Regularity of Conjugacies Between Critical Circle Maps: An Experimental Study

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Abstract

We develop numerical implementations of several criteria to asses the regularity of functions. The criteria are based on finite difference method and harmonic analysis: Littlewood-Paley theory and wavelet analysis.

As a first application of the methods, we study the regularity of conjugacies between critical circle maps (i.e., differentiable homeomorphisms with a critical point) with golden mean rotation number. These maps have a very well developed mathematical theory as well as a wealth of numerical studies.

We compare the results produced by our methods among themselves and with theorems in the mathematical literature.

We confirm that several of the features that are predicted by the mathematical results are indeed observable by numerical computation. Some universal numbers predicted can indeed be computed reliably.

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As a result of our calculations, we obtain that several simple upper bounds seem to be sharp in some cases, but not in others. This indicates that there may be conceptually different mechanisms in play.

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1 Introduction

Classification of circle homeomorphisms under changes of variables is an old and famous problem in mathematics. It was initiated in [Poincaré 1885] motivated by studies in differential equations more than a century ago, and has been actively studied ever since.

Circle maps are important also because of their applications to natural sciences. They appear in the Pomeau-Manneville scenario for transition to turbulence through intermittency [Pommeau and Manneville 1980], second order ODEs with periodic potentials [Moser and Pöschel 1984], cardiac arrhythmias [Glass 1995], oscillations in plasma [Ignatov 1995], electronic devices [Bohr et al. 1984], optical resonators with a periodically moving wall [de la Llave and Petrov 1999], to name just a few. We would like to draw the reader's attention to the collections of reprints [Cvitanović 1989] and [Bai-Lin Hao 1990] which contain many articles devoted to circle maps and their applications.

The main dynamical invariant of homeomorphisms of the circle is the rotation number (see Section 2.1). It was quickly realized that it is an invariant under topological equivalence [Poincaré 1885] and that for C^2 maps it is a complete invariant for topological conjugacy [Denjoy 1932]. The theory of smooth equivalence of smooth diffeomorphisms is by now very well understood [Herman 1979], [Katznelson and Ornstein 1989].

Nevertheless, the theory of smooth equivalence of "critical circle maps", i.e., smooth circle maps that are homeomorphisms but not diffeomorphisms (the simplest one – and the only one that we will consider in this paper – being a smooth map with a critical point), is much less developed. This will be the main subject of our empirical studies.

In the articles [Shenker 1982], [Feigenbaum et al. 1982], [Rand et al. 1982], [Ostlund et al. 1983], it was found numerically that cubic critical circle maps exhibit interesting "universal" properties – for large classes of circle maps there exist numbers and functions that are the same for all functions in the class – similar to the Feigenbaum-Coullet-Tresser universality of unimodal maps of the interval. (Lately similar studies were carried out for maps with critical points of higher degrees [Dixon et al. 1997], [Briggs at al. 1998]). Shortly after the initial numerical studies, a renormalization theory that explains these properties was developed and some parts of the theory were put of a firm mathematical basis [Feigenbaum et al. 1982], [Ostlund et al. 1983], [Shraiman 1984], [Epstein 1986], [Eckmann and Epstein 1986], [Eckmann and Epstein 1987], [Lanford 1986], [Lanford 1987], [Lanford 1988], [Rand 1987], [Rand 1988a], [Kim and Ostlund 1989], [Epstein 1989], [Rand 1988b], [Veerman and Tangerman 1990], [Tangerman and Veerman 1991], [Pinto and Rand 1992], [Rand 1992]. Recently there has been a significant progress in the renormalization theory of critical circle maps [Świątek 1998], [de Melo 1998], [Yampolsky 1999], [de Faria 1999], [de Faria and de Melo 1999], [de Faria and de Melo 2000].

In [Shenker 1982], a one parameter family of smooth circle maps, $\{f_K\}$, all of rotation number equal to the golden mean, $\gamma = \frac{\sqrt{5}-1}{2}$, was studied numerically. It was found that if f_K is a diffeomorphism, the conjugacy between f_K and the rotation by γ is a smooth function (as predicted by the general theory [Herman 1979]). However, if f_K has a cubic critical point, the conjugacy between f_K and the rotation becomes very rough. Moreover, it was noticed that the conjugacy has a self-similar structure (which he found by studying the first 400 Fourier coefficients of the conjugacy).

In the present paper, we study numerically the smoothness of the conjugacies between non-critical, cubic critical, and quintic critical circle maps. To estimate the smoothness of the conjugacies, we use finite difference method and tools from harmonic analysis (Littlewood-Paley theory and wavelet theory). In some cases, we are able to calculate reliably millions of Fourier coefficients, so we hope that our numerical estimates are quite convincing.

We expect that the numerical methodology developed here will be used to study several other problems in the theory of critical phenomena in dynamical systems in which the regularity of functions and their self-similar properties play a role. Since the theory of circle maps has a well developed mathematical literature, it seemed a good starting point to asses the validity of the methods.

We also find evidence that the conjugating functions are asymptotically self-similar confirming by very different methods the results obtained previously for cubic critical maps.

The fact that we have precise numbers for the regularity of the conjugacies predicted in the theorems allows us to observe that in some cases some simple upper bounds for the regularity of the conjugacies appear to be sharp, whereas in other cases they seem to be very far from optimal. This indicates that possibly there are conceptually different phenomena at play. (See Section 8.)

The paper is organized as follows. In Section 2 we review some definitions and rigorous results about circle maps and conjugacies between them. In Section 4 we discuss how the conjugacies are calculated and explore them visually. In Section 5 we collect some results from harmonic analysis and in Section 6 we discuss them from point of view of numerically implementations. In Section 7 we show the results from our analysis. It Section 8, we develop some mathematically rigorous arguments that show that if certain scaling relations are present, the regularity of the conjugacies are bounded from above. The scaling relations can be verified numerically. We compare these upper bounds with the regularities calculated directly. In the Conclusion, we briefly recapitulate our findings.

2 Rigorous Results About Conjugacies Between Circle Maps

In this section we recall briefly some basic definitions from the theory of circle maps (for details see, e.g., [Katok and Hasselblatt 1995], [de Melo and Van Strien 1998], [Herman 1979]). We also collect some recent results in the mathematical literature. We cannot aim for completeness, but just want to set the notations and to give the reader a feeling of the results.

2.1 Rotation Number

If $F : \mathbb{R} \to \mathbb{R}$ is a map satisfying $F(x+1) = F(x) + 1$, and if

$$\pi : \mathbb{R} \to \mathbb{T} \equiv \mathbb{R}/\mathbb{Z} : x \mapsto \pi(x) := x \bmod 1 \ ,$$

then the map

$$f := \pi \circ F : \mathbb{T} \to \mathbb{T}$$

is a map of the circle $\mathbb{T}$, and F is called a *lift* of f. In this paper a “circle map” will always mean orientation-preserving circle homeomorphism.

The most important characteristic for the classification of circle maps is the rotation number. If $f : \mathbb{T} \to \mathbb{T}$ is a circle homeomorphism and F is a lift of f, then the *rotation number* $\tau(f)$ of f is defined by

$$\tau(f) := \left(\lim_{n\to\infty} \frac{F^n(x) - x}{n} \right) \bmod 1 \ . \tag{1}$$

The above limit exists and is independent of the choice of the lift and the point $x \in \mathbb{T}$.

The simplest circle map is the *rotation*

$$r_\rho : \mathbb{T} \to \mathbb{T} : x \mapsto (x+\rho) \bmod 1 \ ,$$

whose rotation number obviously is $\tau(r_\rho) = \rho \bmod 1$. For r_ρ, there are two types of orbits of the points of $\mathbb{T}$. If $\rho = p/q \in \mathbb{Q}$ (with p and q relatively prime), then the orbit $\{r_\rho^n(x)\}_{n\in\mathbb{Z}}$ is periodic of minimal period q. If $\rho \notin \mathbb{Q}$, then there are no periodic orbits for the rotation r_ρ and the orbit of any point $x \in \mathbb{T}$ is dense in $\mathbb{T}$.

2.2 Conjugacies; Existence Theorems

Because of the simplicity of the rotations, it is natural to investigate whether a particular circle map f is "equivalent" in some sense to a rotation, i.e., whether there exists a change of variables $y = h(x)$ such that in the new variables f "looks like" a rotation. The definition is the following.

Definition 2.1 *Two circle maps f and g are* topologically (resp. C^k-, smoothly, analytically) conjugate *if there exists a homeomorphism (resp. C^k, C^∞, or C^ω diffeomorphism, resp.) $h : \mathbb{T} \to \mathbb{T}$ such that*

$$f = h^{-1} \circ g \circ h \ . \tag{2}$$

The map h is called the conjugacy *between f and g.*

If f and g are conjugate, many of their properties are the same – e.g., the possible types of the orbits of the points of $\mathbb{T}$ under f and under g are the same, the rotation number of f is equal to that of g, etc.

Early results about the existence of a conjugacy to a rotation are the following theorems [Poincaré 1885], [Denjoy 1932].

Theorem 2.1 (Poincaré) *Assume that the rotation number of the circle map f is irrational and the orbit of some point $x \in \mathbb{T}$ is dense in $\mathbb{T}$. Then f is topologically conjugate to $r_{\tau(f)}$. (If only the first condition is satisfied, then there exists a noninvertible continuous map $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f = r_{\tau(f)} \circ h$.)*

Theorem 2.2 (Denjoy) *A circle diffeomorphism with irrational rotation number and derivative of bounded variation is topologically conjugate to a rotation.*

Denjoy's theorem implies that every C^2 circle map is topologically conjugate to a rotation. The C^2 condition is rather sharp – Herman [Herman 1979, Sec. X.3.19] constructed examples of $C^{2-\varepsilon}$ circle maps that are not conjugate to a rotation, even if the rotation number satisfies additional restrictions.

In Denjoy's theorem, it is important that f^{-1} is differentiable. If f^{-1} is not differentiable, one cannot guarantee the existence of a conjugacy to a rotation even by assuming that f is C^∞. The article [Hall 1981] contains an example of a C^∞ circle map with rotation number ρ (for any irrational $\rho \in [0, 1)$) which has no dense orbit and therefore cannot be conjugate to r_ρ. This map is onto, has no periodic orbits and has no more than two points where the derivative of the map vanishes (in fact, one can construct such a map with one critical point only).

In striking contrast with the C^∞ Denjoy counterexamples of [Hall 1981], it was shown [Yoccoz 1984a] that the maps exhibiting the behavior of those in [Hall 1981] cannot be real analytic. More precisely,

Theorem 2.3 (Yoccoz) *Any real analytic circle map with no periodic orbits is topologically conjugate to a rotation.*

This theorem guarantees that any two real analytic circle maps with irrational rotation number are topologically conjugate (we will use this fact in Section 4.3).

The above result was extended in [Świątek 1991], [Świątek 1998].

2.3 Smoothness of the Conjugacy

The theorems in Section 2.2 do not assert anything about the smoothness of the conjugacy h. Can one say more about the differentiability properties of the conjugacies in the case of smooth or analytic maps? The answer to this question involves number-theoretic properties of the rotation number.

2.3.1 Arithmetic Properties of the Rotation Number

It turns out that, to guarantee that a smooth circle diffeomorphism f is smoothly conjugate to a rotation, it is not enough that $\tau(f)$ be irrational, but also that it should not be well approximable by rationals.

Definition 2.2 *A number ρ is called* Diophantine *of type (C,ν) (or simply of type ν) for positive C and ν, if for any $p/q \in \mathbb{Q}$*

$$\left|\rho - \frac{p}{q}\right| > \frac{C}{|q|^{2+\nu}} \ . \tag{3}$$

A number that is not Diophantine is called a Liouville number*.*

It is easy to show that there are no numbers that satisfy (3) when $\nu < 0$. Hence, we will assume $\nu \geq 0$.

Each irrational number $\rho \in (0,1)$ can be written in the continued fraction representation as

$$\rho = \langle \rho_0, \rho_1, \rho_2, \ldots \rangle := \cfrac{1}{\rho_0 + \cfrac{1}{\rho_1 + \cfrac{1}{\rho_2 + \cdots}}} \ ,$$

where the positive integers ρ_n are called *partial quotients* of ρ.

In this representation, a number is Diophantine if its partial quotients increase slowly. In this sense, the "most irrational" number between 0 and 1 is the *golden mean*

$$\gamma := \langle 1, 1, 1, \ldots \rangle = \frac{\sqrt{5}-1}{2} = \lim_{n\to\infty} \frac{Q_n}{Q_{n+1}} \ ,$$

where $Q_1 = 1$, $Q_2 = 2$, $Q_{n+1} = Q_n + Q_{n-1}$ are the Fibonacci numbers.

When the partial quotients of ρ are bounded, ρ is called a number of *bounded* (or *constant*) type. This is equivalent to the fact that (3) holds with $\nu = 0$. The *combinatorial type* of ρ is by definition $\sup_n \rho_n$.

2.3.2 Results for Non-Critical Maps

The question about the smoothness of the conjugacy in the case of analytic circle maps close to a rotation was first addressed in [Arnold 1961]. There, it was proved that any analytic circle map whose rotation number ρ is Diophantine of type $\nu \geq 0$ and which is close to a rotation is analytically conjugate to r_ρ. This result was extended to the case of finite differentiability in [Moser 1966].

The first theorem without the hypothesis of closeness to a rotation was proved in [Herman 1979]. The theorem states that there exists a set $\mathcal{K} \subset [0,1]$

of full Lebesgue measure such that if $f \in C^k$ for $3 \leq k \leq \omega$ and $\tau(f) \in \mathcal{K}$, then the conjugacy is $C^{k-2-\varepsilon}$ for any $\varepsilon > 0$. The set $\mathcal{K}$ is characterized in terms of the growth of the partial quotients of the continued fraction expansions of its members; all numbers in $\mathcal{K}$ are Diophantine of order ν for any $\nu > 0$. This result was significantly improved in [Yoccoz 1984b]. Similar theorems were proved by using renormalization group techniques [Sinai and Khanin 1989], which lowered the minimal regularity required. The most comprehensive result we have found in the literature is the one in [Katznelson and Ornstein 1989].

Theorem 2.4 *If f is a C^k circle diffeomorphism whose rotation number is Diophantine of order ν, and $k > \nu + 2$, then the homeomorphism h that conjugates f with the rotation $r_{\tau(f)}$ is of class $C^{k-1-\nu-\varepsilon}$ for any $\varepsilon > 0$.*

2.3.3 Results for Critical Maps

The following result was proved recently ([de Faria and de Melo 1999], [de Faria and de Melo 2000], see also [de Melo 1998], [de Faria 1999], [Yampolsky 1999]):

Theorem 2.5 *Let f and g be real analytic critical circle maps with the same rotation number ρ of bounded type. Then f and g are $C^{1+\alpha}$ conjugate for some $\alpha \in (0,1)$ depending only on the combinatorial type, $N := \sup_n \rho_n$, of ρ.*

It was conjectured in [de Melo 1998] that the Hölder exponent of the conjugacy between two critical circle maps whose critical points are if the same order and which have the same rotation number of bounded type does not depend on N (although in the proof of Theorem 2.5 α does depend on N).

For the regularity of the conjugacies between critical maps and rotations, which exist because of Theorem 2.3, the sharpest result we are aware of is from [Świątek 1998], whose Theorem 1.1 (the theorem and a previous proof are credited to an unpublished manuscript of M. Herman) implies:

Theorem 2.6 *Let f be an analytic critical circle map with an irrational rotation number. The conjugacy between the map and a rotation is quasi-symmetric if and only if the rotation number is of constant type.*

We note that it is a well-known fact in the theory of quasi-conformal maps that the quasi-symmetric maps are Hölder ([Väisälä, Sec. 18], [Ghering and Palka, manuscript]). Hence, we can conclude that the conjugacy between a critical map and a rotation is Hölder. Therefore, the conjugacies between critical circle maps of the same rotation number have to be Hölder with exponent 1.

It is not difficult to show that, when the critical maps have different order, the conjugacy between them cannot have Hölder exponent 1.

Putting together Theorem 2.6, the Hölder regularity of quasi-symmetric maps and Theorem 2.5, we obtain:

Corollary 2.1 *The conjugacy between critical circle maps with golden mean rotation number is Hölder. The Hölder exponent depends only on the order of the critical points of the two functions.*

The above results seem to give very little information about what the actual values of these regularities are. In this paper, we will develop methods that allow us to compute these numbers as well as to explore numerically some geometric properties of the conjugacies.

3 Some General Heuristic Remarks on Renormalization and Conjugacies

A unifying point of view in the study of long term dynamics – especially in one-dimensional systems – has been provided by the scaling and renormalization group ideas.

Formulated somewhat loosely, the unifying idea of renormalization group can be formulated as saying that "highly iterated maps, when observed in small scales, have forms that are largely independent of the map" (the universal properties can be different in sets of maps of positive codimension).

In this section, we want to present a heuristic point of view of the relations of the renormalization and smoothness of conjugacy which seems to be applicable to a wide variety of models.

There are several ways to make this widely applicable heuristic formulation, precise and susceptible of theorems. Nevertheless, the rigorous analysis seems to depend on the models considered. Indeed, in Section 2 we have presented a review of the mathematical literature for circle maps that makes precise and establishes rigorously part of the general renormalization group picture in this case.

In the rest of the paper after this section, we will present the numerical tools that we need to verify some aspects of the renormalization group ideas and also the implementation for the circle maps. Since in this case we have a good mathematical understanding, we can asses the validity of the methods.

Note that in the case of circle maps, besides the mathematically rigorous literature, one also has a very well developed theoretical literature (summarized briefly in Section 1) which could also serve as a guide. We have, however, decided not to present a systematic view of it. These theoretical studies have been carried out in a large variety of contexts, which we also cannot hope to summarize here.

The main connection of the study of regularity properties of conjugacies and renormalization group arises from the fact that the regularity of the conjugacies is a very good test of universality properties.

Note that regularity depends on very fine scales. Moreover, formula (12) that we will establish later, makes it clear that the conjugacies in increasingly small scales are determined by the increasingly long recurrence times.

The fact that some conjugacies examined in very small scales are self-similar leads, at least in an informal way, to several consequences of universality for the conjugacies that we will later explore empirically for the case of circle maps.

(A1) The regularity of the conjugacies between maps of the same universality class is a "universal number".

Note that for the case of golden mean circle maps this is a consequence of Corollary 2.1.

(A2) These universal regularities between maps in the same class are higher than those between maps of different classes.

Note that for circle maps, this follows from the observation that critical circle maps can be only Hölder of exponent less than one conjugate to maps with critical points of different order and that by Theorem 2.5, the conjugacies between maps of the same order is $C^{1+\alpha}$.

(A3) The functions giving the conjugacies are asymptotically self-similar.

If h_1 and h_2 are conjugacies of maps f_1, f_2 to the golden mean rotation, (i.e., $f_1 \circ h_1 = h_1 \circ r_\gamma$, $f_2 \circ h_2 = h_2 \circ r_\gamma$), then $f_1 \circ h_1 \circ h_2^{-1} = h_1 \circ h_2^{-1} \circ f_2$. The fact that we observe that $h_1 \circ h_2^{-1} = k$, $h_1^{-1} \circ h_2 = \ell$ are very smooth means that $h_1 = k \circ h_2$, $h_1 = h_2 \circ \ell^{-1}$, in other words, we can obtain h_1 from h_2 by composing with a very smooth map. This makes precise the notion that h_1 and h_2 are very similar. Even if each of them is rather rough, the roughness of one is very precisely comparable to that of the other.

We hope that the present work could serve as a stimulus for further mathematical investigations. Our numerics are precise enough that we can even study the corrections to (A3). We formulate them as

(A4) The convergence to self-similarity is exponentially fast.

Somewhat more precisely (but still very far from a mathematically rigorous statement), we can write conjugacies in the form

$$h(x) = \sum \lambda_1^n H_1(\alpha^n x) + \sum \lambda_2^n H_2(\alpha^n x) + \cdots \tag{4}$$

for some $|\alpha| > 1$, $1 > \lambda_1 > \lambda_2 > \cdots$, where μ, λ_1, λ_2, $\ldots$ are universal numbers.

Of course, scalings such as those in (4) do not, strictly speaking, make sense in the case that the variables are in the circle. Nevertheless, since (4) is supposed to hold in the asymptotic sense of the very small scales, we can identify the whole circle with the real line.

We also note the following conjecture that seems to be reasonable for many areas in which renormalization applies. For the case of period doubling it is studied extensively in [de la Llave and Schafer, manuscript (1996)].

Conjecture 3.1 *Consider the set $\mathcal{M}_\nu$ (for odd integer $\nu \geq 1$) of analytic maps of the circle such that:*

(a) The maps are homeomorphisms.

(b) They are of the form $f(x) = Ax^\nu + O(x^{\nu+1})$ with A a nonzero constant (hence f is a homeomorphism but not a diffeomorphism).

(c) Their rotation number is the golden mean γ.

Note that the sets $\mathcal{M}_\nu$ are manifolds.

Then in an open set $\mathcal{B}_\nu \in \mathcal{M}_\nu$ we can find foliations $\mathcal{F}^i$ integers d_i and numbers λ_i ($d_i > d_{i+1}$, $\lambda_i < \lambda_{i+1}$, $\lambda_i \to \infty$) such that, if W_f^i is the leaf of $\mathcal{F}^i$ passing through the map f, then

(1) $W_f^i \subset W_f^{i+1}$,

(2) W_f^i *is a smooth submanifold of* $\mathcal{M}_\nu$ *of codimension* d_i,

(3) the foliations $\mathcal{F}^i$ *are Hölder,*

(4) if $f \in W_g^i$, *then* f *is* $C^{\lambda_i - \varepsilon}$ *conjugate to* g,

(5) if $f \notin W_g^i$, *then* f *is not* $C^{\lambda_i + \varepsilon}$ *conjugate to* g.

The most important consequence of this conjecture is that the conjugacies between maps in the classes $\mathcal{M}_\nu$ can only have regularities which belong to a discrete set $\{\lambda_i\}$ (ignoring the ε's which are as small as desired). In particular, if we know that a conjugacy is $C^{\lambda_i+\varepsilon}$, we can conclude that it is $C^{\lambda_{i+1}-\varepsilon}$.

In the rest of the paper, we will present methods that allow us to carry high precision calculation of golden mean circle maps as well as an array of methods that asses the regularity of their conjugacies. By comparing the results of these different methods among themselves and with the results of the mathematical literature, we can asses their validity and hope to apply them to other contexts.

In the process of doing that, we also obtain some information about the relation between the regularity and renormalization. In particular, in Section 8, we obtain indications that the regularities may be limited by other mechanisms that just simple scaling phenomena.

4 Computing the Conjugacies

4.1 Examples Studied

Let f and g be analytic circle maps whose derivatives possibly vanish at one point; without loss of generality we can take this point to be $x = 0$. We studied numerically the following families of analytic circle maps (for values of K for which the maps are invertible):

(i) the noncritical (N) family $(0 \le K < 1)$

$$f^{\mathrm{N}}_{K,\omega}(x) = \left(x + \omega - \tfrac{K}{2\pi} \sin 2\pi x\right) \bmod 1 \; ; \tag{5}$$

(ii) the cubic critical (C) family ($0 \le K < \frac{4}{3}$)

$$f^{\mathrm{C}}_{K,\omega}(x) = \left[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x + \tfrac{1-K}{2} \sin 4\pi x)\right] \bmod 1 \ , \tag{6}$$

where the coefficients are chosen in such a way that for every K,

$$f^{\mathrm{C}}_{K,\omega}(x) = \omega + \tfrac{2\pi^2(4-3K)}{3} x^3 + O(x^5) \ ;$$

(iii) the quintic critical (Q) family ($\frac{1}{2} \le K < \frac{3}{2}$)

$$f^{\mathrm{Q}}_{K,\omega}(x) = \left[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x + \tfrac{9-8K}{10} \sin 4\pi x + \tfrac{3K-4}{15} \sin 6\pi x)\right] \bmod 1 \ , \tag{7}$$

where the coefficients are chosen in such a way that for every K,

$$f^{\mathrm{Q}}_{K,\omega}(x) = \omega + \tfrac{8\pi^4(3-2K)}{5} x^5 + O(x^7) \ .$$

Of course, there are similar formulae for higher order critical points, but the numerics cannot be carried out easily.

We studied the case of rotation number equal to the golden mean γ:

$$\tau(f^{\bullet}_{K,\omega})\big|_{\omega = \Omega^{\bullet}(K)} = \gamma \ , \tag{8}$$

where $\bullet$ stands for N, C, or Q. The golden mean is chosen because its continued fraction expansion is periodic (and simple). Hence, renormalization arguments can be expressed in terms of operators and, since all partial quotients of γ are 1, renormalization operators are as simple as possible.

Since we have to iterate $f^{\bullet}_{K,\omega}$, we need to know the value of $\Omega^{\bullet}(K)$ with a very high precision. To achieve this, we used the C++ software package `doubledouble` [Briggs 2000]. This package allowed us to use about 30 decimal places floating point precision arithmetic, and to find 24–25 digits of the parameter $\Omega^{\bullet}(K)$ in the case of N circle maps and about 16 digits of $\Omega^{\bullet}(K)$ in the C and Q cases.

To double-check the results, we also used the GNU MP library [GMP 2000] – a public domain library for arbitrary precision arithmetic. We wrote subroutines for high precision trigonometric functions, which we did by using local Taylor series expansion, and tested them by using the following elegant and numerically stable method.

For x, y positive, define $x_1 := x$, $y_1 := y$, and

$$x_{n+1} := \frac{x_n + y_n}{2} \ , \quad y_{n+1} := \sqrt{x_n y_n}$$

for $n \in \mathbb{N}$. The sequences $\{x_n\}$ and $\{y_n\}$ converge quadratically to a common limit called the arithmetic-geometric mean (AGM) of x and y, $\mathsf{M}(x, y)$ (see, e.g., [Borwein and Borwein 1987]). The AGM has many remarkable properties, e.g.,

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\zeta}{\sqrt{x^2\cos^2\zeta + y^2\sin^2\zeta}} = \frac{1}{\mathsf{M}(x,y)}$$

(in particular, elliptic integrals can be calculated through the AGM). To calculate precisely trigonometric functions, e.g., sin, one can use the fact that

$$\arcsin x = \frac{x}{\mathsf{M}(\sqrt{1-x^2}, 1)} \ ,$$

and calculate sin as the inverse function by using, for instance, the program `zeroin` (which is quadratically convergent, so each iteration doubles the number of correct digits).

The AGM has been used for precise computations, e.g., in [Brent 1976]; see also the discussion in [Borwein and Borwein 1984] (both papers reprinted in [Berggren et al. 2000]).

4.2 Calculating the Parameters for Rotation Number the Golden Mean

Having chosen some K (for which $f^\bullet_{K,\omega}$ is a homeomorphism), we first have to determine the function $\Omega^\bullet(K)$ such that (8) is satisfied. To achieve this, we use the following method [Greene 1979], [Shenker 1982]. First we determine the phase-locking intervals

$$I^\bullet_{K,n} := \left\{ \omega \in [0,1] \;\middle|\; \tau(f^\bullet_{K,\omega}) = \tfrac{Q_n}{Q_{n+1}} \right\}$$

($n \in \mathbb{N}$). It is guaranteed that the value $\Omega^\bullet(K)$ we are looking for is between $I^\bullet_{K,n}$ and $I^\bullet_{K,n+1}$. If $\Omega^\bullet_n(K)$ is the end of $I^\bullet_{K,n}$ that is closer to $I^\bullet_{K,n+1}$, then we assume that

$$\Omega^\bullet_n(K) = \Omega^\bullet(K) + C\beta^n_\bullet \ , \tag{9}$$

for some constants $-1 < \beta_\bullet < 0$ and C ($\beta_\bullet$ is a universal number that depends only on the degree of the critical point, while C is different for different maps), and find $\Omega^\bullet(K)$ by Aitken extrapolation [Press et al. 1992]. This assumption does not affect the validity of our results, but it speeds up our searches since the rotation numbers of fewer maps are computed. We note that the renormalization group picture also predicts (9). Moreover, this assumption is in an excellent agreement with our numerics:

(a) in the N case, we find

$$\beta_{\mathrm{N}} = -0.381966011250 \pm 10^{-12} = -\gamma^{\kappa_{\mathrm{N}}}$$

for $|\kappa_{\mathrm{N}} - 2| < 6 \times 10^{-12}$ (according to Theorem 2.4, $\kappa_{\mathrm{N}} = 2$);

(b) in the C case, we find

$$\beta_{\mathrm{C}} = -0.3529067 \pm 10^{-7} = -\gamma^{\kappa_{\mathrm{C}}}$$

for $\kappa_{\mathrm{C}} = 2.1644347 \pm 0.0000006$;

(c) in the Q case, we find

$$\beta_{\mathrm{Q}} = -0.32858 \pm 10^{-5} = -\gamma^{\kappa_{\mathrm{Q}}}$$

for $\kappa_{\mathrm{Q}} = 2.31286 \pm 0.00006$.

The values we found are in perfect agreement with those found in [Shenker 1982] (for the C case), and, for different families of circle maps, in [Hu et al. 1990], [Delbourgo and Kenny 1991]. Note that what we call β is called in these papers δ^{-1}).

To find the phase-locking interval $I^{\bullet}_{K,n}$, we used that when ω enters this interval, the map

$$x \mapsto (F^{\bullet}_{K,\omega})^{Q_{n+1}}(x) - Q_n$$

(where $F^{\bullet}_{K,\omega}$ is the lift of $f^{\bullet}_{K,\omega}$, i.e., it is given by the same formula as $f^{\bullet}_{K,\omega}$, but without the $\mathrm{mod}\,1$) undergoes a tangent bifurcation. To determine the values at which bifurcations occur, we used the subroutines `fmin` and `zeroin` from [Forsythe et al. 1977] (translated into C and slightly modified).

We will denote by $f^{\bullet}_{K}$ the map $f^{\bullet}_{K,\Omega^{\bullet}(K)}$ for $\Omega^{\bullet}(K)$ such that $\tau(f^{\bullet}_{K}) = \gamma$. In the table below we give the values of $\Omega^{\bullet}(K)$ for the values of K we studied numerically.

Map	$\Omega^\bullet(K)$
$f^{\mathrm{N}}_{0.2}$	0.617425455584922780978570
$f^{\mathrm{N}}_{0.3}$	0.6166923606057855021928
$f^{\mathrm{N}}_{0.5}$	0.6145263876774487765559862
$f^{\mathrm{N}}_{0.8}$	0.61007440530846512053842071
$f^{\mathrm{C}}_{0.3}$	0.626871059546737818
$f^{\mathrm{C}}_{0.6}$	0.617607758640542315
$f^{\mathrm{C}}_{0.7}$	0.6148131852529150525
$f^{\mathrm{C}}_{1.0}$	0.606661063470112017
$f^{\mathrm{Q}}_{0.6}$	0.633133040895040332
$f^{\mathrm{Q}}_{0.9}$	0.616330501795706578
$f^{\mathrm{Q}}_{1.2}$	0.60250115301615805
$f^{\mathrm{Q}}_{4/3}$	0.59694625982733198

4.3 Calculating the Conjugacies on an Equidistant Grid

Having found appropriate values of the parameters of the maps f and g such that $\tau(f) = \tau(g) = \gamma$, we construct numerically the conjugacy h between them. Instead of h, it is more convenient to study

$$\theta := h - \mathrm{Id}\ . \tag{10}$$

because θ is a periodic function, hence it is better suited for harmonic analysis. For brevity, we will denote the map θ defined by (10) for h being a conjugacy between, say, an N map f and a C map g by θ^{NC}, and will call θ, a "conjugacy".

From (2) we obtain

$$h \circ f^n = g^n \circ h\ . \tag{11}$$

Theorem 2.3 guaranteeing the existence of the conjugacies between f (resp. g) and the rotation r_γ allows us to impose the condition $h(0) = 0$ or, equivalently, $\theta(0) = 0$. This implies that $\theta(f^n(0)) = g^n(0) - f^n(0)$, so the points

$$(f^n(0),\, g^n(0) - f^n(0)) \tag{12}$$

belong to the graph of θ and fill it densely. It is apparent that one can compute the points in (12) by iterating f and g on 0.

One problem with the calculation above is that the points in (12) do not have first coordinates that are distributed on a equidistant grid, and to apply fast Fourier of wavelet transforms, we need to know the values of θ on an equidistant grid. We used the grid

$$x_\ell := 2^{-L}\ell \;, \quad \ell = 0,\, 1,\, \ldots,\, 2^L - 1 \tag{13}$$

for some $L \in \mathbb{N}$ (typically about 20). Since the iterates $\{f^n(0)\}$ are not equidistantly distributed, we used interpolation and calculated the values of the interpolating function at the points $\{x_\ell\}_{\ell=0}^{2^L-1}$. To this end, we used the cubic interpolation subroutines `spline` and `seval` from [Forsythe et al. 1977] (the periodicity of θ was taken into account).

An important source of difficulty for the numerical computation is the fact that the iterates of a C or a Q map are very nonuniformly distributed. To give an idea about the seriousness of the problem, we have shown in Figure 1 the distribution of four million iterates of $f^{\mathrm{Q}}_{0.6}$ and, for comparison, of $f^{\mathrm{N}}_{0.3}$, in 256 bins, each of size $\frac{1}{256}$, between 0 and 1. The number of iterates of $f^{\mathrm{Q}}_{0.6}$ in a bin varies from 15 to 118304; for $f^{\mathrm{N}}_{0.5}$ it varies from 13076 to 18739. The largest gap between the iterates of $f^{\mathrm{Q}}_{0.6}$ is 0.001308.

If the gaps are very large, it is complicated and unstable to compute the values of the interpolating function at the gaps. The way we found to deal with this problem is to use a large number of iterates which, however, is very memory consuming and leads to accumulation of numerical error. This problem becomes more severe when the order of the critical point is higher. This is the main reason why our investigation did not cover critical maps of degree higher than 5.

4.4 Conjugacies – Visual Explorations

Theorem 2.4 guarantees that each θ^{NN} (recall that this means a conjugacy between two N circle maps) is analytic, but does not say anything about critical circle maps. The goal of this paper is to study the conjugacies of critical circle maps to a golden mean rotation and assess their regularity and asymptotic scaling properties.

To motivate our subsequent analysis, we start with some preliminary visual explorations.

In Figure 2 we show two θ^{NC} and one θ^{CC}. Obviously the θ^{NC}'s are less differentiable then the θ^{CC}; visually, θ^{CC} is smoother than C^1.

In Figure 3 we show the conjugacies between a map of type N (resp. C, Q) and a Q map. Again, the conjugacy between two maps of the same type (i.e., of the same degree of flatness of the critical point; in this case θ^{QQ}) is evidently more differentiable than the ones between maps of different types.

Another observation is the self-similar structure of the conjugacy between an N map and a critical (C or Q) map. To illustrate this, in Figure 4 we show magnified regions of the conjugacy between $f^{\mathrm{N}}_{0.8}$ and $f^{\mathrm{Q}}_{0.9}$. The self-similarity of the conjugacies between an N and a C map is one of the predictions of the theory of renormalization for C maps; we observed a self-similar structure in the case of the conjugacy between an N map and a Q map as well.

The self-similarity of the conjugacies of type θ^{NC} and θ^{NQ} can be seen distinctly from their Fourier spectra displayed in log-log form (Figures 5 and 6). The self-similarity manifests itself in the "periodicity" of the Fourier spectrum for large k's.

This effect becomes even more prominent in the plot of $\log_{10}(k^{\lambda}|\hat{\theta}_k|)$ vs. $\log_{10} k$, as shown in Figures 7 (for θ^{NC}, $\lambda = 1.29$) and 8 (for θ^{NQ}, $\lambda = 1.19$). In both cases, the width of the "periodic windows" is approximately equal to $\log_{10}\gamma$, as predicted by renormalization theory.

5 Methods for Studying the Regularity

In this section we describe the function spaces studied and collect the theorems from harmonic analysis we used to compute the regularity of conjugacies.

5.1 Hölder Spaces

Let $C^n(\mathbb{T})$ ($n \in \mathbb{N}$) stand for the space of n times continuously differentiable functions on $\mathbb{T}$.

Definition 5.1 *The* Hölder spaces $\Lambda_\alpha(\mathbb{T})$ *are defined as follows.*

- *For* $\alpha \in (0,1)$*:*

$$\|\theta\|_{\Lambda_\alpha(\mathbb{T})} := \sup_{|y|>0} \frac{|\theta(x+y)-\theta(x)|}{|y|^\alpha} \ ,$$

$$\Lambda_\alpha(\mathbb{T}) := \left\{ \theta \in L^\infty(\mathbb{T}) \ : \ \|\theta\|_{\Lambda_\alpha(\mathbb{T})} < \infty \right\} \ ,$$

- *For* $\alpha = n + \alpha'$ $(n \in \mathbb{N},\ \alpha' \in (0,1))$*:*

$$\Lambda_\alpha(\mathbb{T}) := \left\{ \theta \in C^n(\mathbb{T}) \ : \ \theta^{(n)} \in \Lambda_{\alpha'}(\mathbb{T}) \right\} \ .$$

- *For* $\alpha = 1$*:*

$$\|\theta\|_{\Lambda_1(\mathbb{T})} := \|\theta\|_{L^\infty(\mathbb{T})} + \sup_{|y|>0} \frac{|\theta(x+y) + \theta(x-y) - 2\theta(x)|}{|y|} \ ,$$

$$\Lambda_1(\mathbb{T}) := \left\{ \theta \in L^\infty(\mathbb{T}) \cap C^0(\mathbb{T}) \ : \ \|\theta\|_{\Lambda_1(\mathbb{T})} < \infty \right\} \ .$$

- *For* $\alpha = n \in \{2, 3, 4, \ldots\}$*:*

$$\Lambda_n(\mathbb{T}) := \left\{ \theta \in L^\infty(\mathbb{T}) \cap C^{n-1}(\mathbb{T}) \ : \ \theta^{(n-1)} \in \Lambda_1(\mathbb{T}) \right\} \ .$$

Remark 5.1 1. $C^1(\mathbb{T}) \subset \mathrm{Lip}\,(\mathbb{T}) \subset \Lambda_1(\mathbb{T})$ and $C^n(\mathbb{T}) \subset \Lambda_n(\mathbb{T})$ $(n \geq 2)$; all these inclusions are strict.

2. Every $\theta \in \Lambda_\alpha(\mathbb{T})$ $(0 < \alpha)$ may be modified on a set of measure zero so that it becomes continuous [Stein 1970, Sec. V.4.1].

□

The spaces in these scales have several characterizations some of which lead to algorithms that can be used to assess the regularity of functions. Some of these characterizations will de discussed in Sections 5.2, 5.3, 5.4. The numerical implementation of these methods will be discussed in Section 6.

5.2 Finite Differences Method

In this subsection we discuss the characterization of Hölder spaces by means of finite differences (FD) [Krantz 1983].

Let $\mathcal{D}_y^n$ be the finite difference operator,

$$\left(\mathcal{D}_y^n \theta\right)(x) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} \theta(x + (n-2j)y) \ .$$

Theorem 5.1 (FD) *Let* $\theta \in L^\infty(\mathbb{T}) \cap C^0(\mathbb{T})$ *and* $0 < \alpha < n \in \mathbb{Z}$*. Then* $\theta \in \Lambda_\alpha(\mathbb{T})$ *if and only if* $\exists\ C > 0$ *such that* $\forall\ y \in \mathbb{T}$

$$\|\mathcal{D}_y^n \theta\|_{L^\infty(\mathbb{T})} \leq C|y|^\alpha \qquad (14)$$

for all $y \in \mathbb{T}$*.*

The FD method is simple and convenient to use if one can compute the values of the function in points that are arbitrarily close. Unfortunately, we know the values of the conjugacies only at the points of the type $f^n(0)$, so to apply Theorem 5.1, we have to interpolate. The influence of large gaps between the iterates $f^n(0)$ is discussed in Section 6.3.

5.3 Fourier Methods – Littlewood-Paley Theorem

The trigonometric system $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T}, dx)$, hence, according to the Plancherel's theorem, a function

$$\theta(x) = \sum_{k\in\mathbb{Z}} \hat{\theta}_k \, e^{2\pi ikx} \tag{15}$$

belongs to $L^2(\mathbb{T})$ if and only if

$$\sum_{k\in\mathbb{Z}} |\hat{\theta}_k|^2 < \infty \ .$$

The main result of Littlewood-Paley theory is that similar characterization of $L^p(\mathbb{T})$ $(1 < p < \infty)$ can be obtained by grouping the terms of the Fourier series in dyadic blocks. Define the decomposition

$$\theta = \sum_{M=1}^{\infty} \mathcal{L}_M \theta$$

of $\theta \in L^1(\mathbb{T})$ in *dyadic partial sums*

$$\left(\mathcal{L}_M \theta\right)(x) := \sum_{A^{M-1} \leq |k| < A^M} \hat{\theta}_k \, e^{2\pi ikx} \ ,$$

$(M \in \mathbb{N})$, $\mathcal{L}_0\theta := \hat{\theta}_0$, and $A > 1$.

Remark 5.2 Usually A is taken to be 2, since the precise value does not make any difference for the mathematical treatment.

In the numerical applications, we will find it convenient to use some A's different from 2. Nevertheless, we have not introduced A in the notation, since it will be clear from the context, and we follow the standard practice of calling the decomposition "dyadic". □

The dyadic blocks can be written as

$$\mathcal{L}_M\theta = (\phi_{A^M} - \phi_{A^{M-1}}) * \theta \ ,$$

where

$$\phi_N(x) := \sum_{|k|<N} e^{2\pi i k x} \ . \tag{16}$$

To formulate the celebrated Littlewood-Paley (LP) theorem, let us introduce the Littlewood-Paley d-function,

$$d(\theta)(x) := \left(\sum_{M=0}^{\infty} |\mathcal{L}_M\theta(x)|^2 \right)^{1/2} \ ,$$

and its "continuous" analog, the G-function,

$$G(\theta)(x) := \left(\int_0^1 (1-s) \left| \left(\frac{dP_s}{ds} * \theta \right)(x) \right|^2 ds \right)^{1/2} \ ,$$

where

$$\begin{aligned} P_s(x) &= \sum_{k\in\mathbb{Z}} s^{|k|}\, e^{2\pi i k x} \\ &= \frac{1-s^2}{1-2s\cos 2\pi x + s^2} \ , \quad s \in [0,1) \end{aligned} \tag{17}$$

is the periodic Poisson kernel. Note that if Δ is the Laplacian, then

$$\begin{aligned} P_{\exp(-2\pi t)} * \theta(x) &= e^{-t\sqrt{-\Delta}}\,\theta(x) \\ &= \sum_{k\in\mathbb{Z}} \hat{\theta}_k\, e^{-2\pi t|k|}\, e^{2\pi i k x} \ . \end{aligned}$$

The Poisson kernel can also be considered as defined on the real line. In that case, it can be given by the formula $P_t = e^{-t\sqrt{-\Delta}}$ or as the convolution with the kernel $P_t(x) = \pi^{-1/2}\, t/(x^2+t^2)$.

Note that we can consider a periodic function of period 1 on the real line as a function on the circle. When we apply the real Poisson kernel to a periodic function of period 1, it also produces a periodic function of period 1.

It is well known and not difficult to check (Poisson summation formula) that it is the same to apply the real Poisson kernel to a periodic function of period one defined on $\mathbb{R}$ or to consider the function as defined on the circle and to apply the periodic Poisson kernel (17).

Remark 5.3 On the real line it makes sense to define scaling transformations and to investigate how the Poisson kernel behaves under scalings. It is very easy to check that, for every $\lambda > 0$,

$$P_{\lambda t}(\lambda x) = \lambda^{-1} P_t(x) \tag{18}$$

On the circle, we cannot speak about scaling, therefore the relation (18) does not, strictly speaking, make sense for the Poisson kernel on the circle when λ is not an integer. Nevertheless, for small scales, the circle can be identified with the real line, so that the scalings of the periodic Poisson kernel can be used when examining asymptotic features in small scales. $\square$

Using the heuristic argument that the partial sums, $\phi_n * \theta$, behave like the Abel means, $P_{1-\frac{1}{n}} * \theta$, one easily finds that the $L^p(\mathbb{T})$ norms of $d(\theta)$ and $G(\theta)$ are equivalent for $1 < p < \infty$ if $\hat{\theta}_0 = 0$.

Theorem 5.2 (Littlewood-Paley) *If $\theta \in L^p(\mathbb{T})$, $1 < p < \infty$, then there exist positive constants A_p and B_p such that*

$$A_p \|\theta\|_{L^p(\mathbb{T})} \leq \|d(\theta)\|_{L^p(\mathbb{T})} \leq B_p \|\theta\|_{L^p(\mathbb{T})} \ .$$

Theorem 5.2 has many important implications. In particular, it gives useful characterizations of Sobolev, Hölder, Hardy, Besov spaces – see, e.g., [Stein 1970, Ch. 5], [Hernández and Weiss 1996, Ch. 6], [Meyer 1990, Ch. 6], [Frazier et al. 1991].

In our numerical explorations, we used methods based on the following two corollaries of Theorem 5.2, which we will call "discrete" (DLP) and "continuous" (CLP) versions of the Littlewood-Paley theorem.

Theorem 5.3 (DLP) [Krantz 1983, Theorem 5.9] *The function θ (15) is of class $\Lambda_\alpha(\mathbb{T})$ ($\alpha \in \mathbb{R}_+$) if and only if there exists a $C > 0$ such that for any $M \in \mathbb{N}$*

$$\|\mathcal{L}_M \theta\|_{L^\infty(\mathbb{T})} \leq C\, 2^{-\alpha M} \ . \tag{19}$$

Theorem 5.4 (CLP) [Stein 1970, Ch. 5, Lemma 5] *The function θ (15) is of class $\Lambda_\alpha(\mathbb{T})$ ($\alpha \in \mathbb{R}_+$) if and only if for each $\eta \geq 0$ there exists a $C > 0$ such that for any $t > 0$*

$$\left\| \left(\frac{\partial}{\partial t}\right)^\eta e^{-t\sqrt{-\Delta}}\, \theta \right\|_{L^\infty(\mathbb{T})} \leq C\, t^{\alpha - \eta} \ . \tag{20}$$

5.4 Wavelet Methods

Expansions in wavelet bases are very well suited to studying the local properties of functions because of their localization in space. Wavelet methods are especially appropriate for analyzing self-similar functions like some of the conjugacies between circle maps studied in this paper. Below we introduce the notations and collect the basic theoretical results about regularity of functions expanded in wavelet bases. For more details we refer the reader to [Meyer 1990], [Daubechies 1992], [Mallat 1998], [Hernández and Weiss 1996], [Härdle et al. 1998], [Louis et al. 1997].

Let $L^2(\mathbb{T})_{2^L}$ be the "discrete" version of the space of square integrable circle maps, i.e., the 2^L-dimensional space of the circle maps defined on the grid $x_\ell = 2^{-L}\ell$, $\ell = 0, 1, \ldots, 2^L - 1$. We use the following multiresolution analysis of $L^2(\mathbb{T})_{2^L}$:

$$V_0 \subset V_1 \subset \cdots \subset V_{L-1} \subset V_L = L^2(\mathbb{T})_{2^L} .$$

Let W_j be the orthogonal complement of V_j in V_{j+1}, so that

$$L^2(\mathbb{T})_{2^L} = V_0 \oplus \left(\bigoplus_{j=0}^{L-1} W_j \right) ;$$

$$\dim V_j = \dim W_j = 2^j .$$

The space W_j is spanned by $\{\psi_{jk}\}_{k=0}^{2^j-1}$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$$

and ψ is the "mother wavelet". Let $\theta_{2^L} := \{\theta(x_\ell)\}_{\ell=0}^{2^L-1} \in L^2(\mathbb{T})_{2^L}$ be the discrete representation of the function θ, and

$$\Pi_J : L^2(\mathbb{T})_{2^L} \to V_J : \theta_{2^L} \mapsto \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} \langle \theta, \psi_{jk} \rangle \, \psi_{jk}$$

be the projections onto V_J, $J = 0, 1, \ldots, L$.

Littlewood-Paley theorem can be generalized to bases other than the trigonometric one by observing that the proofs do not use the explicit form of ϕ_N (16) and P_s (17), but only some of their properties, so that the results are valid for larger function classes. In particular, the following theorem holds:

Theorem 5.5 [Hernández and Weiss 1996, Th. 7.16] *If $\psi \in \Lambda_\alpha(\mathbb{T})$, then the function θ is of class $\Lambda_\alpha(\mathbb{T})$ if and only if there exists a $C > 0$ such that for any $j \in \mathbb{N}$*

$$\sup_{0 \le k \le 2^j - 1} |\langle \theta, \psi_{jk} \rangle| \le C\, 2^{-j(\alpha + \frac{1}{2})} \ . \tag{21}$$

Another formulation which is useful for numerical computations is

Theorem 5.6 *If $\psi \in \Lambda_\alpha(\mathbb{T})$, then the function θ is of class $\Lambda_\alpha(\mathbb{T})$ if and only if there exists a $C > 0$ such that for any $j \in \mathbb{N}$*

$$\|\theta - \Pi_j \theta\|_{L^\infty(\mathbb{T})} \le C\, 2^{-j\alpha} \ . \tag{22}$$

For more subtle results on applications of wavelets to studies of local regularity of functions, see [Jaffard and Meyer 1996], [Holschneider and Tchamitchian 1991], [Meyer 1998]. For the study of conjugacies, however, the local regularity is empirically found to be the same at all points. A heuristic explanation of this phenomenon often repeated in the literature is that the conjugacies are obtained by iterating (11) which "spreads" the regularity, i.e., makes it the same at all points. A difficulty with this argument is that the critical points can change the regularity of a function. In this paper we will not explore local regularity.

6 Numerical Implementation

In this subsection we discuss the numerical implementation of the theorems from Section 5.

6.1 General Remarks

The numerical implementation brings us a wealth of issues which we will try to discuss.

One issue is that the characterizations mentioned above involve inequalities that have to be satisfied for an infinite number of integers. Obviously, the numerical calculation can only compute the Fourier and wavelet transform up to a finite order. It is conceivable that the behavior of the functions is different for high Fourier modes than for the values that can be explored.

In spite of the above solipsistic argument, there are good reasons (a renormalization group description) that strongly suggest that the functions we are studying are asymptotically self-similar, so that the the study of a finite number of scales predicts accurately the behavior at all scales. Indeed, we find empirically that the upper bounds giving the regularity become approximately identities. We see that, after a very short transient, the upper bounds become identities up to a small periodic error whose interpretation we discuss in Section 6.5.

Because of this empirical observation and the renormalization group description, we believe that it is reasonable to extrapolate from the observed values and conclude that the upper bounds giving regularity are saturated to all scales.

Another issue that one has to discuss in numerical implementations is the effect of the round off and discretization error. This analysis is very similar to the standard considerations of numerical analysis.

Finding numerically the regularity of functions that are very smooth is difficult because their Fourier/wavelet coefficients decrease faster. That is why we were not able to asses the precise values of the smoothness of the conjugacies of type θ^{CC} and θ^{QQ}, whose smoothness is more than one.

In these two cases, as well as for all conjugacies between f and g for f being critical (C or Q), an important issue is the presence of big gaps between the iterates $f^n(0)$ (see Section 4.3). This is due to the fact that we perform Fast Fourier Transform (FFT) or Discrete Wavelet Transform (DWT) not on the exact values of θ at the points x_ℓ (13), but on the values of the interpolating cubic polynomials at these points, which significantly deteriorates the precision of the spectra.

For the FFT, we used the routines `four1` and `realft` from [Press et al. 1992] (for long double precision).

For the DWT, we used the freely available C routines documented in detail in the book [Wickerhauser 1994].

For the graphing and some of the data analysis, we used the plotting tool `ACE/gr`.

Numerically, the most important restriction on the number of Fourier or wavelet coefficients computed was not the speed, but the memory usage (in some of the cases about 200M).

6.2 Calibration of the Methods

To asses the validity of the numerical methods that have been employed, we have taken an empirical approach, testing them on functions whose regularity is known.

One particularly good class of functions for calibration are the Weierstrass functions

$$w_{a,b}(x) = \sum_{k=1}^{\infty} a^k \sin(2\pi b^k x) \ , \tag{23}$$

where $a < 1$, $b \in \mathbb{N}$. As it is well known, $w_{a,b} \in \Lambda_{-\log_b a}$ and for any $\delta > 0$, $w_{a,b} \notin \Lambda_{-\log_b a+\delta}$.

To calibrate our numerical methods, we have generated the $w_{a,b}$ functions at points obtained by iterating the diffeomorphisms we are studying. Then, we obtained the regularity applying the methods outlined above.

This procedure gave us an idea of the severity of the problem of the lack of equidistribution of the points or of the effects of the interpolation etc.

We think that the use of Weierstrass function to calibrate the methods is appropriate because the working hypothesis (A2) asserts that the functions we are studying are very similar to the functions (23). Hence, one can hope that the problems of interpolation and lack of distribution can be assessed by testing the methods on (23).

6.3 Finite Differences Method

We applied Theorem 5.1 for $y = 2^{-j}$, in which case (14) yields

$$\log_2 \|\mathcal{D}^n_{2^{-j}}\theta\|_{L^\infty(\mathbb{T})} \leq \text{const} - \alpha j$$

(naturally, one can consider the case of arbitrary y's). As examples of the results obtained by applying this method, we show in Figure 9 the plot of $\log_2 \|\mathcal{D}^1_{2^{-j}}\theta\|_{L^\infty(\mathbb{T})}$ as a function of j for four conjugacies of type NC (x's) and four ones of type CQ (circles); to calculate θ^{NC}, we used 10^7 iterates and 2^{22} interpolated values, while for θ^{CQ} these numbers were 2×10^6 and 2^{22}, respectively.

In the favorable case (NC), we see that the numerical results correspond to parallel straight lines that cover the whole range plotted. On the other hand, in the unfavorable case (CQ), the numerical results present two straight lines joined by a break.

This can be clearly explained because the graph presented for the NC case includes computations of in which many of the points in the finite difference operator are included in the gaps. Hence, the finite difference operator is observing the regularity of the interpolating spline.

In the NC case the gaps between the iterates did not exceed 1.5×10^{-7}. At the same time, in the CQ case the maximum gap was about $2 \times 10^{-4} \approx 2^{-12}$, which corresponds quite exactly to the position of the break in the graph.

When we restrict the differences to regions which are larger than the gaps, the method produces results consistent with the other methods.

6.4 DLP Method

Theorem 5.3 implies that

$$\log_A \|\mathcal{L}_M \theta\|_{L^\infty(\mathbb{T})} \leq \text{const} - \alpha M \ ,$$

i.e., the Hölder exponent of θ is the negative of the slope of the graph of $\log_A \|\mathcal{L}_M \theta\|_{L^\infty(\mathbb{T})}$ vs. M.

Graphs of this type for some classes of conjugacies are shown in Figure 10. Each case is represented by two conjugacies, the first one depicted by a big empty shape, and the second one – by a small full shape: $(f^{\rm N}_{0.3}, f^{\rm C}_{0.6})$ and $(f^{\rm N}_{0.3}, f^{\rm C}_{0.7})$ – circles, $(f^{\rm N}_{0.5}, f^{\rm Q}_{0.6})$ and $(f^{\rm N}_{0.5}, f^{\rm Q}_{0.9})$ – squares, $(f^{\rm C}_{0.6}, f^{\rm Q}_{0.6})$ and $(f^{\rm C}_{0.3}, f^{\rm Q}_{0.9})$ – diamonds, $(f^{\rm C}_{0.6}, f^{\rm C}_{0.3})$ and $(f^{\rm C}_{0.7}, f^{\rm C}_{0.6})$ – triangles down, $(f^{\rm Q}_{0.6}, f^{\rm Q}_{0.9})$ and $(f^{\rm Q}_{0.9}, f^{\rm Q}_{1.2})$ – triangles up. Clearly, the smoothness of the conjugacies of different classes is different, but this graph does not allow us to find the smoothness of the conjugacies precisely (and for $\theta^{\rm CC}$ and $\theta^{\rm QQ}$ the results are very poor). The reasons for this are the following.

Firstly, each point on this graph is computed by using not all Fourier coefficients of θ, but only a dyadic block of them, so for small M the points on the graph are based on a small number of Fourier coefficients. For large M the points are based on larger number of Fourier coefficients, but these coefficients are affected by the numerical noise. Also, the number of points in the figure is of order $\log_A$ of the number of Fourier coefficients found, i.e., it is significantly smaller than the number of coefficients. In our explorations we used values of A around 1.5.

6.5 CLP Method

From numerical point of view, CLP method (based on Theorem 5.4) is much better than DLP.

First of all, we can calculate $\|\frac{\partial^\eta}{\partial t^\eta}\, e^{-t\sqrt{-\Delta}} f\|_{L^\infty(\mathbb{T})}$ for as many values of t as we wish. Furthermore, for each value of t, the value of this norm is based on the values of *all* known Fourier coefficients of f. Finally, one can perform calculations for different values of η and check whether they yield the same value of α – this is a very good test of the reliability of the numerical results.

To illustrate how well this method works, we show in Figure 11 plots of $\log_{10}\|\frac{\partial^2}{\partial t^2}\, e^{-t\sqrt{-\Delta}} w_{0.57,3}\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for 2^{22} (circles), 2^{13} (x's), and 2^{10} (pluses) Fourier components based on the values of $w_{0.57,3}$ at the points $(f^{\mathrm{N}}_{0.5})^n(0)$ for $n = 0, \ldots, 2^{21}-1$. Evidently, the position of the plateau for small t's depends on the number of Fourier coefficients used in the computation. Theoretically, the regularity of $w_{0.57,3}$ is $-\log_3 0.57 = 0.5117\ldots$. The slope of the straight line that fits best the full circles in the figure is -1.4908, so the numerically found regularity according to (20) is $2 - 1.4908 = 0.5092$ – a value that differs from the exact one by only 0.002.

Figure 12 shows graphs of $\log_{10}\|\frac{\partial^\eta}{\partial t^\eta}\, e^{-t\sqrt{-\Delta}}\theta\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for $\eta = 1, 2, 3$; θ is the conjugacy between $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{C}}_{0.6}$. The results of the linear regression of these data are the presented in the table. The uncertainties are just the the standard errors of the regression.

η	Range of $\log_{10} t$	Regularity
1	$[-5.0, -4.0]$	0.5247 ± 0.0009
2	$[-3.5, -2.5]$	0.5253 ± 0.0012
3	$[-3.0, -1.5]$	0.5244 ± 0.0008

In Figure 13, we show $\log_{10}\|\frac{\partial^\eta}{\partial t^\eta}\, e^{-t\sqrt{-\Delta}}\theta\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for $\eta = 1, 2$ for all the 16 conjugacies between the four N and four C maps we considered. We call attention to the fact that not only the lines are parallel, but the are also very close.

The CLP method can be used also to test some features of the (4). Since (4) is supposed to hold only in the asymptotic limit of very small scales, we can use Remark 5.3 and the scalings (18). Note that, taking the convolution of (4) with the Poisson kernel and using (18), we obtain:

$$P_t * \left[\sum_n \lambda_1^n (H_1 \circ \alpha^n)(x) \quad + \quad \lambda_2^n (H_2 \circ \alpha^n)(x) + \cdots\right]$$

$$= \sum_n \left[\lambda_1^n \, [P_{\alpha^n t} * H_1](\alpha^n x) + \lambda_2^n \, [P_{\alpha^n t} * H_2](\alpha^n x) + \cdots \right]$$

If we take suprema in x and then logarithms, we obtain that the structure of the main term for the resulting function considered as a function of $\log t$ is a sum of a linear function and a function that is periodic. The slope of the linear function is, of course, according to Theorem 5.4, the degree of differentiability, but if we subtract off the linear part, we should see the periodicity.

This exploration of the first differences is undertaken in Figure 14, where we plot the first differences of the graph of $\log_{10} \| \frac{\partial^n}{\partial t^\eta} e^{-t\sqrt{-\Delta}} \theta \|_{L^\infty(\mathbb{T})}$ as a function of $\log_{10} t$ (for $\eta = 2, 3$) at equally spaced points.

Note that taking first differences turns a linear function into a constant and a periodic function into a periodic function. Higher order differences eliminate the linear function and receive contributions of the periodic part.

In Figure 15, we show the same plot as above for four conjugacies of type NQ and the first and second differences of the plot.

We call attention to the fact that the periodic corrections we plot quickly become independent of the functions we start with which corresponds to the fact that the function H_1 is universal. This is particularly remarkable for the case of second differences since they are very susceptible to numerical errors. Hence, this gives us confidence on the reliability of the methods we have used.

We note that the computation of first differences is one of the data analysis features included in `ACE/gr`, so that it is quite feasible to carry out these explorations in an interactive way for a variety of functions.

6.6 Decay of Wavelet Coefficients

Theorem 5.5 can be used to asses the regularity of the functions we study by examining the decay of the coefficients of wavelet transform.

Nevertheless, we do not think that for our functions it is necessary to appeal to Theorem 5.5.

Note that the working hypothesis (A4) gives a representation of the function.

It is not difficult to show that, for functions of the form (4) in the working hypothesis (A4), the degree of regularity is a simple ratio between the logarithms of λ and the scaling factor α defined in (4).

For functions of this form, the logarithm of the size of the projections on a space V_j should decay linearly with j irrespective of what is the wavelet used. In particular, one does not need to use wavelets which are smoother than the regularity observed to obtain the scaling exponents, which also give the regularity.

In our numerical studies we have used Daubechies wavelets of order 4, 10, 20, which we will denote as D4, D10, D20 respectively. It is known that D4$\in \Lambda_{0.38\ldots}$ and not in any more regular space. For large N, D$2N \in \Lambda_{l_N}$ where $l_N \approx 0.20775\, N$. (See, e.g., [Härdle et al. 1998, Sec. 7.1].)

We note that even if Theorem 5.5 does not apply to the measurements of regularity with D4 in some of the cases we consider, nevertheless, we obtain decays which are extremely similar to those obtained using D10 or D20, for which Theorem 5.5 does apply and also extremely similar to the regularities obtained by other methods.

Moreover, we also note that the upper bounds given by Theorem 5.5 are identities.

We interpret the coincidence of the rates of decays obtained by any wavelets and the saturation of the bounds as (at least circumstantial) evidence that the the asymptotic scalings in (4) indeed hold. As we will discuss later, similar coincidences are observed for other methods.

In Figure 16 we show $\log_2 \sup_k |\langle \theta, \psi_{jk} \rangle|$ vs. j for several θ^{QN} and θ^{NQ} maps. The slope of the straight lines on this graph is $-(\alpha + \frac{1}{2})$.

One of the reasons this method to work much better with wavelet instead of Fourier coefficients (cf. Figure 10) is the following. The cubic interpolation in the large gaps distorts all Fourier coefficients. At the same time, in the case of wavelets it only affects the ones whose support intersects the gap; moreover, the "artificial local smoothing" due to the interpolation decreases the wavelets supported at the gap, which does not change $\sup_k |\langle \theta, \psi_{jk} \rangle|$ for fixed j.

6.7 Approximation with Wavelets

The method based on Theorem 5.5 yields very good results. In Figure 17 we show plots of $\log_2 \|\theta - \Pi_j \theta\|_{L^\infty(\mathbb{T})}$ vs. j for several θ^{NC} and θ^{CN}. The slope of the straight lines in this graph is $-\alpha$.

As in the previous case, we note that we have used D4, D10 and D20. Theorem 5.5 does not apply to D4 in some cases. We, nevertheless, find the same linear decay as with the other methods and we interpret it as a confirmation of the asymptotic scaling of the function.

7 Results

In this section we give the numerical values of the Hölder exponents of the conjugacies.

To determine these values, we used the methods based on Theorems 5.1, 5.3, 5.4, 5.5, 5.6.

To find the smoothness of a particular type of conjugacy, we applied all these methods to study numerically the smoothness of the conjugacies between all possible combinations of circle maps studied (four N, four C,and four Q maps).

As an example, we show in the table below the results of our analysis of the regularity of the conjugacies between N and Q maps as well as the results of the same methods applied to the test functions $w_{0.66745,3}$, whose Hölder exponent, 0.36800..., is close to the one of the conjugacies of type θ^{NQ}.

The column “Function” indicates the function analyzed: “w on $f^{\mathrm{N}}_{0.2}$” means the regularity of the function $w_{0.66745,3}$ calculated at the points $(f^{\mathrm{N}}_{0.2})^n(0)$, and θ $(f^{\mathrm{N}}_{0.2}/f^{\mathrm{Q}}_{4/3})$ means the conjugacy between $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{Q}}_{4/3}$.

The column “Finite diffs” shows the results of the smoothness found by using the finite difference method.

The columns “CLP, $\eta = 1, 2, 3$” display the results of the CLP analysis for different numbers of derivatives.

“Decay D4, D10, D20” contain the results of analysis of the decay rate of the coefficients of Daubechies 4 (resp. 10, 20) wavelets, while “Approx D4, D10, D20” shows the results of the study of the speed of the approximation using these wavelets.

The meaning of the notations below is the following: 0.3661(13) means 0.3661 ± 0.0013. The error is the standard error of the linear regression.

Function	Finite diffs	CLP, $\eta = 1$	CLP, $\eta = 2$	CLP, $\eta = 3$	Decay D4	Decay D10	Decay D20	Approx D4	Approx D10	Approx D20
w on $f^{\rm N}_{0.2}$	0.3508(162)	0.3556(4)	0.3661(13)	0.3683(94)	0.3548(126)	0.3455(340)	0.3558(461)	0.3672(36)	0.3658(102)	0.3636(136)
w on $f^{\rm N}_{0.3}$	0.3511(156)	0.3625(2)	0.3659(13)	0.3680(93)	0.3563(126)	0.3460(350)	0.3554(469)	0.3685(38)	0.3657(101)	0.3650(141)
w on $f^{\rm N}_{0.5}$	0.3483(155)	0.3632(2)	0.3661(13)	0.3681(94)	0.3569(123)	0.3478(344)	0.3550(461)	0.3659(32)	0.3642(100)	0.3645(140)
w on $f^{\rm N}_{0.8}$	0.3486(155)	0.3634(4)	0.3660(13)	0.3682(94)	0.3559(126)	0.3482(345)	0.3514(455)	0.3678(33)	0.3641(101)	0.3620(138)
θ $(f^{\rm N}_{0.2}/f^{\rm Q}_{4/3})$	0.3652(33)	0.3611(10)	0.3682(8)	0.3676(2)	0.3686(104)	0.3667(247)	0.3643(161)	0.3713(12)	0.3706(75)	0.3664(69)
θ $(f^{\rm N}_{0.2}/f^{\rm Q}_{0.6})$	0.3642(34)	0.3622(12)	0.3675(9)	0.3710(6)	0.3674(106)	0.3556(160)	0.3670(161)	0.3717(11)	0.3710(76)	0.3660(72)
θ $(f^{\rm N}_{0.2}/f^{\rm Q}_{1.2})$	0.3649(33)	0.3613(10)	0.3681(8)	0.3670(3)	0.3684(105)	0.3670(249)	0.3647(162)	0.3711(12)	0.3702(75)	0.3661(70)
θ $(f^{\rm N}_{0.2}/f^{\rm Q}_{0.9})$	0.3684(29)	0.3623(11)	0.3680(8)	0.3667(3)	0.3677(105)	0.3744(217)	0.3629(155)	0.3710(13)	0.3672(49)	0.3658(70)
θ $(f^{\rm N}_{0.3}/f^{\rm Q}_{4/3})$	0.3635(27)	0.3616(10)	0.3685(8)	0.3677(3)	0.3692(108)	0.3658(246)	0.3638(155)	0.3701(8)	0.3691(71)	0.3669(67)
θ $(f^{\rm N}_{0.3}/f^{\rm Q}_{0.6})$	0.3626(28)	0.3625(12)	0.3679(10)	0.3703(5)	0.3686(109)	0.3632(270)	0.3661(155)	0.3703(7)	0.3698(73)	0.3665(68)
θ $(f^{\rm N}_{0.3}/f^{\rm Q}_{1.2})$	0.3633(28)	0.3618(10)	0.3684(8)	0.3671(3)	0.3691(108)	0.3695(234)	0.3644(155)	0.3698(8)	0.3688(72)	0.3666(67)
θ $(f^{\rm N}_{0.3}/f^{\rm Q}_{0.9})$	0.3615(29)	0.3628(11)	0.3685(8)	0.3668(3)	0.3684(108)	0.3546(345)	0.3630(169)	0.3700(8)	0.3725(61)	0.3665(67)
θ $(f^{\rm N}_{0.5}/f^{\rm Q}_{4/3})$	0.3646(25)	0.3610(10)	0.3677(9)	0.3671(2)	0.3694(106)	0.3631(249)	0.3735(164)	0.3712(9)	0.3772(77)	0.3728(70)
θ $(f^{\rm N}_{0.5}/f^{\rm Q}_{0.6})$	0.3641(27)	0.3616(11)	0.3672(9)	0.3684(3)	0.3696(107)	0.3641(195)	0.3812(140)	0.3712(9)	0.3780(79)	0.3724(72)
θ $(f^{\rm N}_{0.5}/f^{\rm Q}_{1.2})$	0.3647(27)	0.3612(10)	0.3676(9)	0.3666(3)	0.3694(106)	0.3663(216)	0.3742(165)	0.3710(9)	0.3766(77)	0.3725(70)
θ $(f^{\rm N}_{0.5}/f^{\rm Q}_{0.9})$	0.3757(36)	0.3620(10)	0.3681(9)	0.3664(3)	0.3689(106)	0.3642(308)	0.3847(166)	0.3709(10)	0.3594(58)	0.3723(70)
θ $(f^{\rm N}_{0.8}/f^{\rm Q}_{4/3})$	0.3674(28)	0.3607(10)	0.3680(9)	0.3654(4)	0.3676(105)	0.3682(249)	0.3629(181)	0.3684(7)	0.3687(73)	0.3692(66)
θ $(f^{\rm N}_{0.8}/f^{\rm Q}_{0.6})$	0.3640(26)	0.3615(10)	0.3681(9)	0.3645(3)	0.3673(107)	0.3701(239)	0.3560(186)	0.3685(7)	0.3699(66)	0.3695(68)
θ $(f^{\rm N}_{0.8}/f^{\rm Q}_{1.2})$	0.3666(28)	0.3612(10)	0.3679(9)	0.3649(5)	0.3675(105)	0.3681(250)	0.3634(182)	0.3683(7)	0.3686(73)	0.3691(66)
θ $(f^{\rm N}_{0.8}/f^{\rm Q}_{0.9})$	0.3592(37)	0.3624(10)	0.3683(7)	0.3649(5)	0.3619(101)	0.3355(325)	0.3440(218)	0.3687(7)	0.3550(86)	0.3694(66)

As seen from the table above, the results obtained by using different methods are consistent, the most precise being the ones based on using CLP. In the table below, we give the Hölder exponent of the conjugacy between the maps f and g. The margins of error are determined empirically, and only in very few cases outliers are ignored.

In the case of conjugacies of types CC and QQ for reasons explained in the text, we were not able to determine the smoothness of the conjugacies, but only to give a rough estimates.

$\downarrow f \quad g \rightarrow$	N	C	Q
N	Analytic	0.527 ± 0.003	0.368 ± 0.003
C	0.63 ± 0.02	$1.4^{+0.4}_{-0.2}$	0.71 ± 0.03
Q	0.54 ± 0.05	0.86 ± 0.02	1.7 ± 0.5

8 Some Bounds on the Regularity of Conjugacies

In this section, we derive some bounds for the regularities of the congugacies.

8.1 Some Simple Bounds

It follows directly from the definition of Λ_σ, $0 < \sigma < 1$, that if $h_1 \in \Lambda_{\sigma_1}$, $h_2 \in \Lambda_{\sigma_2}$, then $h_1 \circ h_2 \in \Lambda_{\sigma_1 \sigma_2}$. It is not difficult to produce functions that saturate the above bounds (just take $h_i(x) = |x|^{\sigma_i}$) as well as functions for this bound is not optimal (take $h_1(x) = |x|^{\sigma_1}$, $h_2 = |x - 0.1|^{\sigma_2}$).

We also note that if $h^{1,2} \circ f_1 = f_2 \circ h^{1,2}$, $h^{2,3} \circ f_2 = f_3 \circ h^{2,3}$, and we define $h^{1,3}$ by $h^{1,3} = h^{1,2} \circ h^{2,3}$, we have $h^{1,3} \circ f_1 = f_3 \circ h^{1,3}$.

Denote by $\rho^{a,b}$, where a, b are among N, C, Q, be the regularities of the conjugacy between golden mean circle maps of class a to circle maps of class b, ie. the entries in the table above.

It follows from the regularity of the composition that when a, b, c are such that $a \neq b$, $b \neq c$, we should have

$$\rho^{a,c} \geq \rho^{a,b} \rho^{b,c} \tag{24}$$

The inequality (24) can be verified in two cases in the table that we have. Namely, we can take $a =$ N, $b =$ C, $c =$ Q, or $a =$ Q, $b =$ C, $c =$ N. When we try to carry out this verification, we find out that, up to the error of the calculation, we obtain that (24) becomes an identity.

This is presumably not a coincidence. We believe that it is again a manifestation of the self-similarity of the function at small scales. If we compose two functions that in each small scale have oscillations comparable to those allowed by the Hölder exponent, the resulting function will also have oscillations that are comparable to the product of the Hölder exponents. Note however that this argument does not suggest that there is a simple relation between the regularity of a function and its inverse.

We note that the equation (24) can be described as saying that the regularities of the conjugacies as indexed by the classes form a multiplicative supercocycle. We find empirically it is a cocycle.

In the following section, we will derive upper bounds for the regularity that are cocycles.

8.2 Scalings of the Recurrence and Upper Bounds on Hölder Exponents of Conjugacies

In this section, we discuss the relation between the scaling properties and smoothness of the conjugacy.

Scalings have been studied numerically from the beginning of renormalization theory. Some of them have been probed to hold.

In this section, we will report some rigorous results that show that if certain scalings hold, then there are bounds for the regularity of the conjugacy.

Since these scaling relations – hypothesis to our lemma – are numerically accessible, we can use the rigorous results to obtain numerical upper bounds.

One of the first numerical observations that were made in the study of the golden mean rotation number critical circle maps was that

$$(f^{\bullet})^{Q_n}(0) \approx \alpha_{\bullet}^{-n} \ , \tag{25}$$

where $\bullet$ stands for N, C, Q, and $\alpha_{\bullet}$ are universal constants. The numbers $\alpha_{\bullet}$ play a fundamental role in the fixed point equations.

For non-critical maps, because of Theorem 2.4, α_{N} is the same as for rotations by golden mean, and, by using the well-known relation

$$\frac{Q_n}{Q_{n+1}} = \gamma + C(-\gamma^2)^n + o(\gamma^{2n}) \ ,$$

we obtain $\alpha_{\mathrm{N}} = \gamma^{-1}$.

We note that for the cubic critical case, there are unpublished computer assisted proofs ([Mestel, thesis (1984)], [Lanford and de la Llave, manuscript (1984)]) that establish the existence of the α_{C}, upper and lower bounds for it, and the fact that (25) holds for maps in open sets.

Some relation between the scaling properties of the returns and the regularity of the conjugacy is given by

Lemma 8.1 *Let*

$$\begin{aligned} f_1^{Q_n}(0) &= C_1\alpha_1^{-n} + o(\alpha_1^{-n}) && (26)\\ f_2^{Q_n}(0) &= C_2\alpha_2^{-n} + o(\alpha_2^{-n}) && (27)\end{aligned}$$

and

$$\sigma := \log|\alpha_2|/\log|\alpha_1| \notin \mathbb{N}. \tag{28}$$

Then, if h satisfies $h \circ f_1 = f_2 \circ h$, $h(0) = 0$, then, for every $\delta > 0$, $h \notin \Lambda_{\sigma+\delta}$.

Proof. For any $\chi > 0$ we have

$$\frac{h \circ f_1^{Q_n}(0) - h(0)}{\left[f_1^{Q_n}(0) - 0\right]^{\chi}} = \frac{f_2^{Q_n}(0)}{\left[f_1^{Q_n}(0)\right]^{\chi}} = \frac{C_2\alpha_2^{-n} + o(\alpha_2^{-n})}{C_1^{\chi}\alpha_1^{-\chi n} + o(\alpha_1^{-\chi n})} \ . \tag{29}$$

We argue by contradiction: if $h \in \Lambda_\chi$ for some $\chi > \sigma$, we use (29) to prove by induction that $h^{(n)}(0) = 0$ for all $n \le \sigma$, $n \in \mathbb{N}$. Then we note that

$h \in \Lambda_{\sigma+\delta}$ (for any $\delta > 0$) would imply that if we substitute $\chi = \sigma + \delta$ in (29), the left-hand side is bounded uniformly in n. At the same time, the right-hand side of (29) is unbounded in n. □

We emphasize that Lemma 8.1 does not conclude anything when $\beta \in \mathbb{N}$; in particular, it does not conclude anything in the cases when f_1 and f_2 have the same scaling factor α, which happens when f_1 and f_2 are in the same universality class.

We have verified the relations (25) for the maps we considered and obtained values of α as follows:

$$\alpha_{\mathrm{C}} = -1.2886 \ , \qquad \alpha_{\mathrm{Q}} = -1.194 \ , \tag{30}$$

which agree with the values reported in [Shenker 1982] (for the C case), and those in the papers [Hu et al. 1990], [Delbourgo and Kenny 1991], even though they consider functions that are periodized versions of polynomials which are not C^1.

Taking the values in (30) and the exact value for α_N, we obtain the following upper bounds for the regularity of the conjugacies between the maps f and g.

$\downarrow f \quad g \rightarrow$	N	C	Q
N	?	0.5269	0.368
C	1.898	?	0.70
Q	2.72	1.43	?

We note that the in the cases NC, NQ, CQ, the upper bounds obtained applying Lemma 8.1 agree within the margin of error with the values of the regularity reported. In the other cases, the results are absurd.

We conjecture that indeed the upper bounds produced by applying Lemma 8.1 are sharp for those cases.

Note that the fact that the upper bounds that work in one case are far off in another seem to imply that there are different mechanisms limiting the regularity. In the cases CN, QN, QC, these mechanisms cannot be just a simple scaling argument.

We note that since the σ defined in (28) is the ratio of a quantity depending on the domain and a quantity depending on the range, the upper bounds that we have derived form a multiplicative cocycle.

For the sake of completeness, we also point out that there is a very similar argument that gives upper bounds for the regularity of the conjugacy for the maps in the same class. The following set of ideas was found useful in [de la Llave and Schafer, manuscript (1996)]. There it is shown that this argument produced upper bounds that were sharp in the case of conjugacies of limiting sets of unimodal maps.

Lemma 8.2 *Assume that*

$$\begin{aligned} f_1^{Q_n}(0) &= C_1\alpha^{-n} + D_1\tilde{\alpha}^{-n} + o(\tilde{\alpha}^{-n}) \\ f_2^{Q_n}(0) &= C_2\alpha^{-n} + D_2\tilde{\alpha}^{-n} + o(\tilde{\alpha}^{-n}) \end{aligned}$$

with $|\alpha| < |\tilde{\alpha}|$, $C_1C_2 \neq 0$,

$$D_2 \neq (C_2/C_1)D_1 \tag{31}$$

and $\sigma := \log|\tilde{\alpha}|/\log|\alpha| \notin \mathbb{N}$. *Then, if* h *satisfies* $h \circ f_1 = f_2 \circ h$, $h(0) = 0$, *then* $h \notin \Lambda_{\sigma+\delta}$ *for any* $\delta > 0$.

Proof. The argument is very similar to the argument that proved Lemma 8.1.

Note that the assumptions imply that $\sigma > 1$. Hence we are only excluding regularities higher than C^1.

We argue by contradiction, assuming that $h \in \Lambda_{\sigma+\delta}$. We note that

$$h \circ f_1^{Q_n}(0) - h(0) = f_2^{Q_n}(0)\ . \tag{32}$$

Since $h \in \Lambda_{\sigma+\delta}$, and $\sigma > 1$, we conclude that $h'(0) = C_2/C_1 \neq 0$. Hence h is invertible in a neighborhood of zero and $h^{-1} \in \Lambda_{\sigma+\delta}$.

We note that $h(t) = P(t) + R(t)$ where $P(t)$ is a polynomial of degree $[\sigma]$ and $|R(t)| \leq |t|^{\sigma+\delta}$.

Given the assumptions on the degrees we have made, we have

$$P(C_1\alpha^{-n} + D_1\tilde{\alpha}^{-n} + o(\tilde{\alpha}^{-n})) = h'(0)\tilde{\alpha}^{-n} + \sum_j C_j(\alpha^j)^{-n} + o(\alpha^{-n([\beta]+1)})$$

We also note that $R(C_1\alpha^{-n} + D_1\tilde{\alpha}^{-n} + o(\tilde{\alpha}^{-n})) = o(\tilde{\alpha}^{-n})$.

Equating the coefficients of the $\tilde{\alpha}^{-n}$ in (32), we obtain $D_2 = (C_2/C_1)D_1$, which contradicts our assumption (31). This is the desired contradiction with the assumption that h admitted a Taylor expansions near 0 with Hölder bounds. $\square$

Note that in contrast with Lemma 8.1, Lemma 8.2 contains a hypothesis (31) which could fail for a finite codimension set of maps. This is to be expected since the other hypothesis allows to take $f_1 = f_2$, $h = \mathrm{Id}$. In this situation, the conjugacy h is very regular. The elementary theory of renormalization tells us that (31) indeed occurs in a set which is of positive codimension.

The conjecture 3.1 amounts to the fact that the upper bounds produced by applying Lemma 8.2 are sharp.

At the moment, unfortunately, we do not have accurate enough values for the $\tilde{\alpha}$ for cubic critical or quintic critical maps and, hence, cannot give concrete values for the upper bounds of the regularity in the CC or QQ cases. It is interesting to remark that in the NN case, these bounds are far from optimal.

9 Conclusion

We have studied the problem of the smoothness of maps of the circle with critical points and golden mean rotation number.

The first step was to obtain an extremely precise calculation of the parameters of the function that put us in the correct universality class.

The most important step is to develop a numerical toolkit based on a wide array of methods from harmonic analysis (Littlewood-Paley, wavelets) to study the regularity and the fine scale structure of these functions.

We have used the combination of the methods applied to critical circle maps to asses their range of validity.

We have found indication that the regularity of critical circle maps is limited by considerations other than just scaling of recurrences.

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Figure Captions

1. Density of the iterates of a Q map. The number of iterates in a bin (in thousands) vs. the position of the bin, for four million iterates of $f^{\mathrm{Q}}_{0.6}$ (thin line) and of $f^{\mathrm{N}}_{0.5}$ (thick line), in 256 bins.

2. Conjugacies θ between: $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{C}}_{0.3}$ (thin solid line), $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{C}}_{0.6}$ (thick solid line), and $f^{\mathrm{C}}_{0.6}$ and $f^{\mathrm{C}}_{0.3}$ (dashed line).

3. Conjugacies θ between: $f^{\mathrm{N}}_{0.3}$ and $f^{\mathrm{Q}}_{0.9}$ (thin solid line), $f^{\mathrm{C}}_{0.6}$ and $f^{\mathrm{Q}}_{0.9}$ (thick solid line), and $f^{\mathrm{Q}}_{0.6}$ and $f^{\mathrm{Q}}_{0.9}$ (dashed line).

4. Zooming in the graph of the conjugacy between $f^{\mathrm{N}}_{0.8}$ and $f^{\mathrm{Q}}_{0.9}$.

5. Plot of $\log_{10}|\hat{\theta}_k|$ vs. $\log_{10} k$ where θ is the conjugacy between $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{C}}_{0.6}$.

6. Plot of $\log_{10}|\hat{\theta}_k|$ vs. $\log_{10} k$ where θ is the conjugacy between $f^{\mathrm{N}}_{0.3}$ and $f^{\mathrm{Q}}_{0.9}$.

7. Plot of $\log_{10}(k^{1.29}|\hat{\theta}_k|)$ vs. $\log_{10} k$ where θ is the conjugacy betweeh $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{C}}_{1.0}$.

8. Plot of $\log_{10}(k^{1.19}|\hat{\theta}_k|)$ vs. $\log_{10} k$ where θ is the conjugacy between $f^{\mathrm{N}}_{0.2}$ and $f^{\mathrm{Q}}_{0.6}$.

9. Plot of $\log_2 \|\mathcal{D}^1_{2^{-j}}\theta\|_{L^\infty(\mathbb{T})}$ vs. j for four θ^{NC}'s (x's) and four θ^{CQ}'s (circles).

10. Plot of $\log_{10}\|\mathcal{L}_M\theta\|_{L^\infty(\mathbb{T})}$ vs. M (for $A = 1.4$) for pairs of conjugacies of five different types.

11. Plot of $\log_{10}\|\frac{\partial^2}{\partial t^2}\, e^{-t\sqrt{-\Delta}} w_{0.57,3}\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$.

12. Plot of $\log_{10}\|\frac{\partial^\eta}{\partial t^\eta}\, e^{-t\sqrt{-\Delta}}\theta\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for $\eta = 2$ and $\eta = 3$ of all 12 conjugacies between an N and a C map for four N and three C maps with different parameter values. Each line connects 146 points; to obtain each point, we have used 10^6 iterates and $2^{21} \approx 2^6$ spline points.

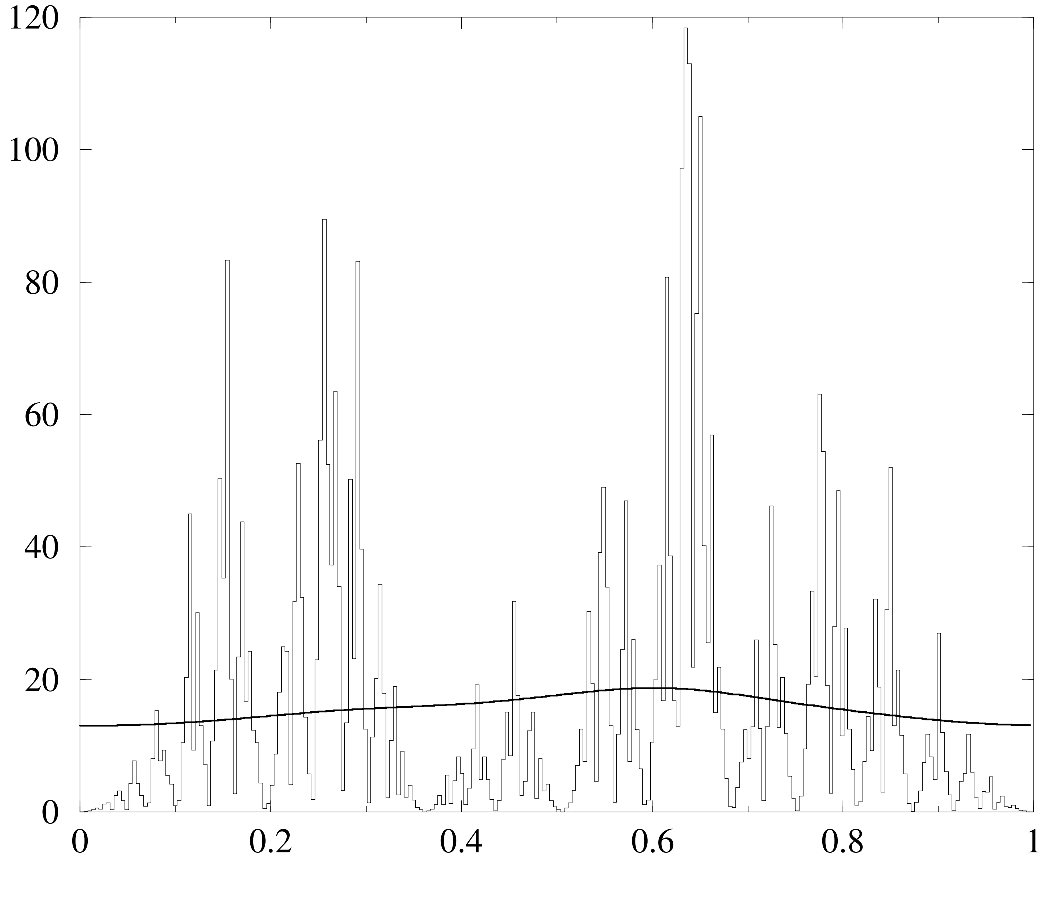
13. Plot of $\log_{10}\|\frac{\partial^\eta}{\partial t^\eta}\, e^{-t\sqrt{-\Delta}}\theta\|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for $\eta = 1$, 2 for 16 conjugacies of type θ^{NC}.

14. Plot of the first differences of the graph of $\log_{10} \| \frac{\partial^\eta}{\partial t^\eta} e^{-t\sqrt{-\Delta}} \theta \|_{L^\infty(\mathbb{T})}$ (in arbitrary units) vs. $\log_{10} t$ for $\eta = 2$ and $\eta = 3$ for four θ of type NC.

15. Plot of $\log_{10} \| \frac{\partial^2}{\partial t^2} e^{-t\sqrt{-\Delta}} \theta \|_{L^\infty(\mathbb{T})}$ vs. $\log_{10} t$ for four θ of type NC and the first and second differences.

16. Plot of $\log_2 \sup_k |\langle \theta, \psi_{jk} \rangle|$ vs. j for 12 conjugacies of type θ^{NQ} (for 2^{22} interpolated values based on 10^7 iterates) and 12 of type θ^{QN} (for 2^{21} interpolated values based on 10^6 iterates).

17. Plot of $\log_2 \|\theta - \Pi_j \theta\|_{L^\infty(\mathbb{T})}$ vs. j for 12 conjugacies of type θ^{NC} (for 2^{22} interpolated values based on 10^7 iterates) and 12 of type θ^{CN} (for 2^{21} interpolated values based on 2×10^6 iterates).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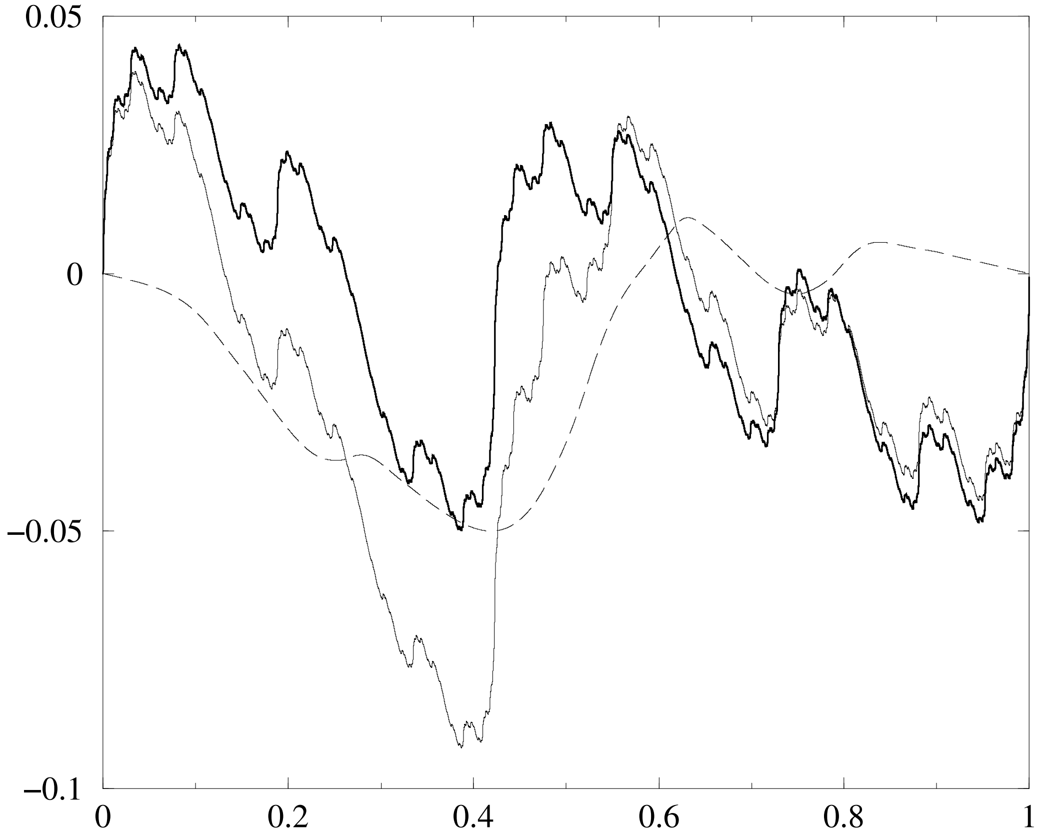

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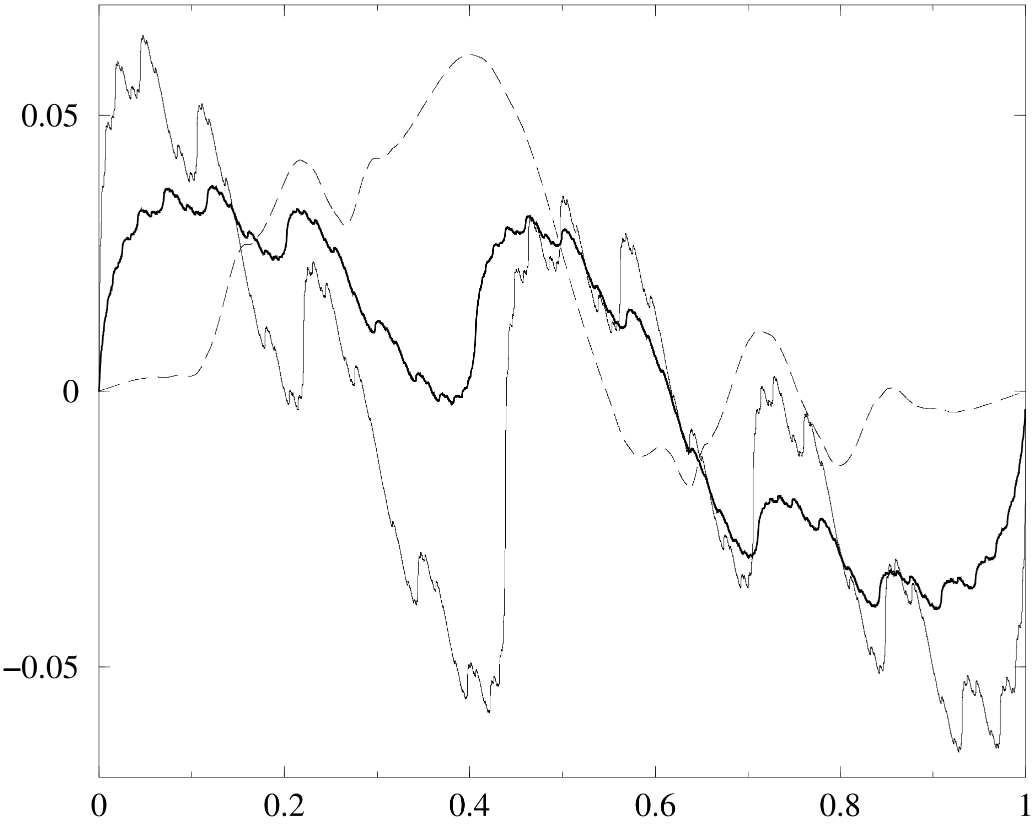
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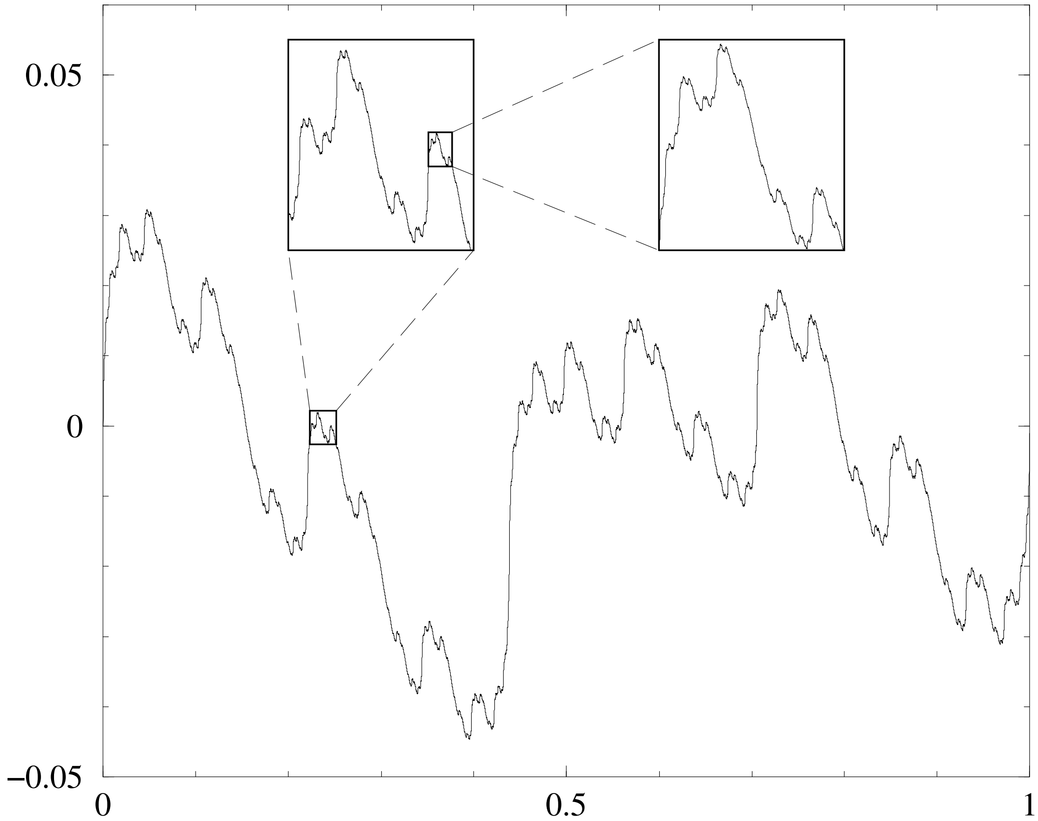
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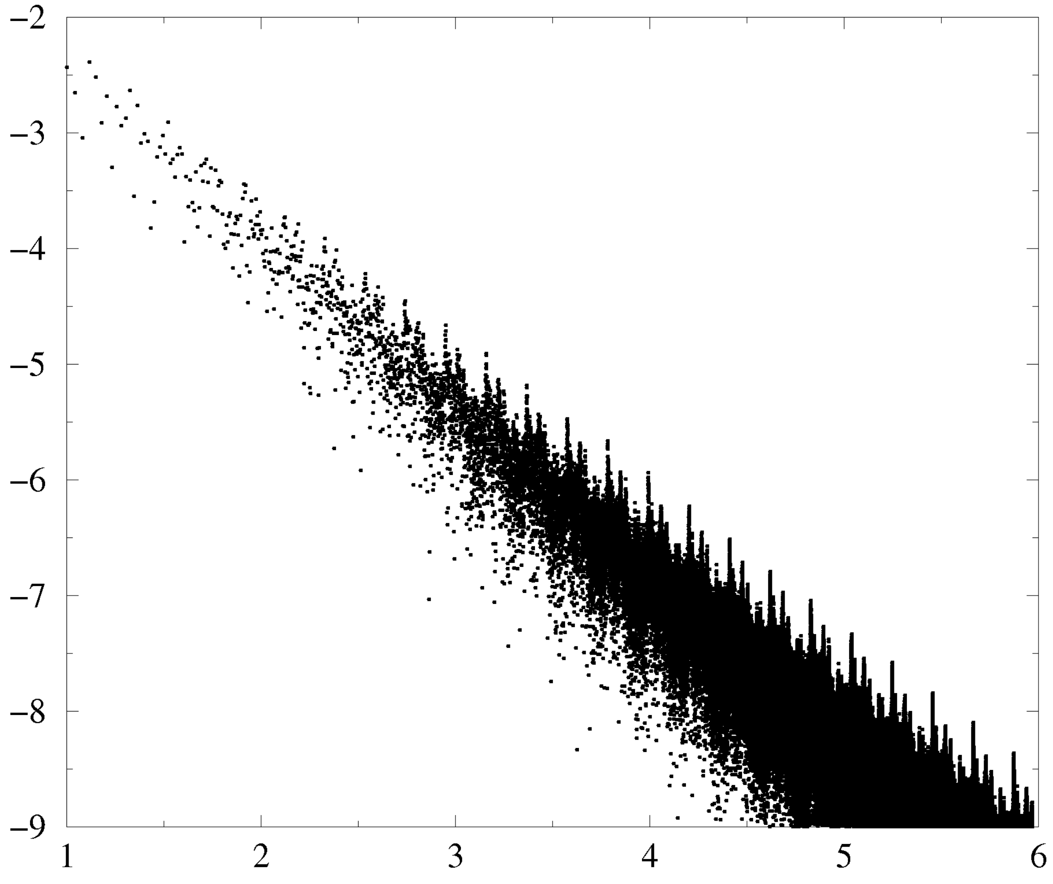

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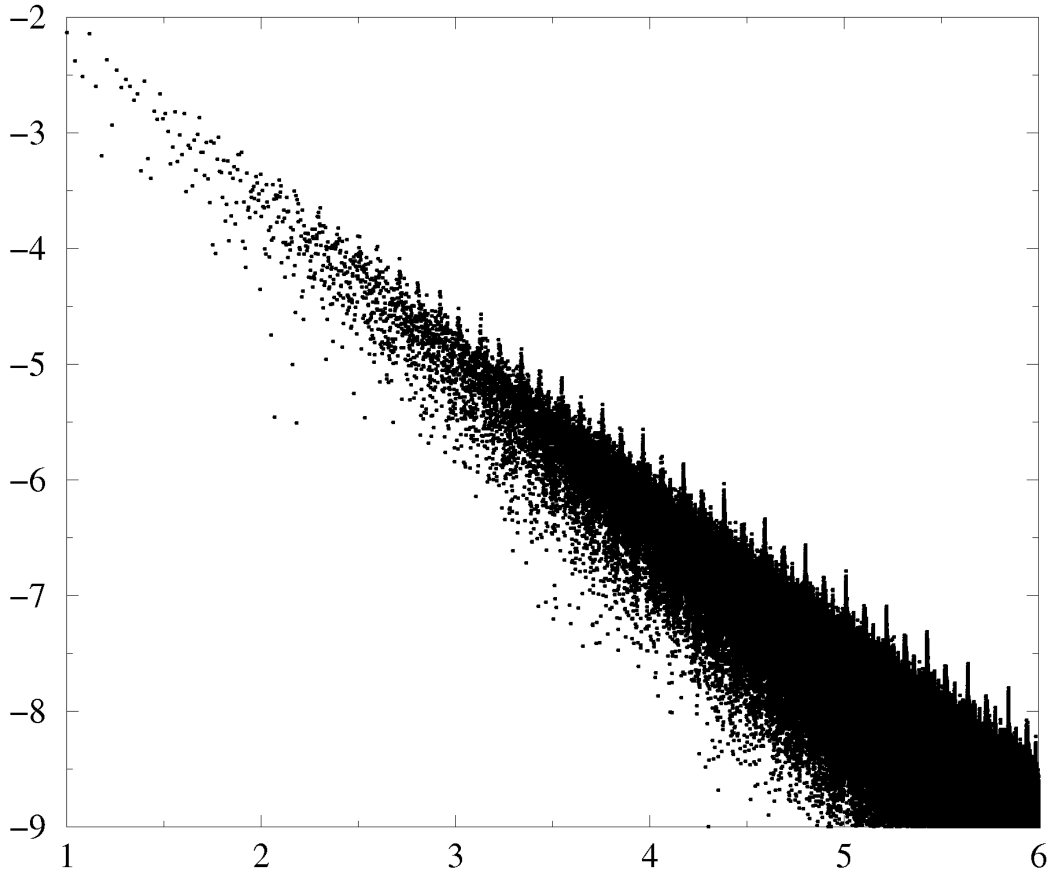

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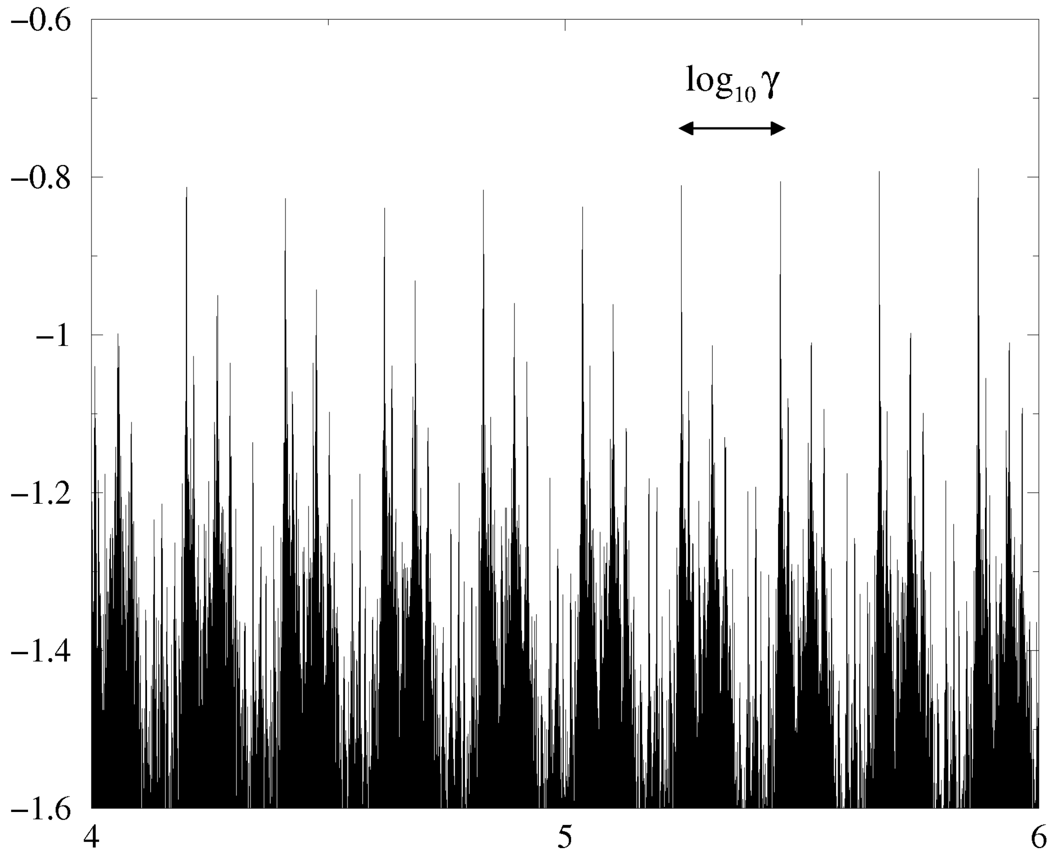

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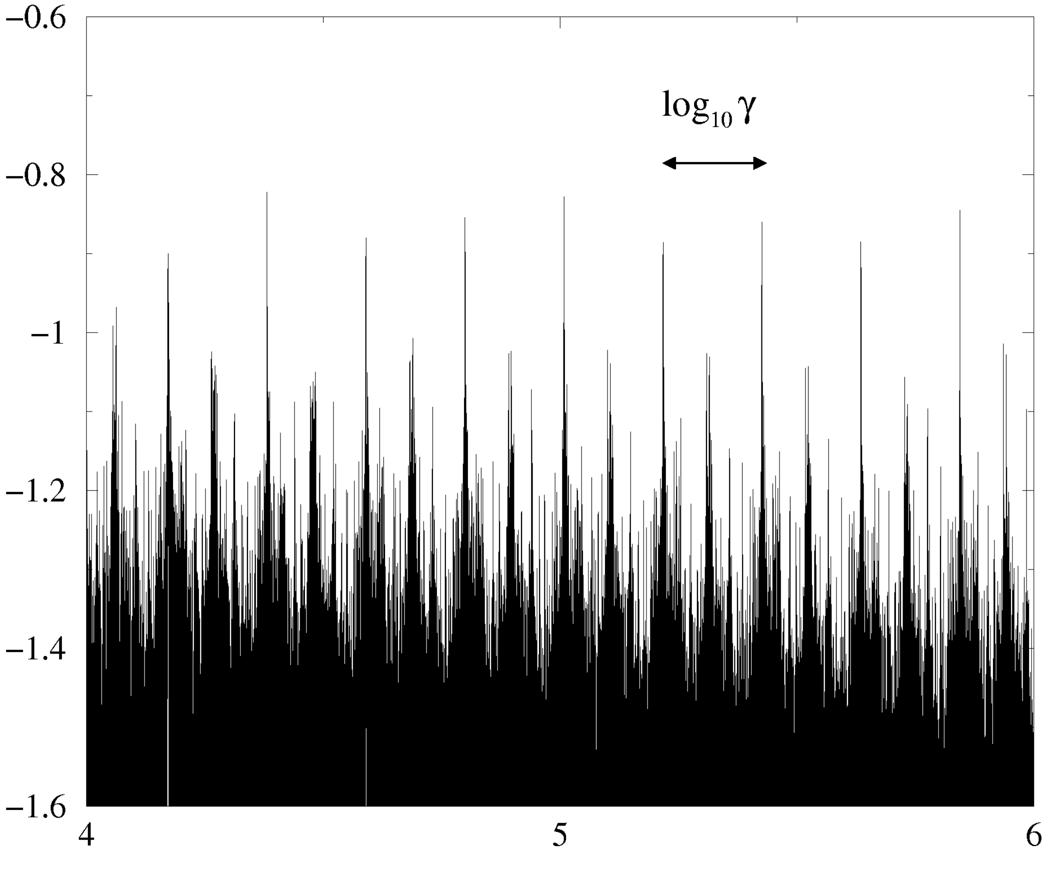

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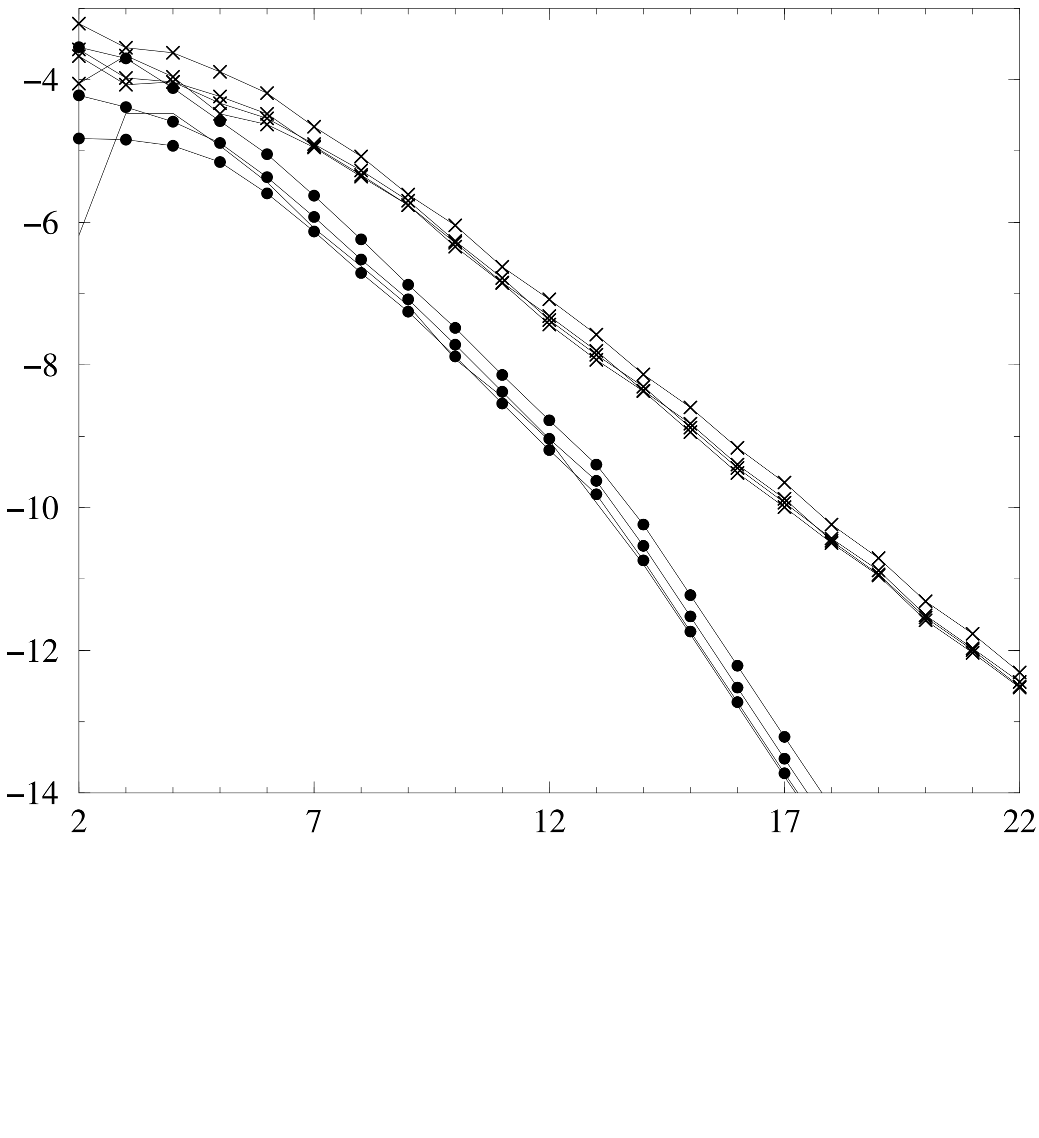
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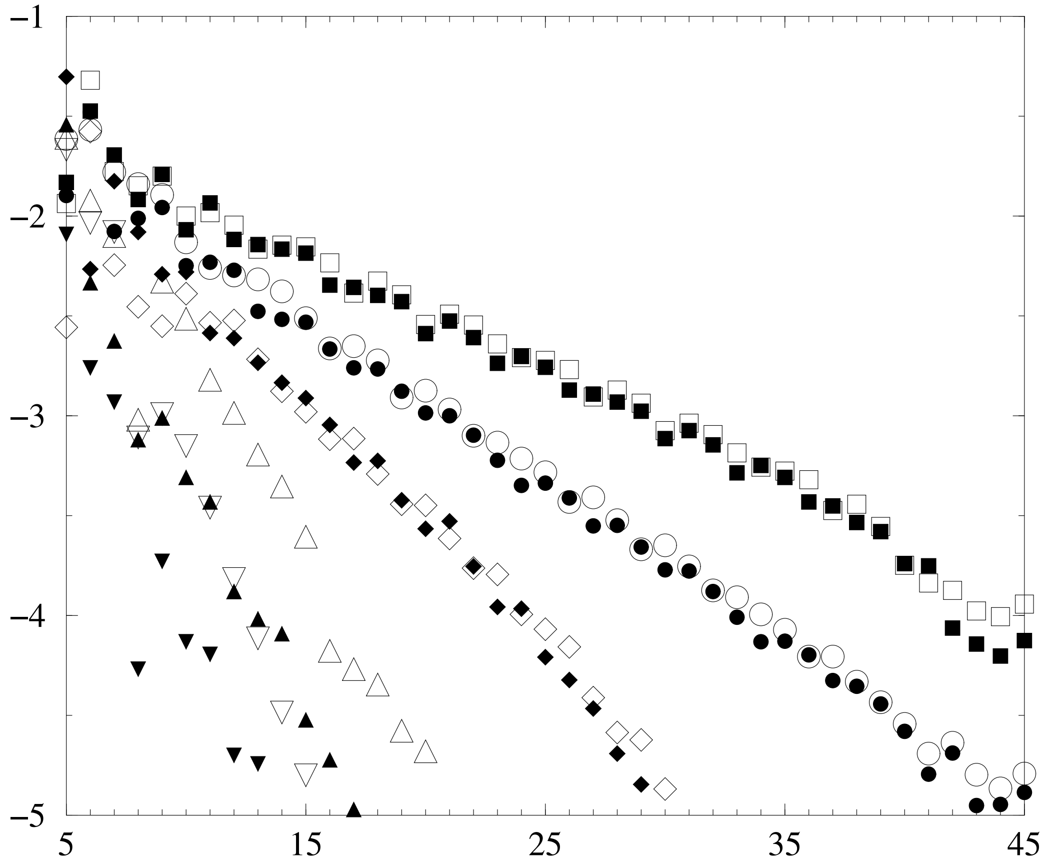
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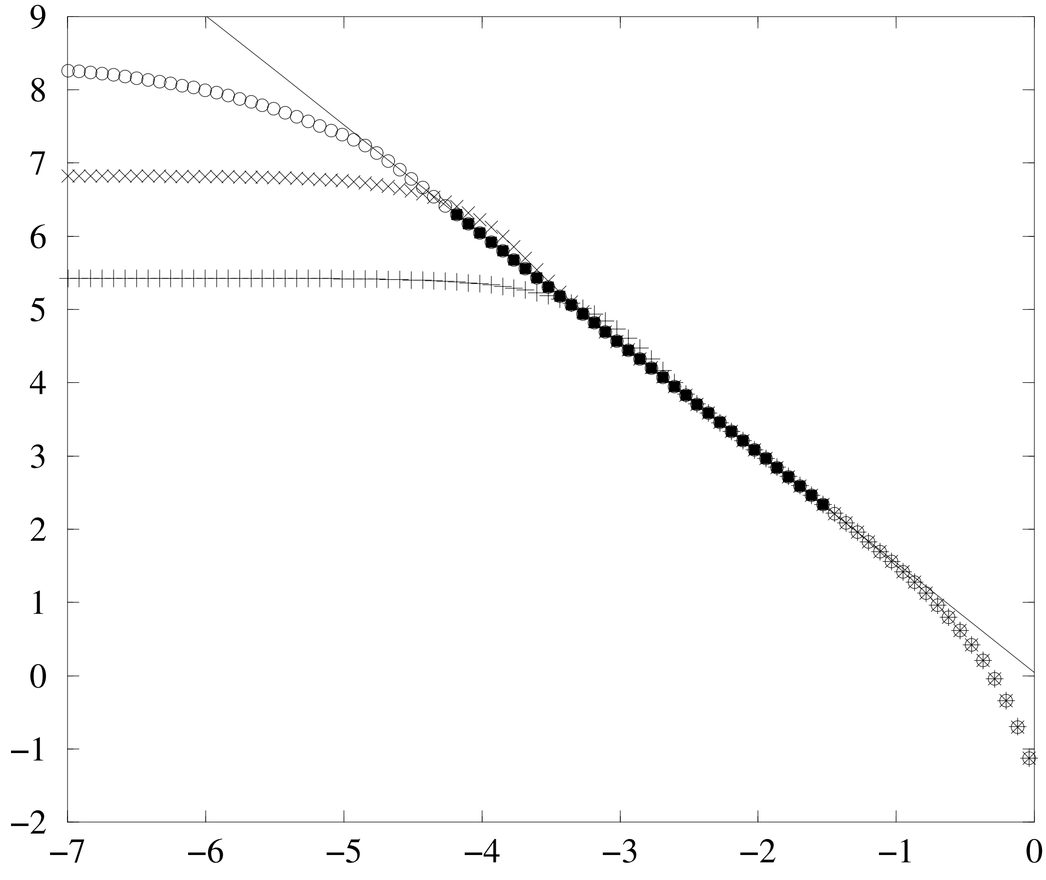
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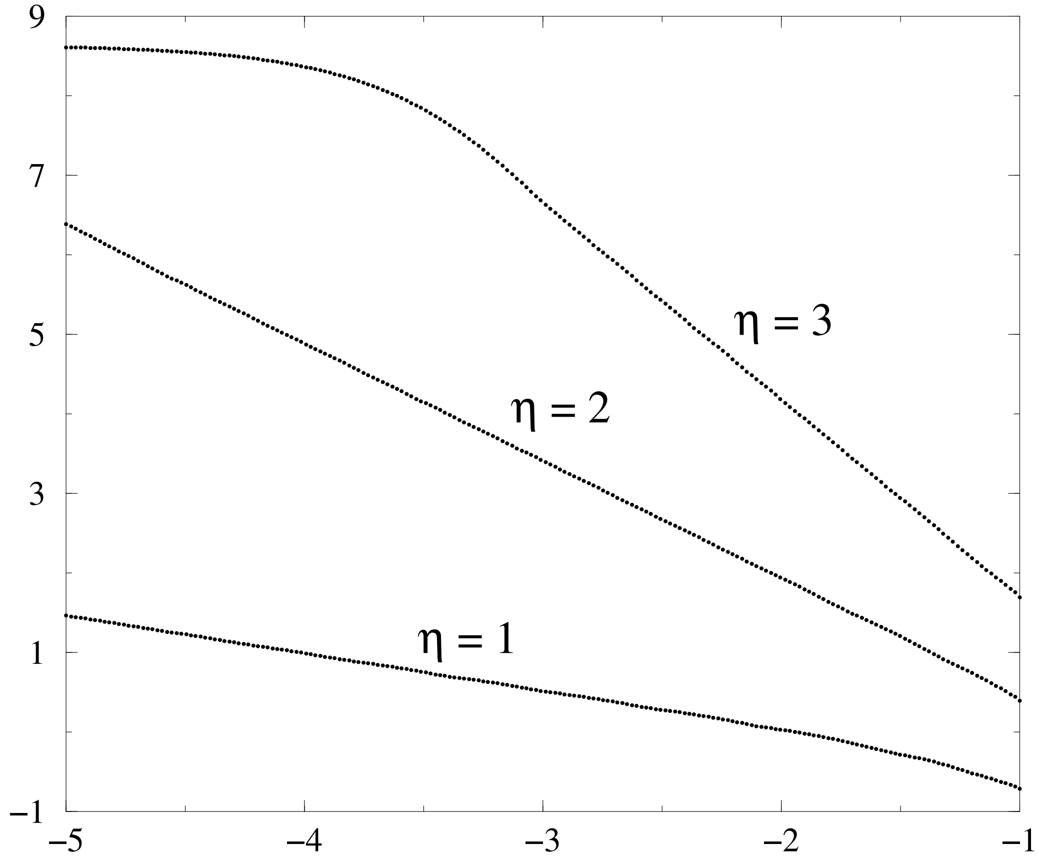
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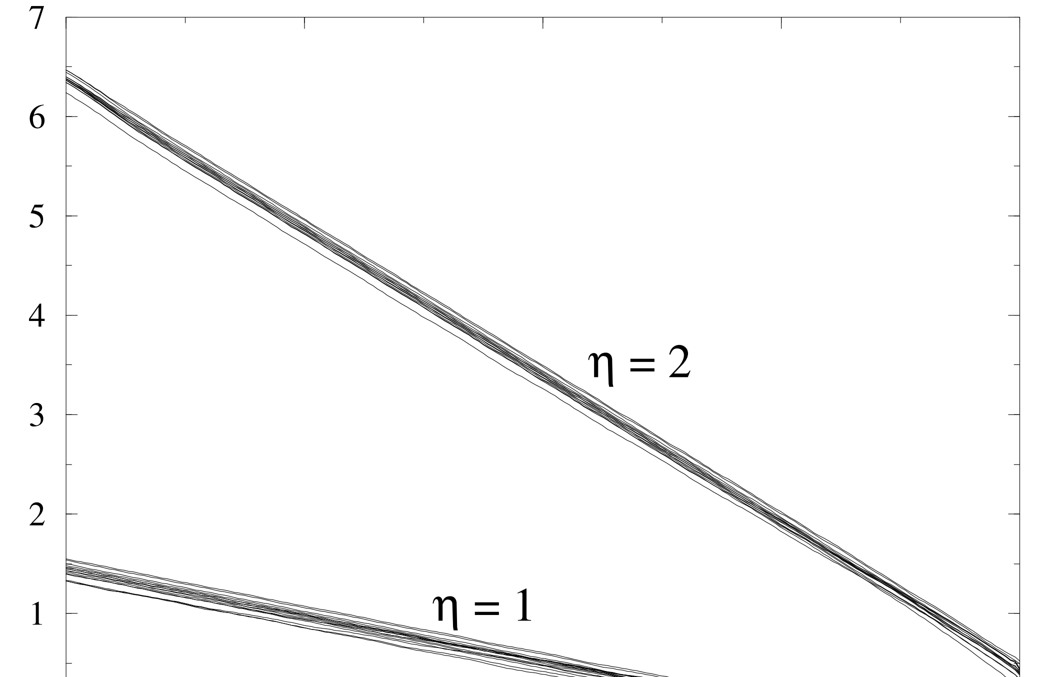

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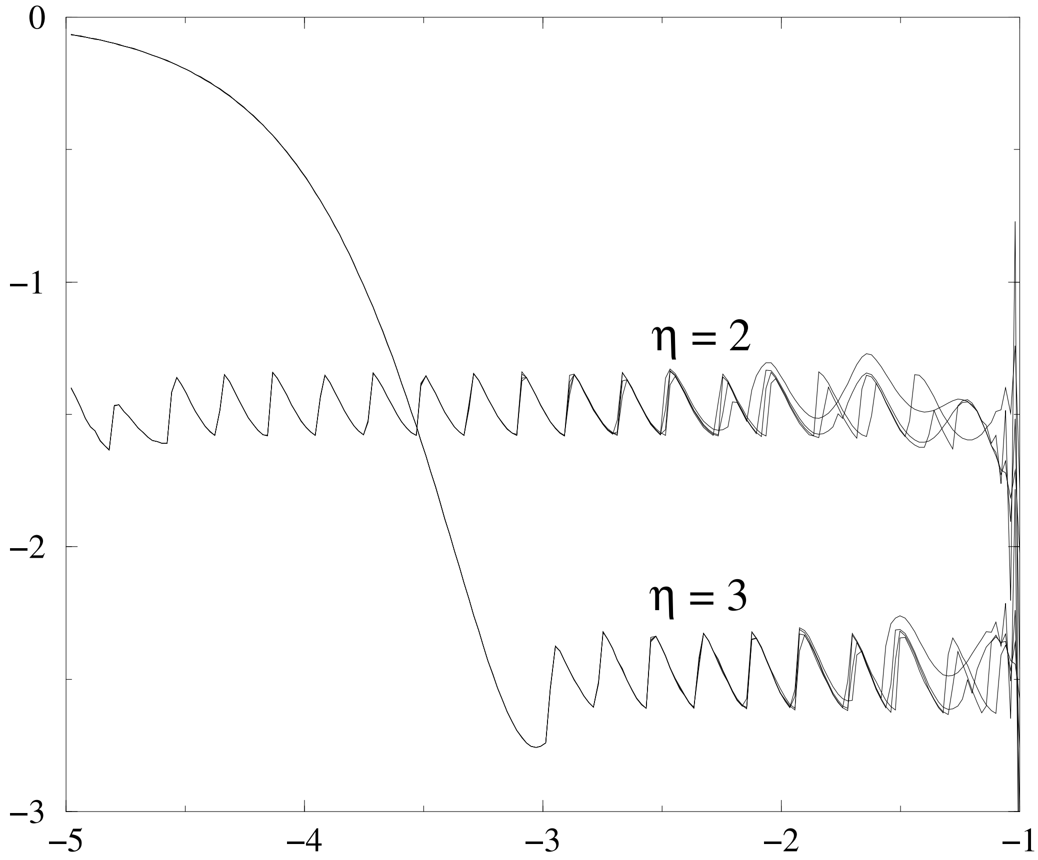
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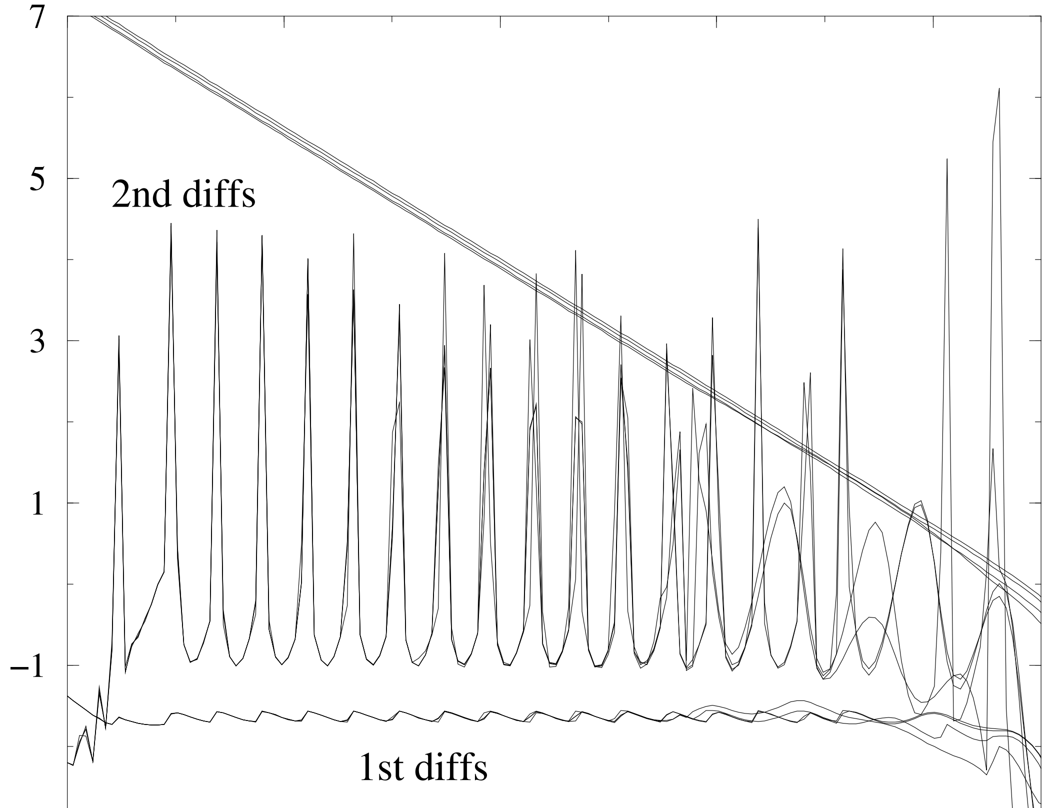
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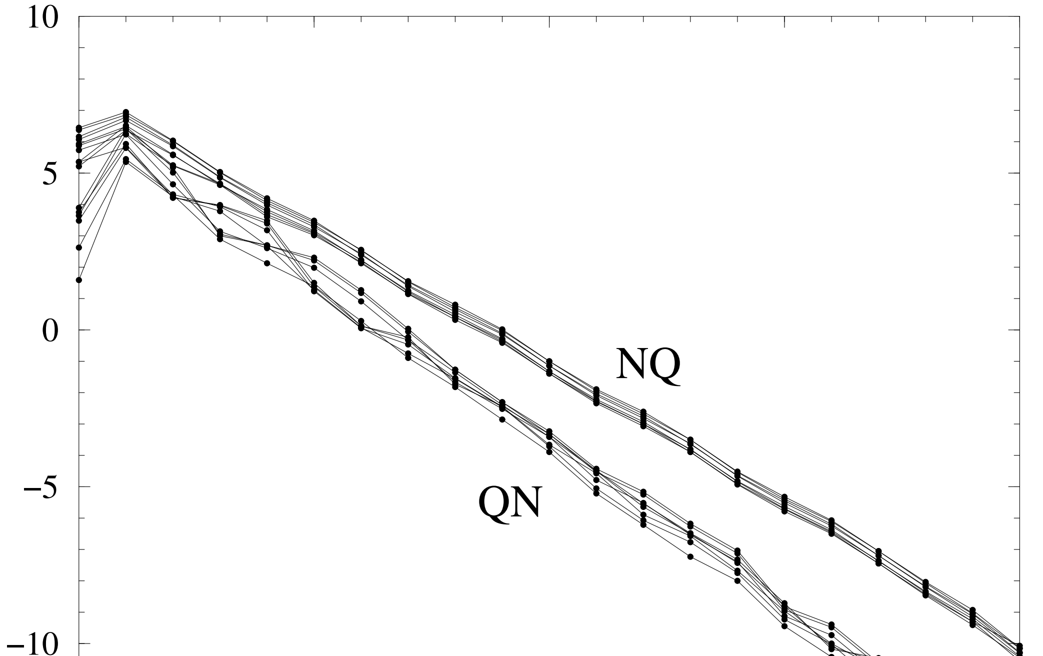
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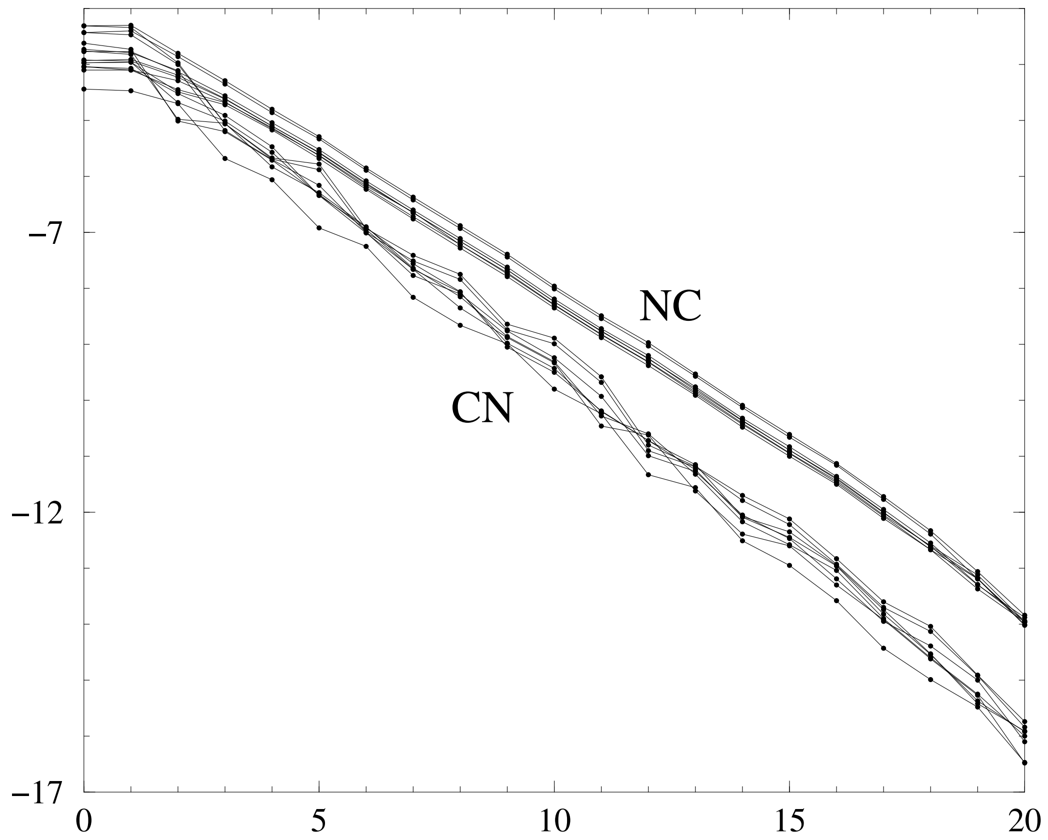

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