PHASE COEXISTENCE FOR THE KAC ISING MODELS

T. BODINEAU

Abstract. We derive the Wulff construction for Kac Ising models with long but finite range interaction in dimensions $d \geqslant 2$. Some open problems concerning the phase coexistence for more general models are also discussed.

1. Introduction

During the last decade important progress has been made in the thorough understanding of the phase coexistence phenomenon. The most precise quantification of the different mechanisms at play and a sharp description of the phase boundaries has been obtained in the context of the 2D Ising model with nearest neighbor interactions [DKS, Pf, ACC, I1, I2, DS, ScS1, ISc, PV1, PV2, DH, Al]. For the 3D Ising model, the results are less precise and the phase segregation is characterized by the concentration in $\mathbb{L}^1$ of coarse grained configurations to the equilibrium crystal shapes [ABCP, BCP, BBBP, BBP, Ce, B1, CePi1, CePi2, BIV2]. For a comprehensive survey on the topic the reader is referred to [BIV1] and the references therein.

In this paper a different approach is implemented in order to obtain the Wulff construction for Kac Ising model with finite range interactions. The motivation behind this strategy is to provide a step towards the understanding of the phase coexistence for a class of models broader than the ferromagnetic Ising models. Before stating the main results of the paper, we discuss some open problems and summarize the difficulties which hinder the extension of the $\mathbb{L}^1$ theory to more general models.

As will be explained later, the actual results on phase coexistence are valid for a limited class of microscopic models, namely the random cluster model with finite range interactions. The description of more subtle phenomena which occur in everyday life, or which are observed in experiments, would require the investigation of more realistic models. One of the most current examples of phase coexistence, the liquid/vapor coexistence, has no rigorous counterpart (with the exception of the Widom Rowlinson model for which the Wulff construction should be a straightforward consequence of the FK correspondence [CCK, GLM]). In fact, even the primary issue of the liquid/vapor phase transition has been derived only for a limited class of models. A discussion on the possible implications of the recent breakthrough [LMP] on the liquid/vapor phase transition is postponed to remark 2.1. More generally, the study of models governed by the Pirogov Sinai Theory would shed light on interesting physical mechanisms which have still not been thoroughly analyzed. For example, for some multi-phase models without symmetric phases, one should

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observe between two phases the occurrence of an intermediate layer of a third phase (see [MMRS] for a discussion of this phenomenon in the case of the large q Potts model at the critical point). Nevertheless, even if the bulk properties are well understood in the context of Pirogov Sinai Theory, the surface properties and, more precisely, a microscopic derivation of the Wulff construction remain an open problem.

In the $\mathbb{L}^1$ approach, the phase coexistence is analyzed by performing a series of localizations from the macroscopic level to the microscopic level in order to relate the macroscopic equilibrium crystal shapes to the surface tension, which is defined in terms of microscopic quantities. This strategy can be essentially implemented on a coarse grained level and therefore does not rely on the microscopic structure of the model. Nevertheless, it requires the validity of a coarse grained representation of the model which keeps track of the local order parameter and ensures some decoupling properties. We refer to [BIV1] for a presentation of the $\mathbb{L}^1$ theory in a general framework.

This semblance of generality suffers from an important exception. Once the interface has been localized on a mesoscopic level, the precise surface tension factor should be extracted from some coarse grained information. Therefore, the structure of the coarse graining becomes essential in order to relate the mesoscopic level to the microscopic information which leads to the surface tension. The previous proofs of the phase coexistence for Bernoulli percolation [Ce]; for finite range Ising model [B1, CePi1, BIV2]; and for Potts model [CePi2] rely crucially on Pisztora's coarse graining [Pi]. This coarse graining is a fundamental tool for the study of the Ising/Potts model in a non perturbative regime (up to the slab percolation threshold). It describes the "backbone" of the Ising/Potts spin configurations in terms of a cluster in the FK representation. Thus, in the previous proofs, the surface tension was redefined in terms of the random cluster model in order to be related to the coarse graining.

This approach would seem to limit the generalization to models for which the FK correspondence does not hold. Furthermore, the previous proofs are based upon correlation inequalities which are not satisfied by general models. Therefore, the analysis of phase coexistence for models in the Pirogov Sinai theory would need to develop different tools and to devise a more robust analysis of surface tension. It would be interesting to introduce a tractable macroscopic definition of surface tension which does not rely on correlation inequalities and on the FK representation.

In this paper more modest questions are addressed. First, an approach of the Wulff construction is presented for the Ising model without using the FK correspondence. This analysis is limited to low temperatures. Nevertheless, since it is mainly based on the Peierls condition, parts of it should be easy to transpose in other contexts. In a final section, this method is generalized in order to derive the Wulff construction for Kac Ising models.

The first step is to introduce a new coarse graining appropriate for the $\mathbb{L}^1$ theory. Then we deduce from a general exponential tightness theorem the phenomenon of phase segregation, i.e. the fact that phase coexistence does not occur by the creation of many small droplets of one phase scattered in the other, but by a macroscopic phase separation. Alternatively, this phenomenon manifests itself by surface order large deviations (see [Sc, Pi]). This first part is only based upon the Peierls condition and should be straightforward to generalize (even in the case of multi-phase models).

In a second step, the equilibrium crystal shapes are derived. The proof follows the general scheme of the $\mathbb{L}^1$ theory, but it is implemented with the new coarse graining directly on the spin level. At this stage, the existence of surface tension (known from correlation inequalities) and the symmetry between the phases (spin flip) come into play.

In the last part, the previous strategy is updated to the context of the Kac Ising models with long but finite range interactions. The first derivation of the Wulff construction for Kac Ising models was obtained by [ABCP, BCP] (see also [AlBe] for anisotropic interactions). In these works, the interaction was rescaled with the number of particles ($N \sim \gamma^{-1-\epsilon}$) and the phase coexistence stemmed from the proof of the Γ-convergence of functionals associated to the continuous limit of the model. The functional framework of the $\mathbb{L}^1$ theory was introduced in these papers and the importance of the geometric measure theory and of related compactness properties was emphasized. In [BBBP, BBP], the prediction of the van der Waals theory of surface tension is recovered when the range of the interaction diverges to infinity. More precisely, the probabilistic cost of a macroscopic interface was proven to be exponentially small with a factor proportional to the perimeter of the interface times the van der Waals surface tension. As a consequence, the equilibrium crystal shapes converge to a sphere in the Lebowitz and Penrose limit. For finite range interaction only a rough description of the crystals was obtained. Building on the previous results, the present paper completes the Wulff construction for finite range interactions. The main ingredient is the derivation of the surface tension from an approximate expression defined only on a coarse grained level. As a consequence of [BBBP], the sequence of surface tensions defined for any range of the interaction $\frac{1}{\gamma}$ converges to the isotropic van der Waals surface tension.

2. Models and results

2.1. **Ising model.** Let Λ be a finite domain of $\mathbb{Z}^d$, for $d \geqslant 2$. Each site i in Λ indexes a spin σ_i which takes values ± 1. The spin configurations $\{\sigma_i\}_{i\in\Lambda}$ have a statistical weight determined by the Hamiltonian

$$H^{\bar\sigma}(\sigma) = -\frac{1}{2}\sum_{i,j\in\Lambda} J(i-j)\sigma_i\sigma_j - \sum_{i\in\Lambda, j\in\Lambda^c} J(i-j)\sigma_i\bar\sigma_j\,,$$

where $\bar\sigma = \{\bar\sigma_i\}_{i\in\Lambda^c}$ are the boundary conditions outside Λ and J are ferromagnetic finite range coupling constants, i.e. $J \geqslant 0$ and there is $R > 0$ such that $J(i) = 0$ if $\|i\|_\infty > R$.

The Gibbs measure associated to the spin system with boundary conditions $\bar\sigma$ is

$$\forall \sigma = \{\sigma_x\}_{x\in\Lambda}, \qquad \mu^{\bar\sigma}_{\beta,\Lambda}(\sigma) = \frac{1}{Z^{\bar\sigma}_{\beta,\Lambda}} \exp\left(-\beta H^{\bar\sigma}(\sigma)\right),$$

where β is the inverse of the temperature ($\beta = \frac{1}{T}$) and $Z^{\bar\sigma}_{\beta,N}$ is the partition function. If the boundary conditions are uniformly equal to 1 (resp -1), the Gibbs measure will be denoted by $\mu^+_{\beta,\Lambda}$ (resp $\mu^-_{\beta,\Lambda}$).

Let $\mathbb{D}_N$ be the subset $\{-N,\dots,N\}^d$ of $\mathbb{Z}^d$. There is a critical value β_c above which a breaking of symmetry occurs in the thermodynamic limit

$$\forall \beta > \beta_c, \qquad \lim_N \mu^+_{\beta,N}(\sigma_0) = -\lim_N \mu^-_{\beta,N}(\sigma_0) = m^*_\beta > 0\,. \tag{2.1}$$

Furthermore, in the thermodynamic limit the measures $\mu^+_{\beta,N}$ and $\mu^-_{\beta,N}$ converge to two distinct Gibbs measures μ^+_β and μ^-_β which are measures on the space $\{\pm 1\}^{\mathbb{Z}^d}$. Each of these measures represents a pure state.

We are going to define the surface tension. Let $\vec{n}$ be a vector in $\mathbb{S}^{d-1}$ such that $\vec{n}\cdot\vec{e}_1 > \frac{1}{\sqrt{d}}$. Let $\Lambda_N = \{i = (i_1,\dots,i_d) \in \mathbb{Z}^d \mid \forall k \geqslant 2, \quad |i_k| \leqslant N\}$ and Δ_N be a sequence of finite subsets of Λ_N such that

$$\{i \in \mathbb{Z}^d \mid \quad |\vec{n}\cdot i| \leqslant f(N)\} \subset \Delta_N,$$

for some function f such that $f(N)$ diverges as N goes to infinity. The mixed boundary conditions $\bar{\sigma}$ are defined as

$$\forall i \in \Delta_N^c, \qquad \bar{\sigma}_i = \begin{cases} +1, & \text{if} \quad \vec{n}\cdot i \geqslant 0, \\ -1, & \text{if} \quad \vec{n}\cdot i < 0. \end{cases}$$

The partition function with mixed boundary conditions is denoted by $Z^{\pm}_{\beta,\Delta_N}(\vec{n})$ and the one with boundary conditions uniformly equal to 1 by Z^{+}_{β,Δ_N}.

Definition 2.1. *The surface tension in the direction* $\vec{n} \in \mathbb{S}^{d-1}$, *with* $\vec{n}\cdot\vec{e}_1 > 0$, *is defined by*

$$\tau_\beta(\vec{n}) = \lim_{N\to\infty} -\frac{(\vec{n},\vec{e}_1)}{N^{d-1}} \log \frac{Z^{\pm}_{\beta,\Delta_N}(\vec{n})}{Z^{+}_{\beta,\Delta_N}}. \tag{2.2}$$

For $\beta > \beta_c$, the surface tension is uniformly positive on $\mathbb{S}^{d-1}$ and its homogeneous extension on $\mathbb{R}^d$

$$\forall x \in \mathbb{R}^d, \qquad \tau_\beta(x) = \|x\|_2\, \tau_\beta\left(\frac{x}{\|x\|_2}\right), \qquad \tau_\beta(0) = 0\,,$$

is convex. The previous properties of surface tension, as well as its existence are derived by using ferromagnetic inequalities. The reader is referred to [MMR] for a comprehensive discussion on surface tension.

2.2. **Kac Ising model.**

The Kac Ising model is a ferromagnetic spin system with interactions tuned by a scaling parameter γ. Let J be a non-negative smooth function supported by $[-1,1]$ and such that $\int_{\mathbb{R}^d} J(|r|)\,dr = 1$. For any $\gamma > 0$ the Kac potentials are defined by

$$\forall i \in \mathbb{Z}^d, \qquad J_\gamma(i) = \gamma^d J(\gamma|i|)\,.$$

For simplicity, γ is chosen as 2^{-g} with $g \in \mathbb{N}$. To emphasize the dependence on the scaling parameter, the Gibbs measure on Λ at inverse temperature β and with boundary conditions $\bar{\sigma}$ is denoted by $\mu^{\bar{\sigma}}_{\beta,\gamma,\Lambda}$.

For any $\gamma > 0$, a critical temperature $\beta_c(\gamma)$ (corresponding to the non-uniqueness of the infinite Gibbs measure) is associated to the system. It is proved in [CaPr, BZ] that

$$\lim_{\gamma\to 0} \beta_c(\gamma) = \beta_c^{\mathrm{mf}} = 1\,,$$

where β_c^{mf} denotes the critical temperature of the mean field system. For any $\beta > 1$, this implies the existence of two distinct Gibbs measures $\mu^+_{\beta,\gamma}$ and $\mu^-_{\beta,\gamma}$ for sufficiently small γ. In particular, the phase transition is characterized by a spontaneous magnetization

$$\forall \beta > 1, \exists \gamma_\beta > 0, \forall \gamma < \gamma_\beta, \qquad \mu^+_{\beta,\gamma}(\sigma_0) = m^*_{\beta,\gamma} > 0\,. \tag{2.3}$$

When γ vanishes, $m^*_{\beta,\gamma}$ converges to the mean field magnetization m^*_β. In passing, notice that a stronger result has been derived in [BMP]

Theorem 2.1. *For any $\beta > 1$, there is $\gamma_\beta > 0$ such that for any $\gamma < \gamma_\beta$, any translation invariant Gibbs measure is a convex combination of $\mu^+_{\beta,\gamma}$ and $\mu^-_{\beta,\gamma}$.*

This assertion shows that, as in the van der Waals Theory, there are only two pure phases for sufficiently long range interactions.

2.3. **The functional setting.** We consider a general ferromagnetic Ising model as defined in Subsections 2.1 and 2.2. In order to enforce phase coexistence, the measure $\mu^+_{\beta,N}$ is conditioned by the event that the averaged magnetization is atypical, i.e. $\big\{\mathbf{M}_N = \frac{1}{(2N+1)^3}\sum_{i\in\mathbb{D}_N}\sigma_i \leqslant m\big\}$, where m is a constant in $]-m^*_\beta, m^*_\beta[$. The problem will be to characterize the typical configurations of the conditional measure $\mu^+_{\beta,N}\big(\,\cdot\,|\,\mathbf{M}_N \leqslant m\big)$. Before stating the results, let us introduce the macroscopic setting.

On the macroscopic level, the system is confined in the cube $\widehat{\mathbb{D}} = [-\frac{1}{2}, \frac{1}{2}]^d$ of $\mathbb{R}^d$ and a macroscopic configuration where the pure phases coexist is described by a function v taking values $\{\pm 1\}$. The function v should be interpreted as a signed indicator representing the local order parameter : if $v_r = 1$ for some $r \in \widehat{\mathbb{D}}$, then the system should be locally at r in equilibrium in the phase m^*_β.

To define the macroscopic interfaces, i.e. the boundary of the set $\{v = -1\}$, a convenient functional setting is the space $\mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\})$ of functions of bounded variation with values ± 1 in $\widehat{\mathbb{D}}$ and uniformly equal to 1 outside $\widehat{\mathbb{D}}$ (see [EG] for a review). For any $v \in \mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\})$, there exists a generalized notion of the boundary of the set $\{v = -1\}$ called reduced boundary and denoted by $\partial^* v$. If $\{v = -1\}$ is a regular set, then $\partial^* v$ coincides with the usual boundary ∂v. The Wulff functional $\mathcal{W}_\beta$ is defined in $\mathbb{L}_1(\widehat{\mathbb{D}})$ as follows

$$\mathcal{W}_\beta(v) = \begin{cases} \int_{\partial^* v} \tau_\beta(\vec{n_x})\, d\mathcal{H}_x, & \text{if} \quad v \in \mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\}), \\ \infty\,, & \text{otherwise.} \end{cases} \tag{2.4}$$

To any measurable subset A of $\widehat{\mathbb{D}}$, we associate the function $\mathbb{1}_A = 1_{A^c} - 1_A$ and simply write $\mathcal{W}_\beta(A) = \mathcal{W}_\beta(\mathbb{1}_A)$.

Let m be in $]-m^*_\beta, m^*_\beta[$. The equilibrium crystal shapes are the solutions of the Wulff variational problem, i.e. they are the minimizers of the functional $\mathcal{W}_\beta$ under a volume constraint

$$\min\left\{\mathcal{W}_\beta(v)\;\middle|\; v \in \mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\}), \qquad \Big|\int_{\widehat{\mathbb{D}}} m^*_\beta\, v_r\, dr\Big| \leqslant m\right\}. \tag{2.5}$$

Let $\mathcal{D}_m$ be the set of minimizers of (2.5). The set of functions of bounded perimeter will also play an important role in the following : for any $a > 0$, we set

$$\mathcal{C}_a = \big\{v \in \mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\}) \;|\quad \mathcal{P}(\{v = -1\}) \leqslant a\big\}, \tag{2.6}$$

where $\mathcal{P}$ denotes the perimeter. This set is compact in the $\mathbb{L}^1$ topology.

The microscopic Ising model is embedded in the continuous setting. Let $\widehat{\mathbb{D}}_N = \frac{1}{N}\mathbb{Z}^d \cap \widehat{\mathbb{D}}$. For simplicity the microscopic size is chosen in the binary form $N = 2^n$ and the mesoscopic scale is denoted by $K = 2^k$. The cube $\widehat{\mathbb{D}}$ is partionned into boxes $\widehat{\mathbb{B}}_{N,K}$, each of them containing K^d sites of $\widehat{\mathbb{D}}_N$

$$j \in \mathbb{Z}^d,\ x = j\frac{K}{N} \in \widehat{\mathbb{D}}_N, \qquad \widehat{\mathbb{B}}_{N,K}(x) = x + \left]-\frac{K}{2N}, \frac{K}{2N}\right]^d.$$

Let $\mathbb{B}_K(Nx)$ be the microscopic counterpart of $\widehat{\mathbb{B}}_{N,K}(x)$, i.e. the sites of $\widehat{\mathbb{D}}_N$ in $\widehat{\mathbb{B}}_{N,K}(x)$. These boxes are centered on the sites of $\widehat{\mathbb{D}}_{N,K} = \frac{K}{N}\mathbb{Z}^d \cap \widehat{\mathbb{D}}$.

Finally, the local magnetization is defined as a piece-wise constant function on the partition $\{\widehat{\mathbb{B}}_{N,K}(x)\}$

$$\forall y \in \widehat{\mathbb{B}}_{N,K}(x), \qquad \mathcal{M}_{N,K}(y) = \frac{1}{K^d} \sum_{i \in \mathbb{B}_K(Nx)} \sigma_i \,. \tag{2.7}$$

The local order parameter $\mathcal{M}_{N,K}$ characterizes the local equilibrium.

2.4. **Main results.** The following results describe the phenomenon of phase coexistence with different accuracies. The system is analyzed on a coarse grained level in the $\mathbb{L}^1$ topology.

The first result implies the occurrence of phase segregation (in a weak form), i.e. without describing explicitly the limiting crystal shapes. This should be interpreted as an intermediate step on the way to the Wulff construction. Let $\mathfrak{B}$ be the set of inverse temperatures for which the Peierls estimate holds : for any β in $\mathfrak{B}$, there is a constant $c_\beta > C(d)$ (where $C(d)$ is a constant large enough depending on the dimension) such that any spin contour Γ of length $|\Gamma|$ has exponentially small probability (uniformly in N)

$$\mu^+_{\beta,N}(\Gamma) \leqslant \exp(-c_\beta |\Gamma|) \,. \tag{2.8}$$

Furthermore, c_β diverges as β goes to infinity.

Theorem 2.2. *Let β be in $\mathcal{B}$ and m be in $]-m^*_\beta, m^*_\beta[$. There is $a > 0$ such that for every $\delta > 0$, there is a scale $K_0 = K_0(\beta, \delta)$ and*

$$\forall K \geqslant K_0, \qquad \lim_{N \to \infty} \mu^+_{\beta,N}\left(\frac{1}{m^*_\beta} \mathcal{M}_{N,K} \in \mathcal{V}(\mathcal{C}_a, \delta) \;\middle|\; \mathbf{M}_N \leqslant m \right) = 1 \,,$$

where $\mathcal{C}_a$ has been defined in (2.6) and $\mathcal{V}(\mathcal{C}_a, \delta)$ is the δ-neighborhood of $\mathcal{C}_a$ in $\mathbb{L}^1$

$$\mathcal{V}(\mathcal{C}_a, \delta) = \left\{ v \in \mathbb{L}^1(\widehat{\mathbb{D}}) \;\middle|\; \exists v' \in \mathcal{C}_a, \quad \|v - v'\|_1 \leqslant \delta \right\}.$$

This result implies that on a macroscopic scale, one observes macroscopic droplets with phase boundaries of perimeter smaller than a. The proof of Theorem 2.2 is based on a general approach developed in [BIV1] and on a new coarse graining. Notice that for Kac Ising models, a similar result was already derived in [BBP].

The argument, implemented in this paper for the nearest neighbor Ising model at low temperature, is robust and should also apply to models in the Pirogov Sinai Theory (possibly containing several phases). This rough description relies on Peierls type estimates.

Another characterization of phase segregation is the occurrence of surface order large deviations (see [Sc, Pi]). This is a direct consequence of the proof of Theorem 2.2.

Corollary 2.1. *Let β be in $\mathcal{B}$ and m in $]-m^*_\beta, m^*_\beta[$. Then, there exists $c_\beta > 0$ such that*

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}(\mathbf{M}_N \leqslant m) \leqslant -c_\beta \,.$$

The computation of sharp asymptotic with the exact surface tension factor enables to go beyond the previous result and to prove the $\mathbb{L}^1$ concentration of the local order parameter $\mathcal{M}_{N,K}$ to the minimizers of the variational problem (2.5). Let us state the result for the Kac Ising model. For a given range of the interactions $\frac{1}{\gamma}$, the magnetization is denoted by

$m^*_{\beta,\gamma}$ (2.3), the surface tension by $\tau_{\beta,\gamma}$ (2.2) and the set of the equilibrium crystal shapes $\mathcal{D}_{m,\gamma}$ (2.5).

Theorem 2.3. *For any $\beta > 1$, let us fix $\gamma \in]0, \gamma_\beta[$ and $m \in] - m^*_{\beta,\gamma}, m^*_{\beta,\gamma}[$. For every $\delta > 0$, one can choose a scale $K_0 = K_0(\beta, \gamma, \delta)$ such that for any $K \geqslant K_0$*

$$\lim_{N \to \infty} \mu^{m^*_\beta}_{\beta,\gamma,N} \left(\inf_{v \in \mathcal{D}_{m,\gamma}} \| \mathcal{M}_{N,K} - m^*_{\beta,\gamma} v \|_1 \leqslant \delta \;\Big|\; \mathbf{M}_N \leqslant m \right) = 1 .$$

The convergence of the surface tension $\tau_{\beta,\gamma}$ to the anisotropic van der Waals surface tension was proven in [BBBP]. This implies that the equilibrium crystal shapes converge to spheres as γ vanishes (when the equilibrium crystal shapes are given by the Wulff construction, i.e. for values of m close enough to m^*_β).

Remark 2.1. *In a groundbreaking work Lebowitz, Mazel, Presutti* [LMP] *proved the liquid/vapor transition for particles in the continuum interacting with Kac potentials. Their proof is based on a renormalization procedure which enables to reduce the system to coarse grained variables with properties similar to the ones of the coarse grained Kac Ising model. Therefore, it is natural to hope that parts of the argument developed for the Kac Ising model could be transposed in the framework investigated in* [LMP]. *For the moment the lack of ferromagnetic inequalities and of symmetry between the phases are major difficulties for the analysis of surface tension. These problems are also encountered for the derivation of surface tension in the Pirogov Sinai Theory. Nevertheless, the basic phase segregation phenomenon (see Theorem 2.2) follows in this context from the argument developed in* [BBP].

Remark 2.2. *Theorem 2.2 could also have been stated for different types of boundary conditions (free, mixed, ...).*

3. THE $\mathbb{L}^1$ THEORY

In this section, the salient features of the proof are outlined. For simplicity, we focus on the Ising model with nearest neighbor interaction at low temperature and postpone the study of Kac Ising model to section 4. The basic assumption is the validity of the Peierls estimate (2.8).

3.1. Phase segregation.

Theorem 2.2 and corollary 2.1 are direct consequences of the following exponential tightness theorem.

Proposition 3.1. *Let β be in $\mathfrak{B}$. Then there exists a constant $C(\beta) > 0$ such that for all δ positive one can find $K_0(\delta)$ such that for $K \geqslant K_0$*

$$\forall a > 0, \qquad \limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N} \left(\frac{1}{m^*_\beta} \mathcal{M}_{N,K} \notin \mathcal{V}(\mathcal{C}_a, \delta) \right) \leqslant - C(\beta)\, a,$$

where $\mathcal{V}(\mathcal{C}_a, \delta)$ is the δ-neighborhood of $\mathcal{C}_a$ in $\mathbb{L}^1(\widehat{\mathbb{D}})$.

This proposition tells us that only the configurations close to the compact set $\mathcal{C}_a$ have a contribution which is of a surface order.

Let us first deduce Theorem 2.2 and Corollary 2.1 from Proposition 3.1. There exists a constant c_d depending on the dimension such that

$$\frac{1}{m^*} \mathcal{M}_{N,K} \in \mathcal{V}(\mathcal{C}_a, \delta) \Rightarrow \mathbf{M}_N = \int_{\widehat{\mathbb{D}}} \mathcal{M}_{N,K}(x)\, dx \geqslant m^* (1 - \delta - c_d a^{d/(d-1)}) .$$

Therefore the spin configurations in $\{\mathbf{M}_N \leqslant m\}$ for $m < m^*$ belong as well to $\big\{\frac{1}{m^*}\mathcal{M}_{N,K} \notin \mathcal{V}(\mathcal{C}_a, \delta)\big\}$ for some $a > 0$ and $\delta > 0$. An application of Proposition 3.1 leads to the surface order deviations of Corollary 2.1.

To prove Theorem 2.2 it is enough to check that there is a constant c_0 such that

$$\liminf_{N \to \infty} \frac{1}{N^{d-1}} \log \mu^{+}_{\beta,N} \left(\mathbf{M}_N \leqslant m\right) \geqslant -c_0 .$$

This follows from the Peierls estimate and the exponential relaxation in the pure phases (for $\beta \in \mathfrak{B}$).

The derivation of Proposition 3.1 is based upon a coarse grained description of the system and a theorem (valid for general coarse graining) which ensures the exponential tightness (see [BIV1]).

Coarse graining :

This coarse grained description is obtained under the Peierls condition (2.8). It is valid in any dimensions $d \geqslant 2$ and does not require correlations inequalities or the FK representation.

The typical spin configurations are defined at the mesoscopic scale $K = 2^k$. Let $\partial\mathbb{B}_K = \mathbb{B}_{K+K^\alpha} \setminus \mathbb{B}_K$ be the enlarged external boundary of the box $\mathbb{B}_K$, where α is in $(0,1)$. The parameter $\zeta > 0$ will control the accuracy of the coarse graining.

Let x be in $\widehat{\mathbb{D}}_{N,K}$. For any $\varepsilon = \pm 1$, the box $\widehat{\mathbb{B}}_{N,K}(x)$ is ε-*good* if the spin configuration inside the enlarged box $\mathbb{B}_{K+K^\alpha}(x)$ is typical, i.e.

(P1) The box $\mathbb{B}_K(x)$ is surrounded by at least a connected surface of spins in $\partial\mathbb{B}_K(x)$ with sign uniformly equal to ε.

(P2) The average magnetization $\mathcal{M}_{N,K}(x)$ inside $\mathbb{B}_K(x)$ is close to the equilibrium value εm^*

$$\left|\mathcal{M}_{N,K}(x) - \varepsilon m^*\right| \leqslant \zeta . \tag{3.1}$$

On the mesoscopic level, each box $\widehat{\mathbb{B}}_{N,K}(x)$ is labelled by a mesoscopic phase label

$$\forall x \in \widehat{\mathbb{D}}_{N,K}, \qquad u^{\zeta}_{N,K}(x) = \begin{cases} \varepsilon, & \text{if } \widehat{\mathbb{B}}_{N,K}(x) \text{ is } \varepsilon\text{-}good\ , \\ 0, & \text{otherwise.} \end{cases}$$

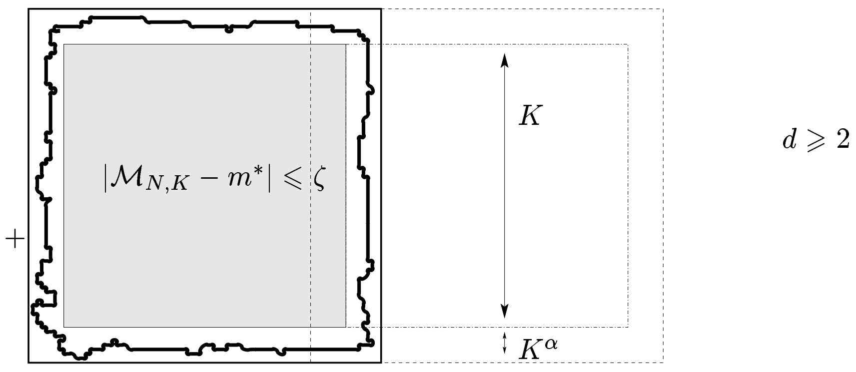

FIGURE 1. Coarse grained configuration with overlapping + good blocks.

For large mesoscopic boxes, the typical spin configurations occur with overwhelming probability.

Theorem 3.1. *Let β be in $\mathfrak{B}$. Then for any $\zeta > 0$, the following holds uniformly in N*

$$\forall \{x_1, \dots, x_\ell\} \in \widehat{\mathbb{D}}_{N,K}, \quad \mu_{\beta,N}^{+}\left(u_{N,K}^{\zeta}(x_1) = 0, \dots, u_{N,K}^{\zeta}(x_\ell) = 0\right) \leqslant \left(\rho_K^{\zeta}\right)^{\ell}, \tag{3.2}$$

where the parameter ρ_K^{ζ} vanishes as K goes to infinity.

Notice that the mesoscopic phase labels are far from being independent variables. In particular, the enlarged boundaries of two neighboring boxes in $\widehat{\mathbb{D}}_{N,K}$ overlap. This implies that two neighboring boxes cannot be labelled by opposite signs. Let us first draw some consequences of this construction. The proof of Theorem 3.1 is postponed to the Appendix.

This coarse graining describes the local order parameter. Given any $\delta > 0$, one can choose the accuracy ζ of the coarse graining and a scale $K_0(\delta, \beta)$ such that for any mesoscopic $K \geqslant K_0$

$$\lim_{N \to \infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}^{+}\left(\|\mathcal{M}_{N,K} - m_\beta^* u_{N,K}^{\zeta}\|_1 > \delta\right) \; = \; -\infty \,. \tag{3.3}$$

In order to check this assertion, we first notice that the property **(P2)** implies

$$\|\mathcal{M}_{N,K} - m^* u_{N,K}^{\zeta}\|_1 \; \leqslant \zeta + \frac{2K^d}{N^d} \sum_{x \in \widehat{\mathbb{D}}_{N,K}} 1_{u_{N,K}^{\zeta}(x)=0} \,.$$

This estimate combined with Theorem 3.1 leads to

$$\begin{aligned} \mu_{\beta,N}^{+}\left(\|\mathcal{M}_{N,K} - m^* u_{N,K}^{\zeta}\|_1 > \delta\right) &\leqslant \mu_{\beta,N}^{+}\left(\frac{1}{|\widehat{\mathbb{D}}_{N,K}|} \sum_{x \in \widehat{\mathbb{D}}_{N,K}} 1_{u_{N,K}^{\zeta}(x)=0} > \frac{\delta - \zeta}{2}\right) \\ &\leqslant \exp\left(-c_1 \frac{N^d}{K^d} \log\left(\frac{\delta - \zeta}{2\rho_K^{\zeta}}\right)\right) . \end{aligned}$$

According to estimate (3.3) the local averaged magnetization can be controlled by the mesoscopic phase labels and in particular the Proposition 3.1 follows from

Theorem 3.2. *Let β be in $\mathfrak{B}$ and $\zeta > 0$. For every $a > 0$ and $\delta > 0$ there exists a finite scale $K_0(\delta)$, such that for all $K \geqslant K_0$*

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}^{+}\left(u_{N,K}^{\zeta} \notin \mathcal{V}(\mathcal{C}_a, 2\delta)\right) \leqslant \; - c(\beta, K) a \,, \tag{3.4}$$

where $c(\beta, K)$ is a positive constant.

This theorem amounts to control the phase of the small contours, i.e. to prove that the macroscopic phase separation is the optimal scenario to shift the magnetization whereas the creation of many small droplets does not contribute. The above theorem has been derived in [BIV1] in an abstract setting (see also [BBP]). As the coarse graining satisfies the hypotheses of Theorem 2.2.1 of [BIV1], the conclusion follows.

Remark 3.1. *It should be noted that in the proof of* [BIV1] *the BK inequality was invoked but a Peierls inequality for the coarse grained contours is sufficient.*

As noticed in [BIV2], a theorem similar to Theorem 3.2 can also be stated for phase labels taking a finite number of values. Thus, similar results (coarse graining, exponential tightness) should be valid as well for multi-phase models which satisfy Peierls type estimates.

3.2. **Equilibrium crystal shapes.** We are going to prove the analogue of Theorem 2.3 for the Ising model at low temperature. The proof relies on the previous coarse graining. Nevertheless, it also uses model dependent arguments, namely the spin flip symmetry of the Ising model and the existence of the surface tension.

Theorem 3.3. *Let β be in $\mathcal{B}$ and m be in $]-m^*_\beta, m^*_\beta[$. For every $\delta > 0$, one can choose a scale $K_0 = K_0(\beta,\delta)$ such that for any $K \geqslant K_0$*

$$\lim_{N\to\infty} \mu^+_{\beta,N}\left(\inf_{v\in\mathcal{D}_m} \|\mathcal{M}_{N,K} - m^*_\beta v\|_1 \leqslant \delta \;\middle|\; \mathbf{M}_N \leqslant m \right) = 1\,.$$

This theorem holds also for a larger range of temperatures ([CePi1, BIV1]), where $\mathcal{B}$ is the domain of validity of Pisztora's coarse graining [Pi]. It is actually conjectured that $\mathcal{B} =]\beta_c, \infty[$.

The concentration in $\mathbb{L}^1$ of $\mathcal{M}_{N,K}$ to the solutions of the variational problem requires the derivation of precise logarithmic asymptotic. More precisely, we are going to check that the probability that $\mathcal{M}_{N,K}$ is close to a configuration $m^*_\beta v$ (with $v \in \mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\})$) decays exponentially with a surface order $N^{d-1}\mathcal{W}_\beta(v)$.

Proposition 3.2. *Let β be in $\mathfrak{B}$ and let v be in $\mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\})$, then one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$*

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}\big(\|\mathcal{M}_{N,K} - m^*_\beta v\|_1 \leqslant \delta \big) \geqslant -\mathcal{W}_\beta(v) - o(\delta)\,,$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

Proposition 3.3. *Let β be in $\mathfrak{B}$. For all v in $\mathrm{BV}(\widehat{\mathbb{D}}, \{\pm 1\})$ such that $\mathcal{W}_\beta(v)$ is finite, one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$*

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}\big(\|\mathcal{M}_{N,K} - m^*_\beta v\|_1 \leqslant \delta \big) \leqslant -\mathcal{W}_\beta(v) + o(\delta)\,,$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

Theorem 3.3 can be obtained by combining Propositions 3.1, 3.2, 3.3. We recall that $\mathcal{C}_a$ is compact with respect to the $\mathbb{L}^1$ topology (see [EG]). Thus the exponential tightness property 3.1 enables us to focus only on a finite number of configurations close to $\mathcal{C}_a$. The precise asymptotic of these configurations is then estimated by Propositions 3.2, 3.3 (see [B1] for details).

In the following sections, Propositions 3.2, 3.3 are implemented in the framework of nearest neighbor Ising model at low temperature.

3.3. **Lower bound.** In order to derive Proposition 3.2, it is enough to consider the typical spin configurations which contain a microscopic contour in a neighborhood of the boundary of $\partial^* v$.

Step 1 : Approximation procedure.

We first start by approximating the boundary $\partial^* v$ by a regular surface $\partial\widehat{V}$. A polyhedral set has a boundary included in the union of a finite number of hyper-planes. The surface $\partial^* v$ can be approximated as follows (see figure 2)

Theorem 3.4. *For any δ positive, there exists a polyhedral set $\widehat{V}$ such that*

$$\|\mathbb{1}_{\widehat{V}} - v\|_1 \leqslant \delta \qquad \text{and} \qquad \big|\mathcal{W}_\beta(\widehat{V}) - \mathcal{W}_\beta(v)\big| \leqslant \delta.$$

For any h small enough there are ℓ disjoint parallelepipeds $\widehat{R}^1, \dots, \widehat{R}^\ell$ with basis $\widehat{B}^1, \dots, \widehat{B}^\ell$ included in $\partial\widehat{V}$ of side length h and height δh. Furthermore, the sets $\widehat{B}^1, \dots, \widehat{B}^\ell$ cover $\partial\widehat{V}$ up to a set of measure less than δ denoted by $\widehat{U}^\delta = \partial\widehat{V} \setminus \bigcup_{i=1}^\ell \widehat{B}^i$ and they satisfy

$$\Big|\sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - \mathcal{W}_\beta(v)\Big| \leqslant \delta,$$

where the normal to $\widehat{B}^i$ is denoted by $\vec{n}_i$.

The proof is a direct application of Reshtnyak's Theorem and can be found in the paper of Alberti, Bellettini [AlBe].

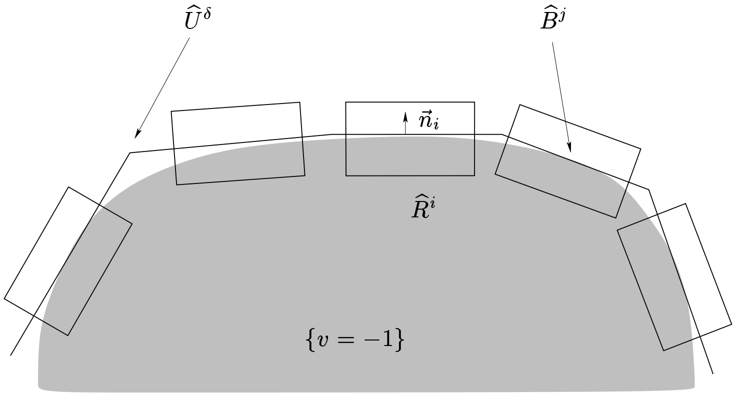

Figure 2. Polyhedral approximation.

Using Theorem 3.4, we can reduce the proof of Proposition 3.2 to the computation of the probability of $\{\|\mathcal{M}_{N,K} - m^*\mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\}$. According to (3.3) the estimates can be restated in terms of the mesoscopic phase labels. It will be enough to show that : for any $\delta > 0$, there exists $\zeta = \zeta(\delta)$ and $K_0(\delta)$ such that for all $K \geqslant K_0$

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}\Big(\|u^\zeta_{N,K} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\Big) \geqslant -\mathcal{W}_\beta(\widehat{V}) - o(\delta), \tag{3.5}$$

where the function $o(\delta)$ vanishes as δ goes to 0.

Step 2 : Localization of the interface.

In order to impose the phase segregation, we will enforce the occurrence of a microscopic interface along the boundary $\partial\widehat{V}$. The images of $\widehat{V}$, $\widehat{R}^i$ and $\widehat{U}^\delta$ in $\mathbb{D}_N$ will be denoted by V_N, R^i_N and U^δ_N. We split R^i_N into $R^{i,-}_N$ and $R^{i,+}_N$ which are the microscopic counterparts of $\widehat{V} \cap \widehat{R}^i$ and $\widehat{R}^i \setminus \widehat{V}$.

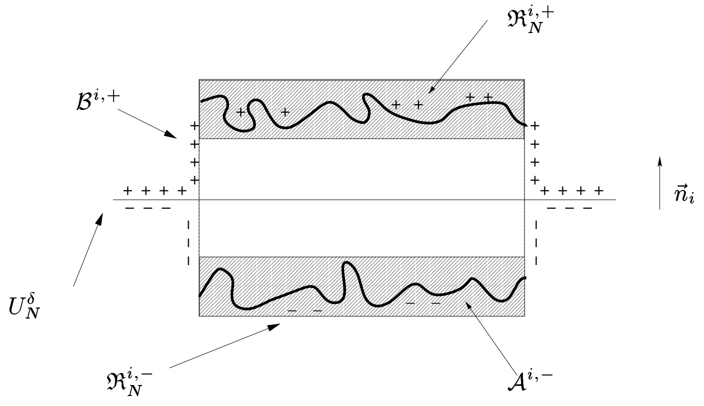

FIGURE 3. Microscopic interface localized close to B^i.

First, we impose the microscopic constraint that there is a $+$ surface of spins in the upper half of $R^{i,+}_N$ (see figure 3). Let $\widehat{Q}^i$ be the parallelepiped included in $\widehat{R}^i$ with basis $\widehat{B}^i$ and height $\frac{\delta h}{2}$. We define the microscopic region $\mathfrak{R}^{i,+}_N$ as $R^{i,+}_N \setminus Q^i_N$. Let $\mathcal{A}^{i,+}$ be the event that there is a surface of spins equal to 1 crossing $\mathfrak{R}^{i,+}_N$ in the direction orthogonal to $\vec{n}_i$. In the same way, we introduce the set $\mathfrak{R}^{i,-}_N$ and the event $\mathcal{A}^{i,-}$ such that there is a crossing surface of spins equal to -1. We set $\mathcal{A} = \bigcap_{i=1}^{\ell} \mathcal{A}^{i,+} \cap \mathcal{A}^{i,-}$.

Finally, we define $\mathcal{B}^{i,+}$ (resp $\mathcal{B}^{i,-}$) the set of spin configurations such that the spins are equal to 1 (resp -1) on the sides of $R^{i,+}_N$ (resp $R^{i,-}_N$) parallel to $\vec{n}_i$. In order to construct a closed contour of spins surrounding V_N, we define $\mathcal{B}$ as the set of configurations in $\mathcal{B}^{i,+}$ and $\mathcal{B}^{i,-}$ such that the spins on one side of U^δ_N are $-$ and $+$ in the other side (see figure 3).

Any spin configuration in $\mathcal{A} \cap \mathcal{B}$ contains a microscopic interface which decouples V_N from its complement. One has

$$\mu^+_{\beta,N}\left(\|u^\zeta_{N,K} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\right) \geqslant \mu^+_{\beta,N}\left(\left\{\|u^\zeta_{N,K} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\right\} \cap \mathcal{A} \cap \mathcal{B}\right). \tag{3.6}$$

The spin configurations inside V_N (resp V^c_N) are surrounded by $-$ (resp $+$) boundary conditions, so that they are in equilibrium in the $-$ (resp $+$) pure phase. A proof similar to the one of Theorem 3.2 implies that one can choose h small enough, $\zeta' = \zeta'(\delta)$ and $K'_0 = K'_0(\delta)$ such that

$$\lim_{N\to\infty} \mu^+_{\beta,N}\left(\int_{\widehat{V}^c} |u^{\zeta'}_{N,K}(x) - 1|\, dx \geqslant \frac{\delta}{2} \text{ or } \int_{\widehat{V}} |u^{\zeta'}_{N,K}(x) + 1|\, dx \geqslant \frac{\delta}{2} \;\Big|\; \mathcal{A} \cap \mathcal{B}\right) = 0\,,$$

So that (3.6) can be rewritten for N large enough as

$$\mu^+_{\beta,N}\left(\|u^{\zeta'}_{N,K} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\right) \geqslant \frac{1}{2}\mu^+_{\beta,N}(\mathcal{A} \cap \mathcal{B})\,. \tag{3.7}$$

Step 3 : Surface tension.

The configurations in the event $\mathcal{A} \cap \mathcal{B}$ contain $+$ and $-$ closed surfaces which split the domain $\mathbb{D}_N$ into 3 regions. Let Λ^- be the region surrounded by the $-$ surface, Λ^+ be the region outside the $+$ surface. To ensure the uniqueness of the decomposition, we choose the $+$ surface (resp $-$) in each $\mathfrak{R}_N^{i,+}$ (resp $\mathfrak{R}_N^{i,-}$) as the closest surface from $(R_N^i)^c$. By definition, the configurations inside Λ^- (resp Λ^+) are surrounded by $-$ (resp $+$) boundary conditions. Finally $\Delta = \mathbb{D}_N \setminus \Lambda^+ \cup \Lambda^-$.

We proceed now to evaluate the RHS of (3.7)

$$\mu_{\beta,N}^+ (\mathcal{A} \cap \mathcal{B}) \geqslant \frac{1}{Z_N^+} \sum_{\Lambda^+,\Lambda^-} Z_{\Lambda^+}^+ Z_{\Lambda^-}^- Z_\Delta^{+,-} ,$$

where Z_D^ω denotes the partition function on the domain D with boundary conditions ω.

By using the spin flip symmetry we get

$$\mu_{\beta,N}^+ (\mathcal{A} \cap \mathcal{B}) \geqslant \min_\Delta \left(\frac{Z_\Delta^{+,-}}{Z_\Delta^{+,+}} \right) \frac{1}{Z_N^+} \sum_{\Lambda^+,\Lambda^-} Z_{\Lambda^+}^+ Z_{\Lambda^-}^+ Z_\Delta^{+,+} . \tag{3.8}$$

The minimum is taken over the sets Δ which can be obtained by the previous construction. In particular, Δ should contain the union of disjoint domains Δ^i such that $Q_N^i \subset \Delta^i \subset R_N^i$. It remains to evaluate the two terms in the RHS.

The partition function $Z_\Delta^{+,-}$ takes into account the interaction on both sides of U_N^δ, as well as inside each Δ^i. As the former interaction involves only the spins in a neighborhood of U_N^δ, we obtain

$$\min_\Delta \frac{Z_\Delta^{+,-}}{Z_\Delta^{+,+}} \geqslant \exp\big(-\beta o(\delta) N^{d-1} \big) \prod_{i=1}^{\ell} \min_{\Delta^i} \frac{Z_{\Delta^i}^{+,-}}{Z_{\Delta^i}^{+,+}} ,$$

where the minimum is taken over the sets Δ^i such that $Q_N^i \subset \Delta^i \subset R_N^i$. Using the definition (2.2), we know that

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \min_{\Delta^i} \left(\log \frac{Z_{\Delta^i}^{+,-}}{Z_{\Delta^i}^{+,+}} \right) \geqslant - \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} . \tag{3.9}$$

We used the fact that Δ^i increases as N diverges.

In the last term in the RHS of (3.8), the constraint on the spins along the set U_N^δ and on the sides of R_N^i parallel to $\vec{n}_i$ can be released up to a small cost wrt the surface order. This comes from the fact that the event $\mathcal{B}$ is supported by at most $c(d,\delta)N^{d-1}$ edges where $c(d,\delta)$ vanishes as δ goes to 0. Therefore the probability of $\mathcal{B}$ is negligible with respect to a surface order.

$$\frac{1}{Z_N^+} \sum_{\Lambda^+,\Lambda^-} Z_{\Lambda^+}^+ Z_{\Lambda^-}^+ Z_\Delta^{+,+} \geqslant \exp\big(-\beta o(\delta) N^{d-1} \big) \mu_{\beta,N}^+ (\tilde{\mathcal{A}}) ,$$

where $\tilde{\mathcal{A}}$ is the event deduced from $\mathcal{A}$ by spin flip symmetry, i.e. such that there are in each set $\mathfrak{R}_N^{i,+}$ and $\mathfrak{R}_N^{i,-}$ a $+$ surface of spins. Using the Peierls argument, we conclude that $\mu_{\beta,N}^+(\tilde{\mathcal{A}})$ is uniformly bounded from below in N.

Thus inequalities (3.7), (3.8), (3.9) imply

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}(\mathcal{A}\cap\mathcal{B}) \geqslant -\sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}^{(d-1)}_x - o(\delta)\,. \tag{3.10}$$

This concludes the proof of Proposition 3.2.

3.4. **Upper bound.** The proof of Proposition 3.3 follows the general scheme of the $\mathbb{L}^1$ Theory. The different steps will be recalled and modified in order to use only the spin representation. First the boundary $\partial^* v$ is approximated; this enables us to reduce the proof to local computations in small regions. Then in each region we localize the interface on the mesoscopic level by using the minimal section argument. Finally in the last step, the surface tension factor is computed.

Step 1 : Approximation procedure.

We approximate $\partial^* v$ with a finite number of parallelepipeds.

Theorem 3.5. *For any δ positive, there exists h positive such that there are ℓ disjoint parallelepipeds $\widehat{R}^1,\dots,\widehat{R}^\ell$ included in $\widehat{\mathbb{D}}$ with basis $\widehat{B}^1,\dots,\widehat{B}^\ell$ of size h and height δh. The basis $\widehat{B}^i$ divides $\widehat{R}^i$ in 2 parallelepipeds $\widehat{R}^{i,+}$ and $\widehat{R}^{i,-}$ and the normal to $\widehat{B}^i$ is denoted by $\vec{n}_i$. Furthermore, the parallelepipeds satisfy the following properties*

$$\int_{\widehat{R}^i} |\mathcal{X}_{\widehat{R}^i}(x) - v(x)|\, dx \leqslant \delta\, vol(\widehat{R}^i) \quad \text{and} \quad \Big|\sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}^{(d-1)}_x - \mathcal{W}_\beta(v)\Big| \leqslant \delta,$$

where $\mathcal{X}_{\widehat{R}^i} = 1_{\widehat{R}^{i,+}} - 1_{\widehat{R}^{i,-}}$ and the volume of $\widehat{R}^i$ is $vol(\widehat{R}^i) = \delta h^d$.

The proof follows from standard arguments of geometric measure theory (see for example [Ce, B1]). Theorem 3.5 enables us to decompose the boundary into regular sets (see figure 4) so that it will be enough to consider events of the type

$$\left\{ \frac{1}{m^*}\mathcal{M}_{N,K} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i)) \right\},$$

where $\mathcal{V}(\widehat{R}^i,\varepsilon)$ is the ε-neighborhood of $\mathcal{X}_{\widehat{R}^i}$

$$\mathcal{V}(\widehat{R}^i,\varepsilon) = \left\{ v' \in \mathbb{L}^1(\widehat{\mathbb{D}}) \;\Big|\; \int_{\widehat{R}^i} |v'(x) - \mathcal{X}_{\widehat{R}^i}(x)|\, dx \leqslant \varepsilon \right\}.$$

According to (3.3), the local averaged magnetization can be replaced by the mesoscopic phase labels. Therefore Proposition 3.3 is equivalent to the following statement : for any δ positive, there exists $K_0 = K_0(\delta,h)$, $\zeta_0 = \zeta_0(\delta,h)$ such that uniformly in $K \geqslant K_0$, $\zeta \leqslant \zeta_0$

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}\left(u^\zeta_{N,K} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta\mathrm{vol}(\widehat{R}^i)) \right) \leqslant -\mathcal{W}_\beta(v) + C(\beta,v)\delta. \tag{3.11}$$

Step 2 : Minimal section argument.

Following, the notation of the subsection 3.3, we consider the partition $(R^{i,+}_N, R^{i,-}_N)$ of R^i_N. At a given mesoscopic scale K, we associate to any spins configuration the set of *bad* boxes which are the boxes $\mathbb{B}_K$ intersecting R^i_N with label 0 and the ones intersecting $R^{i,+}_N$

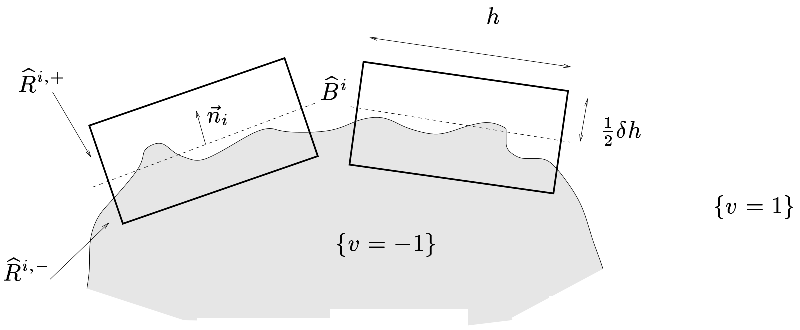

FIGURE 4. Approximation by parallelepipeds.

(resp $R_N^{i,-}$) labeled by -1 (resp 1). For any integer j, we set $\widehat{B}^{i,j} = \widehat{B}^i + j\, c_d \frac{N}{K} \vec{n}_i$ and define

$$B_N^{i,j} = \big\{ j' \in R_N^i \mid \exists x \in \widehat{B}^{i,j}, \qquad \|j' - Nx\|_1 \leqslant 10 \big\}.$$

The sections $\mathcal{B}_j^i$ of the parallelepiped R_N^i are defined as the smallest connected set of boxes $\mathbb{B}_K$ intersecting $B_N^{i,j}$. The parameter c_d is chosen such that the $\mathcal{B}_j^i$ are disjoint surfaces of boxes. For j positive, let $n_i^+(j)$ be the number of *bad* boxes in $\mathcal{B}_j^i$ and define

$$n_i^+ = \min \left\{ n_i^+(j) : \qquad 0 < j < \frac{\delta h}{2 c_d} \frac{K}{N} \right\}.$$

Call j^+ the smallest location where the minimum is achieved and define the minimal section in $R_N^{i,+}$as $\mathcal{B}_{j^+}^i$. For j negative, we denote by $\mathcal{B}_{j^-}^i$ the minimal section in $R_N^{i,-}$ and n_i^- the number of *bad* boxes in $\mathcal{B}_{j^-}^i$ (see figure 5).

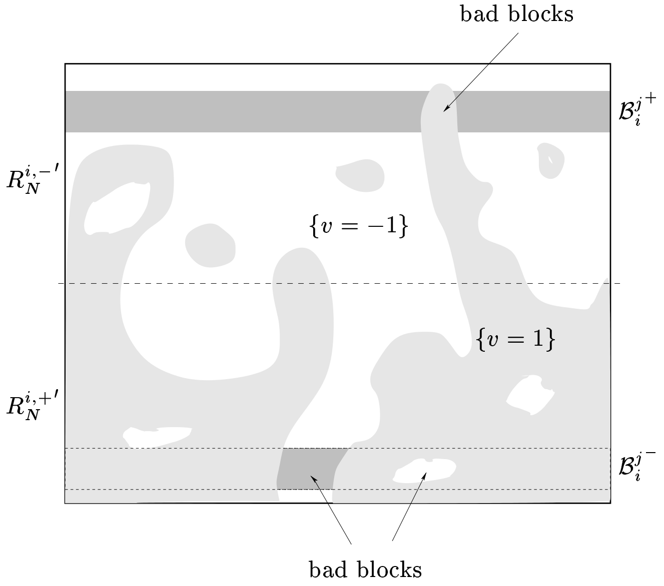

FIGURE 5. Minimal sections.

For any spins configuration such that $u_{N,K}^{\zeta}$ belongs to $\bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))$, the number of *bad* boxes in a minimal section is bounded by

$$n_i^+ + n_i^- \leqslant \delta \mathrm{vol}(\widehat{R}^i) \frac{2c_d}{\delta h} \left(\frac{N}{K}\right)^{d-1} \leqslant 2c_d \delta h^{d-1} \left(\frac{N}{K}\right)^{d-1} .$$

As $\sum_{i=1}^{\ell} |\widehat{B}^i| = \ell h^{d-1}$ can be controlled in terms of the perimeter of $\partial^* v$, the total number of *bad* boxes is bounded by

$$\sum_{i=1}^{\ell} n_i^+ + n_i^- \leqslant \delta\, C(v) \left(\frac{N}{K}\right)^{d-1} . \tag{3.12}$$

The previous estimate implies that a mesoscopic interface is mainly located between the two minimal sections and that the fluctuations of the interface necessarily intersect the *bad* boxes. Using this mesoscopic information, we are going to deduce the existence of a microscopic interface crossing R_N^i in the direction orthogonal to $\vec{n}_i$. Once the microscopic interface is localized, it will be possible to extract the surface tension factor (see Step 3). In order to achieve the localization, the previous proofs for Ising model [B1, CePi1] used the properties of the FK representation and of Pisztora's coarse graining [Pi]. In the low temperature regime, the proof can be simplified thanks to the coarse graining introduced previously which provides a direct correspondence between the microscopic and the mesoscopic scales.

From the very construction of the coarse graining, the + spin surfaces associated to overlapping boxes with label 1 are connected. As each minimal section contains mainly + good or − good blocks, there exist, in each minimal section, surfaces of + and − spins which almost cross R_N^i. This implies that a microscopic interface is sandwiched between the two minimal sections and that the large fluctuations of this interface can occur only through the *bad* boxes. By modifying the spins configurations σ on the *bad* boxes, we will localize the interface between the minimal sections.

More precisely, we associate to any configuration σ the configuration $\bar{\sigma}$ with spins equal to + (resp −) on the boundary of each *bad* box in the minimal section $\mathcal{B}_{j^+}^i$ (resp $\mathcal{B}_{j^-}^i$) and equal to σ otherwise. Let $\mathcal{A}$ be the image of the set

$$\left\{ \sigma \ \middle| \ u_{N,K}^{\zeta}(\sigma) \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i)) \right\}$$

by this mapping. Inequality (3.12) implies that σ and $\bar{\sigma}$ differ on at most $\delta C_1(v) N^{d-1}$ sites, so that the cost of this surgical procedure can be estimated from above

$$\mu_{\beta,N}^+ \left(u_{N,K}^{\zeta} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i)) \right) \leqslant \exp\left(o(\delta)\, C_2(v,\beta) N^{d-1}\right) \mu_{\beta,N}^+(\mathcal{A}) . \tag{3.13}$$

The upper bound takes into account the energetic factor necessary to flip some spins by force, as well as the combinatorial factor which stems from the choice of the minimal sections. In a given box, there are less than N possibilities to locate the height of a minimal section. Once this height is fixed, the number of configurations with different locations of bad boxes is bounded by

$$\binom{(hN/K)^{d-1}}{\delta (hN/K)^{d-1}} \leqslant \exp\left((hN/K)^{d-1} \big(\delta \log \delta + (1-\delta) \log(1-\delta)\big) \right) .$$

Summing over all the configurations leads to the upper bound (3.13).

Remark 3.2. *In the case of Kac Ising model, it is no longer possible to localize a microscopic interface. Therefore a more delicate analysis of surface tension is required (see Section 4).*

Step 3 : Surface tension estimates.

As a consequence of the previous step, any spin configuration in $\mathcal{A}$ contains microscopic contours which cross each cube R_N^i. The statistical weight of such contours will be estimated by the surface tension factor.

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu^+_{\beta,N}(\mathcal{A}) \leqslant -\sum_{i=1}^{\ell} \int_{\widehat{B}_i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x + C_4(\beta,v)\delta. \tag{3.14}$$

Combining the previous inequality with (3.13), we deduce (3.11). We now proceed in deriving (3.14).

Let $\partial^{\mathrm{top}} R_N^i$ and $\partial^{\mathrm{bot}} R_N^i$ be the two faces of R_N^i orthogonal to the vector $\vec{n}_i$. The face contiguous to $R_N^{i,+}$ will be denoted by $\partial^{\mathrm{top}} R_N^i$. For any spins configuration in $\mathcal{A}$, we define the set of sites $\gamma^{i,+}$ as the support of the $+$ crossing surface in $R_N^{i,+}$ which is the closest to $\partial^{\mathrm{top}} R_N^i$. In the same way, $\gamma^{i,-}$ is the location of the surface of $-$ spins crossing $R_N^{i,-}$ which is the closest to $\partial^{\mathrm{bot}} R_N^i$. Contrary to the proof of the lower bound, the localization of the interfaces has been deduced from the $\mathbb{L}^1$ constraint. As a consequence of this, we have no control of the configurations outside the sets $(R_N^i)_{i \leqslant \ell}$. This prevents us to conclude that all the crossing interfaces merge into contours surrounding the connected components of v as it was the case for the lower bound (see figure 6).

We first pin the interfaces on the sides of each R_N^i by imposing that the spins on the boundary of each $R_N^{i,+}$ (resp $R_N^{i,-}$) parallel to $\vec{n}_i$ are equal to 1 (resp -1). Since the height of R_N^i is δh, this procedure requires to flip at most $\delta h^{d-1} N^{d-1}$ spins. Therefore this has no further impact on the evaluation of the statistical weights of the configurations because the cost of flipping these spins is bounded by $\exp(\delta C(v)\, N^{d-1})$.

Let $\{\Gamma_j\}_{j \leqslant k}$ (with $k \leqslant \ell$) be the collection of the contours obtained by extending the microscopic interfaces outside each R_N^i (see figure 6). To each Γ_j, we associate two sets of sites (Γ_j^+, Γ_j^-) defined as follows : If Γ_j crosses the set R_N^i then the restriction of Γ_j^+ (resp Γ_j^-) inside R_N^i coincide with $\gamma^{i,+}$ (resp $\gamma^{i,-}$). Outside $\cup_{i=1}^{\ell} R_N^i$, the sets $\Gamma_j^\pm$ contain the sites which support the $\pm$ spins of the contour Γ_j. Thus (Γ_j^+, Γ_j^-) should be interpreted as a modified "contour" inflated in the regions R_N^i. Let Δ_N^i be the domain in R_N^i between $\gamma^{i,+}, \gamma^{i,-}$.

For simplicity, we first consider the case of a single contour Γ_1. We suppose that Γ_1^+ surrounds Γ_1^- and denote by $\mathrm{ext}(\Gamma_1^+)$ the exterior of Γ_1^+ and by $\mathrm{int}(\Gamma_1^-)$ the interior of Γ_1^- (see figure 6). In this way, the domain $\mathbb{D}_N$ is partionned by (Γ_1^+, Γ_1^-) into

$$\mathbb{D}_N = \mathrm{ext}(\Gamma_1^+) \bigvee \mathrm{int}(\Gamma_1^-) \bigvee_{i=1}^{\ell} \Delta_N^i\,.$$

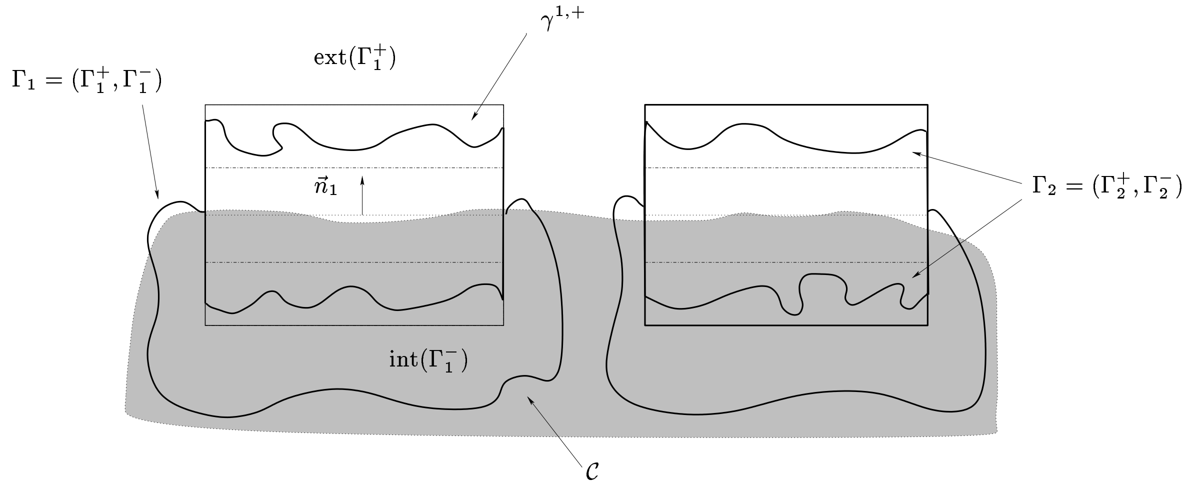

FIGURE 6. Decomposition into contours.

We are going to derive an upper bound for the probability of the configurations compatible with a given set (Γ_1^+, Γ_1^-)

$$\mu^+_{\beta,N}(\Gamma_1^+, \Gamma_1^-) = \frac{1}{Z^+_{\beta,N}} Z^+_{\mathrm{ext}(\Gamma_1^+)} \prod_{i=1}^{\ell} Z^{\pm}_{\Delta_N^i} Z^-_{\mathrm{int}(\Gamma_1^-)} \exp(-\beta|\Gamma_1|),$$

where the length of the contour Γ_1 outside the set $\cup_{i=1}^{\ell} R_N^i$ is denoted by $|\Gamma_1|$. The last term in the RHS takes into account the interaction across Γ_1 outside the sets R_N^i. Using the spin flip symmetry, we see that

$$\mu^+_{\beta,N}(\Gamma_1^+, \Gamma_1^-) = \frac{Z^+_{\mathrm{ext}(\Gamma_1^+)} \prod_{i=1}^{\ell} Z^+_{\Delta_N^i} Z^+_{\mathrm{int}(\Gamma_1^-)}}{Z^+_{\beta,N}} \exp(-\beta|\Gamma_1|) \left(\prod_{i=1}^{\ell} \frac{Z^{\pm}_{\Delta_N^i}}{Z^+_{\Delta_N^i}} \right). \tag{3.15}$$

Let $\tilde{\Gamma}_1$ be the set of spin configurations deduced from the ones compatible with (Γ_1^+, Γ_1^-) by flipping the spins inside $\mathrm{int}(\Gamma_1^-)$. Equation (3.15) can be rewritten as follows

$$\mu^+_{\beta,N}(\Gamma_1^+, \Gamma_1^-) = \mu^+_{\beta,N}(\tilde{\Gamma}_1) \exp(-\beta|\Gamma_1|) \left(\prod_{i=1}^{\ell} \frac{Z^{\pm}_{\Delta_N^i}}{Z^+_{\Delta_N^i}} \right).$$

By the definition of the surface tension, the last term in the RHS is bounded by

$$\prod_{i=1}^{\ell} \frac{Z^{\pm}_{\Delta_N^i}}{Z^+_{\Delta_N^i}} \leqslant \exp\left(-N^{d-1} \left[\sum_{i=1}^{\ell} \int_{\widehat{B}_i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x + o(N) \right] \right). \tag{3.16}$$

Notice that in the particular case of the Ising model, the error term $o(N)$ does not exists (FKG property).

It remains to check that

$$\sum_{(\Gamma_1^+, \Gamma_1^-)} \mu^+_{\beta,N}(\tilde{\Gamma}_1) \exp(-\beta|\Gamma_1|) \leqslant C(\beta)^{\ell}, \tag{3.17}$$

where the summation is taken over the couples (Γ_1^+,Γ_1^-) compatible with the previous construction. As the spin flip correspondence $(\Gamma_1^+,\Gamma_1^-) \Rightarrow \tilde\Gamma_1$ is not a one-to-one the evaluation of (3.17) requires some care.

First we consider the restriction of the spins configurations in each R_N^i, in this case, the correspondence is a one-to-one because the spin flip occurs in regions delimited by $\gamma^{i,+},\gamma^{i,-}$ which are defined unambiguously. For a fixed spin configuration in $\cup_{i=1}^\ell R_N^i$, we sum over the contours Γ_1 attached to the boxes R_N^i. In this case the Peierls argument is fully effective : by construction Γ_1 stems from the boundary of the boxes R_N^i which means that the contour is localized in space. Therefore the term involving $|\Gamma_1|$, i.e. the area of the contour outside $\cup_{i=1}^\ell R_N^i$, is summable.

More precisely, for any σ compatible with (Γ_1^+,Γ_1^-), we write $\sigma \rightsquigarrow (\Gamma_1^+,\Gamma_1^-)$. The image of σ by the previous mapping will be denoted by $\tilde\sigma$. Let $\mathcal{R}_N = \cup_{i=1}^\ell R_N^i$ and set $\sigma_{\mathcal{R}_N}$ (resp $\sigma_{\mathcal{R}_N^c}$) the restriction of σ on $\mathcal{R}_N$ (resp $\mathcal{R}_N^c$). For simplicity, γ will denote the restriction of (Γ_1^+,Γ_1^-) on $\mathcal{R}_N$ and $\mathcal{C}$ the restriction on $\mathcal{R}_N^c$.

$$\sum_{(\Gamma_1^+,\Gamma_1^-)} \mu^+_{\beta,N}(\tilde\Gamma_1)\exp(-\beta|\Gamma_1|) = \sum_{(\Gamma_1^+,\Gamma_1^-)}\ \sum_{\sigma\rightsquigarrow(\Gamma_1^+,\Gamma_1^-)} \mu^+_{\beta,N}(\tilde\sigma)\exp(-\beta|\Gamma_1|)\,,$$

$$\leqslant \sum_{\mathcal{C}}\left(\sum_{\sigma_{\mathcal{R}_N^c}\rightsquigarrow\mathcal{C}}\ \sum_{\gamma}\ \sum_{\sigma_{\mathcal{R}_N}\rightsquigarrow\gamma}\mu^+_{\beta,N}(\tilde\sigma)\right)\exp(-\beta|\Gamma_1|)\,.$$

From the one-to-one correspondence on $\mathcal{R}_N$

$$\sum_{\sigma_{\mathcal{R}_N^c}\rightsquigarrow\mathcal{C}}\ \sum_{\gamma}\ \sum_{\sigma_{\mathcal{R}_N}\rightsquigarrow\gamma}\mu^+_{\beta,N}(\tilde\sigma)\leqslant 1\,.$$

Thus

$$\sum_{(\Gamma_1^+,\Gamma_1^-)} \mu^+_{\beta,N}(\tilde\Gamma_1)\exp(-\beta|\Gamma_1|) \leqslant \sum_{\mathcal{C}}\exp(-\beta|\Gamma_1|)\leqslant C_\beta^\ell\,.$$

The last bound follows from the Peierls argument and the fact that the contour $\mathcal{C}$ can be divided in at most ℓ interfaces.

If the contour Γ_1 is not unique, similar estimates can be performed for each contour. Indeed, up to a combinatorial factor denoted α_ℓ, we associate to each of the k contours the regions R_N^i it will cross. Then we perform the estimates independently and obtain an upper bound depending only on β and ℓ : $\alpha_\ell(C_\beta)^\ell$. After renormalization, this factor vanishes in the thermodynamic limit. Combining this final estimate with (3.17), we deduce (3.14).

4. KAC ISING MODEL

The previous strategy will be adapted to derive the phase coexistence in the Kac Ising model. This requires to set up a multiscale analysis where the spin variables are replaced by coarse grained variables. The proof relies heavily on the model-dependent estimates derived in [CaPr, BMP].

4.1. **Coarse graining.** The long range structure of the Kac Ising model prevents us from using directly the coarse grained estimates of subsection 3.1.

In a first step, the system is renormalized on a scale proportional to the range of the interaction γ^{-1}. On this scale, the renormalized system behaves as an Ising model with effective inverse temperature proportional to γ^{-d}. Thus, these estimates enables us to implement a second renormalization similar to the one introduced in subsection 3.1.

It should be stressed that the difficult estimates to construct this coarse graining were already obtained in [CaPr, BMP].

Level 1.

We introduce two intermediate scales depending on the range of interaction $\gamma^{-1} = 2^g$. Let $L = 2^{-\ell}\gamma^{-1} = 2^{g-\ell}$ and $H = 2^h\gamma^{-1} = 2^{g+h}$, where $\ell, h \in \mathbb{N}$. In the following ℓ will be fixed and γ will go to 0 so that the condition $\ell \leqslant g$ will be always satisfied. By construction each box $\mathbb{B}_H$ is partionned into smaller boxes $\mathbb{B}_L$.

The averaged magnetization in the box $\mathbb{B}_L(x)$ centered in $x \in \mathbb{D}_{N,L}$ was introduced in (2.7) and denoted by $\mathcal{M}_{N,L}(x)$. Given $\varepsilon > 0$ and $\gamma > 0$, the phase labels $\eta^\varepsilon_{N,H}$ at the mesoscopic scale H are defined as follows : the box $\mathbb{B}_H(x)$ centered in $x \in \mathbb{D}_{N,H}$ is labelled by

$$\eta^\varepsilon_{N,H}(x) = \begin{cases} \pm 1, & \text{if } |\mathcal{M}_{N,L}(y) \mp m^*_\beta| \leqslant \varepsilon, \quad \forall \mathbb{B}_L(y) \subset \mathbb{B}_H(x)\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

The parameter ε quantifies the deviation of the averaged magnetization from the mean field magnetization m^*_β (and not $m^*_{\beta,\gamma}$ (2.3)).

Following [CaPr, BMP], we introduce the renormalized notions of boxes, contours and boundary conditions at the mesoscopic scale H.

- A box $\mathbb{B}_H(x)$ is *correct* if $\eta^\varepsilon_{N,H}(x) = \pm 1$ and if all the $\star$-neighboring boxes have the same labels as $\mathbb{B}_H(x)$.
- The support of a contour is a maximal $\star$-connected component of the *incorrect* boxes. A contour is defined by its support and by the specification of the phase labels on the support.
- Contrary to the low temperature Ising model, we are going to consider boundary conditions which are only statistically pure. A region Λ (measurable wrt the partition at the scale H) has + boundary conditions in averaged if the phase labels $\eta^{\varepsilon/2}_{N,H}$ associated to σ_{Λ^c} are uniformly equal to 1. This set of boundary conditions is denoted by $G^{+,\varepsilon}(\Lambda) \subset [-1,1]^{\Lambda^c}$. In the same way, the set of $-$ boundary conditions in averaged is denoted by $G^{-,\varepsilon}(\Lambda)$.

For the renormalized contours, a Peierls estimate of the type (2.8) has been proven in [CaPr]

Theorem 4.1. *There exist functions $\varepsilon^*(\beta) > 0$, $\ell(\beta,\varepsilon) \in \mathbb{N}, h(\beta,\varepsilon) \in \mathbb{N}$, $c(\beta,\varepsilon) > 0$, $c'(\beta,\varepsilon) > 0$ such that the following holds. Let $\gamma > 0, \beta > 1$ and $\varepsilon < \varepsilon^*$. Then for any region Λ (measurable wrt the partition on the H scale) and any generalized contour Γ in Λ*

$$\forall \sigma_{\Lambda^c} \in G^{+,\varepsilon}(\Lambda), \qquad \mu_{\beta,\gamma,\Lambda}\left(\Gamma \,\middle|\, \sigma_{\Lambda^c}\right) \leqslant c'(\beta,\varepsilon) \exp\left(-\frac{c(\beta,\varepsilon)}{\gamma^d}|\Gamma|\right), \tag{4.1}$$

where $|\Gamma|$ denotes the number of boxes in the contour Γ.

After the renormalization procedure, the effective temperature becomes $\frac{c(\beta,\varepsilon)}{\gamma^d}$. Therefore, for γ small enough, the coarse grained system behaves as an Ising model in the low temperature regime.

A more precise description of the pure phases has been derived in [BMP] (Theorem 2.4) : the Gibbs measures with almost pure boundary conditions relax exponentially fast to the corresponding pure phase.

Theorem 4.2. [BMP] *For any $\beta > 1$, $\forall \varepsilon \in]0, \varepsilon^*(\beta)[$ there exists $\gamma(\beta, \varepsilon) > 0, c(\varepsilon) > 0$ such that the following holds. For any subset Δ of $\mathbb{Z}^d$, for any $\mathcal{K} \subset \Delta$, one has $\forall \gamma \leqslant \gamma(\beta,\varepsilon), \forall \sigma_{\Delta^c} \in G^{+,\varepsilon}(\Delta)$*

$$\left| \mu_{\beta,\gamma,\Delta}\big(\sigma_{\mathcal{K}} \big| \sigma_{\Delta^c}\big) - \mu^+_{\beta,\gamma}(\sigma_{\mathcal{K}}) \right| \leqslant |\mathcal{K}| \, \exp\big(- c\gamma^2 \, dist(\mathcal{K}, \Delta^c)\big) \, .$$

Level 2.

The previous estimates are the building blocks for the next renormalization step. The phase labels $\eta^{\varepsilon}_{N,H}$ quantify only the deviations of the averaged magnetization from m^*_β instead of $m^*_{\beta,\gamma}$. Furthermore, the control of these deviations (4.1) induces a dependency between the accuracy ε and the range of the interaction γ^{-1}.

By analogy with Subsection 3.1, we will define a coarse graining at the mesoscopic scale $K \gg H$. The phase labels $\eta^{\varepsilon}_{N,H}$ are going to play the role of the spins. This will enables us to strengthen the control on the deviation of the magnetization in K-boxes from its expected value $m^*_{\beta,\gamma}$ with an arbitrary precision.

For $\beta > 1$, the parameters γ, ε, L and H are fixed such that Theorems 4.1 and 4.2 hold and that the effective temperature is low enough. The parameter $\zeta > 0$ will control the accuracy of the coarse graining at the scale $K = 2^k H$.

Let x be in $\widehat{\mathbb{D}}_{N,K}$ and α be in $(0,1)$. The box $\widehat{\mathbb{B}}_{N,K}(x)$ is + *good* if the spin configuration inside the enlarged box of side length $K + 2^{\alpha k} H$ is typical, i.e.

(P1) The box $\mathbb{B}_K(x)$ is surrounded by a surface of boxes $\mathbb{B}_H$ in $\partial \mathbb{B}_K(x)$ with mesoscopic phase labels $\eta^{\varepsilon}_{N,H}$ uniformly equal to 1.

(P2) The averaged magnetization $\mathcal{M}_{N,K}(x)$ inside $\mathbb{B}_K(x)$ is close to the equilibrium value $m^*_{\beta,\gamma}$

$$\left| \mathcal{M}_{N,K}(x) - m^*_{\beta,\gamma} \right| \leqslant \zeta \, . \tag{4.2}$$

The $-$ *good* boxes are defined in the same way.

On the mesoscopic level, each box $\widehat{\mathbb{B}}_{N,K}(x)$ is labelled by a mesoscopic phase label

$$\forall x \in \widehat{\mathbb{D}}_{N,K}, \qquad u^{\zeta}_{N,K}(x) = \begin{cases} \pm 1, & \text{if } \widehat{\mathbb{B}}_{N,K}(x) \text{ is } \pm \textit{ good }, \\ 0, & \text{otherwise.} \end{cases}$$

The typical behavior follows from

Theorem 4.3. *Let $\beta > 1$ and fix $\gamma_\beta > 0$ and $\varepsilon > 0$ such that the Theorems 4.1 and 4.2 are satisfied. For any $\gamma < \gamma(\beta, \varepsilon)$, $\zeta > 0$, the following holds uniformly for domains Λ (measurable wrt K-partitions) and for any $\sigma_{\Lambda^c} \in G^{+,\varepsilon}(\Lambda)$*

$$\forall \{x_1, \dots, x_\ell\}, \quad \mu_{\beta,\gamma,\Lambda}\left(u^{\zeta}_{N,K}(x_1) = 0, \dots, u^{\zeta}_{N,K}(x_\ell) = 0 \,\big|\, \sigma_{\Lambda^c} \right) \leqslant \left(\rho^{\zeta}_{\gamma,K}\right)^{\ell}, \tag{4.3}$$

where the parameter $\rho^{\zeta}_{\gamma,K}$ vanishes as K goes to infinity.

The proof goes along the lines of Theorem 3.1; the Peierls argument is implied by Theorem 4.1 and the decay of correlations by Theorem 4.2.

4.2. **Structure of the proof.** As explained in Section 3, the proof of Theorem 2.3 is based upon coarse grained estimates. Therefore, the same strategy can be transposed to the Kac Ising model by using the mesoscopic representation described above. The first part of the proof is very similar to the approach introduced in [BBBP, BBP] and therefore, we do not repeat the arguments of Section 3. Nevertheless, the derivation of the precise surface tension factor requires further analysis.

In the case of nearest neighbor Ising model at low temperature, the localization of the interface at the mesoscopic level by the minimal section argument implied directly, thanks to the coarse graining, the localization at the microscopic level. Therefore the ratio of partition functions $\frac{Z^{\pm}}{Z^{+}}$ (see (3.9) or (3.16)) arises from the very construction of the coarse graining. This is no longer the case, both for the lower and the upper bound. Indeed, the localization at the scale K implies only the occurrence of circuits of ± good boxes at the intermediate scale H. As a consequence, the surface tension factor should now be related to the asymptotic of the ratio of partition functions with statistically pure boundary conditions at the scale H.

Let us give now a precise mathematical formulation of the problem. For $N \in \mathbb{N}$ and $\delta > 0$, we define $\Lambda_N^\delta = \{i \in \mathbb{Z}^d \quad | \, |i_1| \leqslant \delta N, \; k = 2, \dots, d, \; |i_k| \leqslant N\}$. Let $G^{\pm,\varepsilon}(\Lambda_N^\delta)$ be the set of mixed boundary conditions (σ^+, σ^-) such that the configuration σ^+ is supported by the domain $\{i_1 > \delta N\}$ and contains mesoscopic phase labels $\eta_{N,H}^\varepsilon$ uniformly equal to 1. Similarly, σ^- is supported by the domain $\{i_1 < -\delta N\}$ and contains mesoscopic phase labels $\eta_{N,H}^\varepsilon$ uniformly equal to -1. The partition function with boundary conditions in $G^{\pm,\varepsilon}(\Lambda_N^\delta)$ is denoted by $Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\sigma^-}$. Let $\tilde{\sigma}^-$ be the configuration deduced from σ^- by spin flip. In the derivation of the lower and the upper bounds, the surface tension should be related to the following approximate quantities

$$\log \left(\frac{Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\sigma^-}}{Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\tilde{\sigma}^-}} \right) \tag{4.4}$$

Ultimately the parameter δ will vanish and therefore the boundary conditions on the faces of Λ_N^δ parallel to $\vec{e}_1$ can be chosen arbitrarily. For convenience we fix these conditions to be equal to m_β^*.

The surface tension is defined for arbitrary directions and general domains (see (2.2)). To simplify the notation, we will consider only the direction $\vec{n} = \vec{e}_1$ and domains Δ_N^δ. In fact, the results also hold for arbitrary directions $\vec{n}$ and domains.

Remark 4.1. *Notice that the mesoscopic phase labels on the scale K are now built with blocks of spins at the scale $H = 2^h \gamma^{-1}$. Therefore the mesoscopic surgery involves blocks of spins and the error term is of order $\exp(\frac{1}{\gamma})$. This is actually not a problem because γ is fixed and these terms disappear in the limit ($\delta \to 0$).*

4.3. **Lower bound.**

Proposition 4.1. *Let $\beta > 1$ and fix $\gamma_\beta > 0$, $\varepsilon > 0$ such that Theorems 4.1 and 4.2 hold. For all $\gamma < \gamma_\beta$,*

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^{d-1}}\inf_{\sigma^+,\sigma^-}\log\left(\frac{Z^{\sigma^+,\sigma^-}_{\gamma,\Lambda_N^\delta}}{Z^{\sigma^+,\tilde\sigma^-}_{\gamma,\Lambda_N^\delta}}\right) \geqslant -\tau_{\beta,\gamma}\,, \tag{4.5}$$

where the infimum is taken over the boundary configurations such that (σ^+,σ^-) belong to $G^{\pm,\varepsilon}(\Lambda_N^\delta)$.

Proof. If the boundary conditions are $\sigma^+ = 1$ and $\sigma^- = -1$ then, by definition of the surface tension, (4.5) holds. We are going to interpolate between the boundary conditions (σ^+,σ^-) and $(1,-1)$. For simplicity, the dependency on γ and δ will be omitted in the notation and the partition function will be denoted by $Z_N^{\sigma^+,\sigma^-}$.

First we are going to check that

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\inf_{\sigma^+,\sigma^-}\left[\log\frac{Z_N^{\sigma^+,\sigma^-}}{Z_N^{\sigma^+,\tilde\sigma^-}} - \log\frac{Z_N^{+,\sigma^-}}{Z_N^{+,\tilde\sigma^-}}\right] \geqslant 0\,. \tag{4.6}$$

For a given boundary condition σ^+, we define $\Lambda_N^{\delta,+}$ as the subset of $\{i_1 > \delta N\}$ containing the sites i such that $\sigma_i^+ = -1$. For any $i \in \Lambda_N^{\delta,+}$, we set

$$\forall s \in [0,1], \qquad \sigma_i^+(s) = (1-\sigma_i^+)s + \sigma_i^+$$

and by integrating

$$\log\frac{Z_N^{+,\sigma^-}}{Z_N^{+,\tilde\sigma^-}} - \log\frac{Z_N^{\sigma^+,\sigma^-}}{Z_N^{\sigma^+,\tilde\sigma^-}} = \sum_{i\in\Lambda_N^{\delta,+},\,j\in\Lambda_N^\delta}\int_0^1 ds\, J_\gamma(i,j)\left(\mu_{\beta,\gamma,N}^{\sigma^+(s),\sigma^-}(\sigma_j) - \mu_{\beta,\gamma,N}^{\sigma^+(s),\tilde\sigma^-}(\sigma_j)\right)\,.$$

The RHS can be split into two terms. The first one is non positive by FKG inequality

$$\sum_{i\in\Lambda_N^{\delta,+},\,j\in\Lambda_N^\delta}\int_0^1 ds\, J_\gamma(i,j)\left(\mu_{\beta,\gamma,N}^{\sigma^+(s),\sigma^-}(\sigma_j) - \mu_{\beta,\gamma,N}^{\sigma^+(s),+}(\sigma_j)\right) \leqslant 0\,.$$

The second term

$$\sum_{i\in\Lambda_N^{\delta,+},\,j\in\Lambda_N^\delta}\int_0^1 ds\, J_\gamma(i,j)\left(\mu_{\beta,\gamma,N}^{\sigma^+(s),+}(\sigma_j) - \mu_{\beta,\gamma,N}^{\sigma^+(s),\tilde\sigma^-}(\sigma_j)\right) \tag{4.7}$$

will be evaluated by using the exponential decay of correlations. We need to state a result slightly stronger than Theorem 4.2, whose proof is implicitly contained in [BMP]. For a given domain Δ, the extended set of boundary conditions $\mathcal{G}^{+,\varepsilon}(\Delta)$ contains the configurations σ_{Δ^c} such that

$$\forall \mathbb{B}_L(y) \subset \Delta^c, \qquad \mathcal{M}_{N,L}(y) - m_\beta^* \geqslant -\frac{\varepsilon}{2}\,.$$

In particular, the boundary conditions uniformly equal to 1 belong to this set.

Theorem 4.4. *For any $\beta > 1$, there exists $\gamma_\beta > 0$ and ε_β^* such that for $\varepsilon \in]0,\varepsilon_\beta^*[$ the following holds. For any subset Δ of $\mathbb{Z}^d$, let (σ,σ') be two configurations in $\mathcal{G}^{+,\varepsilon}(\Delta^c)$ then for $K \subset \Delta$, one has*

$$\forall \gamma \leqslant \gamma_\beta, \qquad \left|\mu_{\beta,\gamma,\Delta}^{\sigma}(\sigma_{\mathcal{K}}) - \mu_{\beta,\gamma,\Delta}^{\sigma'}(\sigma_{\mathcal{K}})\right| \leqslant |\mathcal{K}|\exp\big(-c(\varepsilon)\gamma^2\, dist(\mathcal{K},\mathcal{S})\big)\,,$$

where $\mathcal{S}$ is the subset of Δ^c where σ and σ' differ and $c(\varepsilon)$ is a positive constant.

The boundary conditions $(\sigma^+(s), \tilde{\sigma}^-)$ and $(\sigma^+(s), +)$ differ only in the lower part of $(\Lambda_N^\delta)^c$, thus the exponential relaxation enables us to evaluate (4.7)

$$\log \frac{Z_N^{+,\sigma^-}}{Z_N^{+,\tilde{\sigma}^-}} \leqslant \log \frac{Z_N^{\sigma^+,\sigma^-}}{Z_N^{\sigma^+,\tilde{\sigma}^-}} + \frac{N^{d-1}}{\gamma^d} \exp(-c\gamma^2\, \delta N) .$$

This estimate holds uniformly over the boundary conditions. Letting N go to infinity, we derive (4.6) .

It remains to check that

$$\forall \sigma^-, \qquad \frac{Z_N^{+,\sigma^-}}{Z_N^{+,\tilde{\sigma}^-}} \geqslant \frac{Z_N^{+,-}}{Z_N^{+,+}} . \tag{4.8}$$

This follows from FKG inequality. By interpolating, we get

$$\log \frac{Z_N^{+,\sigma^-}}{Z_N^{+,\tilde{\sigma}^-}} - \log \frac{Z_N^{+,-}}{Z_N^{+,+}} = \sum_{i \in \Lambda_N^{\delta,-}, j \in \Lambda_N^\delta} \int_0^1 ds\, J_\gamma(i,j) \left(\mu_{\beta,\gamma,N}^{+,\sigma^-(s)}(\sigma_j) + \mu_{\beta,\gamma,N}^{+,\tilde{\sigma}^-(s)}(\sigma_j) \right) ,$$

where $\Lambda_N^{\delta,-}$ is defined in a similar way as $\Lambda_N^{\delta,+}$. The spin flip symmetry enables us to rewrite the RHS. As FKG inequality implies that the magnetization is increasing wrt the boundary fields

$$\sum_{i \in \Lambda_N^{\delta,-}, j \in \Lambda_N^\delta} \int_0^1 ds\, J_\gamma(i,j) \left(\mu_{\beta,\gamma,N}^{+,\sigma^-(s)}(\sigma_j) - \mu_{\beta,\gamma,N}^{-,\sigma^-(s)}(\sigma_j) \right) \geqslant 0 .$$

Therefore the proposition is complete. □

Remark 4.2. *A proof of the lower bound without using FKG inequality can be done along the lines of the proof of the upper bound. Nevertheless, this would involve more technicalities because another argument would be required to replace* (4.10). *This can be achieved by considering rough minimal sections* $(\mathcal{S}^+, \mathcal{S}^-)$ *instead of rigid ones.*

4.4. **Upper bound.** For some boundary conditions in $G^{\pm,\varepsilon}(\Lambda_N^\delta)$, the quantity (4.4) is a bad approximation of the surface tension. Some boundary conditions interact with the interface and one should expect the following lower bound

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^{d-1}} \sup_{(\sigma^+,\sigma^-) \in G^{\pm,\varepsilon}(\Lambda_N^\delta)} \log \left(\frac{Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\sigma^-}}{Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\tilde{\sigma}^-}} \right) > -\tau_{\beta,\gamma} .$$

In order to screen the influence of the boundary conditions, one is lead to introduce a different approximation of the surface tension. This is done by localizing the interface away from the boundary with the help of two extra minimal sections which decouple the interface and the boundary conditions.

Mimicking the argument used for the derivation of the upper bound (subsection 3.4), we can find 4 minimal sections in each cube R_N^i of height $\delta h N$. Let us denote by $Q_N^i(s)$ the parallelepiped included in R_N^i with basis B^i and height sN. There is a minimal section in $R_N^{i,+} \cap Q_N^i(\frac{\delta h}{4} N)$ and another in $R_N^{i,+} \setminus Q_N^i(\frac{3\delta h}{2} N)$. By symmetry, there exist also 2 minimal sections in $R_N^{i,-}$.

By applying the surgical procedure, we deduce from the $\mathbb{L}^1$ constraint the existence of 2 surfaces of + good blocks (resp −) at the mesoscopic scale H crossing the domains $R_N^{i,+}$ (resp $R_N^{i,-}$). After conditioning wrt the ± surfaces which are the closest to $(R_N^i)^c$, it remains to consider a partition function with mixed boundary conditions and with the constraint that there are also ± crossing surfaces in $R_N^{i,\pm} \cap Q_N^i(\frac{\delta h}{4}N)$.

For simplicity, let us define the new constrained partition function on the domain Λ_N^δ. Let $\mathcal{S}$ be the event such that there exist two crossing surfaces of good H-blocks. For any spin configuration in $\mathcal{S}$, the surface of + blocks included in $\{i \in \Lambda_N^\delta \,|\, 0 \leqslant i_1 \leqslant \frac{\delta}{4}N\}$ which is the closest to $\{i_1 = 0\}$ is denoted by $\mathcal{S}^+$. In the same way, the surface of − blocks included in $\{i \in \Lambda_N^\delta \,|\, -\frac{\delta}{4}N \leqslant i_1 \leqslant 0\}$ which is the closest to $\{i_1 = 0\}$ is denoted by $\mathcal{S}^-$. The constrained partition function with mixed boundary conditions $(\sigma^+, \sigma^-) \in G^{\pm,\varepsilon}(\Lambda_N^\delta)$ is defined by

$$Z_{\gamma,\Lambda_N^\delta}^{\sigma^+,\sigma^-}(\mathcal{S}) = \sum_\sigma 1_{\mathcal{S}}(\sigma) \exp\left(-\beta H_\gamma(\sigma | (\sigma^+,\sigma^-))\right) .$$

The constraint $\mathcal{S}$ implies that the mesoscopic interface is localized inside the stripe $\{i \in \Lambda_N^\delta \,|\, |i_1| \leqslant \frac{\delta}{4}N\}$. Thus, uniformly over the boundary conditions in $G^{\pm,\varepsilon}(\Lambda_N^\delta)$ the system will relax to equilibrium in each region outside this stripe.

Proposition 4.2. *Let $\beta > 1$ and fix $\gamma_\beta > 0$, $\varepsilon > 0$ such that the Theorems 4.1 and 4.2 hold. Then, for all $\gamma < \gamma_\beta$,*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^{d-1}} \sup_{\sigma^+,\sigma^-} \log \left(\frac{Z_{\Lambda_N^\delta}^{\sigma^+,\sigma^-}(\mathcal{S})}{Z_{\Lambda_N^\delta}^{\sigma^+,\tilde{\sigma}^-}} \right) \leqslant -\tau_{\beta,\gamma} , \tag{4.9}$$

where the supremum is taken over the boundary configurations such that (σ^+, σ^-) belong to $G^{\pm,\varepsilon}(\Lambda_N^\delta)$.

Proof. As before, the proof is based upon an interpolation which enables us to compare the statistically pure boundary conditions with the mixed boundary conditions $(1,-1)$. Nevertheless, the method is quite different because it relies on the relaxation of the spin configurations in the domains outside the stripe $\{i \,|\, |i_1| \leqslant \frac{\delta}{4}N\}$. Notice that

$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \left(\frac{Z_N^{+,-}(\mathcal{S})}{Z_N^{+,+}} \right) \leqslant \limsup_{N \to \infty} \frac{1}{N^{d-1}} \log \left(\frac{Z_N^{+,-}}{Z_N^{+,+}} \right) = -\tau_{\beta,\gamma} .$$

We define 2 regions

$$\begin{aligned} \mathcal{B}^+ &= \left\{ i, \quad i_1 \in [\frac{\delta}{2}N, \frac{\delta}{2}N + \frac{10}{\gamma}] \right\} , \\ \mathcal{B}^- &= \left\{ i, \quad i_1 \in [-\frac{\delta}{2}N - \frac{10}{\gamma}, -\frac{\delta}{2}N] \right\} . \end{aligned}$$

The spins configurations inside $(\mathcal{B}^+, \mathcal{B}^-)$ are interpolated by the constant configuration $(m^*, -m^*)$

$$\begin{aligned} \forall s \in [0,1], \forall i \in \mathcal{B}^+, \qquad \sigma_i(s) &= s\sigma_i + (1-s)m_\beta^* , \\ \forall i \in \mathcal{B}^-, \qquad \sigma_i(s) &= s\sigma_i - (1-s)m_\beta^* . \end{aligned}$$

Outside $(\mathcal{B}^+,\mathcal{B}^-)$, the spins are unchanged. Let $Z^{\sigma^+,\sigma^-}_{N,s}(\mathcal{S})$ be the partition function depending on the spins $\sigma(s)$. The configurations of the partition function in the denominator are interpolated in $(\mathcal{B}^+,\mathcal{B}^-)$ with the configuration (m^*_β, m^*_β) and the corresponding partition function is denoted by $Z^{\sigma^+,\tilde{\sigma}^-}_{N,s}$.

For given boundary conditions $(\sigma^+,\sigma^-) \in G^{\pm,\varepsilon}(\Lambda^\delta_N)$, we introduce

$$\Phi(s) = \log \frac{Z^{+,-}_{N,s}(\mathcal{S})}{Z^{+,+}_{N,s}} - \log \frac{Z^{\sigma^+,\sigma^-}_{N,s}(\mathcal{S})}{Z^{\sigma^+,\tilde{\sigma}^-}_{N,s}} . \tag{4.10}$$

When $s=0$, the spin configurations in $\mathcal{B}^+$ and $\mathcal{B}^-$ decouple the configurations into three independent systems.

$$\Phi(0) = \log \left(\frac{Z^{+,m^*}_N}{Z^{+,m^*}_N} \, \frac{Z^{m^*,-m^*}_N(\mathcal{S})}{Z^{m^*,m^*}_N} \, \frac{Z^{-m^*,-}_N}{Z^{m^*,+}_N} \right) - \log \left(\frac{Z^{\sigma^+,m^*}_N}{Z^{\sigma^+,m^*}_N} \, \frac{Z^{m^*,-m^*}_N(\mathcal{S})}{Z^{m^*,m^*}_N} \, \frac{Z^{-m^*,\sigma^-}_N}{Z^{m^*,\tilde{\sigma}^-}_N} \right) .$$

The symmetries of the model imply the cancellation of all the terms $\Phi(0) = 0$.

All what remains to do is to control the derivative $\Phi'(s)$. We encounter two types of terms in the sections $\mathcal{B}^\pm$

$$\begin{aligned}
\Phi'(s) \; &= \textstyle\sum_{i\in\mathcal{B}^+, j\in\Lambda^\delta_N} J_\gamma(i,j) \quad \Big[\Big(\mu^{+,-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)|\mathcal{S}\big) - \mu^{+,+}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) \Big) \\
&\qquad\qquad - \Big(\mu^{\sigma^+,\sigma^-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)|\mathcal{S}\big) - \mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) \Big) \Big] \\
&+ \textstyle\sum_{i\in\mathcal{B}^-, j\in\Lambda^\delta_N} J_\gamma(i,j) \quad \Big[\Big(\mu^{+,-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i + m^*)|\mathcal{S}\big) - \mu^{+,+}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) \Big) \\
&\qquad\qquad - \Big(\mu^{\sigma^+,\sigma^-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i + m^*)|\mathcal{S}\big) - \mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) \Big) \Big] .
\end{aligned}$$

By using the exponential decay of correlations (Theorem 4.4), we obtain

$$\sum_{i\in\mathcal{B}^+, j\in\Lambda^\delta_N} J_\gamma(i,j) \left| \mu^{+,+}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) - \mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,N,s}\big(\sigma_j(\sigma_i - m^*)\big) \right| \leqslant \frac{N^{d-1}}{\gamma^d} \exp(-c\gamma^2\,\delta N) .$$

This estimate holds uniformly over the boundary conditions $(\sigma^+,\tilde{\sigma}^-) \in \mathcal{G}^{+,\varepsilon}(\Lambda^\delta_N)$.

Remark 4.3. *Notice that Theorem 4.4 holds also for the interpolated measures* $(\mu^{\sigma^+,\tilde{\sigma}^-}_{\beta,\gamma,N,s})_{s\in[0,1]}$. *This fact was pointed to us by E. Presutti. The interpolation by more regular configurations can only improve the properties of the measure, thus modifying the interactions on* $(\mathcal{B}^+,\mathcal{B}^-)$ *does not alter the conclusion of the Theorem.*

The minimal section becomes effective in order to estimate the other terms. We consider the spin correlations in $\mathcal{B}^+$ (the same argument holds with $\mathcal{B}^-$). Theorem 4.4 provides estimates for boundary conditions which are in $\mathcal{G}^{+,\varepsilon}(\Lambda^\delta_N)$ and therefore cannot be applied directly for mixed boundary conditions. Nevertheless the section $\mathcal{S}^+$ screens the effect of the $-$ boundary conditions. Thus the configurations in $\mathcal{B}^+$ relax to the $+$ pure phase. Let $\Delta^+_{\mathcal{S}}$ be the domain above the surface $\mathcal{S}^+$. By conditioning, wrt the configuration $\sigma_{\mathcal{S}^+}$ in the section $\mathcal{S}^+$, we have

$$\mu^{+,-}_{\beta,\gamma,N,s}\big(\sigma_j\sigma_i|\mathcal{S}\big) = \frac{1}{\mu^{+,-}_{\beta,\gamma,N,s}(\mathcal{S})} \mu^{+,-}_{\beta,\gamma,N,s}\big(1_{\mathcal{S}}\, \mu^{+,\sigma_{\mathcal{S}^+}}_{\beta,\gamma,\Delta^+_{\mathcal{S}},s}(\sigma_j\sigma_i)\big) .$$

According to Theorem 4.4 the previous expression can be replaced by the expectation in the infinite stripe $\{i, \quad |i_j| \leqslant N,\ j = 2, \dots, d\}$.

$$\left|\mu^{+,-}_{\beta,\gamma,N,s}(\sigma_j\sigma_i|\mathcal{S}) - \mu^{m^*}_{\beta,\gamma,s}(\sigma_j\sigma_i)\right| \leqslant \sup_{\sigma_{\mathcal{S}^+}} \left|\mu^{+,\sigma_{\mathcal{S}^+}}_{\beta,\gamma,\Delta^+_{\mathcal{S}},s}(\sigma_j\sigma_i) - \mu^{m^*}_{\beta,\gamma,s}(\sigma_j\sigma_i)\right| \leqslant \exp(-c\gamma^2\delta N)\,,$$

where $\mu^{m^*}_{\beta,\gamma,s}$ is the measure on the infinite stripe. The same identity holds for $\mu^{\sigma^+,\sigma^-}_{\beta,\gamma,N,s}(\sigma_j\sigma_i|\mathcal{S})$, therefore

$$\sum_{i\in\mathcal{B}^+, j\in\Lambda^\delta_N} J_\gamma(i,j)\big|\mu^{+,-}_{\beta,\gamma,N,s}(\sigma_j(\sigma_i - m^*)\,|\,\mathcal{S}) \quad - \quad \mu^{\sigma^+,\sigma^-}_{\beta,\gamma,N,s}(\sigma_j(\sigma_i - m^*)\,|\,\mathcal{S})\big|$$

$$\leqslant \frac{N^{d-1}}{\gamma^d}\exp(-c\gamma^2\,\delta N)\,.$$

This conclude the proposition. □

Remark 4.4. *The proof of theorem 4.2 does not use correlation inequalities.*

5. Appendix : Theorem 3.1

The proof of the domination bound for the 0-blocks (3.2) is divided into 3 steps.

Step 1. Let us start with a single box. If $\widehat{\mathbb{B}}_{N,K}(x)$ is not a *good* box then either there is a contour of length at least K^α crossing the enlarged boundary or conditionally on the event that the box $\mathbb{B}_K(x)$ is surrounded by a surface of spins of sign ε_x, the magnetization $\mathcal{M}_{N,K}(x)$ is atypical. These two occurrences can be estimated separately. Applying the Peierls estimate (2.8), we get

$$\mu^+_{\beta,N}\big(\text{there is a contour crossing } \partial\mathbb{B}_K(x)\big) \leqslant K^{d-1}\exp(-c_\beta K^\alpha)\,. \tag{5.1}$$

Conditionally on the occurrence of a connected surface $\mathcal{S}$ of ε_x-spins surrounding the box $\mathbb{B}_K(x)$, the configurations inside $\mathbb{B}_K(x)$ are decoupled from the exterior. We first use Tchebyshev inequality

$$\mu^+_{\beta,N}\left(\{|\mathcal{M}_{N,K}(x) - \varepsilon_x m^*| \geqslant \zeta\}\ \big|\ \mathcal{S}\right) \leqslant \frac{1}{\zeta^2 K^{2d}}\mu^{\varepsilon_x}_{\beta,\mathrm{int}(\mathcal{S})}\left(\Big(\sum_{i\in\mathbb{B}_K(x)}\sigma_i - \varepsilon_x m^*\Big)^2\right)\,.$$

where $\mathrm{int}(\mathcal{S})$ is the region surrounded by $\mathcal{S}$. As $\mathcal{S}$ has been chosen as the closest surface to $(\mathbb{B}_{K+K^\alpha})^c$, the magnetization inside the box $\mathbb{B}_K(x)$ is measurable after the conditioning . As a consequence of the Peierls estimates (low temperature expansions), the correlations decay exponentially in the ε_x-pure phase, so that we obtain

$$\mu^{\varepsilon_x}_{\mathrm{int}(\mathcal{S})}\left(\{|\mathcal{M}_{N,K}(x) - \varepsilon_x m^*| \geqslant \zeta\}\right) \leq \frac{1}{\zeta^2 K^d}\chi\,, \tag{5.2}$$

where the susceptibility $\chi = \sum_{i\in\mathbb{Z}^d}\mu^+_\beta(\sigma_0;\sigma_i)$ is finite.

Step 2. In order to evaluate the probability of the event

$$\left\{u^\zeta_{N,K}(x_1) = 0, \dots, u^\zeta_{N,K}(x_\ell) = 0\right\}$$

the lattice $\widehat{\mathbb{D}}_{N,K}$ is partionned into c_d sub-lattices $(\widehat{\mathbb{D}}^{(i)}_{N,K})_{i\,\leqslant\, c_d}$ such that two cubes of size $K + K^\alpha$ centered on two sites of $\mathbb{D}^{(i)}_{N,K}$ are disjoint. By applying Hölder inequality, the estimate (3.2) is reduced to cubes which are not nearest neighbors.

$$\mu^+_{\beta,N}\left(u^\zeta_{N,K}(x_1)=0,\dots,u^\zeta_{N,K}(x_\ell)=0\right) \leqslant \prod_{i=1}^{c_d}\mu^+_{\beta,N}\left(\forall x_j\in\widehat{\mathbb{D}}^{(i)}_{N,K},\quad u^\zeta_{N,K}(x_j)=0\right)^{\frac{1}{c_d}} .$$

<u>*Step 3.*</u> The event $\left\{u^\zeta_{N,K}(x_1)=0,\dots,u^\zeta_{N,K}(x_\ell)=0\right\}$ can be decomposed into 2 terms : on ℓ' boxes the density is atypical, whereas there are contours crossing the $\ell-\ell'$ enlarged boundaries of the remaining boxes.

For a given collection of j boxes, we define

$$\begin{aligned}\mathcal{A}_j &= \{\text{The } j \text{ boxes are surrounded by } \pm \text{ surfaces, but their averaged magnetizations} \\ &\qquad\quad \text{are non typical}\} \\ \mathcal{B}_j &= \{\text{There are contours crossing the } j \text{ enlarged boundaries of the boxes}\} .\end{aligned}$$

The probabilities of both events can be evaluated as follows. As the j boxes are disjoint and the surfaces of spins decouple the configurations inside each box

$$\mu^+_{\beta,N}(\mathcal{A}_j) \leqslant \left(\mu^+_{\beta,N}(\mathcal{A}_1)\right)^j \leqslant \left(\alpha_K\right)^j ,$$

where the constant $\alpha_K=\frac{\chi}{\zeta^2K^d}$ was introduced in (5.2).

$$\mu^+_{\beta,N}(\mathcal{B}_j)=\sum_{i=1}^{j}\mu^+_{\beta,N}\big(\{\exists\ i \text{ contours crossing the } j \text{ enlarged boundaries}\}\big) .$$

We choose i blocks as starting points of these contours. Then we have to evaluate

$$\sum_{|\Gamma_1|+\dots+|\Gamma_i|\,\geqslant\, jK^\alpha}\mu^+_{\beta,N}(\Gamma_1,\dots,\Gamma_i) ,$$

where the contours $(\Gamma_1,\dots,\Gamma_i)$ have also to cross each boundaries of the j cubes.

Let n_r be the number of boundaries crossed by the contour r

$$\sum_{|\Gamma_1|+\dots+|\Gamma_i|\,\geqslant\, jK^\alpha}\mu^+_{\beta,N}(\Gamma_1,\dots,\Gamma_i) \leqslant \sum_{n_1+\dots+n_i=j}\ \sum_{(\Gamma_r,n_r)}\mu^+_{\beta,N}(\Gamma_1,\dots,\Gamma_i) .$$

If a contour crosses n_r boundaries then it has a length at least $n_rK^\alpha+(n_r-1)K$ because the distance between the boxes is at least K. Thus

$$\begin{aligned}\sum_{|\Gamma_1|+\dots+|\Gamma_i|\,\geqslant\, jK^\alpha}\mu^+_{\beta,N}(\Gamma_1,\dots,\Gamma_i) &\leqslant \sum_{n_1+\dots+n_i=j}\prod_{r=1}^{i}\exp(-c_\beta n_rK^\alpha-c_\beta(n_r-1)K) \\ &\leqslant \exp(-c_\beta jK^\alpha)\left(\sum_{n=1}^{\infty}\exp(-c_\beta(n-1)K)\right)^i \\ &\leqslant C^i\exp(-c_\beta jK^\alpha) .\end{aligned}$$

$$\mu^+_{\beta,N}(\mathcal{B}_j)\leqslant\sum_{i=1}^{j}\binom{j}{i}K^{(d-1)i}C^i\exp(-c_\beta jK^\alpha)\leqslant\exp(-c_\beta\, jK^\alpha)(1+CK^{d-1})^j=(\alpha'_K)^j .$$

where the constant α'_K vanishes as K goes to infinity.

Combining both estimates, we obtain

$$\mu^+_{\beta,N}\left(u^\zeta_{N,K}(x_1)=0,\dots u^\zeta_{N,K}(x_\ell)=0\right) \leqslant \sum_{\ell'=1}^{\ell}\binom{\ell}{\ell'}\mu^+_{\beta,N}(\mathcal{A}_{\ell'})^{1/2}\mu^+_{\beta,N}(\mathcal{B}_{\ell-\ell'})^{1/2} \leqslant \left(\alpha_K+\alpha'_K\right)^{\ell}.$$

This completes the proof.

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Laboratoire de Probabilités et Modèles Aléatoires, CNRS-UMR 7599, Université Paris VI et VII, UFR de Mathématiques, case 7012, 2 place Jussieu, Paris 75251, France.

E-mail address: `bodineau@gauss.math.jussieu.fr`