; \mathbb{E}\Big[\|R_A\, Q_\vartheta \cdots R_A\, Q_1\, R^{(r)} \chi_\Gamma\|_2^2\Big]\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{4.59}$$

Here $s,\ r \in \{0, 1, \ldots, \vartheta\}$ and $R^{(s)}$ denotes some product of s factors each of which either being $R_{A,V_n}(z)$ or its adjoint. Moreover, $\widetilde{Q}_1, \ldots, \widetilde{Q}_\vartheta$ respectively $Q_1, \ldots, Q_\vartheta$ are random potentials suitably chosen from the set $\{1, z - E_2 - V_n, \overline{z} - E_2 - V_n\}$. The estimate in (4.59) is just Hölder's inequality for the trace norm and for the expectation. Thanks to [46, Lemma 5.10] we may use the estimate $\|R^{(r)}\| \le |\mathrm{Im}\, z|^{-r}$ inside the expectation. We therefore obtain the inequality

$$\begin{aligned}
\mathbb{E}\Big[\|R_A\, Q_\vartheta \cdots R_A\, Q_1\, R^{(r)} \chi_\Gamma\|_2^2\Big] &\le |\mathrm{Im}\, z|^{-2r}\; \mathbb{E}\Big[\|R_A\, Q_\vartheta \cdots R_A\, Q_1 \chi_\Gamma\|_2^2\Big] \\
&\le |\mathrm{Im}\, z|^{-2r}\; \mathbb{E}\Big[\|R_0\, |Q_\vartheta| \cdots R_0\, |Q_1|\, \chi_\Gamma\|_2^2\Big],
\end{aligned} \tag{4.60}$$

and analogously for the other factor, involving $R^{(s)}$ instead of $R^{(r)}$. The second inequality in (4.60) is a consequence of [53, Thm. 2.13] and the diamagnetic inequality [52, 28] which upon iteration gives

$$|R_A\, Q_\vartheta \cdots R_A\, Q_1\, \chi_\Gamma \varphi| \le R_0\, |Q_\vartheta| \cdots R_0\, |Q_1|\, \chi_\Gamma\, |\varphi| \tag{4.61}$$

for all $\varphi \in \mathrm{L}^2(\mathbb{R}^d)$. To complete the proof we use the iterated Hölder inequality as in [46, Lemma 5.11(i)]. Taking there $p = 2\vartheta$, $g_j = g_{p+2-j} = |Q_j|$, $t_j = t_{p+2-j} = 2\vartheta$ for $j \in \{1, \ldots, \vartheta\}$ and $g_p = 1$, $t_p = \infty$, we in fact obtain

$$\mathbb{E}\Big[\|R_0\, |Q_\vartheta| \cdots R_0\, |Q_1|\, \chi_\Gamma\|_2^2\Big] \le C_5(E_2)\, |\Gamma| \prod_{j=1}^{\vartheta} \Big\{\mathbb{E}\Big[\,|Q_j(0)|^{2\vartheta}\,\Big]\Big\}^{\frac{1}{\vartheta}}, \tag{4.62}$$

with

$$C_5(E_2) := \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \Big(\frac{p^2}{2} - E_2\Big)^{-2\vartheta} = |E_2|^{d/2-2\vartheta}\; \frac{(2\vartheta - 1 - \frac{d}{2})!}{(2\pi)^{\frac{d}{2}} (2\vartheta - 1)!}, \tag{4.63}$$

see also [46, Lemma 5.9]. Since $\max_j\{|Q_j|, |\widetilde{Q}_j|\} \leq 1 + |z - E_2| + |V|$ for all $j \in \{1, \dots, \vartheta\}$ the proof is complete.

As to b). We let $m, n \in \mathbb{N}$. The resolvent equation for powers of resolvents gives

$$\begin{aligned} &|R_{A,V_m}(z)|^{2\vartheta} - |R_{A,V_n}(z)|^{2\vartheta} \\ &\quad = \sum_{k=1}^{2\vartheta} \left(\prod_{i=1}^{k} R_{A,V_m}(z_i) \right) (V_n - V_m) \left(\prod_{j=k}^{2\vartheta} R_{A,V_n}(z_j) \right), \end{aligned} \tag{4.64}$$

see also [46, Eq. (5.4)], with $z_k = z$ if $k \in \{1, \dots, \vartheta\}$ and $z_k = \overline{z}$ otherwise. Using the resolvent equation (4.58) and its adjoint, we may accumulate in total 2ϑ resolvents R_A, analogously to what was done to obtain (4.59), such that we are confronted with estimating finitely many terms of the form

$$\begin{aligned} &\mathbb{E}\Big\{ \|\chi_\Gamma\, R^{(s)}\, Q_1\, R_A \cdots Q_\vartheta\, R_A\, Q_{\vartheta+1}\, R_A\, Q_{\vartheta+2} \cdots R_A\, Q_{2\vartheta+1} R^{(r)}\, \chi_\Gamma \|_1 \Big\} \\ &\qquad \leq \Big\{ \mathbb{E}\Big[\|\chi_\Gamma\, R^{(s)}\, Q_1\, R_A \cdots Q_\vartheta\, R_A\, |Q_{\vartheta+1}|^{\frac{1}{2}}\, \|_2^2 \Big] \\ &\qquad\qquad \times\, \mathbb{E}\Big[\|\, |Q_{\vartheta+1}|^{\frac{1}{2}}\, R_A\, Q_{\vartheta+2} \cdots R_A\, Q_{2\vartheta+1} R^{(r)} \chi_\Gamma \|_2^2 \Big] \Big\}^{\frac{1}{2}}. \end{aligned} \tag{4.65}$$

Here $s, r \in \{0, 1, \dots, 2\vartheta\}$ and $R^{(s)}$ is some product of s factors each of which either being $R_{A,V_m}(z)$, $R_{A,V_n}(z)$ or one of their adjoints. Moreover, $Q_1, \dots, Q_{2\vartheta+1}$ are random potentials suitably chosen from the set $\{1, z - E_2 - V_n, \overline{z} - E_2 - V_n, z - E_2 - V_m, \overline{z} - E_2 - V_m, V_n - V_m\}$ and exactly one of these is equal to $V_n - V_m$. We now copy the steps between (4.60) and (4.62) and take $p = 2\vartheta$, $g_j = g_{p+2-j} = |Q_j|$, $g_p = |Q_{\vartheta+1}|$ and $t_j = 2\vartheta + 1$ for $j \in \{1, \dots, \vartheta + 1\}$ in [46, Lemma 5.11(i)] to obtain the bound

$$\begin{aligned} &\mathbb{E}\Big[\|\, |Q_{\vartheta+1}|^{\frac{1}{2}}\, R_0\, |Q_\vartheta| \cdots R_0\, |Q_1|\, \chi_\Gamma \|_2^2 \Big] \\ &\quad \leq C_5(E_2)\, |\Gamma|\, \Big\{ \mathbb{E}\Big[|Q_{\vartheta+1}(0)|^{2\vartheta+1} \Big] \Big\}^{\frac{1}{2\vartheta+1}} \prod_{j=1}^{\vartheta} \Big\{ \mathbb{E}\Big[|Q_j(0)|^{2\vartheta+1} \Big] \Big\}^{\frac{2}{2\vartheta+1}} \end{aligned} \tag{4.66}$$

for $|\mathrm{Im}\, z|^s$ times the first expectation on the r.h.s. of (4.65). The second expectation is treated similarly. Since exactly one of the Q_j is equal to $V_n - V_m$ and all others in the above set may be bounded by $1 + |z - E_2| + |V|$, the proof is complete.

As to c). Since $H(A, V_n^{(\omega)})\,\varphi \to H(A, V^{(\omega)})\,\varphi$ as $n \to \infty$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and $\mathscr{C}_0^\infty(\mathbb{R}^d)$ is a common core for all $H(A, V_n^{(\omega)})$ and $H(A, V^{(\omega)})$ for $\mathbb{P}$-almost all $\omega \in \Omega$ by Proposition 2.4, [47, Thm. VIII.25(a)] implies that

$$R_{A,V^{(\omega)}}(z) = \operatorname*{s-lim}_{n\to\infty} R_{A,V_n^{(\omega)}}(z) \tag{4.67}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$ and $\mathbb{P}$-almost all $\omega \in \Omega$. Part (ii) of the present proof together with assumption (I) shows that

$$\lim_{n,m\to\infty} \int_\Omega \mathbb{P}(\mathrm{d}\omega)\, \Big\| \chi_\Gamma \Big[|R_{A,V_n^{(\omega)}}(z)|^{2\vartheta} - |R_{A,V_m^{(\omega)}}(z)|^{2\vartheta} \Big] \chi_\Gamma \Big\|_1 = 0. \tag{4.68}$$

Analogous reasoning as in the proof of Corollary 3.3 yields the existence of some sequence (n_j) of natural numbers such that

$$\lim_{i,j\to\infty} \left\| \chi_\Gamma \left[|R_{A,V_{n_i}^{(\omega)}}(z)|^{2\vartheta} - |R_{A,V_{n_j}^{(\omega)}}(z)|^{2\vartheta} \right] \chi_\Gamma \right\|_1 = 0 \tag{4.69}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. In other words, the subsequence $(\chi_\Gamma |R_{A,V_{n_j}^{(\omega)}}(z)|^{2\vartheta} \chi_\Gamma)_{j\in\mathbb{N}}$ is Cauchy in $\mathscr{J}_1(\mathrm{L}^2(\mathbb{R}^d))$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Thanks to completeness of $\mathscr{J}_1(\mathrm{L}^2(\mathbb{R}^d))$ and the strong convergence (4.67), we have the convergence

$$\lim_{j\to\infty} \left\| \chi_\Gamma \left[|R_{A,V_{n_j}^{(\omega)}}(z)|^{2\vartheta} - |R_{A,V^{(\omega)}}(z)|^{2\vartheta} \right] \chi_\Gamma \right\|_1 = 0 \tag{4.70}$$

in $\mathscr{J}_1(\mathrm{L}^2(\mathbb{R}^d))$ for $\mathbb{P}$-almost all $\omega \in \Omega$. The latter implies

$$\int_\Omega \mathbb{P}(\mathrm{d}\omega) \left\| \chi_\Gamma |R_{A,V^{(\omega)}}(z)|^{2\vartheta} \chi_\Gamma \right\|_1 \leq \liminf_{j\to\infty} \int_\Omega \mathbb{P}(\mathrm{d}\omega) \left\| \chi_\Gamma |R_{A,V_{n_j}^{(\omega)}}(z)|^{2\vartheta} \chi_\Gamma \right\|_1 \tag{4.71}$$

by Fatou's lemma. Since Proposition 4.15(i) holds for all V_{n_j}, the proof of Proposition 4.15(i) with V_n replaced by V is complete. For a proof of Proposition 4.15(ii) we proceed analogously using (4.70), part (ii) of the present proof and again Fatou's lemma. □

It remains to carry over the result for $\mathbb{R}^d$-ergodic potentials to $\mathbb{Z}^d$-ergodic ones using the suspension construction, see [29, 30].

Proof of Proposition 4.15 in case V is $\mathbb{Z}^d$-ergodic. We consider the product of the probability spaces $(\Omega, \mathscr{A}, \mathbb{P})$ and $(\Lambda(0), \mathscr{B}(\Lambda(0)), \mathit{Lebesgue})$. The latter corresponds to a uniform distribution on the open unit cube $\Lambda(0)$. On this enlarged space we define the random potential

$$\mathscr{V} : (\Omega \times \Lambda(0)) \times \mathbb{R}^d \to \mathbb{R}, \qquad (\omega, y, x) \mapsto \mathscr{V}^{(\omega,y)}(x) := V^{(\omega)}(x - y). \tag{4.72}$$

It is $\mathbb{R}^d$-ergodic by construction [29] and enjoys properties (S) and (I). The latter assertion is proven by tracing the claimed properties of $\mathscr{V}$ back to the respective properties of V.

It remains to prove that the validity of Proposition 4.15 for $\mathscr{V}$ implies the one for V. For this purpose, we note that the integral transform (4.27) of the (infinite-volume) density-of-states measure corresponding to $\mathscr{V}$ obeys

$$\begin{aligned} &\frac{1}{|\Gamma|} \int_{\Omega\times\Lambda(0)} \mathbb{P}(\mathrm{d}\omega) \otimes \mathrm{d}^d y \ \mathrm{Tr}\left[\chi_\Gamma \, |H(A, \mathscr{V}^{(\omega,y)}) - z|^{-2\vartheta} \, \chi_\Gamma \right] \\ &= \frac{1}{|\Gamma|} \int_{\Lambda(0)} \mathrm{d}^d y \int_\Omega \mathbb{P}(\mathrm{d}\omega) \ \mathrm{Tr}\left[\chi_{\Gamma - y} \, |H(A, V^{(\omega)}) - z|^{-2\vartheta} \, \chi_{\Gamma - y} \right] \\ &= \frac{1}{|\Gamma|} \int_\Omega \mathbb{P}(\mathrm{d}\omega) \ \mathrm{Tr}\left[\chi_\Gamma \, |H(A, V^{(\omega)}) - z|^{-2\vartheta} \, \chi_\Gamma \right] \end{aligned} \tag{4.73}$$

Here the first equality results from (2.11) and the definitions of $\mathscr{V}$ and the cube $\Gamma - y := \{x - y \in \mathbb{R}^d \ : \ x \in \Gamma\}$ together with Fubini's theorem. To obtain the second equality we have used the fact that the trace does not depend on y after performing the $\mathbb{P}(\mathrm{d}\omega)$-integration. This follows from $\mathbb{Z}^d$-homogeneity as well as from the fact that one may "re-arrange" $\Gamma - y$ in the form of Γ by $\mathbb{Z}^d$-translations since Γ is compatible with the lattice. Moreover, one computes

$$\int_{\Omega\times\Lambda(0)} \mathbb{P}(\mathrm{d}\omega)\otimes \mathrm{d}^d y\;\Big(1+|z-E_2|+|\mathscr{V}^{(\omega,y)}(0)|\Big)^{2\vartheta} = \int_{\Omega}\mathbb{P}(\mathrm{d}\omega)\int_{\Lambda(0)}\mathrm{d}^d x\;\Big(1+|z-E_2|+|V^{(\omega)}(x)|\Big)^{2\vartheta} \tag{4.74}$$

using (4.72) and Fubini's theorem. This completes the proof of Proposition 4.15(i) with V_n replaced by V. The other parts of Proposition 4.15 in the $\mathbb{Z}^d$-ergodic case are proven similarly. □

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