Upper bounds on the density of states of single Landau levels broadened by Gaussian random potentials

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We study a non-relativistic charged particle on the Euclidean plane $\mathbb{R}^2$ subject to a perpendicular constant magnetic field and an $\mathbb{R}^2$-homogeneous random potential in the approximation that the corresponding random Landau Hamiltonian on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$ is restricted to the eigenspace of a single but arbitrary Landau level. For a wide class of $\mathbb{R}^2$-homogeneous Gaussian random potentials we rigorously prove that the associated restricted integrated density of states is absolutely continuous with respect to the Lebesgue measure. We construct explicit upper bounds on the resulting derivative, the restricted density of states. As a consequence, any given energy is seen to be almost surely not an eigenvalue of the restricted random Landau Hamiltonian.

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CONTENTS

I. INTRODUCTION

In recent decades considerable attention has been paid to the physics of quasi-two-dimensional electronic structures.[1–5] Some of the occurring phenomena, like the integer quantum Hall effect,[6] are believed to be microscopically explainable in terms of a Fermi gas of non-interacting electrically charged particles in two dimensions subject to a perpendicular constant magnetic field and a static random potential. For these phenomena it should therefore be sufficient to study a single non-relativistic spinless particle on the Euclidean plane $\mathbb{R}^2$ modeled by the *random Landau Hamiltonian*, which is informally given by

$$H\big(V^{(\omega)}\big) := H(0) + V^{(\omega)}. \tag{1}$$

As a random Schrödinger operator it acts on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$ of Lebesgue square-integrable complex-valued functions on the plane $\mathbb{R}^2$. For any realization $\omega \in \Omega$ of the randomness the potential $V^{(\omega)}$ mimics the disorder present in a real sample. Throughout, we will assume that V is homogeneous on the average with respect to Euclidean translations of $\mathbb{R}^2$. The unperturbed part in (1) is the *Landau Hamiltonian*. It represents the kinetic energy of the particle and is informally given (in the symmetric gauge) by the differential expression

$$H(0) := \frac{1}{2}\left[\left(i\frac{\partial}{\partial x_1} - \frac{B}{2}x_2\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{B}{2}x_1\right)^2\right] = \frac{B}{2}\sum_{l=0}^{\infty}(2l+1)\,P_l, \tag{2}$$

in physical units where the mass and the charge of the particle, and Planck's constant divided by 2π are all equal to one. Moreover, $B > 0$ denotes the strength of the magnetic field and

$i = \sqrt{-1}$ stands for the imaginary unit. The second equality in (2) is the spectral resolution of $H(0)$. It dates back to Fock[7] and Landau.[8] The energy eigenvalue $(l + 1/2)B$ is called the l^{th} *Landau level* and the corresponding orthogonal eigenprojection P_l is an integral operator with continuous complex-valued kernel (in other words: position representation)

$$P_l(x, y) := \frac{B}{2\pi} \exp\left[i\frac{B}{2}(x_2 y_1 - x_1 y_2) - \frac{B}{4}|x-y|^2 \right] \mathrm{L}_l^{(0)}\left(\frac{B}{2}|x-y|^2 \right). \tag{3}$$

Here and in the following, $|x - y|^2 := (x_1 - y_1)^2 + (x_2 - y_2)^2$ denotes the square of the Euclidean distance between the points $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Moreover, $\mathrm{L}_l^{(k)}(\xi) := \sum_{j=0}^{l} (-1)^j \binom{l+k}{l-j} \xi^j / j!$, with $\xi \geq 0$ and $k \in \mathbb{N}_0 - l := \{-l, -l+1, -l+2, \dots\}$, is a generalized Laguerre polynomial, see Sec. 8.97 in Ref. 9. The diagonal $P_l(x, x) = B/2\pi$ is naturally interpreted as the degeneracy per area of the l^{th} Landau level.

A quantity of basic interest in the study of the random Landau Hamiltonian (1) is its *integrated density of states* $\nu(]-\infty, E[)$ as a function of the energy $E \in \mathbb{R}$. The underlying positive Borel measure ν on the real line $\mathbb{R}$ is called the *density-of-states measure* of $H(V)$. If the random potential is not only $\mathbb{R}^2$-homogeneous but also *isotropic*, that is, if all finite-dimensional distributions associated with the probability measure $\mathbb{P}$ on Ω, which governs the randomness, are invariant also under in-plane rotations (with respect to the origin), the density-of-states measure ν can be decomposed according to

$$\nu = \frac{B}{2\pi} \sum_{l=0}^{\infty} \widehat{\nu}_l, \qquad \widehat{\nu}_l(I) := \frac{2\pi}{B} \mathbb{E}\left[\left(P_l \chi_I(H(V)) P_l \right)(x, x) \right], \qquad I \in \mathcal{B}(\mathbb{R}), \tag{4}$$

see Refs. 10, 11, and references therein. Here $\mathbb{E}(\cdot) := \int_\Omega \mathbb{P}(\mathrm{d}\omega)\,(\cdot)$ denotes the expectation induced by $\mathbb{P}$ and $\chi_I(H(V^{(\omega)}))$ is the spectral projection operator of $H(V^{(\omega)})$ associated with the energy regime $I \in \mathcal{B}(\mathbb{R})$. The contribution $\widehat{\nu}_l$ related to the *Landau-level index* l is a probability measure on the Borel sets $\mathcal{B}(\mathbb{R})$ in the real line $\mathbb{R}$. It is actually independent of $x \in \mathbb{R}^2$ due to the homogeneity of V.

In the limit of a strong magnetic field, the spacing B between successive Landau levels approaches infinity and the magnetic length $B^{-1/2}$ tends to zero. Therefore, the effect of so-called level mixing should be negligible if either the strength of the random potential V, typically given by the square root of its single-site variance $\mathbb{E}\left[V(0)^2\right] - \left(\mathbb{E}[V(0)]\right)^2$, is small compared to the level spacing or if the (smallest) correlation length of V is much larger than the magnetic length. In both cases $\widehat{\nu}_l(I)$ should be well approximated by $2\pi\, \mathbb{E}\left[\left(P_l \chi_I(P_l H(V) P_l) P_l \right)(x, x) \right] / B$. Indeed, this approximation is exact[12] if V is a spatially constant random potential $x \mapsto V^{(\omega)}(0)$. Since the first part of the l^{th} *restricted random Landau Hamiltonian*, $P_l H(V) P_l = (l + 1/2) B P_l + P_l V P_l$, causes only a shift in the energy, one may equivalently study the probability measure

$$\nu_l(I) := \frac{2\pi}{B} \mathbb{E}\left[\left(P_l \chi_I(P_l V P_l) P_l \right)(x, x) \right], \qquad I \in \mathcal{B}(\mathbb{R}). \tag{5}$$

We call it the l^{th} *restricted density-of-states measure* and its distribution function $E \mapsto \nu_l(]-\infty, E[)$ the l^{th} *restricted integrated density of states*. Again, they are independent of $x \in \mathbb{R}^2$ due to the homogeneity of V. From the physical point of view most interesting is the restriction to the *lowest* Landau level, corresponding to $l = 0$. If the magnetic field is strong enough, all particles may be accommodated in the lowest level without conflicting with Pauli's exclusion principle, since the degeneracy (per area) $B/2\pi$ increases with B. Up to the energy shift $B/2$, the measure $B\nu_0/2\pi$ should then be a good approximation to ν, since the effects of higher Landau levels are negligible if B is large compared to the strength of the random potential, see Prop. 1 in Ref. 13 in case of a Gaussian random potential.

Neglecting effects of level mixing by only dealing with the sequence of restricted operators $(P_l V P_l)_{l \in \mathbb{N}_0}$ is a simplifying approximation which is often made. The interest in these operators relates to the existence of pure-point components in their spectra[14–17] and, what is simpler, to properties of their restricted density-of-states measures (ν_l). The aim of the present paper is to supply conditions under which ν_l is absolutely continuous with respect to the Lebesgue measure. Actually, in the physics literature this differentiability of the restricted integrated density of states $\nu_l\,(]-\infty,E[)$ with respect to E is usually taken for granted so that one deals from the outset with its derivative, the $l^{\rm th}$ *restricted density of states*

$$E \mapsto w_l(E) := \frac{\mathrm{d}\nu_l\,(]-\infty,E[)}{\mathrm{d}E} = \frac{\nu_l(\mathrm{d}E)}{\mathrm{d}E}, \tag{6}$$

see, for example, Refs. 1–6, 10, 12, 18–25. Due to the involved averaging, the disorder is indeed often believed to broaden each Landau level to a *Landau band* in such a way that the resulting restricted integrated density of states is sufficiently smooth. Example 1 below however, which is taken from Ref. 12, illustrates that this belief is wrong without further assumptions. It even shows that for any given $l \geq 1$ it may happen that there is no broadening at all so that the operator $P_l V P_l$ is zero almost surely (although V is non-zero) and hence ν_l is singular. For the formulation of the example we need some preparations. Without losing generality, we will always assume that the homogeneous (but not necessarily isotropic) random potential V has zero mean, $\mathbb{E}[\,V(0)] = 0$. The *variance* σ_l^2 of ν_l is then given by[12]

$$\sigma_l^2 := \int_{\mathbb{R}} \nu_l(\mathrm{d}E)\, E^2 = \frac{2\pi}{B}\,\mathbb{E}\left[(P_l V P_l)^2\,(0,0)\right] = \frac{2\pi}{B}\,(P_l C P_l)\,(0,0) \leq C(0), \tag{7}$$

where $x \mapsto C(x) := \mathbb{E}\,[V(x)V(0)]$ is the *covariance function* of V. When sandwiched between two projections, C is understood as a (bounded) multiplication operator acting on $\mathrm{L}^2(\mathbb{R}^2)$. The standard deviation $\sigma_l := \sqrt{\sigma_l^2}$ may physically be interpreted as the width of the $l^{\rm th}$ Landau band. We note that the width σ_0 of the lowest Landau band is always strictly positive, provided that the covariance function is continuous and obeys $C(0) > 0$. This follows from the formula $\sigma_l^2 = \int_{\mathbb{R}^2} \widetilde{C}(\mathrm{d}^2k)\, \exp\big(-|k|^2/2B\big)\big[\mathrm{L}_l^{(0)}\big(|k|^2/2B\big)\big]^2$. Here the so-called *spectral measure* $\widetilde{C}$ which, according to the Bochner-Khintchine theorem (Thm. IX.9 in Ref. 26), is the unique finite positive (and even) Borel measure on $\mathbb{R}^2$ yielding the Fourier representation $C(x) = \int_{\mathbb{R}^2} \widetilde{C}(\mathrm{d}^2k)\, \exp(ik\cdot x)$ where $k \cdot x := k_1x_1 + k_2x_2$ denotes the standard scalar product on $\mathbb{R}^2$.

Example 1. If V possesses the oscillating covariance function $C(x) = C(0)\,J_0\big(\sqrt{2}|x|/\tau\big)$, where $\tau > 0$ and J_0 is the Bessel function of order zero,[9] then

$$\sigma_l^2 = C(0)\,\exp\left(-\frac{1}{B\tau^2}\right)\left[\mathrm{L}_l^{(0)}\left(\frac{1}{B\tau^2}\right)\right]^2. \tag{8}$$

Choosing the squared length ratio $1/(B\tau^2)$ equal to a zero of $\mathrm{L}_l^{(0)}$, which exists if $l \geq 1$, one achieves that $\sigma_l^2 = 0$. Chebyshev's inequality then implies that ν_l is Dirac's point measure at the origin, informally $w_l(E) = \delta(E)$. Therefore, $P_l V^{(\omega)} P_l = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

In the present paper we provide conditions under which exotic situations as in Example 1 cannot occur. More precisely, we prove that (6) indeed defines w_l as a bounded probability density for a wide class of *Gaussian* random potentials, see Theorem 1 and Theorem 2 below. Moreover, we construct explicit upper bounds on w_l for these potentials. As an implication, we prove that for any $B > 0$ and the class of Gaussian random potentials considered, any given energy $E \in \mathbb{R}$ is almost surely not an eigenvalue of the operator $P_l V P_l$. In particular, these Gaussian random potentials completely lift the infinite degeneracy of the Landau-level energy (here shifted to zero) for any strength of the magnetic field. This stands in contrast to situations with random point impurities considered in Refs. 19, 16, 27, 28, 17.

The present paper was partially motivated by results of Ref. 29 where the (unrestricted) density-of-states measure ν of the random Landau Hamiltonian is proven to be absolutely continuous with a locally bounded density for a certain class of random potentials (including the Gaussian ones considered in Theorem 1) and where any given energy $E \in \mathbb{R}$ is shown to be almost surely not an eigenvalue of $H(V)$. While absolute continuity of ν immediately implies by (4) that of $\widehat{\nu}_l$ for all $l \in \mathbb{N}_0$ (if V is isotropic), in itself it does not imply that of ν_l.

II. THE DENSITY OF STATES OF A SINGLE BROADENED LANDAU LEVEL

A. The restricted random Landau Hamiltonian and its integrated density of states

Let $\|F\| := \sup\big\{\,|\langle\varphi, F\varphi\rangle| \;:\; \varphi \in \mathrm{L}^2(\mathbb{R}^2)\,,\, \langle\varphi,\varphi\rangle = 1\big\} < \infty$ denote the (uniform) norm of a self-adjoint bounded operator F acting on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$. The restriction $P_l F P_l$ of F to the eigenspace $P_l\mathrm{L}^2(\mathbb{R}^2) \subset \mathrm{L}^2(\mathbb{R}^2)$ corresponding to the l^{th} Landau level is an integral operator with kernel $(x,y) \mapsto (P_l F P_l)\,(x,y) := B\,\langle\psi_{l,x}\,, F\,\psi_{l,y}\rangle/2\pi$ which is jointly continuous thanks to the continuity of the usual scalar product $\langle\cdot,\cdot\rangle$ on $\mathrm{L}^2(\mathbb{R}^2)$ and the strong continuity of the mapping $\mathbb{R}^2 \ni x \mapsto \psi_{l,x} \in P_l\mathrm{L}^2(\mathbb{R}^2)$. Here, the two-parameter family of normalized, complex-valued functions ("coherent states") is defined by

$$y \mapsto \psi_{l,x}(y) := \sqrt{\frac{2\pi}{B}}\, P_l(y,x), \qquad x \in \mathbb{R}^2, \qquad \langle\psi_{l,x}\,,\,\psi_{l,x}\rangle = 1. \tag{9}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. By a *random potential* we mean a random field $V : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $(\omega, x) \mapsto V^{(\omega)}(x)$ which is jointly measurable with respect to the sigma-algebra $\mathcal{A}$ of event sets in Ω and the sigma-algebra $\mathcal{B}(\mathbb{R}^2)$ of Borel sets in the Euclidean plane $\mathbb{R}^2$.

The next proposition provides conditions under which the (in general unbounded) integral operator $P_l V P_l$, the (shifted) l^{th} *restricted random Landau Hamiltonian*, is almost surely essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of arbitrarily often differentiable complex-valued functions on $\mathbb{R}^2$ with rapid decrease (Def. on p. 133 in Ref. 26).

Proposition 1. *Let V be an $\mathbb{R}^2$-homogeneous random potential and assume there exists a constant $M < \infty$ such that $\mathbb{E}\left[|V(0)|^{2k}\right] \leq (2k)!\,M^{2k}$ for all $k \in \mathbb{N}$. Then for all $l \in \mathbb{N}_0$ it holds:*

(1) The restricted operator *$P_l V^{(\omega)} P_l$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ for all ω in some subset $\Omega_0 \in \mathcal{A}$ of Ω with full probability, $\mathbb{P}(\Omega_0) = 1$.*

(2) The mapping $\Omega_0 \ni \omega \mapsto P_l V^{(\omega)} P_l$ is measurable in the sense of Def. V.1.3 in Ref. 30.

(3) The restricted density-of-states measure *ν_l, defined in (5), is a probability measure on the sigma-algebra $\mathcal{B}(\mathbb{R})$ of Borel sets in the real line. Moreover, the following (weak) operator identity holds*

$$\mathbb{E}\left[P_l \chi_I(P_l V P_l) P_l\right] = \nu_l(I)\,P_l, \qquad I \in \mathcal{B}(\mathbb{R}). \tag{10}$$

Remark. As far as we know, the operator identity (10) for general $\mathbb{R}^2$-homogeneous V and general $l \in \mathbb{N}_0$ was first shown in Ref. 20, also see Ref. 24.

Proof of Proposition 1. (1) The assumed limitation on the growth of the even moments $\mathbb{E}\left[|V(0)|^{2k}\right]$ as a criterion for the almost-sure essential self-adjointness of $P_l V P_l$ is taken from the proof of Thm. 2.1 in Ref. 14, which is based on Nelson's analytic-vector theorem (Thm. X.39 in Ref. 31), also see Ref. 32.

(2) By truncating the large fluctuations of the random potential we construct a sequence of restricted random operators $\big(P_l V_n^{(\omega)} P_l\big)_{n\in\mathbb{N}}$, where $V_n^{(\omega)}(x) := V^{(\omega)}(x)\,\Theta\big(n - \big|V^{(\omega)}(x)\big|\big)$ is bounded and measurable for all n. Here $\Theta := \chi_{]0,\infty[}$ denotes Heaviside's unit-step function. For $\mathbb{P}$-almost all $\omega \in \Omega$ we have the strong convergence $P_l V_n^{(\omega)} P_l \varphi \to P_l V^{(\omega)} P_l \varphi$ as $n \to \infty$ for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$. Consequently, Thm. VIII.25 in Ref. 26 implies that the sequence converges towards $P_l V^{(\omega)} P_l$ in the strong resolvent sense implying that the latter operator is also measurable thanks to Prop. V.1.4 in Ref. 30.

(3) Since $0 \leq \big\langle\psi_{l,x}\,, \chi_I\big(P_l V^{(\omega)} P_l\big)\,\psi_{l,x}\big\rangle \leq \big\langle\psi_{l,x}\,, \chi_{\mathbb{R}}\big(P_l V^{(\omega)} P_l\big)\,\psi_{l,x}\big\rangle = 1$ for all

$x \in \mathbb{R}^2$, all $I \in \mathcal{B}(\mathbb{R})$, and $\mathbb{P}$-almost all $\omega \in \Omega$, the r.h.s. of (5) indeed defines a probability measure on $\mathcal{B}(\mathbb{R})$. For the proof of (10) we introduce the family of *magnetic translation operators* $\{T_x\}_{x\in\mathbb{R}^2}$ which are unitary on $\mathrm{L}^2(\mathbb{R}^2)$ and defined by

$$(T_x\varphi)\,(y) := \exp\left[i\frac{B}{2}\,(x_1y_2 - x_2y_1)\right]\,\varphi(y-x), \qquad \varphi \in \mathrm{L}^2(\mathbb{R}^2). \tag{11}$$

They constitute an irreducible representation of the Heisenberg-Weyl group on $P_l\mathrm{L}^2(\mathbb{R}^2)$.[20] Since V is $\mathbb{R}^2$-homogeneous, it follows that

$$T_x^\dagger\,\mathbb{E}\left[P_l\chi_I(P_lVP_l)P_l\right]T_x = \mathbb{E}\left[P_l\chi_I(P_lVP_l)P_l\right] \tag{12}$$

for all $x \in \mathbb{R}^2$ and all $I \in \mathcal{B}(\mathbb{R})$. A suitable variant of Schur's lemma (Prop. 4 of §3, Ch. 5 in Ref. 33) then gives the claimed result. ☐

In Theorem 1 and Theorem 2 below, we will assume that V is a Gaussian random potential in the sense of

Definition 1. A *Gaussian random potential* is a Gaussian[34,35] random field V which is $\mathbb{R}^2$-homogeneous, has zero mean, $\mathbb{E}[V(0)] = 0$, and is characterized by a covariance function $\mathbb{R}^2 \ni x \mapsto C(x) := \mathbb{E}[V(x)V(0)]$ which is continuous at the origin where it obeys $0 < C(0) < \infty$.

Remarks. (1) Our continuity requirement for the covariance function C of a Gaussian random potential implies that C is uniformly continuous and bounded by $C(0)$. Consequently, there exists a separable version of V which is jointly measurable, see Thm. 3.2.2 in Ref. 36. When speaking about a Gaussian random potential, we will tacitly assume that only this version is dealt with.

(2) A Gaussian random potential fulfills the assumption of Proposition 1 with $M = \sqrt{C(0)}$ by the usual "Gaussian combinatorics" to be found, for example, in Lemma 5.3.1 of Ref. 34.

B. Existence and boundedness of the restricted density of states

Wegner estimates[37] have turned out to be an efficient tool for proving the absolute continuity of density-of-states measures for certain random operators and for deriving upper bounds on their respective Lebesgue densities. One method to derive estimates of this genre uses *one-parameter spectral averaging*. It provides upper bounds on the averaged spectral projections of a self-adjoint operator which is perturbed by a bounded positive operator with fluctuating coupling strength. The abstract version of such an averaging, which we will use, is due to Combes and Hislop.[38] It is rephrased as

Lemma 1. *Let K, L, and M be three self-adjoint operators acting on a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$. Moreover, let K and M be bounded such that $\kappa := \inf_{\varphi\in\mathcal{H},\,K\varphi\neq 0}\langle\varphi\,,\,M\,\varphi\rangle/\langle\varphi\,,\,K^2\,\varphi\rangle > 0$ is strictly positive. Finally, let $g \in \mathrm{L}^\infty(\mathbb{R})$ be a Lebesgue-essentially bounded function on the real line, $\|g\|_\infty := \operatorname{ess\,sup}_{\xi\in\mathbb{R}}|g(\xi)| < \infty$. Then the inequality*

$$\int_{\mathbb{R}} \mathrm{d}\xi\;\; |g(\xi)|\,\langle\varphi\,,\,K\,\chi_I(L+\xi M)\,K\,\varphi\rangle \leq |I|\,\frac{\|g\|_\infty}{\kappa}\,\langle\varphi,\varphi\rangle \tag{13}$$

holds for all $\varphi \in \mathcal{H}$ and all $I \in \mathcal{B}(\mathbb{R})$.

Proof. See Cor. 4.2 in Ref. 38 and Lemma 3.1 in Ref. 29. ☐

If one focuses only on the absolute continuity of the measure ν_l without aspiring after sharp upper bounds on the resulting Lebesgue density, a straightforward application of Lemma 1 yields the following

Theorem 1. *Let V be a Gaussian random potential in the sense of Definition 1. Suppose that there exists a finite signed Borel measure μ on $\mathbb{R}^2$ such that the covariance function C of V obeys*

$$0 \leq C_\mu(x) := \int_{\mathbb{R}^2} \mu(\mathrm{d}^2 y)\, C(x-y) < \infty, \qquad \int_{\mathbb{R}^2} \mu(\mathrm{d}^2 y)\, C_\mu(y) = 1 \tag{14}$$

for all $x \in \mathbb{R}^2$. Then the l^{th} restricted density-of-states measure ν_l is absolutely continuous with respect to the Lebesgue measure and the resulting Lebesgue probability density w_l, the restricted density of states*, is uniformly bounded according to*

$$w_l(E) := \frac{\nu_l(\mathrm{d}E)}{\mathrm{d}E} \leq \frac{1}{\sqrt{2\pi}\; \|P_l C_\mu P_l\|} \tag{15}$$

for Lebesgue-almost all energies $E \in \mathbb{R}$. Moreover, any given $E \in \mathbb{R}$ is not an eigenvalue of $P_l\, V^{(\omega)}\, P_l$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Remarks. (1) The second equality in (14) is just a convenient normalization of μ. The measure μ allows one to optimize the upper bound in (15) (see Example 2 below) as well as to apply Theorem 1 to Gaussian random potentials with certain covariance functions C taking on also negative values. One such example is $C(x) = C(0)\, \exp\left[-|x|^2/(2\tau^2)\right]\, \left[1 - 7|x|^2/(16\tau^2) + |x|^4/(32\tau^4)\right]$ with arbitrary length scale $\tau > 0$. This may be seen by choosing the Gaussian measure $\mu\left(\mathrm{d}^2 x\right) = \mathrm{d}^2 x\, N\, \exp\left[-|x|^2/(8\tau^2)\right]$ with a suitable normalization factor $N > 0$. Of course, for the oscillating covariance function of Example 1 no μ exists yielding the positivity condition (14).

(2) We note that $0 < \langle \psi_{l,0}\,, C_\mu\, \psi_{l,0}\rangle \leq \|P_l C_\mu P_l\| < \infty$. The first (strict) inequality follows from the assumptions on C in Definition 1, Eq. (14), and the explicit form (9) of $\psi_{l,0}$.

Proof of Theorem 1. The proof consists of two parts. In the first part, we use the fact that the Gaussian random potential V admits a one-parameter decomposition into a standard Gaussian random variable λ and a non-homogeneous zero-mean Gaussian random field U which are defined by

$$\lambda^{(\omega)} := \int_{\mathbb{R}^2} \mu(\mathrm{d}^d y)\, V^{(\omega)}(y), \qquad U^{(\omega)}(x) := V^{(\omega)}(x) - \lambda^{(\omega)}\, C_\mu(x). \tag{16}$$

The positive bounded function C_μ is defined in (14). Since $\mathbb{E}\left[\lambda\, U(x)\right] = 0$ for all $x \in \mathbb{R}^2$, λ and U are stochastically independent. We now multiply both sides of (10) from the left and right by $K := (P_l C_\mu P_l)^{1/2}$ and take the quantum-mechanical expectation with respect to an arbitrary non-zero $\varphi \in P_l \mathrm{L}^2(\mathbb{R}^2)$ to obtain

$$\begin{aligned} \nu_l(I)\ \langle \varphi\,, K^2 \varphi\rangle &= \mathbb{E}\Big[\langle \varphi\,,\, K\, \chi_I(P_l V P_l)\, K\varphi\rangle\Big] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \mathrm{d}\xi\, \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, \langle \varphi\,,\, K\, \chi_I(P_l U P_l + \xi K^2)\, K\, \varphi\rangle\right] \leq |I|\, \frac{\langle \varphi\,,\, \varphi\rangle}{\sqrt{2\pi}}. \end{aligned} \tag{17}$$

For the second equality we used the one-parameter decomposition (16) and the stochastic independence of λ and U. The Lebesgue integral in (17), which constitutes a partial averaging, is then bounded with the help of (13) uniformly in ω. The absolute continuity of ν_l with respect to the Lebesgue measure is now a consequence of (17) and the Radón-Nikodým theorem. Minimizing the upper bound on $\nu_l(I)$, coming from (17), with respect to $\varphi \in P_l \mathrm{L}^2(\mathbb{R}^2)$ yields the claimed inequality (15).

In the second part, we note that (10) implies the equivalence: ν_l has no pure points, that is, $\nu_l(\{E\}) = 0$ for all $E \in \mathbb{R}$, if and only if $\mathbb{E}\left[\langle\varphi,\, \chi_{\{E\}}(P_l V P_l)\,\varphi\rangle\right] = 0$ for all $E \in \mathbb{R}$ and all $\varphi \in P_l\mathrm{L}^2(\mathbb{R}^2)$. Given an orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}}$ in $P_l\mathrm{L}^2(\mathbb{R}^2)$, there hence exists for every $k \in \mathbb{N}$ some $\Omega_k \in \mathcal{A}$ with $\mathbb{P}(\Omega_k) = 1$ such that $\langle\varphi_k,\, \chi_{\{E\}}(P_l V^{(\omega)} P_l)\varphi_k\rangle = 0$ for all $\omega \in \Omega_k$. As a consequence, $\chi_{\{E\}}(P_l V^{(\omega)} P_l) = 0$ for all $\omega \in \cap_{k\in\mathbb{N}}\Omega_k$, hence $\mathbb{P}$-almost all $\omega \in \Omega$. ☐

Remarks. (1) If the spectral measure $\widetilde{C}$ has a (positive) Lebesgue density, C admits the representation $C(x) = \int_{\mathbb{R}^2}\mathrm{d}^2y\ \gamma(x+y)\,\gamma(y)$ with some $\gamma \in \mathrm{L}^2(\mathbb{R}^2)$. If furthermore there exists some $f \in \mathrm{L}^2(\mathbb{R}^2)$ with $\langle f, f\rangle = 1$ such that $0 \leq u(x) := \int_{\mathbb{R}^2}\mathrm{d}^2y\ f(x+y)\,\gamma(y) < \infty$ and $u \neq 0$, one may replace C_μ in (15) by u to obtain another upper bound on w_l for the given Gaussian random potential V. Roughly speaking, the idea is to write V as $V^{(\omega)}(x) = \int_{\mathbb{R}^2}\mathrm{d}^2y\,\gamma(x+y)\,W^{(\omega)}(y)$, where W is the standard delta-correlated (generalized) Gaussian random field on $\mathbb{R}^2$ informally characterized by $\mathbb{E}\left[\,W(x)\right] = 0$ and $\mathbb{E}\left[\,W(x)W(y)\right] = \delta(x-y)$. The Gaussian random potential V hence admits a one-parameter decomposition into the standard Gaussian random variable $\lambda^{(\omega)} := \int_{\mathbb{R}^2}\mathrm{d}^2y\, f(y)\,W^{(\omega)}(y)$ and the non-homogeneous Gaussian random field $U^{(\omega)}(x) := V^{(\omega)}(x) - \lambda^{(\omega)}\,u(x)$, which is stochastically independent of λ.

(2) The essential ingredients of the above proof are the operator identity (10) and the fact that the (not necessarily Gaussian) random potential admits a one-parameter decomposition[39,29] $V^{(\omega)}(x) = U^{(\omega)}(x) + \lambda^{(\omega)}u(x)$ into a positive function u, a random field U, and a random variable λ whose conditional probability measure with respect to the sub-sigma-algebra generated by the family of random variables $\{U(x)\}_{x\in\mathbb{R}^2}$ has a bounded Lebesgue density ϱ. Following the lines of reasoning of the above proof, the restricted density of states may then be shown to be bounded according to

$$w_l(E) = \frac{\nu_l(\mathrm{d}E)}{\mathrm{d}E} \leq \frac{\|\varrho\|_\infty}{\|P_l u P_l\|} \tag{18}$$

for Lebesgue-almost all $E \in \mathbb{R}$. Moreover, any given $E \in \mathbb{R}$ is not an eigenvalue of $P_l V^{(\omega)} P_l$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

(3) The energy-independent estimate (15) is rather rough, because one expects $w_l(E)$ to fall off to zero for energies E approaching the edges $\pm\infty$ (if $\sigma_l^2 > 0$) of the almost-sure spectrum of $P_l V P_l$. More precisely, in the present case of a Gaussian random potential V it follows from arguments in Ref. 12 that the leading asymptotic behavior of the restricted integrated density of states for $|E| \to \infty$ is Gaussian according to

$$\lim_{E\to-\infty}\frac{\ln\nu_l(]-\infty,E[)}{E^2} = \lim_{E\to\infty}\frac{\ln\nu_l(]E,\infty[)}{E^2} = -\frac{1}{2\Gamma_l^2}, \tag{19}$$

where the *decay energy* Γ_l is the solution of the maximization problem

$$\Gamma_l^2 := \sup_{\substack{\varphi\in P_l\mathrm{L}^2(\mathbb{R}^2)\\ \langle\varphi,\varphi\rangle=1}} \gamma^2(\varphi), \quad \gamma^2(\varphi) := \mathbb{E}\left[\langle\varphi, V\varphi\rangle^2\right] = \int_{\mathbb{R}^2}\mathrm{d}^2x\int_{\mathbb{R}^2}\mathrm{d}^2y\,|\varphi(x)|^2\,|\varphi(y)|^2\,C(x-y). \tag{20}$$

We recall from Ref. 12 the inequalities $\sigma_l^4/C(0) \leq \Gamma_l^2 \leq \sigma_l^2$.

Gaussian random potentials with positive covariance functions nicely illustrate Theorem 1.

Example 2. If $0 \leq C(x) < \infty$ for all $x \in \mathbb{R}^2$, the optimal μ in (15) belongs to the class of positive measures of the form $\mu(\mathrm{d}^2x)\,\gamma(\varphi) = \mathrm{d}^2x\ |\varphi(x)|^2$ with $\varphi \in P_l\mathrm{L}^2(\mathbb{R}^2)$, $\langle\varphi,\varphi\rangle = 1$, and $\gamma(\varphi)$ as defined in (20). Optimizing with respect to φ yields

$$w_l(E) \leq \frac{1}{\sqrt{2\pi\Gamma_l^2}}, \tag{21}$$

where Γ_l is the decay energy of the restricted integrated density of states. In particular, for the Gaussian covariance function $C(x) = \alpha^2 \exp\left[-|x|^2/(2\tau^2)\right]/(2\pi\tau^2)$ with correlation length $\tau > 0$ and single-site variance $C(0) = \alpha^2/(2\pi\tau^2) > 0$, one has explicitly[12]

$$\Gamma_l^2 = \gamma^2(\varphi_{l,-l}) = \frac{\alpha^2}{2\pi\tau^2}\left(\frac{B\tau^2}{B\tau^2+2}\right)^{l+1} \mathrm{P}_l\left(\frac{(B\tau^2+1)^2+1}{(B\tau^2+1)^2-1}\right), \tag{22}$$

where the maximizer $\varphi_{l,-l}$ is given in (31) below and $\mathrm{P}_l(\xi) := (1/l!\,2^l)(\mathrm{d}^l/\mathrm{d}\xi^l)(\xi^2-1)^l$ is the l^{th} Legendre polynomial.[9]

Remarks. (1) That the class of measures referred to in Example 2 contains indeed the optimal one, derives from the Fourier representation

$$\langle \varphi\,, C_\mu\, \varphi\rangle = \int_{\mathbb{R}^2} \widetilde{C}\left(\mathrm{d}^2k\right) \left(\int_{\mathbb{R}^2} \mathrm{d}^2x\, |\varphi(x)|^2\, e^{ik\cdot x}\right) \left(\int_{\mathbb{R}^2} \mu\left(\mathrm{d}^2y\right)\, e^{-ik\cdot y}\right) \tag{23}$$

valid for all $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$. Since $\widetilde{C}$ is positive, the claim follows from (23) with the help of the Cauchy-Schwarz inequality and the positivity of C.

(2) In the physics literature one often considers the limit of a *delta-correlated* Gaussian random potential informally characterized by $C(x) = \alpha^2\,\delta(x)$ with some $\alpha > 0$. It emerges from the Gaussian random potential with the Gaussian covariance function given between (21) and (22) in the limit $\tau \downarrow 0$. In this limit (22) reduces to $\Gamma_l^2 = \left(\alpha^2 B/4\pi\right)(2l)!/(l!\,2^l)^2$ and the variance of ν_l becomes, by (7), independent of the Landau-level index, $\sigma_l^2 = \sigma_0^2 = \alpha^2 B/(2\pi)$. Remarkably, in this limit explicit expressions for w_0 and w_l, in the additional high Landau-level limit $l \to \infty$, are available. The first result is due to Wegner[18] and reads

$$w_0\,(E) = \frac{2}{\pi^{3/2}\sigma_0}\,\frac{\exp(\eta^2)}{1+\left[2\pi^{-1/2}\int_0^\eta \mathrm{d}\xi\, \exp(\xi^2)\right]^2}, \qquad \eta := \frac{E}{\sigma_0}, \tag{24}$$

also see Refs. 19, 20, 13. Of course, when specializing the bound in (21), it is consistent with (24) because $2 < \pi$. As for the second result, it is known[21,22] that w_l approaches for $l \to \infty$ a semi-elliptic probability density,

$$\lim_{l\to\infty} w_l(E) = \frac{1}{2\pi\sigma_0}\,\Theta\left(4-\eta^2\right)\,\sqrt{4-\eta^2}, \qquad \eta = \frac{E}{\sigma_0}. \tag{25}$$

Unfortunately, in the delta-correlated limit the bound in (21) diverges asymptotically like $l^{1/4}/(\pi^{1/4}\sigma_0)$ as $l \to \infty$.

(3) Different from (25), for the above Gaussian covariance function with a strictly positive correlation length $\tau > 0$, the high Landau-level limit (informally) reads $\lim_{l\to\infty} w_l(E) = \delta(E)$. This follows from Chebyshev's inequality and the fact that the Landau-level broadening vanishes in this limit if $\tau > 0$: $\lim_{l\to\infty} \sigma_l^2 = 0$.[12] In agreement with that, the bound in (21) diverges in this case, as may be seen either from $0 \le \Gamma_l^2 \le \sigma_l^2$, valid[12] for any covariance function, or directly from (22).

(4) The existence of a bounded w_0 in the delta-correlated limit of a Gaussian random potential stands in contrast to situations with random point impurities, $V^{(\omega)}(x) = \sum_j \lambda_j^{(\omega)}\,\delta\big(x - p_j^{(\omega)}\big)$. To our knowledge, the following four cases have been considered so far:

(a) the *impurity positions* $p_j \in \mathbb{R}^2$ randomly located according to Poisson's distribution and the *coupling strengths* $\lambda_j \in \mathbb{R}$ non-random, strictly positive, and all equal.[19,28]

(b) $p_j \in \mathbb{R}^2$ randomly located according to Poisson's distribution and $\lambda_j \in \mathbb{R}$ independently, identically distributed[19] according to a probability measure whose support is a compact interval containing the origin.[27]

(c) $p_j \in \mathbb{Z}^2$ non-random and $\lambda_j \in \mathbb{R}$ independently, identically distributed according to a bounded probability density whose support is a compact interval containing the origin.[16]

(d) $p_j = j + d_j$ with $j \in \mathbb{Z}^2$ non-random and the *displacements* $d_j \in \mathbb{R}^2$ independently, identically distributed according to a bounded probability density with support contained in the unit square $]-1/2, 1/2[^2 \subset \mathbb{R}^2$. Moreover, $\lambda_j \in \mathbb{R}$ as in the previous case.[17]

In either of these cases, it has been shown that $P_0 V^{(\omega)} P_0$ has an infinitely degenerate eigenvalue at zero energy for $\mathbb{P}$-almost all $\omega \in \Omega$, if the magnetic-field strength B is sufficiently large.

C. Gaussian upper bound on the restricted density of states

As already pointed out, the estimate (15) is rather rough, because it does not depend on the energy. Fortunately, under an additional isotropy assumption and with more effort one may construct an energy-dependent estimate.

Theorem 2. *Suppose the situation of Theorem 1 and that the there defined convolution C_μ is spherically symmetric (with respect to the origin). Then the l^{th} restricted density of states w_l is bounded by a Gaussian in the sense that*

$$w_l(E) \le \frac{1}{\sqrt{2\pi}\, \langle \psi_{l,0}\, , \, C_\mu\, \psi_{l,0} \rangle} \exp\left(-\frac{E^2}{2C(0)} \right) \tag{26}$$

for Lebesgue-almost all energies $E \in \mathbb{R}$. [*Here $\psi_{l,0}$ is defined in (9).*]

Remarks. (1) Equality holds in (26) (and (27) below) with $\langle \psi_{l,0}\, , \, C_\mu\, \psi_{l,0} \rangle = \sqrt{C(0)} = \sigma_l = \Gamma_l$ in the simple extreme case of a spatially constant Gaussian random potential V, that is, if $C(x) = C(0)$ for all $x \in \mathbb{R}^2$. Of course, V is not ergodic in this case. For a lucid discussion of ergodicity and related notions in the theory of random (Schrödinger) operators, see Ref. 40.

(2) In view of (19), we conjecture the true leading decay of $w_l(E) = d\nu_l(]-\infty, E[)/dE$ for $|E| \to \infty$ to be Gaussian with decay energy Γ_l. This energy is strictly smaller than $\sqrt{C(0)}$, if not $C(x) = C(0)$ for all $x \in \mathbb{R}^2$.

(3) Using $\exp\big(-E^2/2C(0)\big) \le 1$ in (26), one obtains an energy-independent estimate which in general is weaker than (15) because $\langle \psi_{l,0}\, , \, C_\mu\, \psi_{l,0} \rangle \le \|P_l C_\mu P_l\|$. In particular this is true in the delta-correlated limit in which the energy-dependence of the bound in (26) disappears anyway.

Gaussian random potentials with positive, spherically symmetric covariance functions illustrate Theorem 2.

Example 3. For a positive covariance function $0 \le C(x) < \infty$, which is additionally spherically symmetric, the prefactor of the Gaussian in (26) is minimized by taking $\mu(\mathrm{d}^2x)\, \gamma(\psi_{l,0}) = \mathrm{d}^2x\, |\psi_{l,0}(x)|^2$ so that $\langle \psi_{l,0}\, , \, C_\mu\, \psi_{l,0} \rangle = \gamma(\psi_{l,0})$. By the Fourier representation (23) and Jensen's inequality, with $\widetilde{C}/C(0)$ as the underlying Borel probability measure on $\mathbb{R}^2$, one finds that $\gamma^2(\psi_{l,0}) \ge \sigma_l^4/C(0)$. Therefore, the estimate (26) may be weakened to the following more explicit one

$$w_l(E) \le \frac{C(0)}{\sigma_l^2} \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{E^2}{2C(0)} \right), \tag{27}$$

where σ_l^2 is the variance of ν_l, see (7). Alternatively, (27) may be obtained directly from (26) by choosing $\mu(\mathrm{d}^2x)\, \sqrt{C(0)} = \mathrm{d}^2x\, \delta(x)$ so that $C_\mu(x) = C(x)/\sqrt{C(0)}$.

Remark. For the Gaussian covariance function $C(x) = C(0) \exp\big[-|x|^2/(2\tau^2)\big]$, it is known[23,12] that $\gamma^2(\psi_{0,0}) = \Gamma_0^2 = C(0)\, B\tau^2/(B\tau^2 + 2)$, also see (22). Theorem 2 together with the minimizing result mentioned in Example 3 therefore gives the estimate

$$w_0(E) \le \sqrt{\frac{B\tau^2 + 2}{B\tau^2}} \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{E^2}{2C(0)} \right) \tag{28}$$

for the restricted density of states of the lowest Landau band. In this setting w_0 has been approximately constructed using a continued-fraction approach.[25] In accordance with the first remark below (26), a comparison with this approximation supports the fact that the estimates (28), (27), and (26) are the sharper, the longer the distance is over which the fluctuations of the Gaussian random potential are significantly correlated, more precisely, the larger the squared length ratio $B\tau^2$ is.

III. PROOF OF THE GAUSSIAN UPPER BOUND

The proof of Theorem 2 requires two major ingredients, an approximation result (Proposition 2) and a Wegner-type of estimate (Proposition 3). We defer the details and proofs of these results to Subsections III.B and III.C. Taking these results for granted, the arguments for the validity of Theorem 2 are as follows.

Proof of Theorem 2. Since the restricted density-of-states measure is even for a (zero-mean) Gaussian random potential, that is, $\nu_l(I) = \nu_l(-I)$ for all $I \in \mathcal{B}(\mathbb{R})$, and since we already know from Theorem 1 that the density of states w_l exists and is bounded by a constant which does not exceed the prefactor of the Gaussian in (26), it is sufficient to consider ν_l on the strictly negative half-line $]-\infty, 0[$.

We now use Proposition 2 to show that a suitably defined sequence of probability measures $(\nu_{l,n})_{n\in\mathbb{N}}$ (see (45) below) converges weakly to ν_l as $n \to \infty$. Given $E_1 < E_2 \leq 0$, we introduce the open interval $I :=]E_1, E_2[$. Then we have

$$\nu_l(I) \leq \liminf_{n\to\infty} \nu_{l,n}(I), \tag{29}$$

by the portmanteau theorem (Thm. 30.10 in Ref. 41). We now use Proposition 3 to estimate the prelimit expression and obtain

$$\nu_l(I) \leq \frac{|I|}{\sqrt{2\pi}\,\langle\psi_{l,0}\,,\, C_\mu\,\psi_{l,0}\rangle} \exp\left(\beta E + \frac{\beta^2}{2} C(0)\right), \tag{30}$$

for all $E \in [E_2, 0]$ and all $\beta \geq 0$. Choosing $\beta = -E/C(0) \geq 0$ gives the claimed upper bound on w_l for $E < 0$. □

Before we proceed with the proofs of the approximation result and the Wegner-type of estimate, which were needed in the above proof, we collect some preparations in the next subsection.

A. Angular-momentum eigenfunctions

The functions

$$x \mapsto \varphi_{l,k}(x) := \sqrt{\frac{l!}{(l+k)!}} \left[\sqrt{\frac{B}{2}}\,(x_1 + ix_2)\right]^k \mathrm{L}_l^{(k)}\left(\frac{B\,|x|^2}{2}\right) \sqrt{\frac{B}{2\pi}} \exp\left(-\frac{B\,|x|^2}{4}\right) \tag{31}$$

constitute[7] with $k \in \{-l, -l+1, \dots\}$ an orthonormal basis in the l^{th} Landau-level eigenspace $P_l\mathrm{L}^2(\mathbb{R}^2)$. In fact, $\varphi_{l,k}$ is an *eigenfunction of the (perpendicular component of the canonical) angular-momentum* operator $L_3 := i\,(x_2\partial/\partial x_1 - x_1\partial/\partial x_2)$ corresponding to the eigenvalue k, that is, $L_3\varphi_{l,k} = k\,\varphi_{l,k}$.

Lemma 2. *Let* $u : \mathbb{R}^2 \to [0, \infty[$ *be a measurable, positive, bounded, and spherically symmetric function. Then the operator inequality*

$$P_l u_x P_l \geq \langle\psi_{l,0}\,,\, u\,\psi_{l,0}\rangle\, \psi_{l,x}\langle\psi_{l,x}\,,\cdot\,\rangle \tag{32}$$

holds for all $x \in \mathbb{R}^2$. *Here the function* $u_x(\,\cdot\,) := u(\,\cdot\, - x)$ *is the* x*-translate of* u *and* $\psi_{l,x}\langle\psi_{l,x}\,,\cdot\,\rangle$ *denotes the orthogonal projection operator onto the one-dimensional subspace spanned by* $\psi_{l,x}$*, see (9).*

Proof. Since the function u is spherically symmetric, the operator $P_l u P_l$ is diagonal in the angular-momentum basis such that

$$P_l u P_l = \sum_{k=-l}^{\infty} \langle \varphi_{l,k}, u\,\varphi_{l,k}\rangle\, \varphi_{l,k}\langle\varphi_{l,k}\,,\cdot\,\rangle \geq \langle \varphi_{l,0}, u\,\varphi_{l,0}\rangle\, \varphi_{l,0}\langle\varphi_{l,0}\,,\cdot\,\rangle. \tag{33}$$

The shifted operator $P_l u_x P_l = T_x\, P_l\, u\, P_l\, T_x^{\dagger}$ results from the l.h.s. of (33) by a unitary transformation with the magnetic translation T_x, see (11). The proof is hence completed by observing that $\varphi_{l,0} = \psi_{l,0}$ and $\psi_{l,x} = T_x \psi_{l,0}$. ☐

Subsequently, we will consider the n-dimensional subspaces $P_{l,n}\mathrm{L}^2(\mathbb{R}^2) \subset P_l\mathrm{L}^2(\mathbb{R}^2)$ spanned by the first n angular-momentum eigenfunctions. The orthogonal projection $P_{l,n}$ is therefore defined by

$$P_{l,n} := \sum_{k=-l}^{n-l-1} \varphi_{l,k}\langle\varphi_{l,k}\,,\cdot\,\rangle, \qquad n \in \mathbb{N}. \tag{34}$$

The completeness of $\{\varphi_{l,k}\}$ in $P_l\mathrm{L}^2(\mathbb{R}^2)$ implies the strong-limit relation $\text{s-}\lim_{n\to\infty} P_{l,n} = P_l$ on $\mathrm{L}^2(\mathbb{R}^2)$. The projections $P_{l,n}$ are integral operators with (continuous) kernels $P_{l,n}(x,y)$, whose diagonals are given by $P_{l,n}(x,x) = B\, G_{l,n}\big(B\,|x|^2/2\big)\,/2\pi \leq B/2\pi$. Here the function

$$G_{l,n}\,(\xi) := e^{-\xi} \sum_{k=-l}^{n-l-1} \frac{l!}{(k+l)!}\, \xi^k \left(\mathrm{L}_l^{(k)}(\xi)\right)^2, \qquad \xi \geq 0, \qquad n \in \mathbb{N}, \tag{35}$$

is approximately one and approximately zero for ξ smaller and larger than $n - 1/2$, respectively. Moreover, the length of the interval on which its values differ significantly from one and zero does not depend on n, also see the remark after the following

Lemma 3. *Let* $G_{l,n}$ *be defined by (35). Then the following scaling-limit relation holds*

$$\lim_{n\to\infty} G_{l,n}(n\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi < 1 \\ 0 & \text{if} \quad 1 < \xi < \infty. \end{cases} \tag{36}$$

Moreover, for every $l \in \mathbb{N}_0$ *there exist an* $N_l \in \mathbb{N}$ *and a real* $A_l > 0$*, such that* $0 \leq G_{l,n}(n\xi) \leq A_l\, e^{-\xi}$ *for all* $\xi \geq 0$ *and all* $n \geq N_l$*.*

Remark. With more effort one may even prove that for every $l \in \mathbb{N}_0$ there exists some polynomial $\zeta \mapsto \mathrm{Pol}(\zeta, l)$ of maximal degree $2l + 1$ such that

$$0 \leq G_{l,n}\Big((\sqrt{n - 1/2} + \zeta)^2\Big) \leq e^{-\zeta^2}\, \mathrm{Pol}(\zeta, l) \tag{37}$$

for all $n \in \mathbb{N}$ and all $\zeta \geq 0$. Moreover,

$$1 - e^{-\zeta^2}\, \mathrm{Pol}(-\zeta, l) \leq G_{l,n}\Big((\sqrt{n - 1/2} - \zeta)^2\Big) \leq 1 \tag{38}$$

for all $n \in \mathbb{N} + l$ and all $0 \leq \zeta \leq \sqrt{n - 1/2}$.

Proof of Lemma 3. The proof is based on the following recurrence relation

$$G_{l,n}(\xi) - G_{l-1,n}(\xi) = -\,e^{-\xi}\, \frac{(l-1)!}{(n-1)!}\, \xi^{n-l}\, \mathrm{L}_{l-1}^{(n-l)}(\xi)\, \mathrm{L}_l^{(n-l)}(\xi) =: D_{l,n}(\xi) \tag{39}$$

for all $l \geq 1$. It follows from the fact that $D_{l,n}$ may be written as a telescope sum according to

$$D_{l,n}(\xi) = e^{-\xi} \sum_{k=-l+1}^{n-l} \frac{(l-1)!}{(k+l-1)!}\, \xi^k \Big[\frac{k+l-1}{\xi}\, \mathrm{L}_{l-1}^{(k-1)}(\xi)\, \mathrm{L}_{l}^{(k-1)}(\xi) - \mathrm{L}_{l-1}^{(k)}(\xi)\, \mathrm{L}_{l}^{(k)}(\xi) \Big]. \tag{40}$$

Equation 8.971(4) in Ref. 9 may be written as $(k+l-1)\, \mathrm{L}_{l-1}^{(k-1)}(\xi) = \xi\, \mathrm{L}_{l-1}^{(k)}(\xi) + l\, \mathrm{L}_{l}^{(k-1)}(\xi)$. Using this together with Eq. 8.971(5) in Ref. 9, the difference in the square bracket in (40) is seen to be equal to $\big(\mathrm{L}_{l}^{(k-1)}(\xi)\big)^2\, l/\xi - \big(\mathrm{L}_{l-1}^{(k)}(\xi)\big)^2$. Splitting the sum into two parts yields (39). The proof of (36) then follows by mathematical induction on $l \in \mathbb{N}_0$. In case $l = 0$ we write

$$G_{0,n}(\xi) = e^{-\xi} \sum_{k=0}^{n-1} \frac{\xi^k}{k!} =: e^{-\xi}\, e_n(\xi) - e^{-\xi}\, \frac{\xi^n}{n!} \tag{41}$$

in terms of the incomplete exponential function e_n, (Eq. 6.5.11 in Ref. 42). By Stirling's estimate $n! \geq \sqrt{2\pi n}\, n^n e^{-n}$ for the factorial (Eq. 6.1.38 in Ref. 42) and the elementary inequality $\xi - 1 - \ln \xi \geq 0$, the second term on the r.h.s. of (41) vanishes in the scaling limit (36) such that the claim reduces to the content of Eq. 6.5.34 in Ref. 42 for $l = 0$. For the induction clause, we use the following exponential, hence rough, growth limitation for Laguerre polynomials

$$\big|\mathrm{L}_{l}^{(k)}(\xi)\big| = \Big| \sum_{j=0}^{l} (-1)^j \binom{l+k}{l-j} \frac{\xi^j}{j!} \Big| \leq \sum_{j=0}^{l} (l+k)^{l-j} \frac{\xi^j}{j!} \leq (l+k)^l\, e^{\xi/(l+k)} \tag{42}$$

which is valid for $k \geq 1 - l$ and obtained by bounding the binomial coefficients. Using again Stirling's estimate, this yields the inequality

$$|D_{l,n}(n\xi)| \leq (l-1)!\, e^{3\xi} \left(\frac{n}{\xi}\right)^l e^{-(\xi - 1 - \ln \xi)\, n} \tag{43}$$

for all $l \geq 1$ and all $n \geq 2$. Since $\xi - 1 - \ln \xi > 0$ for all $\xi \neq 1$, we have $\lim_{n\to\infty} D_{l,n}(n\xi) = 0$ and hence $\lim_{n\to\infty} G_{l,n}(n\xi) = \lim_{n\to\infty} G_{l-1,n}(n\xi)$ for all $\xi \neq 1$, which completes the proof of (36).

For a proof of the exponential bound $0 \leq G_{l,n}(n\xi) \leq A_l e^{-\xi}$ with some $A_l > 0$ and n large enough, we first recall that

$$0 \leq G_{l,n}(\xi) \leq G_{l,\infty}(\xi) = 1 \tag{44}$$

for all $\xi \geq 0$, $l \in \mathbb{N}_0$, and $n \in \mathbb{N}$. Using $n^k \leq (n-1)^k\, e$ for $0 \leq k \leq n-1$ in (41), one obtains $G_{0,n}(n\xi) \leq e^{1-\xi}$ for all $\xi \geq 0$. The claimed exponential bound for all $l \in \mathbb{N}_0$ then follows from (43) and (39). ☐

B. Approximating sequence of probability measures on the real line

Employing the $n \times n$ random Hermitian matrices $P_{l,n} V^{(\omega)} P_{l,n}$, we define a sequence $(\nu_{l,n})_{n\in\mathbb{N}}$ of probability measures by

$$\nu_{l,n}(I) := \frac{1}{n}\, \mathbb{E}\left\{ \mathrm{Tr}\left[P_{l,n}\, \chi_I(P_{l,n} V P_{l,n})\, P_{l,n} \right] \right\}, \quad I \in \mathcal{B}(\mathbb{R}). \tag{45}$$

Here the trace $\mathrm{Tr}\left[P_{l,n}\, \chi_I(P_{l,n} V^{(\omega)} P_{l,n})\, P_{l,n} \right]$ is equal to the (random) number of eigenvalues (counting multiplicity) of $P_{l,n} V^{(\omega)} P_{l,n}$ in the Borel set I. For rather general random potentials the sequence $(\nu_{l,n})$ approximates the restricted density-of-states measure ν_l. This is the first ingredient of the proof of Theorem 2.

Proposition 2. *Let* V *be an* $\mathbb{R}^2$*-homogeneous random potential with* $\mathbb{E}\{|V(0)|\} < \infty$. *Moreover, assume that* $P_l V^{(\omega)} P_l$ *and* $P_{l,n} V^{(\omega)} P_{l,n}$ *for all* $n \in \mathbb{N}$ *are self-adjoint on* $\mathrm{L}^2(\mathbb{R}^2)$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$. *Then*

$$\nu_l = \lim_{n\to\infty} \nu_{l,n} \tag{46}$$

in the sense of weak convergence of finite measures.

Remark. The assumptions of the proposition are fulfilled for a Gaussian random potential in the sense of Definition 1, because $P_l V P_l$ is almost surely essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$ by Proposition 1. Moreover, the random matrix operator $P_{l,n} V P_{l,n}$ is almost surely self-adjoint for all $n \in \mathbb{N}$, because of the almost-sure finiteness $|\langle \varphi_{l,j}, V\varphi_{l,k}\rangle| < \infty$ for all $j, k \in \mathbb{N}_0 - l$.

Proof of Proposition 2. The claimed weak convergence of (finite) measures is equivalent to pointwise convergence of their Stieltjes transforms, that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\nu_{l,n}(\mathrm{d}E)}{E-z} = \int_{\mathbb{R}} \frac{\nu_l(\mathrm{d}E)}{E-z} \tag{47}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$, see, for example, Prop. 4.9 in Ref. 43. The spectral theorem and (10) show that the latter convergence follows from

$$\lim_{n\to\infty} \frac{1}{n}\, \mathbb{E}\left\{\left|\mathrm{Tr}\,\left[P_{l,n}\left((P_{l,n}VP_{l,n}-z)^{-1} - (P_lVP_l-z)^{-1}\right)P_{l,n}\right]\right|\right\} = 0. \tag{48}$$

As a self-adjoint operator of finite rank, $P_{l,n}V^{(\omega)}P_{l,n}$ is defined on the whole space $\mathrm{L}^2(\mathbb{R}^2)$ for $\mathbb{P}$-almost all $\omega \in \Omega$, so that we may use the (second) resolvent equation.[44] Together with the fact that $(P_{l,n}V^{(\omega)}P_{l,n}-z)^{-1}$ and $P_{l,n}$ commute with each other, the absolute value of the trace in (48) is hence seen to be equal to

$$\begin{aligned}
&\left|\mathrm{Tr}\,\left[P_{l,n}(P_{l,n}V^{(\omega)}P_{l,n}-z)^{-1}P_{l,n}V^{(\omega)}(P_l-P_{l,n})(P_lV^{(\omega)}P_l-z)^{-1}P_{l,n}\right]\right| \\
&\quad\le \left\|(P_{l,n}V^{(\omega)}P_{l,n}-z)^{-1}\right\| \left\|(P_lV^{(\omega)}P_l-z)^{-1}\right\| \left\|P_{l,n}V^{(\omega)}(P_l-P_{l,n})\right\|_1 \\
&\quad\le |\,\mathrm{Im}\,z|^{-2} \left\|P_{l,n}V^{(\omega)}(P_l-P_{l,n})\right\|_1 .
\end{aligned} \tag{49}$$

Here we employed Hölder's inequality for the trace norm $\|A\|_1 := \mathrm{Tr}\,\left(A^\dagger A\right)^{1/2}$ and $\mathrm{Im}\,z$ denotes the imaginary part of z. The trace norm in (49) is in turn estimated as follows

$$\begin{aligned}
\left\|P_{l,n}V^{(\omega)}(P_l-P_{l,n})\right\|_1 &= \left\|\,\frac{B}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}^2x\; V^{(\omega)}(x)\,\psi_{l,x,n}\,\langle\psi_{l,x}-\psi_{l,x,n},\cdot\,\rangle\,\right\|_1 \\
&\le \frac{B}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}^2x\left|V^{(\omega)}(x)\right|\,\left\|\,\psi_{l,x,n}\,\langle\psi_{l,x}-\psi_{l,x,n},\cdot\,\rangle\,\right\|_1 \\
&= \int_{\mathbb{R}^2}\mathrm{d}^2x\left|V^{(\omega)}(x)\right|\sqrt{P_{l,n}(x,x)\,[P_l(x,x)-P_{l,n}(x,x)]},
\end{aligned} \tag{50}$$

where we introduced the sequence of two-parameter families of complex-valued functions

$$y \mapsto \psi_{l,x,n}(y) := (P_{l,n}\,\psi_{l,x})\,(y), \qquad x \in \mathbb{R}^2. \tag{51}$$

Note that these functions are not normalized, $\langle\psi_{l,x,n}\,,\psi_{l,x,n}\rangle = 2\pi P_{l,n}(x,x)/B = G_{l,n}(B|x|^2/2) \le 1$. Combining (49) and (50), using Fubini's theorem and the homogeneity of the random potential, the l.h.s. of (48) is seen to be bounded from above by

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\mathbb{E}\,[|V(0)|]}{n\,|\mathrm{Im}\,z|^2}\int_{\mathbb{R}^2}\mathrm{d}^2x\sqrt{P_{l,n}(x,x)\,[P_l(x,x)-P_{l,n}(x,x)]} \\
&\qquad = \frac{\mathbb{E}\,[|V(0)|]}{|\mathrm{Im}\,z|^2}\,\lim_{n\to\infty}\int_0^\infty \mathrm{d}\xi\,\sqrt{G_{l,n}(n\xi)\,[1-G_{l,n}(n\xi)]} = 0.
\end{aligned} \tag{52}$$

Here we employed the definition of $G_{l,n}$ (see text above (35)), performed the angular integration, changed variables $n\xi := B|x|^2/2$ in the remaining integral, and used Lemma 3 and the dominated-convergence theorem. ☐

C. A Wegner-type of estimate

The second ingredient for the proof of Theorem 2 is the following

Proposition 3. *In the situation of Theorem 2 let* $E_1 < E_2 \le E \le 0$ *and put* $I :=]E_1, E_2[$. *Then*

$$\nu_{l,n}(I) \le \left(\frac{|I|}{\sqrt{2\pi}\, \langle \psi_{l,0}\,,\, C_\mu\, \psi_{l,0} \rangle} + s_{l,n} \right) \exp\left(\beta E + \frac{\beta^2}{2} C(0) \right), \tag{53}$$

for all $\beta \ge 0$. *Here* $s_{l,n} := \int_1^\infty \mathrm{d}\xi\, G_{l,n}(n\xi)$ *converges to zero as* $n \to \infty$.

Proof. By the definition of $\nu_{l,n}$ and the spectral theorem one has

$$\nu_{l,n}(I) \le \frac{e^{\beta E}}{n}\, \mathbb{E}\left\{ \mathrm{Tr}\left[P_{l,n}\, e^{-\beta P_{l,n} V P_{l,n}} \chi_I(P_{l,n} V P_{l,n})\, P_{l,n} \right] \right\}. \tag{54}$$

We evaluate the trace in an orthonormal eigenbasis of $P_{l,n} V^{(\omega)} P_{l,n}$ and apply the Jensen-Peierls inequality[45] to bound the probabilistic expectation in (54) from above by

$$\begin{aligned} &\mathbb{E}\left\{ \mathrm{Tr}\left[P_{l,n}\, e^{-\beta V} P_{l,n}\, \chi_I(P_{l,n} V P_{l,n})\, P_{l,n} \right] \right\} \\ &\qquad = \frac{B}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}^2 x\; \mathbb{E}\left\{ e^{-\beta V(0)} \langle \psi_{l,x,n}\,,\, \chi_I(P_{l,n} V(\,\cdot\, - x) P_{l,n})\, \psi_{l,x,n} \rangle \right\} \end{aligned} \tag{55}$$

where we used Fubini's theorem and the $\mathbb{R}^2$-homogeneity of V. The Lebesgue integral in (55) over the plane may be split into two parts with domains of integration inside and outside a disk centered at the origin and with radius $\sqrt{2n/B}$.

To estimate the inner part, we use the one-parameter decomposition (16) of the Gaussian random potential V. Since U and λ are stochastically independent, we may estimate the conditional expectation of the integrand in (55) relative to the sub-sigma-algebra generated by $\{U(y)\}_{y \in \mathbb{R}^2}$ with the help of Lemma 1. Taking there $g(\xi) = \exp\big(-\beta\xi\, C_\mu(0) - \xi^2/2 \big)/\sqrt{2\pi}$, $K = \psi_{l,x,n} \langle \psi_{l,x,n}\,,\, \cdot \rangle / \langle \psi_{l,x,n}, \psi_{l,x,n} \rangle$, and $M = P_{l,n} C_\mu(\,\cdot\, - x) P_{l,n} \ge \langle \psi_{l,0}\,,\, C_\mu\, \psi_{l,0} \rangle \langle \psi_{l,x,n}, \psi_{l,x,n} \rangle K^2$, where the last inequality follows from Lemma 2 and the positivity as well as the spherical symmetry of C_μ, we obtain an ω- and x-independent bound according to

$$\begin{aligned} &\int_{\mathbb{R}} \mathrm{d}\xi\, \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, e^{-\beta\xi\, C_\mu(0)} \langle \psi_{l,x,n}\,,\, \chi_I \Big(P_{l,n} U^{(\omega)}(\,\cdot\, - x) P_{l,n} + \xi\, P_{l,n} C_\mu(\,\cdot\, - x) P_{l,n} \Big)\, \psi_{l,x,n} \rangle \\ &\qquad\qquad \le \frac{|I|}{\sqrt{2\pi}\, \langle \psi_{l,0}\,,\, C_\mu\, \psi_{l,0} \rangle} \exp\left(\frac{\beta^2}{2} \left(C_\mu(0) \right)^2 \right). \end{aligned} \tag{56}$$

Using $\mathbb{E}\big[\exp(-\beta U(0)) \big] = \exp\big(\beta^2 \big(C(0) - (C_\mu(0))^2 \big)/2 \big)$, the inner part of the integral in (55), may hence be estimated as follows

$$\begin{aligned} &\int_{|x|^2 \le 2n/B} \mathrm{d}^2 x\; \mathbb{E}\left\{ e^{-\beta V(0)} \langle \psi_{l,x,n}\,,\, \chi_I(P_{l,n} V(\,\cdot - x) P_{l,n})\, \psi_{l,x,n} \rangle \right\} \\ &\qquad\qquad \le \frac{2\pi n}{B} \frac{|I|}{\sqrt{2\pi}\, \langle \psi_{l,0}\,,\, C_\mu\, \psi_{l,0} \rangle} \exp\left(\frac{\beta^2}{2} C(0) \right). \end{aligned} \tag{57}$$

This gives the first part of the claimed inequality (53).

To complete the proof, we estimate the outer part of the integral in (55) as follows

$$\begin{aligned} &\int_{|x|^2 \ge 2n/B} \mathrm{d}^2 x\; \mathbb{E}\Big\{ e^{-\beta V(0)} \langle \psi_{l,x,n}\,, \chi_I(P_{l,n} V(\,\cdot - x) P_{l,n})\, \psi_{l,x,n} \rangle \Big\} \\ &\le \mathbb{E}\left\{ e^{-\beta V(0)} \right\} \int_{|x|^2 \ge 2n/B} \mathrm{d}^2 x\; \langle \psi_{l,x,n}\,,\, \psi_{l,x,n} \rangle = \frac{2\pi n}{B}\, \mathbb{E}\left\{ e^{-\beta V(0)} \right\} \int_1^\infty \mathrm{d}\xi\, G_{l,n}(n\xi). \end{aligned} \tag{58}$$

Here we employed (51) and changed variables $n\xi := B\,|x|^2/2$ to obtain the last equality. Thanks to Lemma 3, the last integral in (58), and hence $s_{l,n}$, converges to zero as $n \to \infty$ by the dominated-convergence theorem. □

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