A NOTE ON THE QUANTUM WIDOM-ROWLISON MODEL

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Abstract. Using the renormalization methods we show that the symmetry breaking in the quantum Widom-Rowlison model of particles obeying Boltzmann statistics occurs at any value of the inverse temperature $\beta > 0$ once the activity of the particles is sufficiently large.

Key Words: Continuous systems, quantum Widom-Rowlison model, path integrals, phase transition, renormalization.

1. Introduction

Rigorous analysis of phase transitions in the continuous particle systems of statistical mechanics is an almost entirely open problem. The symmetry breaking has been proven for the Widom-Rowlison model in [Ru],[CCK] and, in a recent ground breaking work [LMP], for continuous Kac models with four-body short range repulsive interactions. Most of the conventional lattice tools, such as ferromagnetic inequalities and related renormalization procedures or the Pirogov-Sinai theory cannot be readily extended to the continuum setup, and new ideas and techniques are pending.

In this note we prove the existence of symmetry breaking in the quantum version of the Widom-Rowlison model. Our research has been motivated by the work [CP], and, in the case of Boltzmann statistics which we study here, we improve the results of the latter paper. While [CP] is an extension of Ruelle's original Peierls type arguments [Ru] to the quantum case, our approach is built on the alternative geometric analysis developed in [CCK].

The classical Widom-Rowlison model [WR] describes a gas of two types of particles, which we shall denote **A** and **B**. The free distribution of each particle type is Poisson with the same activity λ, and the only interaction is the hard core exclusion between particles of different types. It is conceivable that at large activities λ equilibrium states should be characterized by a "squeezing out" of one of the two particle types, or, equivalently, by putting weight on predominantly **A** or **B** particle configurations. The results of [Ru] and [CCK] furnish a rigorous version of this picture.

The quantum counterpart of the Widom-Rowlison model proposed in [CP] is based on the sample path representation of quantum statistical mechanics (see [Gi] or [BR], Chapter 6.3). The free reference measures describe Poisson gases of **A** and **B** non-interacting Brownian loops, and the hard core exclusion condition acts time-wise on the loops of different colour. The situation here is much more complicated than in the classical case due to a certain loss of locality: in principal arbitrary many loops of both colours can visit a given part of the space without violating the hard core constraint. Consequently, the renormalization procedures we employ are not only on the level of the space, but also take into account various loop lengths via an introduction of appropriate cut-offs.

THE RESEARCH WAS SUPPORTED BY THE FUND FOR THE PROMOTION OF RESEARCH AT THE TECHNION AND BY THE NATO GRANT PST.CLG.976552.

In the next section we define the model and formulate the main result of the paper, which asserts symmetry breaking at any $\beta > 0$ once the activity λ is large enough. The crucial stochastic geometric representation (gray representation in the language of [CCK]) of the system is adjusted to the quantum case in Subsection 3.1. In the rest of Section 3 we set up the renormalization notation and formulate the decoupling estimates which are used to prove the phase separation phenomenon. Finally, our basic stochastic domination result Lemma 3.1 is proved in the concluding Section 4 along with the short and long loop renormalization upper bounds.

Aknowledgement I would like to thank Errico Presutti for for introducing me to the problem studied in the paper and Pier Picco, who has kindly provided a preliminary version of [CP].

2. THE RESULT

2.1. **The reference measure.** It is convenient to describe non-interacting loops in terms of the Poisson point process of excursions: Namely, given the value of the inverse temperature β, the excursion set $\mathbb{U}_\beta$ is the space of Brownian loops of the time duration β:

$$\mathbb{U}_\beta \;=\; \mathbb{C}_{0,0}\left([0,\beta],\mathbb{R}^d\right) \;\overset{\Delta}{=}\; \left\{\omega \in \mathbb{C}\left([0,\beta],\mathbb{R}^d\right)\,:\,\omega(0)=\omega(\beta)=0\right\},$$

and, respectively, the excursion measure $\mathbb{P}_\beta$ is the Brownian bridge measure on $\mathbb{U}_\beta$, that is $\omega(\cdot)$ is distributed under $\mathbb{P}_\beta$ as

$$\overline{\omega}(t) \;-\; \frac{t}{\beta}\overline{\omega}(\beta), \tag{2.1}$$

where $\overline{\omega}$ is the standard Brownian motion on $\mathbb{R}^d$.

Thus, the distribution of the non-interacting gas of both **A** and **B** loops is given by the following *unnormalized* measure ν_β^λ: Using the notation $\{\underline{x},\underline{\omega}\} = \{(x_1,\omega_1),...,(x_n,\omega_n)\}$,

$$\nu_\beta^\lambda\left(\mathcal{X} \in \mathrm{d}\left\{\underline{\mathrm{x}},\underline{\omega}\right\}\right) \;=\; \nu_\beta^\lambda\left(\mathrm{d}\left\{\underline{\mathrm{x}},\underline{\omega}\right\}\right) \;=\; \lambda^n \prod_1^n \mathbb{P}_\beta(\mathrm{d}\omega_\mathrm{k}) \prod_1^\mathrm{n} \mathrm{dx_i}. \tag{2.2}$$

In other words, given measurable subsets $\Lambda \subset \mathbb{R}^d$ and $A \subset \mathbb{U}_\beta$, the random variable

$$N(\Lambda, A) \;=\; \#\left\{(x,\omega)\in\mathcal{X}\,:\,x\in\Lambda \text{ and } \omega\in A\right\}$$

is distributed Poisson$\left(\lambda|\Lambda|\mathbb{P}_\beta(A)\right)$.

2.2. **The quantum Widom-Rowlison model.** To set up the notation, use $\Xi_{\Lambda,\beta}$ to denote the space of Brownian loops originating in $\Lambda \subset \mathbb{R}^d$,

$$\Xi_{\Lambda,\beta} \;=\; \left\{(x,\omega)\,:\,x\in\Lambda \text{ and } \omega\in\mathbb{U}_\beta\right\}.$$

Quantum particle configurations $\mathcal{X}$ on a bounded Λ are, then, finite subsets of $\Xi_{\Lambda,\beta}$. As we have already mentioned, the weights (2.2) give the distribution of the non-interacting gas of loops both in the case of particles of the type **A** and of the type **B**. For two given configurations $\left\{\underline{x}^\mathbf{A},\underline{\omega}^\mathbf{A}\right\}$ and $\left\{\underline{x}^\mathbf{B},\underline{\omega}^\mathbf{B}\right\}$ of **A** and **B** type particles the formal hard-core exclusion interaction potential $\mathcal{H}_a^\beta$ with the interaction radius $a > 0$ is given by

$$\mathcal{H}_\beta^a\left(\left\{\underline{x}^\mathbf{A},\underline{\omega}^\mathbf{A}\right\},\left\{\underline{x}^\mathbf{B},\underline{\omega}^\mathbf{B}\right\}\right) \;=\; \sum_{k=1}^n\sum_{l=1}^m \int_0^\beta \chi_a\left(x_k^\mathbf{A}+\omega_k^\mathbf{A}(t)-x_l^\mathbf{B}-\omega_l^\mathbf{B}(t)\right)\mathrm{d}t,$$

where

$$\chi_a(r) \;=\; \begin{cases} 0, & \text{if } |r| > 2a \\ \infty, & \text{otherwise} \end{cases}$$

Accordingly, the Boltzmann statistics for the Widom-Rowlison gas of two types of particles $\mathbf{A}$ and $\mathbf{B}$ with the hard core exclusion in the vessel $\Lambda \subset \mathbb{R}^d$ is specified by the weights

$$\begin{aligned} &\nu^{\lambda,a}_{\Lambda,\beta}\left(\mathcal{X}^{\mathbf{A}} \in \mathrm{d}\left\{\underline{\mathrm{x}}^{\mathbf{A}}, \underline{\omega}^{\mathbf{A}}\right\}, \mathcal{X}^{\mathbf{B}} \in \mathrm{d}\left\{\underline{\mathrm{x}}^{\mathbf{B}}, \underline{\omega}^{\mathbf{B}}\right\}\right) \\ &\quad = \mathrm{e}^{-\mathcal{H}^a_\beta(\mathcal{X}^{\mathbf{A}},\mathcal{X}^{\mathbf{B}})} \nu^\lambda_\beta\left(\mathrm{d}\left\{\underline{\mathrm{x}}^{\mathbf{A}}, \underline{\omega}^{\mathbf{A}}\right\}\right) \nu^\lambda_\beta\left(\mathrm{d}\left\{\underline{\mathrm{x}}^{\mathbf{B}}, \underline{\omega}^{\mathbf{B}}\right\}\right). \end{aligned} \tag{2.3}$$

where $\mathcal{X}^{\mathbf{A}}, \mathcal{X}^{\mathbf{B}} \subset \Xi_{\Lambda,\beta}$ are the random configurations of, respectively, $\mathbf{A}$ and $\mathbf{B}$ loops.

Let $\mathbb{P}^{\lambda,a}_{\Lambda,\beta}$ denote the corresponding (normalized) probability measure. We shall use the shortcut n for the domains $\Lambda_n = [-n/2, n/2]^d$.

Theorem 2.1. *For every $\beta > 0$ and every $\rho > 0$ there exists $\lambda_0 = \lambda_0(\beta, a, \rho)$ and positive constants c_1,$c_2 > 0$, such that uniformly in n and $\lambda \geqslant \lambda_0$,*

$$\mathbb{P}^{\lambda,a}_{n,\beta}\left(\max\{\mathcal{N}_{\mathbf{A}}, \mathcal{N}_{\mathbf{B}}\} - \min\{\mathcal{N}^{\mathbf{A}}, \mathcal{N}^{\mathbf{B}}\} < \rho n^d\right) \;\leqslant\; c_1 \mathrm{e}^{-c_2 n^{d-1}}, \tag{2.4}$$

where $\mathcal{N}^{\mathbf{A}} = \#(\mathcal{X}^{\mathbf{A}})$ (respectively $\mathcal{N}^{\mathbf{B}} = \#(\mathcal{X}^{\mathbf{B}})$) is the number of $\mathbf{A}$-loops (respectively $\mathbf{B}$-loops) originating in Λ_n.

3. STRUCTURE OF THE PROOF

3.1. **The gray representation.** Following the approach of [CCK] to the classical Widom-Rowlison model, let us consider the induced distribution of $\mathcal{X} = \mathcal{X}^{\mathbf{A}} \cup \mathcal{X}^{\mathbf{B}}$, which is specified by the (*unnormalized*) weights

$$\nu^{\lambda,a}_{n,\beta}\left(\mathcal{X} \in \mathrm{d}\left\{\underline{\mathrm{x}}, \underline{\omega}\right\}\right) \;=\; 2^{\mathcal{C}_a(\mathcal{X})} \nu^\lambda_\beta(\mathcal{X} \in \mathrm{d}\left\{\underline{\mathrm{x}}, \underline{\omega}\right\}), \tag{3.1}$$

where $\mathcal{C}_a(\mathcal{X})$ is the number of the maximal connected components of $\mathcal{X}$: Two loops (x_k, ω_k) and (x_l, ω_l) are said to be connected if,

$$\min_{0 \leqslant t \leqslant \beta} |x_k + \omega_k(t) - x_l - \omega_l(t)| \;\leqslant\; 2a.$$

The joint configuration $\left(\mathcal{X}^{\mathbf{A}}, \mathcal{X}^{\mathbf{B}}\right)$ could be recovered from the gray configuration $\mathcal{X}$ via the independent coloring maximal connected components of $\mathcal{X}$ into $\mathbf{A}$ or $\mathbf{B}$ with the probability $1/2$ each.

It has been observed in [CCK] that the gray representation of the classical Widom-Rowlison model is reminiscent of the Fortuin-Kasteleyn random cluster representation of lattice ferromagnetic systems. In particular, appropriate versions of various FKG inequalities hold in the classical case, which lead not only to the proof of the symmetry breaking but also to a meaningful definition of the surface tension. As we have already remarked, due to the loss of locality the quantum situation is more complicated, and one needs more care in devising the corresponding renormalization procedures. Nevertheless, to a certain extent such a "ferromagnetic" approach goes through, and clustering properties of the field of gray loops can be stochastically compared with those of the Bernoulli site percolation process. This is our main technical tool to prove Theorem 2.1.

3.2. **Comparison with the Bernoulli site percolation.** We shall prove that for large values of the activity λ most of the vessel Λ will be, with overwhelming probabilities, covered by a large connected component of the gray configuration $\mathcal{X}$. Roughly speaking the symmetry breaking excess density ρ in Theorem 2.1 is the density of this largest cluster and the dominant phase corresponds to its color which, according to the reconstruction procedure of coloured loops from the gray ones, could be either **A** or **B** with probability 1/2 each. This is essentially the strategy of [CCK]. However, the straightforward realization-wise FKG comparison approach of [CCK] does not apply in the quantum case we consider here. Indeed, unlike the classical case, adding an extra loop to the configuration $\mathcal{X}$ could drastically reduce the number of disjoint clusters $\mathcal{C}_a(\mathcal{X})$. This, of course, happens because loops can be arbitrary long. On the other hand very long loops are improbable under the reference excursion measure $\mathbb{P}_\beta$. Accordingly let us fix $h \in \mathbb{R}_+$ and split the space of loops $\Xi_{n,\beta}$ into the h-short and h-long parts,

$$\Xi_{n,\beta} \;=\; \Xi^s_{n,\beta} \bigvee \Xi^l_{n,\beta} \quad \text{where} \quad \Xi^s_{n,\beta} = \left\{ (x,\omega) \in \Xi_{n,\beta} \,:\, \mathrm{diam}(\omega) \leqslant h \right\}.$$

Subsequently, the realization $\mathcal{X}$ of the (gray) point process of loops splits as

$$\mathcal{X} \;=\; \left(\mathcal{X} \cap \Xi^s_{n,\beta}\right) \bigvee \left(\mathcal{X} \cap \Xi^l_{n,\beta}\right) \;\stackrel{\Delta}{=}\; \mathcal{X}^s \vee \mathcal{X}^l.$$

A simple but important observation is that the point processes $\mathcal{X}^s$ and $\mathcal{X}^l$ are independent under the reference measure (2.2).

Without loss of generality let us assume that the radius of the interaction a divides n and that n/a is odd. Then, we split Λ_n into smaller boxes of the linear size a as

$$\Lambda_n \;=\; \bigcup_{t \in \Lambda^a_n} \Lambda_a(t), \tag{3.2}$$

where $\Lambda^a_n = a\mathbb{Z}^d \cap \Lambda_n$ and $\Lambda_a(t) \stackrel{\Delta}{=} t + \Lambda_a$.

Given a number $k \in \mathbb{N}$ let us define the dependent percolation process X^k_a on $\{0,1\}^{\Lambda^a_n}$ as

$$X^k_a(t) \;=\; \begin{cases} 1, & \text{if } \# \left\{ \mathcal{X}^s \cap \{ (x,\omega) \in \Xi_{n,\beta} \,:\, x \in \Lambda_a(t) \} \right\} \geqslant k \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

In other words, $X^s_a(t)$ is the indicator function of the event that at least k h-short loops originate in $\Lambda_a(t)$. Theorem 2.1 is an immediate consequence of the following

Proposition 3.1. *For every $\beta > 0$, $k \in \mathbb{N}$ and $p < 1$ there exist the short loop parameter $h = h(\beta, a, k, p)$ and the activity $\lambda_0 = \lambda_0(\beta, a, k, p)$ such that for every $\lambda \geqslant \lambda_0$ the distribution of $\{X^k_a(t)\}$ under $\mathbb{P}^{\lambda,a}_{n,\beta}$ stochastically dominates the independent Bernoulli site percolation measure $\mathbb{P}^{\mathrm{perc}}_p$ on $\{0,1\}^{\Lambda^a_n}$.*

The proof of Proposition 3.1 is based on the following version of the strong FKG lattice condition (c.f. [ACCN]): Let S be a finite set and $\mathbb{P}$, $\mathbb{Q}$ two probability measures on $\{0,1\}^S$. Then $\mathbb{P}$ stochastically dominates $\mathbb{Q}$ whenever

$$\frac{\mathbb{P}(\eta \vee \delta_s)}{\mathbb{P}(\eta)} \;\geqslant\; \frac{\mathbb{Q}(\xi \vee \delta_s)}{\mathbb{Q}(\xi)} \tag{3.4}$$

is satisfied for every ordered couple of configurations $\eta, \xi \in \{0,1\}^S$; $\eta \geqslant \xi$ and every $s \in S$.

3.3. **Partition of Λ_n into small boxes.** The FKG comparison techniques suggested by (3.4) yield better results when applied on smaller scales. Let $r \ll a$ and $a/r \in \mathbb{N}$ is odd. As in (3.2) we split Λ_n into the union of small r-boxes

$$\Lambda_n \;=\; \bigcup_{i\in\Lambda_n^r} \Lambda_r(i), \tag{3.5}$$

where $\Lambda_n^r = r\mathbb{Z}^d \cap \Lambda_n$ and $\Lambda_r(i) = i + \Lambda_r$. Given the short-loop parameter h define

$$X_r(i) \;=\; \begin{cases} 1, & \text{if } \{\mathcal{X}^s \cap \{(x,\omega)\in\Xi_{n,\beta} \,:\, x\in\Lambda_r(i)\}\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \tag{3.6}$$

Thus, $X_r(i)$ is the indicator function of the event that there is at least one h-short loop of $\mathcal{X}^s$ originating from $\Lambda_r(i)$.

Lemma 3.1. *For every $\beta > 0$, $r > 0$ and $q < 1/2$ there exist $h = h(\beta,a,r,p)$ and $\lambda_0 = \lambda_0(\beta,a,r,p)$ such that for every $\lambda \geqslant \lambda_0$ the distribution of $\{X_r(i)\}$ under $\mathbb{P}_{n,\beta}^{\lambda,a}$ stochastically dominates the independent Bernoulli site percolation measure $\mathbb{P}_q^{\mathrm{perc}}$ on $\{0,1\}^{\Lambda_n^r}$.*

Since for every $t\in\Lambda_n^k$ the variable $X_a^k(t)$ is a monotone function of $(a/r)^d$ variables $X_r(i)$ with $\Lambda_r(i)\subset\Lambda_a(t)$, the field $\{X_a^k(t)\}$ stochastically dominates the Bernoulli site percolation measure $\mathbb{P}_p^{\mathrm{perc}}$ on Λ_n^k, where

$$p \;=\; \mathbb{P}_q^{\mathrm{perc}}\left(\sum_{i\,:\,\Lambda_r(i)\subset\Lambda_a(t)} X_r(i) \geqslant k \right),$$

or, in other words, the claim of the Proposition 3.1 follows with ($k \ll (a/r)^d$ in the inequality below)

$$p = 1 - \sum_{j=0}^{k-1} \binom{(a/r)^d}{k} q^j (1-q)^{(a/r)^d - j} \geqslant 1 - \exp\{-\frac{1}{2}\left(\frac{a}{r}\right)^d q\}.$$

4. Proof of Lemma 3.1

4.1. **Decomposition of $\mathcal{X}$.** By the strong FKG lattice condition (3.4) it is enough to show that for every $i\in\Lambda_n^r$ and each vector $b\in\{0,1\}^{\Lambda_n^r}$,

$$\frac{\nu_{n,\beta}^{\lambda,a}\left(X_r(i)=1\ ;\ X_r(j)=b_j \text{ for } j\neq i\right)}{\nu_{n,\beta}^{\lambda}\left(X_r(i)=0\ ;\ X_r(j)=b_j \text{ for } j\neq i\right)} \;\geqslant\; \frac{q}{1-q}. \tag{4.1}$$

At this stage the long-loop cutoff h becomes functional: Given $i\in\Lambda_n^r$ we decompose the set of all loops $\Xi_{n,\beta}$ into four subsets,

$$\Xi_{n,\beta} \;=\; \Xi_{n,\beta}^{s,i} \vee \Xi_{n,\beta}^{s,\Lambda_n\setminus i} \vee \Xi_{n,\beta}^{l,i} \vee \Xi_{n,\beta}^{l,\Lambda_n\setminus i}, \tag{4.2}$$

where $\Xi_{n,\beta}^{s,i}$ is the set of all h-short loops originating from $\Lambda_r(i)$,

$$\Xi_{n,\beta}^{s,i} \;=\; \left\{(x,\omega)\in\Xi_{n,\beta}^{s} \,:\, x\in\Lambda_r(i)\right\},$$

whereas $\Xi^{s,\Lambda_n\setminus i}_{n,\beta}$ contains the remaining h-short loops which originate in $\Lambda_n\setminus\Lambda_r(i)$. As for the long loops, the set $\Xi^{l,i}_{n,\beta}$ comprises those which reach $\Lambda_{2a+r+h}(i)$,

$$\Xi^{l,i}_{n,\beta} \;=\; \left\{(x,\omega)\in\Xi^{l}_{n,\beta}\;:\;\bigcup_{t\in[0,\beta]}(x+\omega(t))\cap\Lambda_{4a+2h+r}(i)\neq\emptyset\right\},$$

whereas the rest of the long loops are contained in the set $\Xi^{l,\Lambda_n\setminus i}_{n,\beta}$. Notice that that the definitions are designed in such a way that the short loops from $\Xi^{s,i}_{n,\beta}$ cannot interact with the long loops from $\Xi^{l,\Lambda_n\setminus i}_{n,\beta}$. Thus, the non-local contribution to the left hand side of (4.1) can come only from the long loops from $\Xi^{l,i}_{n,\beta}$. The choice of the cutoff h below will be designed to make the probability having such loops small.
In view of (4.2) any loop configuration $\mathcal{X}\subset\Xi_{n,\beta}$ could be decomposed into

$$\mathcal{X} \;=\; \mathcal{X}^s_i\vee\mathcal{X}^s_{\Lambda_n\setminus i}\vee\mathcal{X}^l_i\vee\mathcal{X}^l_{\Lambda_n\setminus i}.$$

Under the reference measure ν^λ_β the point processes $\mathcal{X}^s_i$, $\mathcal{X}^s_{\Lambda_n\setminus i}$,$\mathcal{X}^l_i$ and $\mathcal{X}^l_{\Lambda_n\setminus i}$ are independent and Poisson. Furthermore, substituting the trivial bound

$$\mathcal{C}_a(\mathcal{X}) \;\leqslant\; \mathcal{C}_a\left(\mathcal{X}^s_i\vee\mathcal{X}^s_{\Lambda_n\setminus i}\vee\mathcal{X}^l_{\Lambda_n\setminus i}\right)+\#\left(\mathcal{X}^l_i\right),$$

into the representation formula (3.1), we infer that for every (measurable) subset $A\subset\Xi^s_{n,\beta}$,

$$\nu^{\lambda,a}_{n,\beta}\left(\mathcal{X}^s\in A\right) \;\leqslant\; \nu^{\lambda,a}_{n,\beta}\left(\mathcal{X}^s\in A;\mathcal{X}^l_i=\emptyset\right)\sum_{k=0}^{\infty}2^k\frac{\nu^\lambda_\beta\left(\#(\mathcal{X}^l_i)=k\right)}{\nu^\lambda_\beta\left(\mathcal{X}^l_i=\emptyset\right)}. \tag{4.3}$$

4.2. **Long loops: the bound on $\sum_{k=0}^\infty 2^k\nu^\lambda_\beta\left(\#(\mathcal{X}^l_i)=k\right)$.** To simplify the notation we shall consider $i=0$. By (2.1), there exists $c_3=c_3(d)$, such that uniformly in $R>0$,

$$\mathbb{P}_\beta\left(\mathrm{diam}(\omega)>R\right) \;\leqslant\; \exp\{-c_3\frac{R^2}{\beta}\}. \tag{4.4}$$

Pick $R=8a+4h+2r$ and consider the decomposition of $\mathbb{R}^d$,

$$\mathbb{R}^d \;=\; \bigcup_{\tau\in R\mathbb{Z}^d}\Lambda_R(\tau) \;=\; \bigcup_{\tau\in R\mathbb{Z}^d}\left(\tau+[-\frac{R}{2},\frac{R}{2}]^d\right). \tag{4.5}$$

Decomposing the h-long loops with respect to the sets in the decomposition (4.5) from which these loops originate we infer that the total number $\#(\mathcal{X}^l_0)$ of h-long loops reaching $\Lambda_{R/2}$ is, under the reference measure ν^λ_β, distributed Poisson with the intensity $\mu=\mu(\lambda,\beta,R)$ bounded above as

$$\mu \;\leqslant\; c_4\lambda\sum_{k=1}^{\infty}k^{d-1}R^d\mathrm{e}^{-c_3k^2R^2/\beta} \;\leqslant\; c_5\lambda\mathrm{e}^{-c_3h^2/\beta}, \tag{4.6}$$

where the constants c_4,c_5 depend only on the dimension d. As a result,

$$\sum_{k=0}^{\infty}2^k\frac{\nu^\lambda_\beta\left(\#(\mathcal{X}^l_i)=k\right)}{\nu^\lambda_\beta\left(\mathcal{X}^l_i=\emptyset\right)} \;=\; \mathrm{e}^{2\mu} \;\leqslant\; \exp\{2c_5\lambda\mathrm{e}^{-c_3h^2/\beta}\}. \tag{4.7}$$

4.3. **Short loops.** The estimates (4.3) enable to control and ignore long loops from $\Xi^{l,i}_{n,\beta}$. Since the loops from $\Xi^{s,i}_{n,\beta}$ and $\Xi^{l,\Lambda_n\setminus i}_{n,\beta}$ do not interact, and any collection of lops from $\Xi^{s,i}_{n,\beta}$ is capable of hitting at most $c_6(h/a)^d$ mutually disconnected short loops from $\Xi^{s,\Lambda_n\setminus i}_{n,\beta}$, the following bound always holds:

$$\mathcal{C}_a\left(\mathcal{X}^s_i\vee\mathcal{X}^s_{\Lambda_n\setminus i}\vee\mathcal{X}^l_{\Lambda_n\setminus i}\right)\geqslant\mathcal{C}_a\left(\mathcal{X}^s_{\Lambda_n\setminus i}\vee\mathcal{X}^l_{\Lambda_n\setminus i}\right)-c_6(h/a)^d. \tag{4.8}$$

Returning to the right hand side of (4.1) we estimate:

$$\begin{aligned}
&\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=1\ ;\ X_r(j)=b_j\text{ for }j\neq i\right)\\
&\geqslant\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=1\ ;\ X_r(j)=b_j\text{ for }j\neq i;\mathcal{X}^l_i=\emptyset\right)\\
&\geqslant\frac{\nu^\lambda_\beta\left(\mathcal{X}^s_i\neq\emptyset\right)}{\nu^\lambda_\beta\left(\mathcal{X}^s_i=\emptyset\right)}2^{-c_6\left(\frac{h}{a}\right)^d}\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=0\ ;\ X_r(j)=b_j\text{ for }j\neq i;\mathcal{X}^l_i=\emptyset\right),
\end{aligned} \tag{4.9}$$

where the last inequality follows by (4.8). On the other hand, by (4.3) and (4.7) the following lower bound holds:

$$\begin{aligned}
&\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=0\ ;\ X_r(j)=b_j\text{ for }j\neq i;\mathcal{X}^l_i=\emptyset\right)\\
&\qquad\geqslant\exp\{-2c_5\lambda\mathrm{e}^{-c_3h^2/\beta}\}\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=0\ ;\ X_r(j)=b_j\text{ for }j\neq i\right)
\end{aligned}$$

Finally,

$$\frac{\nu^\lambda_\beta\left(\mathcal{X}^s_i\neq\emptyset\right)}{\nu^\lambda_\beta\left(\mathcal{X}^s_i=\emptyset\right)}=\sum_{k=1}^\infty\frac{\left(\lambda_s r^d\right)^k}{k!}=\mathrm{e}^{\lambda_s r^d}-1,$$

where $\lambda_s\triangleq\lambda\mathbb{P}_\beta\left(\mathrm{diam}(\omega)\leqslant h\right)$. By (4.4) $\lambda_s\geqslant\left(1-\mathrm{e}^{-c_3h^2/\beta}\right)\lambda$. As a result, the right hand side of (4.9) is bounded below by

$$\exp\left\{\lambda(1-\mathrm{e}^{-c_3h^2/\beta})r^d-c_6\left(\frac{h}{a}\right)^d-2c_5\lambda\mathrm{e}^{-c_3h^2/\beta}\right\}\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=0\ ;\ X_r(j)=b_j\text{ for }j\neq i\right),$$

and, consequently, we arrive to the following estimate on the ratio of probability weights in the lattice FKG condition (4.1):

$$\frac{\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=1\ ;\ X_r(j)=b_j\text{ for }j\neq i\right)}{\nu^{\lambda,a}_{n,\beta}\left(X_r(i)=0\ ;\ X_r(j)=b_j\text{ for }j\neq i\right)}\geqslant\exp\left\{\lambda\left(r^d-c_7\mathrm{e}^{-c_3h^2/\beta}\right)-c_6\left(\frac{h}{a}\right)^d\right\},$$

with the constants c_3,c_6 and c_7 being dependent only on the dimension d.

This implies the claim of Lemma 3.1.

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