WEAK DISORDER LOCALIZATION AND LIFSHITZ TAILS

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ABSTRACT. This paper is devoted to the study of localization of discrete random Schrödinger Hamiltonians in the weak disorder regime. Consider an i.i.d. Anderson model and assume that its left spectral edge is 0. Let γ be the coupling constant measuring the strength of the disorder. For γ small, we prove a Lifshitz tail type estimate and use it to derive localization in a band starting at 0 going up to a distance $\gamma^{1+\eta}$ $(0<\eta<\eta_0)$ of the average of the potential. In this energy region, we show that the localization length at energy E is bounded from above by a constant times the square root of the distance between E and the average of the potential.

RÉSUMÉ. Dans cet article, nous étudions la localisation à faible désordre pour des opérateurs de Schrödinger aléatoires discrets. Considérons un modèle d'Anderson i.i.d. dont le bord spectral gauche vaut 0. Soit γ la constante de couplage mesurant le désordre. Pour γ petit, nous démontrons une estimée de type estimée de Lifshitz pour la densité d'états, et nous utilisons cette estimée pour prouver que le spectre de cet opérateur est localisé dans un intervalle allant de l'énergie 0 jusqu'à une distance de l'ordre de $\gamma^{1+\eta}$ $(0<\eta<\eta_0)$ de la moyenne du potentiel. Dans cette région d'énergie, la longueur de localisation à une énergie E est majorée par une constante fois la racine de la distance séparant E de la moyenne du potentiel.

0. INTRODUCTION

The main purpose of the present paper is to study the spectral behavior of discrete random Schrödinger operators in the weak disorder regime. We prove exponential decay for the Green's function of such operators in a neighborhood of finite spectral edges and give a lower bound on the rate of decay of the eigenfunctions associated to eigenvalues in this neighborhood. As a consequence of such a bound, we derive exponential localization and dynamical localization at finite band edges as well as Poissonian statistics for the eigenvalue spacing.

Let us start with an example. Consider the discrete Anderson model acting on $\ell^2(\mathbb{Z}^d)$ defined by

$$H_{\omega,\gamma} = -\Delta_d + \gamma V_\omega \tag{0.1}$$

where $-\Delta_d \geq 0$ is the free discrete Laplace operator

$$(-\Delta_d u)_n = \sum_{(m,n)\text{ nearest neighbors in }\mathbb{Z}^d} (u_m - u_n) \quad \text{for } u=(u_n)_{n\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$$

γ is a positive coupling constant and V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u=(u_n)_{n\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d). \tag{0.2}$$

We assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent identically distributed, bounded from below and non trivial. It is well known (see e.g. [6, 21]) that, under these assumption, there exists

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$\Sigma_\gamma \subset \mathbb{R}$ such that, for almost every $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$, the spectrum of $H_{\omega,\gamma}$ is equal to Σ_γ. Moreover, Σ_γ is given by

$$\Sigma_\gamma = \sigma(-\Delta_d) + \gamma \cdot \mathrm{supp}(\omega_0) = [0, 4d] + \gamma \cdot \mathrm{supp}(\omega_0).$$

Let Σ_- be the infimum of Σ_γ. To fix ideas, let us assume that $\inf(\omega_0) = 0$. This is no restriction on generality as it can be achieved by shifting energy by a constant. In this case, one has $\Sigma_- = 0$. Our purpose is to study the spectral properties of $H_{\omega,\gamma}$ near 0. Therefore, we also assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are square integrable i.e. $\mathbb{E}(|\omega_0|^2) < +\infty$ and that their common distribution admits a bounded density, say g. In the sequel, $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ respectively denote the probability and expectation with respect to the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$. Define $\overline{\omega} = \mathbb{E}(\omega_0)$.

We prove

Theorem 0.1. *Fix $\eta \in (0, d/(6d+6))$ and $s \in (0,1)$. There exists $\gamma_{\eta,s} > 0$ and $a > 0$ such that, for $\gamma \in (0, \gamma_{\eta,s})$, the Green's function of H_ω satisfies, for $(m,n) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $E \in I_{\gamma,\eta} := [0, \gamma(\overline{\omega} - \gamma^\eta)]$,*

$$\sup_{\varepsilon\in\mathbb{R}} \mathbb{E}\left(\left|\langle \delta_n, (H_{\omega,\gamma} - E - i\varepsilon)^{-1}\delta_m\rangle\right|^s\right) \leq \frac{1}{a} e^{-a\sqrt{|E-\gamma\overline{\omega}|}|m-n|}. \tag{0.3}$$

Here, δ_n is the vector in $\ell^2(\mathbb{Z}^d)$ with all coordinates equal to 0 except the n-th that is equal to 1.

It is well known (see e.g. [24, 2, 4]) that the bound (0.3) has strong implications on the spectral properties of $H_{\omega,\gamma}$ in the energy interval $I_{\gamma,\eta}$; in particular, it is known that it implies

- *Exponential localization ([24, 3]):* ω-almost surely, the spectrum of $H_{\omega,\gamma}$ in the interval $I_{\gamma,\eta}$ is purely punctual, and the corresponding eigenfunctions are exponentially localized. Moreover, the decay rate of the off-diagonal behavior of the Green's function gives an estimate on the rate of decay of the eigenfunctions i.e. an eigenfunction ψ corresponding to an eigenvalue E in $I_{\gamma,\eta}$ satisfies

$$\liminf_{|n|\to+\infty} -\frac{\log|\psi(n)|}{|n|} \geq a\sqrt{|E-\gamma\overline{\omega}|}. \tag{0.4}$$

  Estimate (0.4) gives a lower bound on the Lyapunov exponent, hence, an upper bound on the localization length; we see that this upper bound increases as E approaches $\gamma\overline{\omega}$. This is expected as it is expected that the nature of the spectrum changes at an energy between 0 and $\gamma\overline{\omega}$ called the mobility edge (see e.g. [1]). In this respect, it would be interesting to show a lower bound on the localization length having the same behavior as the upper bound obtained here.

  In the recent paper [23], the localization length in the bulk of the spectrum of $H_{\omega,\gamma}$ was analyzed for γ small in dimensions 1 and 2; in dimension 1, see also [21]. It was proved that it is of size γ^{-2}. It would be interesting to understand the transition between that regime and the one described in the present paper.
- *Dynamical localization ([1, 2]):* if $\Pi_\eta(H_{\omega,\gamma})$ denotes the spectral projector of $H_{\omega,\gamma}$ on the interval $I_{\gamma,\eta}$, then, for some $\tilde{a} > 0$, one has

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left|\langle \delta_n, e^{itH_{\omega,\gamma}}\Pi_\eta(H_{\omega,\gamma})\delta_m\rangle\right|\right) \leq \frac{1}{\tilde{a}} e^{-\tilde{a}\sqrt{|E-\gamma\overline{\omega}|}|m-n|}. \tag{0.5}$$

  That is, wave packets with energy in $I_{\gamma,\eta}$ do not spread.
- *Absence of level repulsion ([20]):* in the energy range $I_{\gamma,\eta}$, the local level spacings for $H_{\omega,\gamma}$ have Poisson statistics.

  More precisely, let Λ_L be the cube of center 0 and side length $2L+1$ in $\mathbb{Z}^d$, and let $H_{\omega,\gamma|\Lambda_L}$ be the Hamiltonian $H_{\omega,\gamma}$ restricted to the cube Λ_L with Dirichlet boundary conditions. $H_{\omega,\gamma|\Lambda_L}$ is a $(2L+1)^d$-dimensional Hermitian matrix. Let $E_1(L) \leq \cdots \leq E_{(2L+1)^d}(L)$ be its eigenvalues. Define the integrated density of states of H_ω by

$$\mathcal{N}_\gamma(E) = \lim_{\#\Lambda \to +\infty} \frac{\#\{\text{eigenvalues of } H_{\omega,\gamma|\Lambda} \leq E\}}{\#\Lambda}. \tag{0.6}$$

The limit in (0.6) exists ω-a.e.; it is non-random and non-decreasing ([6, 21]). Its derivative with respect to E is a positive measure; it is the density of states of H_ω and we denote it by $d\mathcal{N}_\gamma$. As the common distribution of the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ admits a bounded density, the measure $d\mathcal{N}_\gamma$ admits a bounded density i.e. $d\mathcal{N}_\gamma = n_\gamma(E)dE$; this is a consequence of the celebrated Wegner estimate (see [26, 6])).

To define the local level spacings, we fix an energy E and assume $n_\gamma(E) > 0$. We rescale the eigenvalues

$$\tilde{E}_j(L,E) = (2L+1)^d \frac{(E_j(L) - E)}{n_\gamma(E)}$$

and define the point process $\xi(L,E)$ by

$$\xi(L,E) = \sum_{j=1}^{(2L+1)^d} \delta_{\tilde{E}_j(L,E)}$$

where δ_x is the Dirac measure on $\mathbb{R}$.

Theorem 0.1 and the main result of [20] imply that, for γ sufficiently small, for Lebesgue almost every energy in $\Sigma_\gamma \cap I_{\gamma,\eta}$, the point process $\xi(L,E)$ converges in distribution to the standard Poisson point process.

In the recent paper [25] that triggered our interest in this question, W.M. Wang proved that the result of Theorem 0.1 and its consequences hold in a smaller energy interval. Using a weak disorder expansion of the resolvent, she proved that, for $\eta > 0$ fixed, for γ sufficiently small, (0.3) holds true for energies in the range $[0, \gamma^{1+\eta}]$.

Our method of proof is completely different from the one used in [25]. It relies on an estimate of the integrated density of states near the spectral edges that is related to Lifshitz tails; namely, under the assumptions made above, we prove

Theorem 0.2. *Fix $\eta \in (0, d/(6d+4))$. Then, there exists $\varepsilon > 0$ and $\gamma_\eta > 0$ such that, for $\gamma \in (0, \gamma_\eta)$, one has*

$$\mathcal{N}_\gamma(\gamma(\overline{\omega} - \gamma^\eta)) \leq e^{-\gamma^{-\varepsilon}}.$$

In the sequel, we prove a analogue of Theorems 0.2 and 0.1 for more general operators than H_ω. Theorem 0.2 can also be extended to continuous random Schrödinger operators [15] (and, possibly, to other random models) and, thus, can be used to derive localization results and bounds of the type (0.4) or (0.5) for these classes of random operators.

To complete this section, let us now briefly sketch the proof of Theorem 0.2. To simplify the argument, we restrict ourselves to the case when the random variables are bounded. One first reduces the question of estimating the integrated density of states to computing the probability of finding an eigenvalue for H_ω restricted to a cube Λ_N of sidelength N (N is not too large e.g. it is of size an inverse power of γ) below the energy $\gamma(\overline{\omega} - \gamma^\eta)$. So, we want to estimate the probability that there exists a normalized vector φ supported in Λ_L such that

$$\langle -\Delta\varphi, \varphi\rangle + \gamma\langle V_\omega\varphi, \varphi\rangle \leq \gamma(\overline{\omega} - \gamma^\eta).$$

As both $-\Delta$ and V_ω are non-negative, this implies

$$\langle -\Delta\varphi, \varphi\rangle \leq C\gamma, \tag{0.7}$$

$$\langle V_\omega\varphi, \varphi\rangle \leq \overline{\omega} - \gamma^\eta. \tag{0.8}$$

If we cut φ off in momentum space outside of a ball of radius L^{-1}, by (0.7), the part of $\hat{\varphi}$ outside of that ball is of L^2-norm $\lesssim (\gamma L^2)^{1/2}$ (here and in the rest of the paper, $a \lesssim b$ means $a \leq Cb$ for some constant C). We pick L so that $\gamma^{1/2}L \ll \gamma^\eta$ (here and in the rest of the paper, $a \ll b$ means $a/b \to 0$). So, in (0.8), we can assume that the "Fourier transform" of φ has support in a ball of radius L^{-1}. The uncertainty principle then states that, modulo a small error, φ can be replaced by a function that is constant over cube of size $L' \ll L$; the error (estimated by means of Lemma 6.2) is L'/L. We pick L' so that the error term $L'/L \ll \gamma^\eta$. This now enables us to transform condition (0.8) into

$$\sum_j \left(\frac{1}{(L')^d} \sum_{|k-k_j|\leq L'} (\omega_k - \overline{\omega}) \right) a_j \leq -\gamma^\eta \text{ and } \sum_j a_j = 1 + O(\gamma^\eta).$$

The probability of this event is then estimated using large deviations techniques; the exponential estimate one gets is $P \lesssim e^{-\gamma^{2\eta}(L')^d}$. Saturating the two error estimates given above, we get $L \asymp \gamma^{-1/2+\eta}$ and $L' \asymp \gamma^{-1/2+2\eta}$ (here and in the rest of the paper, $a \asymp b$ means $C^{-1}b \leq a \leq Cb$ for some constant C). Hence, P, the probability we want to estimate, is small if $\gamma^{2\eta+2d\eta-d/2} \to +\infty$ when $\gamma \to 0$. This is the case if $\eta < d/(4d+4)$. In Theorem 0.2, the limit value for η is $d/(4d+6)$; the difference comes from the fact that we only assumed that the random variables are square integrable; hence, one has to take their possible unboundedness into account in the error estimates and in the large deviation estimation.

We now describe our results in their full generality.

1. THE RESULTS

Let H be a translational invariant Jacobi matrix with exponential off-diagonal decay that is $H = ((h_{k-k'}))_{k,k'\in\mathbb{Z}^d}$ such that,

H0:
- $h_{-k} = \overline{h_k}$ for $k \in \mathbb{Z}^d$ and for some $k \neq 0$, $h_k \neq 0$.
- there exists $c > 0$ such that, for $k \in \mathbb{Z}^d$,

$$|h_k| \leq \frac{1}{c} e^{-c|k|}. \tag{1.1}$$

The infinite matrix H defines a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Using the Fourier transform, it is easily seen that H is unitarily equivalent to the multiplication by the function $\theta \mapsto h(\theta)$ defined by

$$h(\theta) = \sum_{k\in\mathbb{Z}} h_k e^{ik\theta} \text{ where } \theta = (\theta_1, \dots, \theta_d),$$

acting as an operator on $L^2(\mathbb{T}^d)$ where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ (the Lebesgue measure on $\mathbb{T}^d$ is normalized so that the constant function 1 has norm 1). The function h is real analytic on $\mathbb{T}^d$.
We assume

H1: the minima of $h : \mathbb{T}^d \to \mathbb{R}$ are quadratic non-degenerate.

If H is the free Laplace operator, then $h(\theta) = \sum_{1\leq j\leq d} (1+\cos(\theta_i))$. In this case, assumption (H1) is satisfied.

Let V_ω be the diagonal matrix defined by (0.2) with entries the independent identically distributed real valued random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ that satisfy

H2: the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are non trivial, are lower semibounded and for some $r \in (1, +\infty]$, they satisfy

$$\forall r' < r, \quad \mathbb{E}(|\omega_0|^{r'})^{1/r'} < +\infty. \tag{1.2}$$

At the cost of adding a constant to H, we may assume that 0 is the essential infimum of the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$. To state our results, we define the constant $\eta(r)$

$$0 < \eta(r) = \frac{d}{4(d+1+c(r)))} \leq \frac{d}{4(d+1)} \text{ where } c(r) = \frac{d}{2(r-1)} + \sup\left(\frac{2-r}{r-1}, 0\right). \tag{1.3}$$

Pick $\gamma > 0$ and define the self-adjoint operator

$$H_{\omega,\gamma} = H + \gamma V_\omega. \tag{1.4}$$

Let $\mathcal{N}_\gamma(E)$ be the integrated density of states of $H_{\omega,\gamma}$; it is defined by (0.6).
Our main result is

Theorem 1.1. *Assume (H1) and (H2). Fix* $\eta \in (0, \eta(r))$. *Then, there exists* $\gamma_\eta > 0$ *and* $\varepsilon > 0$ *such that, for* $\gamma \in (0, \gamma_\eta)$, *one has*

$$\mathcal{N}_\gamma(\gamma(\overline{\omega} - \gamma^\eta)) \leq e^{-\gamma^{-\varepsilon}}. \tag{1.5}$$

Let us now comment on our results. Since the work of I.M. Lifshitz (see [16, 17], and also [18]), it is well known that, near the bottom of the spectrum of random Schrödinger operators, the eigenvalue distribution i.e. the density of states has a large deviation regime, the so-called “Lifshitz tails”. In Theorem 1.1, we show that, in the weak disorder limit, the zone where the large deviation regime holds goes quite far, almost up to the average of the potential. It would be very interesting to understand what happens if one goes further up in energy i.e. when one gets closer to $\gamma\overline{\omega}$. One can expect to enter some kind of “central limit theorem” zone where the behavior of the density of states (and certainly of other spectral quantities) changes dramatically. This is the energy region where the mobility edge is expected.

Our second result concerns the decay of the Green’s function. To deduce estimates on the Green’s function of H_ω from Theorem 1.1, we use the results of [4]; hence, we need a regularity condition on the distribution of the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$, namely we assume

H3: the common distribution of the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ is Hölder continuous i.e. that there exists $\tau \in (0,1)$ and $C > 0$ such that, for $a < b$, one has

$$\mathbb{P}(\{\omega_0 \in [a,b]\}) \leq C|b-a|^\tau.$$

Theorem 1.1 implies

Theorem 1.2. *Assume (H1), (H2) and (H3). Fix* $\eta \in (0, \eta(r))$ *and* $s \in (0, \tau/4)$. *There exists* $\gamma_{\eta,s} > 0$ *and* $a > 0$ *such that, for* $\gamma \in (0, \gamma_{\eta,s})$, *the Green’s function satisfies, for* $(m,n) \in \mathbb{Z}^d \times \mathbb{Z}^d$ *and for* $E \in [0, \gamma(\overline{\omega} - \gamma^\eta)]$,

$$\sup_{\varepsilon\in\mathbb{R}} \mathbb{E}\left(\left|\langle \delta_n, (H_{\omega,\gamma} - E - i\varepsilon)^{-1}\delta_m\rangle\right|^s\right) \leq \frac{1}{a} e^{-a\sqrt{|E-\gamma\overline{\omega}|}|m-n|}. \tag{1.6}$$

As a consequence of estimate (1.6), we obtain exponential and dynamical localization, as well as an upper bound on the localization length (see the paragraph following Theorem 0.1 in the introduction, as well as [24, 3, 1, 2]). As for the Poisson behavior of the level spacings, though in [20], the argument is presented in the case when $H = -\Delta_d$, it extends as is to the case we are dealing with here. Hence, when (H3) holds with $\tau = 1$, the Wegner estimate (see e.g. [6]) ensures the existence of the density of states $n_\gamma(E)$; if $n_\gamma(E) > 0$, the level spacings follow a Poisson distribution asymptotically.

We note here that Theorem 0.1 (resp. Theorem 0.2) is a special case of Theorem 1.1 (resp. Theorem 1.2).

Let us shortly comment on our assumptions. First, one can weaken the decay assumption on the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ and only assume that they are integrable; i.e. we assume that

H2': the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are non trivial, lower semibounded and satisfy $\mathbb{E}(|\omega_0|) < +\infty$.
Then, one proves

Theorem 1.3. *Assume (H1) and (H2'). Fix* $\eta \in (0,1)$. *There exists* $\gamma_\eta > 0$ *and* $\varepsilon > 0$ *such that, for* $\gamma \in (0, \gamma_\eta)$, *one has*

$$\mathcal{N}_\gamma(\gamma(\overline{\omega} - \eta))) \leq e^{-\gamma^{-\varepsilon}}.$$

This implies

Theorem 1.4. *Assume (H1), (H2') and (H3). Fix* $\eta \in (0,1)$ *and* $s > 0$. *There exists* $\gamma_{\eta,s} > 0$ *and* $a > 0$ *such that, for* $\gamma \in (0, \gamma_{\eta,s})$, *the Green's function satisfies, for* $(m,n) \in \mathbb{Z}^d \times \mathbb{Z}^d$ *and for* $E \in [0, \gamma(\overline{\omega} - \eta)]$,

$$\sup_{\varepsilon\in\mathbb{R}} \mathbb{E}\left(\left|\langle \delta_n, (H_{\omega,\gamma} - E - i\varepsilon)^{-1}\delta_m\rangle\right|^s\right) \leq \frac{1}{a} e^{-a\sqrt{|E-\gamma\overline{\omega}|}|m-n|}.$$

One can also relax assumption (H1) on the background operator h. From the proof given in the present paper, it is clear that, at the expense of changing $\eta(r)$, Theorems 1.1 and 1.3 still hold if one assumes that h assumes its minimum at finitely many points. In dimension 2, using the techniques developed in [12], one can prove that Theorems 1.1 and 1.3 hold for any non constant, analytic function h.

1.1. **The layout of the paper.** To end this section, we describe the layout of the paper. In section 2, we show how to deduce Theorem 1.2 from Theorem 1.1. In section 3, we reduce Theorem 1.1 to an equivalent statement for bounded random variables depending on γ. Section 4 is devoted to the proof of that statement. The section begins with a description of the periodic approximations, section 4.1; we recall some facts of Floquet theory in section 4.2. Then, we estimate the density of states of the periodic approximations in two steps; first, we reduce this question to estimating the probability of certain events (expressed only in terms of the random variables) in sections 4.3 and 4.4. In section 4.5, we carry out this estimation following the ideas outlined at the end of the introduction. The paper is completed by an appendix, section 6, where we gather some useful technical results.

2. THE PROOF OF THEOREM 1.2

To deduce Theorem 1.2 from Theorem 1.1, we use the finite volume localization criterion developed in Theorem 3.1 of [4]. To be able to use this criterion, we first need to check that, for some $0 < s < \tau$, our random variables are s-regular in the sense of definition 3.1 in [4] i.e. the s-decoupling property (1.7) in [4]; this is done in section 6.3. Let us note that, if h is a trigonometric polynomial, the s-regularity is not necessary as we can use Theorem 1.1 of [4]; this applies in particular when H is the free Laplace operator.

Let us now describe the finite volume criterion. Therefore, we consider the resolvent of $H_{\omega,\gamma}$ restricted to a cube with Dirichlet boundary conditions. More precisely, pick $\Lambda_L \subset \mathbb{Z}^d$, the cube centered at 0 of sidelength $2L+1$. Let $H^D_{\omega,\gamma|\Lambda_L}$ be the random Hamiltonian $H_{\omega,\gamma}$ restricted to the box Λ_L with Dirichlet boundary conditions, i.e., $H^D_{\omega,\gamma|\Lambda_L} = \Pi_{\Lambda_L} H_{\omega,\gamma} \Pi_{\Lambda_L}$ where Π_{Λ_L} is the projector onto the sites in Λ_L. Rephrased with our notations, to check the localization criterion developed in [4], we need to check that, for γ sufficiently small, for some cube Λ_L, one has

$$C_0 L^{2d} \gamma^{-2s} \sum_{\substack{m\in\Lambda_L \\ n\in\mathbb{Z}^d\setminus\Lambda_L}} e^{-c|m-n|} \mathbb{E}(|\langle \delta_0, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s) e^{\delta(E)|n|/C_0} < 1 \tag{2.1}$$

where C_0 is a constant depending only on h and on the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$.
Let us prove the bound (2.1); therefore, we use

Theorem 2.1 ([12])**.** *There exists $C > 0$ such that, for $L \geq 1$, $\gamma \in [0,1]$ and $E \in \mathbb{R}$, one has*

$$\mathbb{P}(\{H^D_{\omega,\gamma|\Lambda_L} \text{ admits an eigenvalue } \leq E) \leq CL^d \mathcal{N}_\gamma(E).$$

In [12], Theorem 2.1 was proved in dimension 2; the proof carries over to arbitrary dimension without a change. We do not repeat it here.

Fix η as in Theorem 1.2 and fix $\eta < \eta' < \eta(r)$. Define $\Omega_{\gamma,\eta',L} = \{$there exists an eigenvalue of $H_{\omega,\gamma|\Lambda_L} \leq \gamma(\varpi - \gamma^{\eta'})\}$. Using Theorem 1.1 in conjunction with Theorem 2.1 we obtain that, there exists $\gamma_\eta > 0$ and $\varepsilon > 0$ such that, for $0 < \gamma < \gamma_\eta$, $1 \leq L \leq e^{\gamma^{-\varepsilon/2}}$, one has

$$\mathbb{P}(\Omega_{\gamma,\eta',L}) \leq C e^{d\gamma^{-\varepsilon/2}} e^{-\gamma^{-\varepsilon}} \leq e^{-\frac{1}{2}\gamma^{-\varepsilon}}. \tag{2.2}$$

By a Combes-Thomas estimate, Lemma 6.1, we get that, for $\omega \notin \Omega_{\gamma,\eta,L}$ and $E \in [0, \gamma(\varpi - \gamma^\eta)]$, one has

$$\left|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle\right| \leq C\gamma^{-1-\eta} e^{-\sqrt{|E-\gamma(\varpi-\gamma^{\eta'})|}|m-n|/C}, \tag{2.3}$$

for $C \geq \tilde{C}_0$ where $\tilde{C}_0 > 0$ is a constant independent of γ and η. Define $\delta(E) := \sqrt{|E - \gamma(\varpi - \gamma^{\eta'})|}$. We notice that, as $E - \gamma\varpi \geq \gamma^{1+\eta}$ and $\eta < \eta'$, for γ sufficiently small, one has

$$\delta(E) \geq \frac{1}{2}\sqrt{|E - \gamma\varpi|}. \tag{2.4}$$

Pick $0 < s < s' < 1$. Using Hölder's inequality and the bound (2.10) in [4], namely

$$\mathbb{E}(|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^{s'}) \leq C\gamma^{-s'},$$

equation (2.3) gives, for some $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}(|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s) &= \mathbb{E}(|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s \mathbf{1}_{\{\omega\notin\Omega_{\gamma,\eta,L}\}}) \\
&\quad + \mathbb{E}(|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s \mathbf{1}_{\{\omega\in\Omega_{\gamma,\eta,L}\}}) \\
&\leq C\gamma^{-1-\eta} e^{-\delta(E)|m-n|/C} \\
&\quad + \mathbb{E}(|\langle \delta_n, (E+i\varepsilon - H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^{s'})^{s/s'} \mathbb{P}(\Omega_{\gamma,\eta,L})^{(s'-s)/s'} \\
&\leq C\gamma^{-1-\eta} e^{-\delta(E)|m-n|/C} + C\gamma^{-s} e^{-\frac{1}{2}\gamma^{-\varepsilon}}.
\end{aligned} \tag{2.5}$$

Summing (2.5) over $m \in \Lambda_L$ for $n \in \mathbb{Z}^d \setminus \Lambda_L$, and taking Λ_L of side-length $1 \le L \le e^{\gamma^{-\varepsilon/2}}$, for γ sufficiently small, we obtain

$$\begin{aligned} CL^{2d}\gamma^{-2s} \sum_{\substack{m\in\Lambda_L \\ n\in\mathbb{Z}^d\setminus\Lambda_L}} & e^{-c|m-n|}\mathbb{E}(|\langle\delta_0,(E+i\varepsilon-H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s)e^{\delta(E)|n|/(8C)} \\ & \le CL^{3d}\gamma^{-3s}e^{\delta(E)L/(8C)}e^{-\gamma^{-\varepsilon}} + CL^{2d}\gamma^{-2s-1-\eta}\cdot S \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} S &:= \sum_{\substack{m\in\Lambda_L \\ n\in\mathbb{Z}^d\setminus\Lambda_L}} e^{-c|m-n|}e^{-\delta(E)|m|/C}e^{\delta(E)|n|/(8C)} \\ &= \sum_{\substack{|m|\le L \\ |n|\ge 2L}} + \sum_{\substack{|m|\le L/2 \\ |n|\ge L}} + \sum_{\substack{L/2\le|m|\le L \\ L\le|n|\le 2L}} e^{-c|m-n|}e^{-\delta(E)|m|/C}e^{\delta(E)|n|/(8C)} \\ &\le Ce^{-L/C}L^{d-1} + \sum_{\substack{L/2\le|m|\le L \\ L\le|n|\le 2L}} e^{-c|m-n|}e^{-\delta(E)|m|/(8C)} \le Ce^{-L/C}L^{d-1} + CL^de^{-\delta(E)L/(8C)}. \end{aligned} \tag{2.7}$$

If we take $\gamma^{-\rho} \le L \le e^{\gamma^{-\varepsilon/2}}$ for $2\rho > 1+\eta(r)$, for $E \in [0, \gamma(\overline{\omega}-\gamma^\eta)]$, one has $\delta(E)L \ge \gamma^{-\varepsilon}$ for some $\varepsilon > 0$ and γ sufficiently small. Hence, using this in (2.7) and (2.6), for γ sufficiently small, we obtain

$$CL^{2d}\gamma^{-2s} \sum_{\substack{m\in\Lambda_L \\ n\in\mathbb{Z}^d\setminus\Lambda_L}} e^{-c|m-n|}\mathbb{E}(|\langle\delta_0,(E+i\varepsilon-H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s)e^{\delta(E)|n|/(8C)} < 1/16.$$

Taking (2.4) into account, we obtain, for $C \ge \tilde{C}_0$, there exists $\gamma_{C,\eta} > 0$ such that, for $0 < \gamma < \gamma_{C,\eta}$,

$$(8C)L^{2d}\gamma^{-2s} \sum_{\substack{m\in\Lambda_L \\ n\in\mathbb{Z}^d\setminus\Lambda_L}} e^{-c|m-n|}\mathbb{E}(|\langle\delta_0,(E+i\varepsilon-H_{\omega,\gamma|\Lambda_L})^{-1}\delta_m\rangle|^s)e^{\sqrt{|E-\gamma\overline{\omega}|}|n|/(8C)} < 1.$$

So, the finite volume criterion (2.1) is satisfied if we take C so that $8C > C_0$. Then, Theorem 3.1 of [4] implies Theorem 1.2.

3. THE PROOF OF THEOREM 1.1

To prove Theorem 1.1, we first bring ourselves back to the case when the random variables are bounded. Of course, except if the $(\omega_n)_{n\in\mathbb{Z}^d}$ are bounded, the new random variables and, a fortiori, their supremum depend on γ.

Let $\eta \in (0, \eta(r)$ be fixed. As there exists $r' < r$ such that $\eta < \eta(r') < \eta(r)$ and $\mathbb{E}(|\omega_0|^{r'})^{1/r'} < +\infty$, we may assume that

$$\mathbb{E}(|\omega_0|^r)^{1/r} < +\infty.$$

Pick $\lambda > 0$ and, for $n \in \mathbb{Z}^d$, consider the random variables $\omega_n(\lambda) = \inf(\omega_n, \lambda)$. They are independent, identically distributed and bounded. Let us estimate the difference $\mathbb{E}(\omega_0 - \omega_0(\lambda))$. Therefore, we write

$$\begin{aligned} 0 \leq \mathbb{E}(\omega_0 - \omega_0(\lambda)) = \mathbb{E}\left((\omega_0 - \lambda)\mathbf{1}_{\{\omega_0 \geq \lambda\}}\right) &\leq \sum_{N \geq 1} \lambda N \mathbb{P}(N\lambda < \omega_0 \leq (N+1)\lambda) \\ &= \lambda \lim_{M \to +\infty} \sum_{N=1}^{M} N \left[\mathbb{P}(\omega_0 > N\lambda) - \mathbb{P}(\omega_0 > (N+1)\lambda)\right] \\ &= \lambda(\mathbb{P}(\omega_0 > \lambda) + \lambda \lim_{M \to +\infty} \left[M\mathbb{P}(\omega_0 > (M+1)\lambda) - \sum_{N=2}^{M} \mathbb{P}(\omega_0 > N\lambda)\right] \\ &= \lambda(\mathbb{P}(\omega_0 > \lambda) - \lambda \sum_{N \geq 2} \mathbb{P}(\omega_0 > N\lambda). \end{aligned} \tag{3.1}$$

The convergence in (3.1) is guaranteed by the Markov bound

$$\mathbb{P}(\omega_0 > N\lambda) \leq \frac{\mathbb{E}(\omega_0^r)}{(N\lambda)^r}.$$

This and (3.1) imply that, if we set $\overline{\omega(\lambda)} := \mathbb{E}(\omega_0(\lambda))$, then

$$0 \leq \overline{\omega} - \overline{\omega(\lambda)}) \leq \frac{\mathbb{E}(\omega_0^r)}{\lambda^{r-1}}. \tag{3.2}$$

On $\ell^2(\mathbb{Z}^d)$, consider the random operator $H_{\omega,\gamma}(\lambda) = H + \gamma V_\omega(\lambda)$ where $V_\omega(\lambda)$ is the diagonal matrix having entries $(\omega_n(\lambda))_{n \in \mathbb{Z}^d}$. One has $H_{\omega,\gamma}(\lambda) \leq H_{\omega,\gamma}$. Hence, if $\mathcal{N}_\gamma(E;\lambda)$ denotes the integrated density of states of $H_{\omega,\gamma}(\lambda)$, we have

$$\mathcal{N}_\gamma(\gamma(\overline{\omega} - \gamma^\eta)) \leq \mathcal{N}_\gamma(\gamma(\overline{\omega} - \gamma^\eta);\lambda) \leq \mathcal{N}_\gamma\left(\gamma(\overline{\omega(\lambda)} - \gamma^\eta + C/\lambda^{r-1});\lambda\right).$$

In the last step, we have used (3.2). In particular, if we set $\lambda = \lambda(\gamma) = \gamma^{-\eta'/(r-1)}$ where $\eta < \eta' < \eta(r)$, for γ sufficiently small, we get

$$\mathcal{N}_\gamma(\gamma(\overline{\omega} - \gamma^\eta)) \leq \mathcal{N}_\gamma\left(\gamma(\overline{\omega(\lambda)}1 - \gamma^{\eta'});\lambda(\gamma)\right). \tag{3.3}$$

Hence, Theorem 1.1 is an immediate corollary of

Theorem 3.1. *Fix $\eta < \eta' < \eta(r)$ and set $\lambda = \lambda(\gamma) = \gamma^{-\eta'/(r-1)}$ as above. Then, there exists $\gamma_\eta > 0$ and $\varepsilon > 0$ such that, for $\gamma \in (0,\gamma_\eta)$, one has*

$$\mathcal{N}_\gamma\left(\gamma(\overline{\omega(\lambda)} - \gamma^\eta);\lambda(\gamma)\right) \leq e^{-\gamma^{-\varepsilon}}.$$

4. THE PROOF OF THEOREM 3.1

From now on, to simplify the notations, except in the case of possible confusion, we do not explicitly write the dependence of the random variables $(\omega_n(\lambda))_{n \in \mathbb{Z}^d}$ in λ; we call them $(\omega_n)_{n \in \mathbb{Z}^d}$; we just keep in mind the fact that they are bounded by $\lambda = \lambda(\gamma)$. We also do so for all the objects involving these random variables; e.g., the operator $H_{\omega,\gamma}(\lambda)$ is called $H_{\omega,\gamma}$, etc.

Let us now briefly explain the proof of Theorem 3.1. First, we introduce a periodic approximation scheme (that has proved useful for related issues in [11, 14]). This scheme delivers a sequence of very quickly converging finite volume approximations to the density of states. To analyze the finite volume density of states, we recall the Floquet theory of periodic operators. We use it to reduce the problem of estimating the density of states to the probability of estimating a well chosen event.

4.1. **Periodic approximations.** Let $(\omega_n)_{n\in\mathbb{Z}^d}$ be a realization of the random variables defined above. Fix $N\in\mathbb{N}^*$. We define $H^N_{\omega,\gamma}$, a periodic operator acting on $\ell^2(\mathbb{Z}^d)$ by

$$H^N_{\omega,\gamma}=H+V^N_{\omega,\gamma}=H+\sum_{n\in\mathbb{Z}^d_{2N+1}}\omega_n\sum_{l\in(2N+1)\mathbb{Z}^d}|\delta_{l+n}\rangle\langle\delta_{l+n}|.$$

Here, $\mathbb{Z}^d_{2N+1}=\mathbb{Z}^d/(2N+1)\mathbb{Z}^d$, $\delta_l=(\delta_{jl})_{j\in\mathbb{Z}^d}$ (δ_{jl} is the Kronecker symbol) and $|u\rangle\langle u|$ is the orthogonal projection on a unit vector u.
For the periodic operator $H^N_{\omega,\gamma}$, we define the integrated density of states denoted by $\mathcal{N}^N_{\omega,\gamma}$ by (0.6) (see e.g. [22]); it is a non-decreasing function. Its derivative, $d\mathcal{N}^N_{\omega,\gamma}$, is a positive measure and, it satisfies, for $\varphi\in\mathcal{C}^\infty_0(\mathbb{R})$,

$$(\varphi,d\mathcal{N}^N_{\omega,\gamma})=\int_{\mathbb{R}}\varphi(x)d\mathcal{N}^N_{\omega,\gamma}(x)=\frac{1}{2N+1}\sum_{k\in\mathbb{Z}^d_{2N+1}}\langle\delta_k,\varphi(H^N_{\omega,\gamma})\delta_k\rangle. \tag{4.1}$$

One has

Lemma 4.1. *Pick $\alpha>0$. There exists $\nu_0>0$ and $\rho>0$ such that, for $\gamma\in[0,1]$, $E\in\mathbb{R}$, $\nu\in(0,\nu_0)$ and $N\geq\nu^{-\rho}$, one has*

$$\mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E-\nu))-\lambda(\gamma)e^{-\nu^{-\alpha}}\leq\mathcal{N}_\gamma(E)\leq\mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E+\nu))+\lambda(\gamma)e^{-\nu^{-\alpha}}. \tag{4.2}$$

Proof. of Lemma 4.1 In [11], we have proved a result similar to Lemma 4.1 in dimension 1.
To prove Lemma 4.1, we first prove a "smoothed out" version of the same result, namely

Lemma 4.2. *There exists $C>1$ such that, for $\varphi\in\mathcal{C}^\infty_0(\mathbb{R})$, for $K\in\mathbb{N}$, $K\geq4$, and $N\in\mathbb{N}^*$, we have*

$$|\mathbb{E}((\varphi,d\mathcal{N}^N_{\omega,\gamma}))-(\varphi,d\mathcal{N})|\leq\lambda(\gamma)\left(\frac{CK}{N}\right)^{K-3}\sup_{\substack{x\in\mathbb{R}\\0\leq J\leq K+3}}\left|(1+x^2)\frac{d^J\varphi}{d^Jx}(x)\right|. \tag{4.3}$$

Proof. of Lemma 4.2 The proof follows the lines of the proof of Lemma 1.1 in [13] (and corrects some misprints). For the reader's convenience, we reproduce it here. We know (see [21, 6]) that

$$(\varphi,d\mathcal{N})=\int_{\mathbb{R}}\varphi(\lambda)d\mathcal{N}=\mathbb{E}(\langle\delta_0,\varphi(H_{\omega,\gamma})\delta_0\rangle).$$

Averaging (4.1) in ω and using the fact that the random variables $(\omega_n)_{n\in\mathbb{Z}}$ are i.i.d and the translation invariance of H, we get

$$\mathbb{E}((\varphi,d\mathcal{N}^N_{\omega,\gamma}))=\mathbb{E}(\langle\delta_0,\varphi(H^n_{\omega,\gamma})\delta_0\rangle). \tag{4.4}$$

So that we want to estimate $\langle\delta_0,\varphi(H^n_{\omega,\gamma})\delta_0\rangle-\langle\delta_0,\varphi(H_{\omega,\gamma})\delta_0\rangle$. Therefore we use Helffer-Sjöstrand's formula ([8]) that reads

$$\varphi(H_\omega)=\frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde\varphi}{\partial\overline{z}}(z)\cdot(z-H_\omega)^{-1}dz\wedge d\overline{z}. \tag{4.5}$$

where $\tilde\varphi$ is an almost analytic extension of φ (see [19]) i.e. a function satisfying

1. for $z\in\mathbb{R}$, $\tilde\varphi(z)=\varphi(z)$;
2. $\mathrm{supp}(\tilde\varphi)\subset\{z\in\mathbb{C};\ |\mathrm{Im}(z)|<1\}$;
3. $\tilde\varphi\in\mathcal{S}(\{z\in\mathbb{C};\ |\mathrm{Im}(z)|<1\})$;

4. the family of functions $x \mapsto \dfrac{\partial\tilde{\varphi}}{\partial\bar{z}}(x+iy)\cdot|y|^{-n}$ (for $0<|y|<1$) is bounded in $\mathcal{S}(\mathbb{R})$ for any $n\in\mathbb{N}$; more precisely, there exists $C>1$ such that, for all $p,q,r\in\mathbb{N}$, there exists $C_{p,q}>0$ such that

$$\sup_{0<|y|\leq 1}\sup_{x\in\mathbb{R}}\left|x^p\frac{\partial^q}{\partial x^q}\left(|y|^{-r}\cdot\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x+iy)\right)\right|\leq C^r C_{p,q}\sup_{\substack{q'\leq r+q+2\\ p'\leq p}}\sup_{x\in\mathbb{R}}\left|x^{p'}\frac{\partial^{q'}\varphi}{\partial x^{q'}}(x)\right|. \tag{4.6}$$

Equations (4.4) and (4.5) then yield

$$\begin{aligned}|\mathbb{E}((\varphi,d\mathcal{N}^N_{\omega,\gamma}))-(\varphi,d\mathcal{N})|&\leq\frac{1}{4\pi}\mathbb{E}\left(\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right||\langle\delta_0,\left((z-H^n_{\omega,\gamma})^{-1}-(z-H_{\omega,\gamma})^{-1}\right)\delta_0\rangle|dxdy\right)\\ &\leq\frac{1}{4\pi}\mathbb{E}\left(\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right|\sum_{|k|\geq n}|\omega_k-\omega_{[k]_n}||\langle\delta_0,(z-H^n_{\omega,\gamma})^{-1}\delta_k\rangle||\langle\delta_k,(z-H_{\omega,\gamma})^{-1}\delta_0\rangle|dxdy\right)\end{aligned} \tag{4.7}$$

where $z=x+iy$ and $[k]_n\equiv k \bmod (2n+1)$ (the point being chosen in $[-n,n]^d$).
By a Combes-Thomas argument (see section 6.1), we know that there exists $C>1$ such that, uniformly in $(\omega_n)_{n\in\mathbb{Z}}$ and $N\geq 1$, we have, for $\mathrm{Im}(z)\neq 0$,

$$|\langle\delta_0,(z-H^n_{\omega,\gamma})^{-1}\delta_k\rangle|+|\langle\delta_k,(z-H_{\omega,\gamma})^{-1}\delta_0\rangle|\leq\frac{C}{|\mathrm{Im}(z)|}e^{-|\mathrm{Im}(z)k|/C}$$

Hence, as the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are bounded by $\lambda(\gamma)$, (4.7) gives, for some $C>1$,

$$|\mathbb{E}((\varphi,d\mathcal{N}^N_{\omega,\gamma}))-(\varphi,d\mathcal{N})|\leq C\lambda(\gamma)\int_{\mathbb{C}}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\right|\frac{1}{|\mathrm{Im}(z)|^3}e^{-|\mathrm{Im}(z)n|/C}dxdy.$$

Taking into account the properties of almost analytic extensions (4.6), for some $C>1$, for $K\geq 3$ and $N\geq 1$, we get

$$\begin{aligned}|\mathbb{E}((\varphi,d\mathcal{N}^N_{\omega,\gamma}))-(\varphi,d\mathcal{N})|&\leq\lambda(\gamma)C^{K+1}\int_{\mathbb{R}\times[-1,1]}\frac{1}{1+x^2}|y|^{K-3}e^{-|yN|/C}dxdy\sup_{\substack{x\in\mathbb{R}\\ 0\leq J\leq K+3}}\left|(1+x^2)\frac{d^J\varphi}{d^Jx}(x)\right|\\ &\leq\lambda(\gamma)\left(\frac{CK}{N}\right)^{K-3}\sup_{\substack{x\in\mathbb{R}\\ 0\leq J\leq K+3}}\left|(1+x^2)\frac{d^J\varphi}{d^Jx}(x)\right|.\end{aligned}$$

This completes the proof of the Lemma 4.2. □

Let us now complete the proof of Lemma 4.1. First note that, as $d\mathcal{N}_{\omega,\gamma}$ and $d\mathcal{N}_\gamma$ are supported in $I:=[0,C+\lambda(\gamma)]$ (for some $C>0$), we only need to prove Lemma 4.1 for energies $E\in I$. Pick φ be a Gevrey class function of Gevrey exponent $\alpha>1$ (see [10]); assume, moreover, that φ has support in $(-1,1)$, that $0\leq\varphi\leq 1$ and that $\varphi\equiv 1$ on $(-1/2,1/2]$. Let $E\in I$ and $\nu\in(0,1)$,and set

$$\varphi_{E,\nu}(\cdot)=\mathbf{1}_{[0,E]}*\varphi\left(\frac{\cdot}{\nu}\right).$$

Then, by Lemma 4.2 and the Gevrey estimates on the derivatives of φ, we get that there exist $C>1$ such that, for $N\geq 1$, $K\geq 1$ and $0<\nu<1$, we have

$$|\mathbb{E}((\varphi_{E,\nu},d\mathcal{N}^N_{\omega,\gamma}))-(\varphi_{E,\nu},d\mathcal{N}_\gamma)|\leq\lambda(\gamma)(N\nu)^3\left(\frac{CK^{1+\alpha}}{N\nu}\right)^K. \tag{4.8}$$

We optimize the right hand side of (4.8) in K and get that, there exist $C > 1$ such that, for $N \geq 1$ and $0 < \nu < 1$, we have

$$|\mathbb{E}((\varphi_{E,\nu}, d\mathcal{N}^N_{\omega,\gamma})) - (\varphi_{E,\nu}, d\mathcal{N}_\gamma)| \leq C\lambda(\gamma)(N + \nu^{-1})^3 e^{-(N\nu/C)^{1/(1+\alpha)} + C(N\nu/C)^{-1/(1+\alpha)}}$$

Now if we choose $N \geq \nu^{-1-\eta}$, we get that there exist $\nu_0 > 0$ such that for $0 < \nu < \nu_0$, we have

$$|\mathbb{E}((\varphi_{E,\nu}, d\mathcal{N}^N_{\omega,\gamma})) - (\varphi_{E,\nu}, d\mathcal{N}_\gamma)| \leq \lambda(\gamma) e^{-\nu^{-\eta/(2\alpha)}}. \tag{4.9}$$

By definition, $\varphi_{E,\nu} \equiv 1$ on $[0, E]$, and $\varphi_{E,\nu}$ has support in $[-\nu, E + \nu]$ and is bounded by 1. As $dN^N_{\omega,\gamma}$ and $d\mathcal{N}_\gamma$ are positive measures, we have

$$\mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E)) \leq \mathbb{E}((\varphi_{E,\nu}, d\mathcal{N}^N_{\omega,\gamma})) \leq \mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E + \nu)). \tag{4.10}$$

Hence, by (4.9) and (4.10), we obtain

$$\begin{aligned}\mathcal{N}_\gamma(E) &\leq (\varphi_{E,\nu}, d\mathcal{N}_\gamma) = \mathbb{E}((\varphi_{E,\nu}, d\mathcal{N}^N_{\omega,\gamma})) + \left[(\varphi_{E,\nu}, d\mathcal{N}_\gamma) - \mathbb{E}((\varphi_{E,\nu}, d\mathcal{N}^N_{\omega,\gamma}))\right] \\ &\leq \mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E + \nu)) + \lambda(\gamma) e^{-\nu^{-\eta/(2\alpha)}}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_\gamma(E) &\geq (\varphi_{E-\nu,\nu}, d\mathcal{N}_\gamma) = \mathbb{E}((\varphi_{E-\nu,\nu}, d\mathcal{N}^N_{\omega,\gamma})) + \left[(\varphi_{E-\nu,\nu}, d\mathcal{N}_\gamma) - \mathbb{E}((\varphi_{E-\nu,\nu}, d\mathcal{N}^N_{\omega,\gamma}))\right] \\ &\geq \mathbb{E}(\mathcal{N}^N_{\omega,\gamma}(E - \nu)) - \lambda(\gamma) e^{-\nu^{-\eta/(2\alpha)}}\end{aligned}$$

This completes the proof of Lemma 4.1. □

4.2. **Some Floquet theory.** To analyze $H^N_{\omega,\gamma}$, we need some Floquet theory that we develop now. We denote by $\mathcal{F}: L^2([-\pi,\pi]^d) \to \ell^2(\mathbb{Z}^d)$ the standard Fourier series transform. Then, we have, for $u \in L^2([-\pi,\pi]^d)$,

$$(\hat{H}_{\omega,\gamma} u)(\theta) = (\mathcal{F} H_{\omega,\gamma} \mathcal{F}^* u)(\theta) = h(\theta) u(\theta) + \gamma \sum_{j \in \mathbb{Z}^d} \omega_j (\Pi_j u)(\theta)$$

$$\text{where } (\Pi_j u)(\theta) = \frac{1}{(2\pi)^d} e^{ij\theta} \int_{[-\pi,\pi]^d} e^{-ij\theta} u(\theta) d\theta.$$

Define the unitary equivalence $U: L^2([-\pi,\pi]^d) \to L^2([-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d) \otimes \ell^2(\mathbb{Z}^d_{2N+1})$ by $(Uu)(\theta) = (u_k(\theta))_{k \in \mathbb{Z}^d_{2N+1}}$ where the $(u_k(\theta))_{k \in \mathbb{Z}^d_{2N+1}}$ are defined by

$$\begin{gathered} u(\theta) = \sum_{k \in \mathbb{Z}^d_{2N+1}} e^{ik\theta} u_k(\theta) \\ \text{and the functions } (\theta \mapsto u_k(\theta))_{k \in \mathbb{Z}^d_{2N+1}} \text{ are } \frac{2\pi}{2N+1}\mathbb{Z}^d \text{ periodic.} \end{gathered} \tag{4.11}$$

The functions $(u_k)_{k \in \mathbb{Z}^d_{2N+1}}$ are computed easily; if the Fourier coefficients of u are denoted by $(\hat{u}_l)_{l \in \mathbb{Z}^d}$, then, one gets

$$u_k(\theta) = \sum_{l \in \mathbb{Z}^d} \hat{u}_{k+(2N+1)l} e^{i(2N+1)l\theta}. \tag{4.12}$$

The operator $U\mathcal{F}H^N_{\omega,\gamma}\mathcal{F}^*U^*$ acts on $L^2([-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d) \otimes \ell^2(\mathbb{Z}^d_{2N+1})$; it is the multiplication by the matrix

$$M^N_{\omega,\gamma}(\theta) = H^N(\theta) + \gamma V^N_\omega \tag{4.13}$$

where

$$H^N(\theta) = ((h_{j-j'}(\theta)))_{(j,j')\in(\mathbb{Z}^d_{2N+1})^2} \text{ and } V^N_\omega = ((\omega_j \delta_{jj'}))_{(j,j')\in(\mathbb{Z}^d_{2N+1})^2}. \tag{4.14}$$

Here, the functions $(h_k)_{k\in\mathbb{Z}^d_{2N+1}}$ are the components of h decomposed according to (4.11). The $(2N+1)^d \times (2N+1)^d$-matrices $H^N(\theta)$ and V^N_ω are non-negative matrices.

This immediately tells us that the Floquet eigenvalues and eigenvectors of $H^N_{\omega,\gamma}$ with Floquet quasi-momentum θ (i.e. the vectors, $u = (u_j)_{j\in\mathbb{Z}^d}$), solution to the problem

$$\begin{cases} H^N_{\omega,\gamma} u & = \lambda u, \\ u_{j+k} & = e^{-ik\theta} u_j \text{ for } j \in \mathbb{Z}^d,\ k \in (2N+1)\mathbb{Z}^d) \end{cases}$$

are the eigenvalues and eigenvectors (continued quasi-periodically) of the $(2N+1)^d \times (2N+1)^d$ matrix $M^N_{\omega,\gamma}(\theta)$. For $E \in \mathbb{R}$, one has

$$\mathcal{N}^N_{\omega,\gamma}(E) = \int_{[-\frac{\pi}{2N+1},\frac{\pi}{2N+1}]^d} \sharp\{\text{eigenvalues of } M^N_{\omega,\gamma}(\theta) \text{ in } [0,E]\} d\theta. \tag{4.15}$$

Considering H as $2N+1$-periodic on $\mathbb{Z}^d$, we see that the Floquet eigenvalues of H (for the quasi-momentum θ) are $(h(\theta + \frac{2\pi k}{2N+1}))_{k\in\mathbb{Z}^d_{2N+1}}$; the Floquet eigenvalue $h(\theta + \frac{2\pi k}{2N+1})$ is associated to the Floquet eigenvector $u_k(\theta)$, $k \in \mathbb{Z}^d_{2N+1}$ defined by

$$u_k(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{-i(\theta+\frac{2\pi k}{2N+1})j})_{j\in\mathbb{Z}^d_{2N+1}}.$$

In the sequel, the vectors in $\ell^2(\mathbb{Z}^d_{2n+1})$ are given by their components in the orthonormal basis $(u_k(\theta))_{k\in\mathbb{Z}^d_{2N+1}}$. The vectors of the canonical basis denoted by $(v_l(\theta))_{l\in\mathbb{Z}^d_{2N+1}}$ have the following components in this basis

$$v_l(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{i(\theta+\frac{2\pi k}{2N+1})l})_{k\in\mathbb{Z}^d_{2N+1}}.$$

We define the vectors $(v_l)_{l\in\mathbb{Z}^d_{2N+1}}$ by

$$v_l = e^{-il\theta} v_l(\theta) = \frac{1}{(2N+1)^{d/2}} (e^{i\frac{2\pi kl}{2N+1}})_{k\in\mathbb{Z}^d_{2N+1}}. \tag{4.16}$$

4.3. Estimating the density of states of the periodic approximations.

Define the event

$$\Omega(E,\gamma,N) = \{\omega;\ \exists\theta \in \mathbb{R}^d \text{ such that } M^N_{\omega,\gamma}(\theta) \text{ has an eigenvalue in } [0,E]\}.$$

By Lemma 4.1 and (4.15), Theorem 3.1 is a consequence of

Proposition 4.1. *Fix $0 < \eta < \eta' < \eta(r)$ and $\rho > d$. Then, there exists $\gamma_{\eta,\rho} > 0$ and $\varepsilon > 0$ such that, for $\gamma \in (0,\gamma_{\eta,\rho})$, one has*

$$\mathbb{P}[\Omega(\gamma(\overline{\omega} - \gamma^\eta),\gamma,N)] \leq e^{-\gamma^{-\varepsilon}}. \tag{4.17}$$

where

$$2N+1 = [\gamma^{-1/2+2\eta'+\eta'/(r-1)}]_o \cdot [\gamma^{-\eta'-\eta'/(r-1)}]_o \cdot [\gamma^{-\rho}]_o \quad .$$

Here, $[\cdot]_o$ denotes the largest odd integer smaller than $\cdot$.

Notice that, by the definition of $\eta(r)$, as $\eta < \eta' < \eta(r)$, one has $-\eta' - \eta'/(r-1) < 0$ and $-1/2 + 2\eta' + \eta'/(r-1) < 0$.
Let us now derive Theorem 1.1 from Proposition 4.1. Pick η as in Theorem 1.1 and $\eta < \eta' < \eta(r)$. Let $\rho > d$ be larger than the exponent ρ obtained in Lemma 4.1 for $\alpha = 1$. By (4.15) and (4.17), one has

$$\begin{aligned}\mathbb{E}[\mathcal{N}^N_{\omega,\gamma}(\gamma(\varpi - \gamma^{\eta'}))] &= \int_{[-\frac{\pi}{2N+1},\frac{\pi}{2N+1}]^d} \mathbb{E}[\sharp\{\text{eigenvalues of } M^N_{\omega,\gamma}(\theta) \text{ in } [0,\gamma(\varpi-\gamma^{\eta'})]\}]d\theta \\ &\le \int_{[-\frac{\pi}{2N+1},\frac{\pi}{2N+1}]^d} \mathbb{P}[\Omega(\gamma(\varpi-\gamma^{\eta'}),\gamma,N)](2N+1)^d d\theta \\ &\le C\mathbb{P}[\Omega(\gamma(\varpi-\gamma^{\eta'}),\gamma,N)].\end{aligned}$$

Here, we used the fact that $M^N_{\omega,\gamma}(\theta)$ has rank $(2N+1)^d$. Applying Proposition 4.1, for γ sufficiently small, we get

$$\mathbb{E}[\mathcal{N}^N_{\omega,\gamma}(\gamma(\varpi - \gamma^{\eta'}))] \le Ce^{-2\gamma^{-\varepsilon}}.$$

Plugging this into (4.2) for $E = \gamma(\varpi - \gamma^{\eta})$ and $\nu = \gamma^{1+\eta'}$, as $\mathcal{N}_\gamma$ is increasing and as $\eta < \eta'$, for γ sufficiently small, we obtain

$$\mathcal{N}_\gamma[\gamma(\varpi-\gamma^{\eta})] \le \mathbb{E}[\mathcal{N}^N_{\omega,\gamma}(\gamma(\varpi - \gamma^{\eta'}))] + e^{-\gamma^{-\varepsilon}} \le e^{-\gamma^{-\varepsilon'}}$$

with $\varepsilon' < \varepsilon$. This completes the proof of Theorem 3.1. ☐

4.4. **The proof of Propositions 4.1.** The proof of Proposition 4.1 follows roughly the lines of the proof of the upper bound in [11], section 2.3. The main difference comes from the fact that, in the present case, the supremum norm of the random variables depends on the energy interval on which we want to control the density of states or the probability of the presence of eigenvalues.
Pick η, ρ as in Proposition 4.1. Define

$$2L'+1 = [\gamma^{-1/2+2\eta'+\eta'/(r-1)}]_o \quad \text{and} \quad 2K'+1 = [\gamma^{-\eta'-\eta'/(r-1)}]_o \cdot [\gamma^{-\rho}]_o.$$

Notice that $2N+1 = (2K'+1)(2L'+1)$.
By assumption (H0), h is real analytic on the compact torus $\mathbb{T}^d$; it admits a compact set of minima. As the minima are quadratic non-degenerate, they are isolated points. Let Z be the finite set of these minima i.e. $Z = h^{-1}(\{0\}) = \{\theta_1,\dots,\theta_M\}$. By assumption (H1), we know that, there exists $C > 0$ such that, for $\theta \in \mathbb{T}^d$,

$$h(\theta) \ge C \inf_{1\le J\le M} |\theta - \theta_J|^2. \tag{4.18}$$

Then, we prove

Lemma 4.3. *Pick $0 < \eta < \eta' < \eta(r)$ and N as in Proposition 4.1. Let L' and K' be defined as above. There exists $C > 0$ and $\gamma_0 > 0$ such that, for $0 < \gamma < \gamma_0$, we have*

$$\Omega(\gamma(\varpi-\gamma^{\eta}),\gamma,N) \subset \bigcup_{|k'|\le K'} \left(\bigcup_{1\le J\le J'\le M} \Omega^{J,J',k'} \right) \tag{4.19}$$

where, for $1 \le J \le J' \le M$ and $|k'| \le K'$, we define the events

$$\Omega^{J,J',k'} = \left\{ \omega;\ \frac{1}{(2L'+1)^d} \left| \sum_{|l'|\le L'} (\omega_{k'(2L'+1)+l'} - \varpi) e^{i(\theta_J - \theta_{J'})l'} \right| \ge \frac{\gamma^{\eta}}{16(M+1)^2} \right\}.$$

Before proving this lemma, let us use it to complete the proof of Proposition 4.1. Therefore, we need to estimate the probability of the events $\Omega^{J,J',k'}$. This is done in

Proposition 4.2. *There exists $\varepsilon > 0$ such that, for γ sufficiently small, for $1 \leq J, J' \leq M$ and for $|k'| \leq K'$, we have*

$$\mathbb{P}(\Omega^{J,J',k'}) \leq e^{-\gamma^{-\varepsilon}}. \tag{4.20}$$

As $K' \leq \gamma^{-\rho-1}$ and $1 \leq J, J' \leq M$, using Lemma 4.3, summing the estimates (4.20) over (J, J', k'), we obtain (4.17).

Proof of Proposition 4.2. Proposition 4.2 is a large deviation estimate. The only difficulty comes from the fact that, in the present case, the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ depend on γ, hence, on the size of the sample of random variables we are considering. Nevertheless, the standard ideas to obtain large deviation estimates do apply (see, e.g. [7]).

Recall that $\omega_0 = \omega_0(\lambda)$ and $\lambda = \gamma^{-\eta'/(r-1)}$. Let us first estimate the variance of $(\omega_n)_{n\in\mathbb{Z}^d}$. One needs to consider two cases. If the exponent r in assumption (H2) satisfies $r \geq 2$, one has

$$\sup_{\lambda>1} \mathbb{E}((\omega_0(\lambda) - \overline{\omega(\lambda)})^2) < +\infty \tag{4.21}$$

If $1 < r < 2$, we compute

$$\mathbb{E}(\omega_0(\lambda)^2) = \mathbb{E}(\omega_0 \mathbf{1}_{\{\omega_0 \leq \lambda\}}) + \lambda^2 \mathbb{P}(\omega_0 > \lambda) \leq \lambda^{2-r}\mathbb{E}(\omega_0^r) + \lambda^2\mathbb{E}(\omega_0^r)/\lambda^{-r} \leq 2\lambda^{2-r}\mathbb{E}(\omega_0^r) \tag{4.22}$$

Let us now estimate $\mathbb{P}(\Omega^{J,J',k'})$. We only do the explicit computations for $J \neq J'$; the case $J = J'$ is treated in the same way. As the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are real, we only have to estimate the probability that

$$\frac{1}{(2L'+1)^d} \left| \sum_{|l'|\leq L'} (\omega_{k'(2K'+1)+l'} - \overline{\omega}) \cos((\theta_J - \theta_{J'})l') \right| \geq \frac{\gamma^\eta}{32(M+1)^2}$$

and the probability that

$$\frac{1}{(2L'+1)^d} \left| \sum_{|l'|\leq L'} (\omega_{k'(2K'+1)+l'} - \overline{\omega}) \sin((\theta_J - \theta_{J'})l') \right| \geq \frac{\gamma^\eta}{32(M+1)^2}.$$

We estimate the first probability; the second estimate is done in the same way. Therefore, we have to estimate the probability that

$$\frac{1}{(2L'+1)^d} \sum_{|l'|\leq L'} (\omega_{k'(2K'+1)+l'} - \overline{\omega}) \cos((\theta_J - \theta_{J'})l') \geq \frac{\gamma^\eta}{32(M+1)^2} \tag{4.23}$$

and the probability that

$$\frac{1}{(2L'+1)^d} \sum_{|l'|\leq L'} (\omega_{k'(2K'+1)+l'} - \overline{\omega}) \cos((\theta_J - \theta_{J'})l') \leq -\frac{\gamma^\eta}{32(M+1)^2}$$

Both estimates being done in the same way, we only do the computations for (4.23).

To simplify the notations, let $R = (2L'+1)^d$ and, reindex the points $\{l';\ |l'| \leq L'\}$ as $\{l'(U);\ 1 \leq U \leq R\}$. Define

$$\tilde{\omega}_U = (\omega_{k'(2K'+1)+l'(U)} - \overline{\omega}) \cos((\theta_J - \theta_{J'})l'(U)).$$

The random variables $(\tilde{\omega}_U)_{1\leq U\leq R}$ are independent, centered (i.e. $\mathbb{E}(\tilde{\omega}_U)=0$) and bounded by λ. Estimating the probability of (4.23) is estimating the probability of

$$\frac{1}{R}\sum_{U=1}^{R}\tilde{\omega}_U \geq \gamma^\eta/C_0$$

where $C_0 = 32(M+1)^2$.

For $t>0$, as the random variables $(\tilde{\omega}_U)_{1\leq U\leq R}$ are independent, we compute

(4.24)

$$\mathbb{P}\left(\frac{1}{R}\sum_{U=1}^{R}\tilde{\omega}_U \geq \gamma^\eta/C_0\right) \leq \mathbb{E}\left(\exp\left[t\left(-\gamma^\eta+\frac{C_0}{R}\sum_{U=1}^{R}\tilde{\omega}_U\right)\right]\right) = e^{-t\gamma^\eta}\prod_{U=1}^{R}\mathbb{E}\left(\exp\left(\frac{tC_0}{R}\tilde{\omega}_U\right)\right).$$

Now, pick t such that $C_0t\lambda\leq R$; then, for $1\leq U\leq R$, one has

$$\begin{aligned}\mathbb{E}\left(\exp\left(\frac{tC_0}{R}\tilde{\omega}_U\right)\right) &\leq \mathbb{E}\left(1+\frac{tC_0}{R}\tilde{\omega}_U+\left(\frac{tC_0}{R}\tilde{\omega}_U\right)^2\right)\\ &\leq 1+\left(\frac{tC_0}{R}\right)^2C(\lambda)\leq \exp\left(\left(\frac{tC_0}{R}\right)^2C(\lambda)\right),\end{aligned}\tag{4.25}$$

where $C(\lambda)$ satisfies

$$C(\lambda)\geq \sup_{1\leq U\leq R}\mathbb{E}((\tilde{\omega}_U)^2).\tag{4.26}$$

Plugging (4.25) into (4.24), we obtain

$$\mathbb{P}\left(\frac{1}{R}\sum_{U=1}^{R}\tilde{\omega}_U \geq \gamma^\eta/C_0\right) \leq \exp\left(-t\left(\gamma^\eta-\frac{tC_0}{R}C(\lambda)\right)\right).$$

Let us now distinguish the two cases:

1. If $1<r<2$, one can take
$$C(\lambda)=C\lambda^{2-r}.$$
Recalling that
$$R\sim\gamma^{-d/2+d\eta'(2+1/(r-1))},\quad \lambda\sim\gamma^{-\eta'/(r-1)},$$
and taking $2C_0t\lambda=R$ i.e. $t\sim\gamma^{-d/2+2d\eta'+(d+1)\eta'/(r-1)}$, for γ sufficiently small, we obtain
$$t\left(\gamma^\eta-\frac{tC_0}{R}C(\lambda)\right)\geq t\gamma^\eta\left(1-\gamma^{-\eta+\eta'}\right)\geq\frac{1}{2}\gamma^{-d/2+2d\eta'+(d+1)\eta'/(r-1)+\eta}.$$
By definition, $-d/2+2d\eta(r)+(d+1)\eta(r)/(r-1)+\eta(r)=0$; hence, as $0<\eta<\eta'<\eta(r)$, one has
$$-d/2+2d\eta'+(d+1)\eta'/(r-1)+\eta<0.$$
This completes the proof of Proposition 4.2 in the case $1<r<2$.
2. If $r\geq 2$, the variance of $\tilde{\omega}_U$ is bounded by C uniformly in $1\leq U\leq R$. Taking $2CC_0t\lambda=R\gamma^\eta$ i.e. $t\sim\gamma^{-d/2+2d\eta'+(d+1)\eta'/(r-1)+\eta}$, for γ sufficiently small, we obtain
$$t\left(\gamma^\eta-\frac{tC_0}{R}C\right)\geq t\gamma^\eta\left(1-1/2\right)\geq\frac{1}{2}\gamma^{-d/2+2d\eta'+d\eta'/(r-1)+2\eta}.$$

By definition, $-d/2 + 2d\eta(r) + d\eta(r)/(r-1) + 2\eta(r) = 0$; hence, as $0 < \eta < \eta' < \eta(r)$, one has

$$-d/2 + 2d\eta' + d\eta'/(r-1) + 2\eta < 0.$$

This completes the proof of Proposition 4.2 in the case $2 \leq r$.

$\square$

4.5. **The proof of Lemma 4.3.** To prove this Lemma 4.3, we follow the argument sketched at the end of the introduction.
We define

$$2L+1 = [\gamma^{-1/2+2\eta'+\eta'/(r-1)}]_o \cdot [\gamma^{-\eta'-\eta'/(r-1)}]_o \quad \text{and} \quad 2K+1 = [\gamma^{-\rho}]_o$$

Notice that $2N+1 = (2K+1)(2L+1)$.
Pick $\omega \in \Omega_\gamma(\theta)$ so there exists $a = \sum_{k \in \mathbb{Z}^d_{2n+1}} a_k u_k(\theta)$ such that

- $\|a\|_{\ell^2(\mathbb{Z}^d_{2N+1})} = \sqrt{\sum_{k\in\mathbb{Z}^d_{2N+1}} |a_k|^2} = 1$,
- $\langle M^N_\omega(\theta)a, a\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq \gamma(\overline{\omega} - \gamma^\eta)$,
- $u_k(\theta)$ is defined in section 4.2.

Using (4.13) and the positivity of $H^N(\theta)$ and V^N_ω, we obtain that

$$\langle H^N(\theta)a, a\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq \gamma(\overline{\omega} - \gamma^\eta) \quad \text{and} \quad \langle V^N_\omega a, a\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq \overline{\omega} - \gamma^\eta. \tag{4.27}$$

As a is normalized, the second inequality in (4.27) yields

$$\langle (V^N_\omega - \overline{\omega} Id^N)a, a\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq -\gamma^\eta \tag{4.28}$$

where Id^N is the identity matrix.
We now use the first inequality in (4.27) to get more information on a. For $1 \leq J \leq M$ and $\theta \in [-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]^d$, by (4.18), we know that, for some $C > 0$ and for γ small enough

$$\left|\frac{2\pi k}{2N+1} - \theta_J\right| \geq \frac{1}{2L+1} \implies \left(h\left(\theta + \frac{2\pi k}{2N+1}\right) \geq \gamma^{1-2\eta'}/C\right) \tag{4.29}$$

For $1 \leq J \leq m$, let $k_J \in \mathbb{Z}^d$ be the unique vector satisfying $2\pi k_J - (2N+1)\theta_J \in [-\pi, \pi)^d$. Define a^J by

$$(a^J)_k = \begin{cases} a_k \text{ if } |k - k_J| \leq K \\ 0 \text{ if not} \end{cases}$$

The $(a^J)_{1\leq J\leq M}$ are pairwise orthogonal. By the first estimate in (4.27) and by (4.29), we get

$$\left\| a - \sum_{J=1}^M a^J \right\|^2_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq C\gamma \cdot \gamma^{-1+2\eta'} = C\gamma^{2\eta'}. \tag{4.30}$$

Plugging this into (4.28), we get that, for some $C > 0$ and γ small enough

$$\left\langle (V^N_\omega - \overline{\omega} Id^N)\left(\sum_{J=1}^M a^J\right), \left(\sum_{J=1}^M a^J\right)\right\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq -\gamma^\eta(1 - C\gamma^{\eta'-\eta}).$$

Expanding the vectors $(a_J)_{1\leq J\leq M}$ in the basis $(v_l)_l$ defined in (4.16), we get

(4.31)
$$\sum_{J=1}^{M}\sum_{l\in\mathbb{Z}^d_{2N+1}}(\omega_l-\overline{\omega})|\langle a^J,v_l\rangle|^2+2\mathrm{Re}\left(\sum_{1\leq J<J'\leq M}\sum_{l\in\mathbb{Z}^d_{2N+1}}(\omega_l-\overline{\omega})\langle a^J,v_l\rangle\overline{\langle a^{J'},v_l\rangle}\right)\leq -\gamma^\eta(1-C\gamma^{\eta'-\eta}).$$

We now translate each of the a^J by k_J so as to center its support at 0. The vector thus obtained we again call a^J; then, (4.31) becomes

$$\sum_{J=1}^{M}\sum_{l\in\mathbb{Z}^d_{2N+1}}(\omega_l-\overline{\omega})|\langle a^J,v_l\rangle|^2+$$
$$+2\mathrm{Re}\left(\sum_{1\leq J<J'\leq M}\sum_{l\in\mathbb{Z}^d_{2N+1}}e^{\frac{2i\pi(k_J-k_{J'})l}{2N+1}}(\omega_l-\overline{\omega})\langle a^J,v_l\rangle\overline{\langle a^{J'},v_l\rangle}\right)\leq -\gamma^\eta(1-C\gamma^{\eta'-\eta}).$$

We apply Lemma 6.2 to each a^J; as $K/K'\sim\gamma^{\eta+\eta'/(r-1)}$, we thus obtain $\tilde a^J\in\ell^2(\mathbb{Z}^d_{2N+1})$ such that

- $\|a^J-\tilde a^J\|_{\ell^2(\mathbb{Z}^d_{2N+1})}\leq C\gamma^{\eta'+\eta'/(r-1)}\|a^J\|_{\ell^2(\mathbb{Z}^d_{2N+1})}$;
- for $l'\in\mathbb{Z}^d_{2L'+1}$ and $k'\in\mathbb{Z}^d_{2K'+1}$, we have

(4.32)
$$\langle\tilde a^J,v_{l'+k'(2L'+1)}\rangle=\langle\tilde a^J,v_{k'(2L'+1)}\rangle;$$

- $\|a^J\|_{\ell^2(\mathbb{Z}^d_{2N+1})}=\|\tilde a^J\|_{\ell^2(\mathbb{Z}^d_{2N+1})}$.

Recalling that the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are bounded by $C\gamma^{-\eta'/(r-1)}$ and that M is fixed, we get that, for some $C>0$ and γ small enough

$$\sum_{J=1}^{M}\sum_{l\in\mathbb{Z}^d_{2N+1}}(\omega_l-\overline{\omega})|\langle\tilde a^J,v_l\rangle|^2+$$
$$2\mathrm{Re}\left(\sum_{1\leq J<J'\leq M}\sum_{l\in\mathbb{Z}^d_{2N+1}}e^{\frac{2i\pi(k_J-k_{J'})l}{2N+1}}(\omega_l-\overline{\omega})\langle\tilde a^J,v_l\rangle\overline{\langle\tilde a^{J'},v_l\rangle}\right)\leq -\gamma^\eta(1-C\gamma^{\eta'-\eta}).$$

Hence, if $\varepsilon:=(\eta'-\eta)/2$, for γ sufficiently small, one has

$$\sum_{J=1}^{M}\sum_{l\in\mathbb{Z}^d_{2N+1}}(\omega_l-\overline{\omega})|\langle\tilde a^J,v_l\rangle|^2+$$
$$2\mathrm{Re}\left(\sum_{1\leq J<J'\leq M}\sum_{l\in\mathbb{Z}^d_{2N+1}}e^{\frac{2i\pi(k_J-k_{J'})l}{2N+1}}(\omega_l-\overline{\omega})\langle\tilde a^J,v_l\rangle\overline{\langle\tilde a^{J'},v_l\rangle}\right)\leq -\gamma^\eta(1-\gamma^\varepsilon).$$

So that, by (4.32), we get

$$\sum_{J=1}^{M}\sum_{k'\in\mathbb{Z}^d_{2K'+1}}\left(\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})\right)(2L'+1)^d|\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle|^2+$$

$$+\sum_{\substack{1\leq J<J'\leq m\\ k'\in\mathbb{Z}^d_{2K'+1}}}2\mathrm{Re}\left(S(J,J',k')e^{\frac{2i\pi(k_J-k_{J'})k'}{2K'+1}}\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle\overline{\langle\tilde{a}^{J'},v_{k'(2L'+1)}\rangle}\right)\leq-\gamma^\eta(1-\gamma^\varepsilon)$$

where

$$S(J,J',k')=\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})e^{\frac{2i\pi(k_J-k_{J'})l'}{2N+1}}.$$

Hence,

$$\sum_{J=1}^{M}\sum_{k'\in\mathbb{Z}^d_{2K'+1}}\left(\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})\right)(2L'+1)^d|\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle|^2$$
$$\leq 2\sum_{\substack{1\leq J<J'\leq m\\ k'\in\mathbb{Z}^d_{2K'+1}}}|S(J,J',k')|\,(2L'+1)^d|\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle\langle\tilde{a}^{J'},v_{k'(2L'+1)}\rangle|-\gamma^\eta(1-C\gamma^\varepsilon). \tag{4.33}$$

If we define

$$\Sigma(J,J',k')=\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})e^{i(\theta_J-\theta_{J'})l'},$$

as $|\frac{2i\pi k_J}{2N+1}-\theta_J|\leq\frac{1}{2N+1}$, we get that

$$|\Sigma(J,J',k')-S(J,J',k')|=O(\gamma^\rho).$$

So, as $\|\tilde{a}^J\|=\|a^J\|\leq 2$, by (4.33), for γ sufficiently small, we obtain

$$\sum_{J=1}^{M}\sum_{k'\in\mathbb{Z}^d_{2K'+1}}\left(\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})\right)(2L'+1)^d|\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle|^2$$
$$\leq 2\sum_{\substack{1\leq J<J'\leq M\\ k'\in\mathbb{Z}^d_{2K'+1}}}|\Sigma(J,J',k')|\,(2L'+1)^d|\langle\tilde{a}^J,v_{k'(2L'+1)}\rangle\langle\tilde{a}^{J'},v_{k'(2L'+1)}\rangle|-\gamma^\eta/2. \tag{4.34}$$

This implies that, for some $1\leq J\leq J'\leq M$ and $k'\in\mathbb{Z}^d_{2K'+1}$, we have

$$\left|\frac{1}{(2L'+1)^d}\sum_{l'\in\mathbb{Z}^d_{2L'+1}}(\omega_{l'+k'(2L'+1)}-\overline{\omega})e^{i(\theta_J-\theta_{J'})l'}\right|\geq\frac{\gamma^\eta}{16(M+1)^2}$$

that is $\omega\in\Omega^{J,J',k'}$. This completes the proof of Lemma 4.3. ☐

5. The proof of Theorems 1.3 and 1.4

To obtain Theorem 1.3, one uses Theorem 1.1 when $r = +\infty$ i.e. when the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are bounded. Do as in section 2; pick λ large enough so that the random variables $(\omega_n(\lambda))_{n\in\mathbb{Z}^d}$ defined in section 2 satisfy $\overline{\omega}(1-\eta/2) \leq \mathbb{E}(\omega_0(\lambda)) = \overline{\omega(\lambda)}$. Then, with the same notations as in section 2, we get

$$\mathcal{N}_\gamma(\gamma(\overline{\omega}-\eta)) \leq \mathcal{N}_\gamma(\gamma(\overline{\omega}-\eta);\lambda) \leq \mathcal{N}_\gamma\left(\gamma\overline{\omega(\lambda)}(1-\eta/2);\lambda\right).$$

Theorem 1.3 then immediately follows from Theorem 1.1 with $r = +\infty$.

Theorem 1.4 is deduced from Theorem 1.3 in the same way as Theorem 1.2 is deduced from Theorem 1.1; we do not repeat the proof.

6. Appendix

This section is devoted to results used in the course of this paper.

6.1. **Combes-Thomas estimates.** Combes-Thomas estimates are estimates on the decay of the Green's kernel of an operator H, see [5, 1, 9]. The ones we need in the present paper are stated in

Lemma 6.1. *Let $H_{\omega,\gamma}$ be as in section 1. Then, there exists $C > 0$ (depending only on h) such that, for any ω, $\gamma > 0$ and any $\Lambda \subset \mathbb{Z}^d$, one has*

- *if $E \in \mathbb{C}\setminus\mathbb{R}$, $|Im(E)| \leq 1$, for $m, n \in \Lambda$,*

$$\left|\langle\delta_n, (E - H_{\omega,\gamma|\Lambda})^{-1}\delta_m\rangle\right| \leq \frac{C}{|Im(E)|}e^{-|Im(E)||m-n|/C}$$

- *if $E < \inf(\sigma(H_{\omega,\gamma|\Lambda}))$ or $E > \sup(\sigma(H_{\omega,\gamma|\Lambda}))$, for $m, n \in \Lambda$, one has*

$$\left|\langle\delta_n, (E - H_{\omega,\gamma|\Lambda})^{-1}\delta_m\rangle\right| \leq \left(\frac{C}{\rho(E)}\right)^2 e^{-\rho(E)|m-n|/C} \tag{6.1}$$

where

$$\rho(E) = \inf\left(\sqrt{dist(E,\sigma(H_{\omega,\gamma|\Lambda}))}, c/2\right).$$

Here, c is defined in assumption (H0).

Proof. The proof of the first part of this lemma is well known (see e.g. [1]). The proof of the second part may be less well known; we give it for the reader's convenience.

To fix ideas assume $E < \inf(\sigma(H_{\omega,\gamma|\Lambda}))$. Pick $a \in \mathbb{R}^d$, $|a| < c$ where c is defined in (H0). On the domain $\ell_c(\mathbb{Z}^d) = \{u = (u_n)_{n\in\mathbb{Z}^d};\ \exists N > 0,\ u_n = 0 \text{ if } |n| > N\}$, consider the operator T_a defined by $(T_a u)_n = e^{a\cdot n}u_n$ $(u = (u_n)_{n\in\mathbb{Z}^d})$. Recall that Π_Λ is the projector on the subset $\Lambda \subset \mathbb{Z}^d$. We note that

$$T_a\Pi_\Lambda = \Pi_\Lambda T_a, \quad T_a^{-1} = T_{-a} \quad \text{and} \quad T_a^* = T_a. \tag{6.2}$$

It is convenient to consider the operator T_a on the Fourier side i.e. to conjugate them by the discrete Fourier transformation. We keep the same notations. For u, a trigonometric polynomial, one has $(T_a u)(\theta) = u(\theta - ia)$. T_a can be extended to the space of functions analytic in the strip $|\mathrm{Im}\theta| < c$. By (6.2), one has

$$T_{-a}(h + \gamma V_\omega)T_a = h(\cdot + ia) + \gamma V_\omega.$$

As h is real analytic, one has

$$h(\theta + ia) = h(\theta) + a^2 g_r(\theta, a) + i(a\cdot\nabla h(\theta) + a^2 g_i(\theta, a)) \tag{6.3}$$

where g_i and g_r are bounded uniformly for $|a| < c/2$.
Equations (6.2) and (6.3) implies that, for $u = \Pi_\Lambda u \in \ell^2(\mathbb{Z}^d)$,

$$\begin{aligned}
|\langle T_{-a}\Pi_\Lambda(h+V_\omega-E)\Pi_\Lambda T_a u, u\rangle|^2 &= |\langle T_{-a}(h+V_\omega-E)T_a u, u\rangle|^2 \\
&= |\langle T_{-a}(h+V_\omega-E+a^2 g_r(\cdot,a))T_a u, u\rangle|^2 \\
&\qquad + |\langle T_{-a}(a\cdot\nabla h(\theta)+a^2 g_i(\cdot,a))T_a|u,u\rangle|^2 \\
&\geq |\langle T_{-a}\Pi_\Lambda(h+V_\omega-E+a^2 g_r(\cdot,a))\Pi_\Lambda T_a u, u\rangle|^2.
\end{aligned} \tag{6.4}$$

Pick a such that $|a| = \rho(E)/C$ where $\rho(E)$ is defined in Lemma 6.1 and $C > 0$ is such that

$$\rho(E) \sup_{\substack{|a|\leq c/2 \\ \theta\in\mathbb{R}^d}} |g_r(\theta,a)| \leq C^2/2,$$

then

$$\Pi_\Lambda(h+V_\omega-E+a^2 g_r(\cdot,a))\Pi_\Lambda \geq \frac{1}{2}\Pi_\Lambda\rho(E)^2\Pi_\Lambda.$$

Hence,

$$|\langle T_{-a}\Pi_\Lambda(h+V_\omega-E)\Pi_\Lambda T_a u, u\rangle|^2 \geq \left(\frac{\rho(E)}{2}\right)^4 \|u\|^2.$$

This implies that $T_{-a}\Pi_\Lambda(h+V_\omega-E)\Pi_\Lambda T_a$ is invertible and its inverse is bounded by $4/\rho^2(E)$. Let us now compute

$$\begin{aligned}
|\langle\delta_n, (\Pi_\Lambda(h+V_\omega-E)\Pi_\Lambda)^{-1}\delta_m\rangle| &= |\langle T_a\delta_n, (T_{-a}(\Pi_\Lambda(h+V_\omega-E)\Pi_\Lambda)T_a)^{-1}T_{-a}\delta_m\rangle| \\
&\leq \left(\frac{2}{\rho(E)}\right)^2 \|T_{-a}\delta_m\|\|T_a\delta_n\| = \left(\frac{2}{\rho(E)}\right)^2 e^{-a(m-n)}.
\end{aligned}$$

Taking $a = \dfrac{\rho(E)}{C|m-n|}(m-n)$, we obtain (6.1). This completes the proof of Lemma 6.1. □

6.2. **The key lemma.** The following lemma is a very simple but quite convenient quantitative version of the uncertainty principle for the Fourier transformation on finite Abelian discrete groups.

Lemma 6.2 ([11])**.** *Assume N, L, K, K' L' are positive integers such that*

- $2N+1 = (2K+1)(2L+1) = (2K'+1)(2L'+1)$ *such*
- $K < K'$ *and* $L' < L$.

Pick $a = (a_n)_{n\in\mathbb{Z}^d_{2N+1}} \in \ell^2(\mathbb{Z}^d_{2N+1})$ such that,

$$\text{for } |n| > K,\ a_n = 0.$$

Then, there exists $\tilde{a} \in \ell^2(\mathbb{Z}^d_{2N+1})$ such that

1. $\|a-\tilde{a}\|_{\ell^2(\mathbb{Z}^d_{2N+1})} \leq C_{K,K'}\|a\|_{\ell^2(\mathbb{Z}^d_{2N+1})}$ *where* $C_{K,K'} \underset{K/K'\to 0}{\asymp} K/K'$.
2. *for* $l' \in \mathbb{Z}^d_{2L'+1}$ *and* $k' \in \mathbb{Z}^d_{2K'+1}$*, we have*
$$\langle\tilde{a}, v_{l'+k'(2L'+1)}\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} = \langle\tilde{a}, v_{k'(2L'+1)}\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})}.$$
3. $\|a\|_{\ell^2(\mathbb{Z}^d_{2N+1})} = \|\tilde{a}\|_{\ell^2(\mathbb{Z}^d_{2N+1})}$.

A one-dimensional version of this result was proved in [11]. The proof in arbitrary dimension is very similar to that in dimension 1; for the reader's convenience, as its proof is quite short, we reproduce it here.

Proof. By definition of (v_m) (see (4.16)), for $m \in \mathbb{Z}^d_{2N+1}$, one has

$$\langle a, v_m\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} = \frac{1}{(2N+1)^{d/2}} \sum_{n\in\mathbb{Z}^d_{2K+1}} a_n e^{-\frac{2i\pi nm}{2N+1}}.$$

We decompose $m = l' + k'(2L'+1)$ where $k' \in \mathbb{Z}^d_{2K'+1}$ and $l' \in \mathbb{Z}^d_{2L'+1}$ so that

$$\begin{aligned}
\langle a, v_m\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} &= \frac{1}{(2K'+1)^{d/2}(2L'+1)^{d/2}} \sum_{n\in\mathbb{Z}^d_{2K+1}} a_n e^{-\frac{2i\pi nk'}{2K'+1}} e^{-\frac{2i\pi nl'}{2N+1}} \\
&= \frac{1}{(2L'+1)^{d/2}} \left(\frac{1}{(2K'+1)^{d/2}} \sum_{n\in\mathbb{Z}^d_{2K'+1}} a_n e^{-\frac{2i\pi nk'}{2K'+1}} \right. \\
&\qquad \left. + \frac{1}{(2K'+1)^{d/2}} \sum_{n\in\mathbb{Z}^d_{2K'+1}} (e^{-\frac{2i\pi nl'}{2N+1}} - 1) a_n e^{-\frac{2i\pi nk'}{2K'+1}} \right) \\
&= \frac{1}{(2L'+1)^{d/2}} \langle a, v^{K'}_{k'}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})} + \frac{1}{(2L'+1)^{d/2}} \langle D^{l'} a, v^{K'}_{k'}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})}.
\end{aligned}$$

where

- a is seen as an element of $\ell^2(\mathbb{Z}^d_{2K'+1})$
- $(v^{K'}_{k'})_{k'\in\mathbb{Z}^d_{2K'+1}}$ is the orthonormal basis of $\ell^2(\mathbb{Z}^d_{2K'+1})$ defined by

$$v^{K'}_{k'} = \frac{1}{(2K'+1)^{d/2}} (e^{\frac{2i\pi kk'}{2K'+1}})_{k\in\mathbb{Z}^d_{2K'+1}}$$

- for $l' \in \mathbb{Z}^d_{2L'+1}$, $D^{l'}$ is the $(2K'+1)^d \times (2K'+1)^d$ diagonal matrix (acting on $\ell^2(\mathbb{Z}^d_{2K'+1})$) with the diagonal entries

$$d^{l'}_{nn} = \begin{cases} e^{-\frac{2i\pi nl'}{2N+1}} - 1 \text{ if } |n| \leq K \\ 0 \text{ if not.} \end{cases}$$

Obviously,

$$\sup_{l'\in\mathbb{Z}^d_{2L'+1}} \|D^{l'}\|_{\mathcal{L}(\ell^2(\mathbb{Z}^d_{2K'+1}))} \leq \sup_{\substack{k\in\mathbb{Z}^d_{2K+1} \\ l'\in\mathbb{Z}^d_{2L'+1}}} \left| e^{-\frac{2i\pi kl'}{2N+1}} - 1 \right| = C_{K,K'}$$

and

$$C_{K,K'} \underset{K/K'\to 0}{\asymp} \frac{K}{K'}.$$

We define

$$\tilde{a} = \sum_{n\in\mathbb{Z}^d_{2N+1}} \frac{1}{(2L'+1)^{d/2}} \langle a, v^{K'}_{[n]_{L'}}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})} v_n$$

where $[n]_{L'}$ is the unique point $k' \in \mathbb{Z}^d$ such that $n - k'(2L'+1) \in (-L', L']^d$. Then, $\tilde{a}$ satisfies Lemma 6.2. Indeed, as the vectors $(v_n)_{n\in\mathbb{Z}^d_{2N+1}}$ form an orthonormal basis of $\ell^2(\mathbb{Z}^d_{2N+1})$, point 2 is obvious by the definition of $\tilde{a}$. Point 3 follows from

$$\begin{aligned}\|\tilde{a}\|^2_{\ell^2(\mathbb{Z}^d_{2N+1})} &= \sum_{n\in\mathbb{Z}^d_{2N+1}} \frac{1}{(2L'+1)^d} |\langle a, v^{K'}_{[n]_{L'}}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})}|^2 \\ &= \frac{1}{(2L'+1)^d} \cdot (2L'+1)^d \sum_{n\in\mathbb{Z}^d_{2K'+1}} |\langle a, v^{K'}_{k'}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})}|^2 = \|a\|^2_{\ell^2(\mathbb{Z}^d_{2N+1})}.\end{aligned}$$

Let us check point 1; we compute

$$\begin{aligned}\|a-\tilde{a}\|^2_{\ell^2(\mathbb{Z}^d_{2N+1})} &= \sum_{N\in\mathbb{Z}^d_{2N+1}} \left| \frac{1}{(2L'+1)^{d/2}} \langle a, v^{K'}_{[n]_{L'}}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})} - \langle a, v_n\rangle_{\ell^2(\mathbb{Z}^d_{2N+1})} \right|^2 \\ &= \sum_{l'\in\mathbb{Z}^d_{2L'+1}} \frac{1}{(2L'+1)^d} \sum_{k'\in\mathbb{Z}^d_{2K'+1}} |\langle D^{l'} a, v^{K'}_{k'}\rangle_{\ell^2(\mathbb{Z}^d_{2K'+1})}|^2 \\ &= \sum_{l'\in\mathbb{Z}^d_{2L'+1}} \frac{1}{(2L'+1)^d} \|D^{l'} a\|^2_{\ell^2(\mathbb{Z}^d_{2K'+1})} \\ &\le C^2_{K,K'} \|a\|^2_{\ell^2(\mathbb{Z}^d_{2N+1})}.\end{aligned}$$

This completes the proof of Lemma 6.2. □

6.3. **The s-decoupling property.** Let us recall the s-decoupling property (1.7) in [4]. A probability distribution dP satisfies the s-decoupling property with some $0 < s < 1$, if there exists $C > 0$ such that

$$\forall a,b,c \in \mathbb{C}, \quad \int_{\mathbb{R}} \frac{1}{|x-a|^s} \left|\frac{x-b}{x-c}\right|^s dP(x) \le C \int_{\mathbb{R}} \frac{1}{|x-a|^s} dP(x) \int_{\mathbb{R}} \left|\frac{x-b}{x-c}\right|^s dP(x). \tag{6.5}$$

We prove

Lemma 6.3. *If dP, the common probability distribution of the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ satisfies assumption (H2') and (H3) and the essential infimum of dP is 0, it is s-regular for $0 < s < \tau/4$ (where τ is the exponent defined in assumption (H3)).*

The purpose of this result is only to be used in section 2; we did not try to extend it to its maximal generality.

Proof. Define

$$\phi_s(a) = \int_{\mathbb{R}} \frac{1}{|x-a|^s} dP(x) \quad \text{and} \quad \psi_s(b,c) = \int_{\mathbb{R}} \left|\frac{x-b}{x-c}\right|^s dP(x).$$

By the Cauchy-Schwartz inequality, as in Lemma C.3 of [4], it suffices to show that the continuous functions $\sqrt{\phi_{2s}(a)}/\phi_s(a)$ and $\sqrt{\psi_{2s}(b,c)}/\psi_s(b,c)$ are bounded. To prove this, we only need to study the behavior of the ϕ_s and ψ_s at infinity.
Let us start with ϕ_s. We may assume $|a|$ large. One has

$$\int_0^{+\infty} \frac{1}{|x-a|^s} dP(x) \ge \int_0^1 \frac{1}{|x-a|^s} dP(x) \ge \frac{1}{C} |a|^{-s} \tag{6.6}$$

where $C > 0$ is independent of a, $|a| > 1$. On the other hand,

$$\int_0^{+\infty} \frac{1}{|x-a|^s} dP(x) = I_1 + I_2 + I_3$$

where, by assumption (H2'),

$$I_1 = \int_0^{|a|/2} \frac{1}{|x-a|^s} dP(x) \leq C|a|^{-s}, \quad I_3 = \int_{2|a|}^{+\infty} \frac{1}{|x-a|^s} dP(x) \leq C|a|^{-s-1},$$
$$I_2 = \int_{|a|/2}^{2|a|} \frac{1}{|x-a|^s} dP(x).$$

Let us estimate I_2. By (H2') and (H3), we know that, for $0 \leq u \leq v$,

$$P([u,+\infty)) \leq C|u|^{-1} \quad \text{and} \quad P([u,v]) \leq C|u-v|^{\tau}.$$

Interpolating between these two estimates, for $\mu \in [0,1]$, we get

$$P([u,v]) \leq C|u|^{-\mu}|u-v|^{\tau(1-\mu)}. \tag{6.7}$$

We rewrite

$$\begin{aligned} I_2 &= \int_{|a|/2}^{|a|} \frac{1}{|x-a|^s} dP(x) + \int_{|a|}^{2|a|} \frac{1}{|x-a|^s} dP(x) \\ &= \sum_{n=1}^{+\infty} \int_{(1-2^{-k})|a|}^{(1-2^{-k-1})|a|} \frac{1}{|x-a|^s} dP(x) + \sum_{n=1}^{+\infty} \int_{(1+2^{-k})|a|}^{(1+2^{-k+1})|a|} \frac{1}{|x-a|^s} dP(x) \\ &\leq \sum_{n=1}^{+\infty} \int_{(1-2^{-k})|a|}^{(1-2^{-k-1})|a|} \frac{1}{|x-|a||^s} dP(x) + \sum_{n=1}^{+\infty} \int_{(1+2^{-k})|a|}^{(1+2^{-k+1})|a|} \frac{1}{|x-|a||^s} dP(x) \end{aligned}$$

Hence, using (6.7), if $-s + \tau(1-\mu) < 0$, we get

$$\begin{aligned} I_2 &\leq C \sum_{n=1}^{+\infty} (2^{-k-1}|a|)^{-s}((1-2^{-k})|a|)^{-\mu}(2^{-k-1}|a|)^{\tau(1-\mu)} \\ &\qquad + C \sum_{n=1}^{+\infty} (2^{-k}|a|)^{-s}((1+2^{-k})|a|)^{-\mu}(2^{-k}|a|)^{\tau(1-\mu)} \\ &\leq C|a|^{-s-\mu+\tau(1-\mu)} \frac{1}{1-2^{-s+\tau(1-\mu)}} \end{aligned}$$

We pick $s < \tau/2$ and $\mu > 1/2$ such that $-s + \tau(1-\mu) < 0$ and $-\mu + \tau(1-\mu) < 0$; hence, we get $I_2 \leq C|a|^{-s}$. Hence,

$$\int_0^{+\infty} \frac{1}{|x-a|^s} dP(x) = I_1 + I_2 + I_3 \leq C|a|^{-s}. \tag{6.8}$$

This and (6.6) proves that, if $s < \tau/2$, for some $C_s > 1$ independent of a, $|a| \geq 1$, one has

$$\frac{1}{C_s}|a|^{-s} \leq \phi_s(a) \leq C_s|a|^{-s}.$$

So, for $s < \tau/4$, the continuous function $\sqrt{\phi_{2s}(a)}/\phi_s(a)$ is bounded.

Let us now study ψ_s. The study is quite similar to the one done for ϕ_s. Let us first that $|c| \geq 1$. For the upper bound, we assume $s < \tau/2 < 1/2$ and use the Cauchy-Schwartz inequality to get

$$\begin{aligned}\psi_s(b,c) = \int_{\mathbb{R}} \left|\frac{x-b}{x-c}\right|^s dP(x) &\leq \left(\int_{\mathbb{R}} |x-b|^{2s} dP(x)\right)^{1/2} \left(\int_{\mathbb{R}} \left|\frac{1}{x-c}\right|^{2s} dP(x)\right)^{1/2} \\ &\leq C\frac{1}{|c|^s}\left(\int_{\mathbb{R}} |x|^{2s} + |b|^{2s} dP(x)\right)^{1/2} \\ &\leq C\frac{1}{|c|^s}(1+|b|^s).\end{aligned} \tag{6.9}$$

In the second step, we have used estimate (6.8). To get the lower bound, we write

$$\psi_s(b,c) \geq \int_0^{1/2} \left|\frac{x-b}{x-c}\right|^s dP(x) \geq \frac{1}{C}\left(\left|\frac{b}{c}\right|^s + \frac{1}{|c|^s}\right). \tag{6.10}$$

This and (6.9) proves that, if $s < \tau/2$, for some $C_s > 1$ independent of b, c, $|c| \geq 1$, one has

$$\frac{1}{C_s}\left(\left|\frac{b}{c}\right|^s + \frac{1}{|c|^s}\right) \leq \psi_s(b,c) \leq C_s\left(\left|\frac{b}{c}\right|^s + \frac{1}{|c|^s}\right).$$

For $|c|$ small, say, $|c| \leq 1$, using the same computations as in (6.9) and (6.10), we get

$$C(|b|^s + 1) \geq \int_{\mathbb{R}} \left|\frac{x-b}{x-c}\right|^s dP(x) \geq \frac{1}{C}(|b|^s + 1).$$

This proves that, for $s < \tau/4$, the continuous function $\sqrt{\psi_{2s}(b,c)}/\psi_s(b,c)$ is bounded. We have thus completed the proof of Lemma 6.3. ☐

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