Mathematical Analysis of the Photoelectric Effect

Volker Bach
FB Mathematik; Johannes Gutenberg-Universität;
D-55099 Mainz; Germany;
email: vbach@mathematik.uni-mainz.de

Frédéric Klopp
LAGA, Institut Galilée; Université Paris-Nord;
F-93430 Villetaneuse; France
email: klopp@math.univ-paris13.fr

Heribert Zenk
FB Mathematik; Johannes Gutenberg-Universität;
D-55099 Mainz; Germany;
email: zenk@mathematik.uni-mainz.de

Aug 31, 2001

Abstract

We study the photoelectric effect on the example of a simplified model of an atom with a single bound state, coupled to the quantized electromagnetic field.

For this model, we show that Einstein's prediction for the photoelectric effect is qualitatively and quantitatively correct to leading order in the coupling parameter. More specifically, considering the ionization of the atom by an incident photon cloud consisting of N photons, we prove that the total ionized charge is additive in the N involved photons. Furthermore, if the photon cloud is approaching the atom from a large distance or is monochromatic, the kinetic energy of the ejected electron is shown to be given by the difference of the photon energy of each single photon in the photon cloud and the ionization energy.

MSC: 81Q10, 81V10, 47N50.

Keywords: Photoelectric Effect, Scattering Theory, QED.

I Introduction

The photoelectric effect was discovered in increasingly precise experiments by Hertz [15, 1887], Hallwachs [14, 1888], Lenard [16, 1902][1], and Millikan [19, 18, 1916].

It was observed that, when light is incident on a metal surface, electrons are ejected from the surface. The striking fact about this phenomenon is the seemingly odd dependence of the maximal kinetic energy T_{max} of the ejected electrons on the frequency of the light and its independence of the light intensity. The latter contradicts the principles of classical physics, and in 1905, Einstein suggested an explanation of this phenomenon [8, 1905] which explicitly involves the *quantum* nature of electromagnetic radiation. He found that

$$T_{max} \;=\; h\,\nu \;-\; \Delta E\,, \tag{I.1}$$

provided that the frequency ν of the light times Planck's constant h, i.e., the photon energy $h\nu$ is larger than the (material dependent) work function ΔE. Conversely, if

$$h\,\nu \;<\; \Delta E\,, \tag{I.2}$$

then no electrons leave the metal surface. Our ultimate goal is the derivation of Einstein's predictions, Eqs. (I.1)–(I.2), from first principles of quantum mechanics and quantum field theory.

In the present paper, we analyze a simplified model which is far from the appropriate model for a metal interacting with electromagnetic radiation and can, at best, be regarded as a caricature of a hydrogen atom interacting with the radiation field. Yet, it contains many of the mathematical difficulties we expect to encounter in the analysis of a more realistic model, and we prove Eqs. (I.1)–(I.2) for this simplified model. Our emphasis lies in the following aspects:

- Given the model as described in Sect. I.1, below, our derivation is mathematically rigorous, and no unjustified approximations are used. To our knowledge, the present paper is the first to treat the photoelectric effect with mathematical rigor.

  We draw from many facts about nonrelativistic quantum electrodynamics which have been previously established in [2, 3, 4, 5].

  Our results are related to scattering theory and the asymptotics of the unitary time evolution operator for models of nonrelativistic quantum electrodynamics. Results in this context, but on other aspects can be found in [1, 6, 9, 10, 11, 20].

[1] In some physics textbooks, e.g., [13], Lenard is not mentioned, and it seems that in 1905, Einstein derived his famous theory from nothing but a Gedankenexperiment.

- While our model for the particle system, i.e., the metal or atom, is a crude model involving only a single, spinless electron, the particle system is coupled to a *quantized* scalar field. (The difference between the quantized [vector] electromagnetic and a quantized scalar field is irrelevant, for the scope of this work. For certain other facts in nonrelativistic quantum electrodynamics, however, this difference is crucial, see, e.g., [4].)

- Our model describes a single atom rather than a metal or even a gas interacting with the electromagnetic field. Thus our derivation of Eqs. (I.1)–(I.2) shows that the photoelectric effect is not a collective, statistical phenomenon, visible only if *many* particles or *many* photons are involved. Most texts on laser theory or quantum optics, e.g., [7, 12], immediately proceed to a statistical description of both the metal or gas of atoms, say, and the photon field, and the question, whether this is really necessary or merely a matter of mathematical convenience, is left open.

- Within our framework, we prove Eqs. (I.1)–(I.2) to be correct in leading nonvanishing order in the coupling parameter g which, in appropriate units, equals $\sqrt{2\pi}\,\alpha^{3/2}$, where $\alpha \approx 1/137$ is the fine structure constant. More precisely, given the interacting atom at rest plus an incident photon cloud consisting of N photons, we show that in leading order the contribution to the charge ejected from the atom is *additive* in each photon and *independent* of all other photons of the incident photon cloud. Moreover, we prove that this contribution is in accordance with (I.1)–(I.2) in case that the photons are in an incoming scattering state or in the limit of *monochromatic light*, i.e., when the support of each photon is sharply localized on energy shells in photon momentum space.

- While it is customary to restrict the analysis of the photoelectric effect to a single photon scattering off the atom or metal, we point out that it is important to consider more than one photon, $N \geq 2$, because for a photon state consisting of a single photon only, total energy and energy of each single photon involved agree. Hence, if studying a single photon state, it is impossible to say whether the ejection of electrons is proportional to the total energy of the photon cloud or depends on the maximal energy of all photons in the photon cloud.

Our paper is organized as follows. In Sect. I.1 below we introduce the mathematical model for the photoelectric effect. This includes a precise description of the atom, the photons, and their interaction in terms of a semibounded, selfadjoint generator of the dynamics, the Hamiltonian.

In Sect. I.2 we then describe our main results. We derive the asymptotics to leading order of the quantity of our interest: the charge transported to infinity, and we discuss the limit of monochromatic light. The proof of the theorems in Sect. I.2 are given in detail in Sects. II and IV.

Finally, our paper contains two appendices. In Appendix A we construct a Bogoliubov transformation that eliminates a single, arbitrary matrix element in the interaction, and in Appendix B we show that bound or negative energy states do not contribute to the transported charge.

Acknowledgement: We thank S. De Bièvre, J. Fröhlich, Ch. Gérard, M. Griesemer, V. Jakšić, V. Kostrykin, T. Paul, B. Schlein, R. Schrader, I. M. Sigal, H. Spohn, and S. Teufel for helpful discussions and remarks.

We gratefully acknowledge the support of our project by the TMR network ERBFMRXCT960001 of the European Union and the Gerhard Hess-grant no. Ba 1477/3-1 of the german science foundation DFG.

I.1 Mathematical Model for the Photoelectric Effect

Our goal is to analyze the photoelectric effect on the example of an atom coupled to the quantized radiation field. The mathematical model for our analysis of the photoelectric effect is a variant of the *standard model of nonrelativistic quantum electrodynamics* as given in [2] which we briefly recall and adapt to our problem at hand.

I.1.1 Atom with a Single Bound State

For the atom to be described we make the simplifying assumptions that it consists of a single, spinless electron bound in a potential well which admits exactly one bound state of energy $e_0 < 0$. More specifically, we assume the Hamiltonian generating the dynamics of the electron to be given in diagonal form as

$$H_{el} \; = \; -\Delta \oplus e_0 \; = \; \begin{pmatrix} -\Delta & 0 \\ 0 & e_0 \end{pmatrix} , \tag{I.3}$$

acting on

$$\mathcal{H}_{el} \; := \; L^2(\mathbb{R}^3) \oplus \mathbb{C} \; = \; \mathcal{H}_{ac}(H_{el}) \oplus \mathcal{H}_d(H_{el}) \, , \tag{I.4}$$

and $\mathcal{H}_{sc}(H_{el}) = \{0\}$. We further denote by

$$P_c \; := \; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_d \; := \; P_c^{\perp} \; = \; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{I.5}$$

the projections onto the continuous subspace and the discrete subspace (of dimension one), respectively. The Hamiltonian H_{el} can be derived from a Schrödinger operator $-\Delta - V(x)$ with a short-range potential V. An appropriate choice of V guarantees that $-\Delta - V(x)$ has a single bound state only, and then $-\Delta - V(x)$ and H_{el} are unitarily equivalent, as can be seen by conjugating $-\Delta - V(x)$ with the wave operators.

I.1.2 Photons

We couple the atom described above to the quantized photon field which, for notational convenience, is assumed to be scalar. For our study of the photoelectric effect, the difference is not relevant. The Hilbert space $\mathcal{F}$, carrying the photon degrees of freedom is the bosonic Fock space $\mathcal{F} = \mathcal{F}_b[L^2(\mathbb{R}^3)]$ over the one-photon Hilbert space $L^2(\mathbb{R}^3)$, i.e.,

$$\mathcal{F} \; = \; \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)} \, , \tag{I.6}$$

where $\mathcal{F}^{(n)}$ is the state space of all n-photon states, the n-photon sector. The vacuum sector, $\mathcal{F}^{(0)}$, is one-dimensional and spanned by the normalized vacuum vector, Ω, i.e., $\mathcal{F}^{(0)} := \mathbb{C}\,\Omega$. For $n \geq 1$, the n-photon sector is the subspace of $L^2[(\mathbb{R}^3)^n]$ containing all totally symmetric vectors. Note that we are making the simplifying assumption of *scalar* photons, since the polarization degrees of freedom play no role in our analysis. The Hamiltonian on $\mathcal{F}$ representing the energy of the free photon field is given by

$$H_f \;=\; \int d^3k\,\omega(k)\,a^*(k)a(k)\;, \tag{I.7}$$

where $\omega(k) := |k|$ is the photon dispersion, and a^*, a are the usual standard creation- and annihilation operators on $\mathcal{F}$ representing the canonical commutation relations, $[a(k), a(k')] = [a^*(k), a^*(k')] = 0$, $[a(k), a^*(k')] = \delta(k - k')$, $a(k)\Omega = 0$, in the sense of operator-valued distributions.

I.1.3 The Atom-Photon System

The Hilbert space of states of the atom-photon system is the tensor product space

$$\mathcal{H} \;:=\; \mathcal{H}_{el} \otimes \mathcal{F}\;, \tag{I.8}$$

and its dynamics is generated by the Hamiltonian

$$H_g \;:=\; H_0 \;+\; g\,W\;, \tag{I.9}$$

where

$$H_0 \;:=\; H_{el} \otimes \mathbf{1}_f \;+\; \mathbf{1}_{el} \otimes H_f \tag{I.10}$$

is the non-interacting Hamiltonian, $0 < g \ll 1$ is a small coupling parameter, and

$$W \;:=\; \int d^3k \Big\{ G(k) \otimes a^*(k) \;+\; G^*(k) \otimes a(k) \Big\} \tag{I.11}$$

is the interaction operator. Here, $G \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{el})]$ is a square-integrable function with values in the bounded operators on $\mathcal{H}_{el}$, given by

$$G(k) \;:=\; \begin{pmatrix} B(k) & p_\uparrow(k) \\ p_\downarrow(k) & 0 \end{pmatrix}\;, \tag{I.12}$$

for $k \in \mathbb{R}^3$, a.e. For the formulation of assumptions about G it is convenient to introduce

$$J_{K,\gamma}(k) \;:=\; \max_{|\alpha| \leq K} \big\| \partial_k^\alpha G(k) \big\| \;+\; \big\| (|x|^\gamma \oplus 1)\, G(k) \big\| \;+\; \big\| G(k)\, (|x|^\gamma \oplus 1) \big\|\;. \tag{I.13}$$

Note that, for $J_{K,\gamma}(k) < \infty$, we necessarily assume a decay of the coupling matrix $G(k)$ as least as $|x|^{-\gamma}$, as $x \to infty$. This assumption is not satisfied, e.g., if we consider electrons minimally coupled to the quantized radiation field. The extension of our results to this case of main physical interest will be the subject of a forthcoming paper.

Further note that

$$\begin{aligned} J_{K,\gamma}(k) \;\le\; J'_{K,\gamma}(k) \;:=\; & \max_{|\alpha|\le K} \big\| \partial_k^\alpha B(k) \big\| \;+\; \big\| \, |x|^\gamma \, B(k) \big\| \;+\; \big\| B(k) \, |x|^\gamma \, \big\| \quad \text{(I.14)} \\ & + \max_{|\alpha|\le K} \big\| \partial_k^\alpha p_\uparrow(k) \big\| \;+\; \max_{|\alpha|\le K} \big\| \partial_k^\alpha p_\downarrow(k) \big\| \;+\; \big\| \, |x|^\gamma \, p_\uparrow(k) \big\| \;+\; \big\| p_\downarrow(k) \, |x|^\gamma \, \big\| \, , \end{aligned}$$

and, conversely, $J'_{K,\gamma}(k) \le 3 J_{K,\gamma}(k)$. We shall make use of the following Hypothesis throughout the paper.

Hypothesis 1. *There is an integer $K > 1$ and a real number $\gamma > 3/2$ such that $G \in C^K[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{el})]$ is K times differentiable and satisfies the following estimate,*

$$\int \Big(1 + \omega(k)^{-1} \Big) \, |J_{K,\gamma}(k)|^2 \, d^3k \;\le\; 1 \, . \tag{I.15}$$

We remark that, although the requirement (I.15) with fractional derivatives of G, for real $K > 1$, is presumably sufficient, we do not try to optimize our result in this respect and work with classical derivatives in this paper.

Since, for a.e. $k \in \mathbb{R}^3$,

$$p_\uparrow(k) : \mathbb{C} \;\to\; L^2(\mathbb{R}^3) \quad \text{and} \quad p_\downarrow(k) : L^2(\mathbb{R}^3) \;\to\; \mathbb{C} \, , \tag{I.16}$$

there exist $\rho(\,\cdot\,, k), \eta(\,\cdot\, k) \in L^2(\mathbb{R}^3)$, such that

$$\big[p_\uparrow(k) z \big](x) \;=\; z \, \eta(x,k) \quad \text{and} \quad p_\downarrow(k)\psi \;=\; \langle \rho(\,\cdot\,, k) | \psi \rangle \;=\; \int \overline{\rho(x,k)} \psi(x) \, d^3x \, , \tag{I.17}$$

for all $z \in \mathbb{C}$, $\psi \in L^2(\mathbb{R}^3)$, and $(x,k) \in \mathbb{R}^3 \times \mathbb{R}^3$, a.e. Furthermore, in many applications (e.g., the Dipole approximation) $B(k) \in \mathcal{B}[L^2(\mathbb{R}^3)]$ acts in the Schrödinger representation as a multiplication operator. That is, there is a function $M \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ such that

$$\big[B(k)\psi \big](x) \;=\; M(x,k) \, \psi(x) \, , \tag{I.18}$$

for $(x,k) \in \mathbb{R}^3 \times \mathbb{R}^3$, a.e. In this case, M, $p_\uparrow$, and $p_\downarrow$ determine G, and we observe that

$$\begin{aligned} J'_{K,\gamma}(k) \;=\; & \sup_{x \in \mathbb{R}^3, \, |\alpha| \le K} \big| \partial_k^\alpha M(x,k) \big| \;+\; 2 \sup_{x \in \mathbb{R}^3} \big| \, |x|^\gamma \, M(x,k) \big| \\ & + \max_{|\alpha|\le K} \Big\{ \int \big| \partial_k^\alpha \rho(x,k) \big|^2 \, d^3x \Big\}^{1/2} \;+\; \max_{|\alpha|\le K} \Big\{ \int \big| \partial_k^\alpha \eta(x,k) \big|^2 \, d^3x \Big\}^{1/2} \\ & + \Big\{ \int |x|^{2\gamma} \, |\rho(x,k)|^2 \, d^3x \Big\}^{1/2} \;+\; \Big\{ \int |x|^{2\gamma} \, |\eta(x,k)|^2 \, d^3x \Big\}^{1/2} \end{aligned} \tag{I.19}$$

We remark that, for a given function M, a physically natural choice for η and ρ is

$$\eta(x,k) \ := \ \rho(x,k) \ := \ M(x,k)\,\varphi_{el}(x)\,, \tag{I.20}$$

where $\varphi_{el} \in L^2(\mathbb{R}^3)$, $\|\varphi_{el}\| = 1$, is the normalized wave function of the atomic bound state. In this case, it is easy to see that (I.19) can be reduced to the following simplified condition,

$$J'_{K,\gamma}(k) \ \leq \ J''_{K,\gamma}(k) \ := \ 3 \sup_{x\in\mathbb{R}^3,\,|\alpha|\leq K} \left|\partial_k^\alpha M(x,k)\right| \ + \ 4 \sup_{x\in\mathbb{R}^3} \left|\, |x|^\gamma \, M(x,k)\right| . \tag{I.21}$$

Note that the special form (I.12) of $G(k)$ implies that

$$(P_d \otimes \mathbf{1})\; W \; (P_d \otimes \mathbf{1}) \ = \ 0\,, \tag{I.22}$$

since the lower right matrix element of $G(k)$ vanishes.

We remark that if the electron Hamiltonian is a Schrödinger operator, as described below Eq. (I.5), a *single* diagonal matrix element can always be "gauged away" by means of a Bogoliubov transformation. The construction of this Bogoliubov transformation is given in Appendix A. That is, for our model with a *single* atomic bound state, we do not loose any generality by assuming (I.22). As (I.22) is an important assumption for the present paper, we point out that, in case the atom has more than one bound state, P_d would be the orthogonal projection onto these, and (I.22) would mean that we assume these bound states not to interact at all.

I.1.4 Selfadjointness, Semiboundedness, and Binding

Next, we discuss selfadjointness, semiboundedness and existence of a ground state (binding) of the Hamiltonian H_g of the system. We basically invoke the theorems and methods for their proof established in [2, Sects. II and III].

Assuming Hypothesis 1, selfadjointness and semiboundedness, for any value of the coupling parameter $g > 0$, are a simple consequence of standard Kato perturbation theory; we appeal to [2, Cor. I.7 and Lemma I.5] which yield that H_g is selfadjoint on its natural domain, $\mathrm{dom}[H_g] = \mathrm{dom}[H_0]$, and that the ground state energy is given by

$$E_0 \ := \ \inf \sigma[H_g] \ \geq \ e_0 + \mathcal{O}(g^2)\,. \tag{I.23}$$

In the present paper we additionally assume that E_0 is an eigenvalue of H_g and that the corresponding eigenvectors have a large component in the vacuum sector $\mathcal{F}^{(0)}$. More precisely, we require

Hypothesis 2. *The Hamiltonian H_g possesses a (normalizable) ground state $\Phi_{\rm gs} \in \mathcal{H}$, $\|\Phi_{\rm gs}\| = 1$, i.e., E_0 is an eigenvalue with corresponding eigenvector $\Phi_{\rm gs}$,*

$$H_g \, \Phi_{\rm gs} \;=\; E_0 \, \Phi_{\rm gs} \, . \tag{I.24}$$

Moreover, denoting $P_\Omega := |\Omega\rangle\langle\Omega| \equiv \mathbf{1} \otimes |\Omega\rangle\langle\Omega|$, the ground state $\Phi_{\rm gs}$ obeys

$$\big\| P_\Omega^\perp \, \Phi_{\rm gs} \big\| \;\leq\; C \, g \, , \tag{I.25}$$

for some constant $C < \infty$.

The existence (I.24) of a ground state $\Phi_{\rm gs}$ and the overlap bound (I.25) is proved in [2, Thm. I.1] under the assumption of a somewhat stronger bound than Eq. (I.15) in Hypothesis 1, namely,

$$\int \Big(1 + \omega(k)^{-2} \Big) \, |J_{0,\gamma}(k)|^2 \, d^3k \;\leq\; 1 \, . \tag{I.26}$$

In this paper, we do not use this estimate but only its consequence in form of Hypothesis 2.

We remark that the noninteracting Hamiltonian H_0 has the unique (non-degenerate) ground state $0 \oplus \Omega \in \mathcal{H}$ corresponding to its ground state energy e_0, and we note in parentheses that, by the results of [2], Eq. (I.25) holds for *any* ground state of H_g. Consequently, the interacting ground state $\Phi_{\rm gs}$ of H_g is unique, provided $g > 0$ is sufficiently small.

In Lemma IV.2 we strengthen (I.25) and show that, for any $\alpha \geq 1$ and any $\Lambda < \infty$,

$$\big\| (H_f^{(\Lambda)} + \mathbf{1})^\alpha \, P_c \, \Phi_{\rm gs} \big\| \;+\; \big\| (H_f^{(\Lambda)} + \mathbf{1})^\alpha \, P_\Omega^\perp \, \Phi_{\rm gs} \big\| \;\leq\; \mathcal{O}(g) \, . \tag{I.27}$$

Here, $H_f^{(\Lambda)}$ denotes a free photon Hamiltonian acting only on photon states with energy less than $\Lambda < \infty$,

$$H_f^{(\Lambda)} \;=\; \int d^3k \, \omega_\Lambda(k) \, a^*(k) a(k) \;=\; \int_{\{\omega(k) \leq \Lambda\}} d^3k \, \omega(k) \, a^*(k) a(k) \, , \tag{I.28}$$

where $\omega_\Lambda(k) := \omega(k) \, \mathbf{1}_{\{\omega(k) < \Lambda\}}$.

I.2 Main Results and Discussion

I.2.1 Charge transported to Infinity

Having introduced the quantum mechanical framework in terms of Hilbert spaces and Hamiltonians and established some basic facts such as selfadjointness, semiboundedness, and binding, we proceed to defining the *charge transported to infinity* or simply the *transported charge*. To this end, we introduce the projection F_R

onto the functions with support outside the ball of radius $R > 0$. More precisely,

$$F_R \ := \ \begin{pmatrix} \mathbf{1}_{\{|x|\geq R\}} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_f \,, \tag{I.29}$$

where $\mathbf{1}_{\{|x|\geq R\}} := \mathbf{1}_{\mathbb{R}^3\setminus B(0,R)}[x]$. Similarly, we introduce the projection $T_{\mathcal{T}}$ onto the particle states with momentum in a measurable set $\mathcal{T} \subset \mathbb{R}^3$, i.e., the functions whose Fourier transform is supported in $\mathcal{T}$. That is,

$$T_{\mathcal{T}} \ := \ \begin{pmatrix} \mathbf{1}_{\{p\in\mathcal{T}\}} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_f \,. \tag{I.30}$$

where $\mathbf{1}_{\{p\in\mathcal{T}\}}$ acts as a Fourier multiplier with $\mathbf{1}_{\mathcal{T}}[p]$. Given a state $\Psi \in \mathcal{H}$, the corresponding (least and most) transported charges with momentum in $\mathcal{T} \subset \mathbb{R}^3$ are defined to be

$$Q_{\mathcal{T}}^{\inf}(\Psi) \ := \ \liminf_{R\to\infty} \liminf_{t\to\infty} \left\| T_{\mathcal{T}}\, F_R\, e^{-itH_g}\, \Psi \right\|^2 , \tag{I.31}$$

$$Q_{\mathcal{T}}^{\sup}(\Psi) \ := \ \limsup_{R\to\infty} \limsup_{t\to\infty} \left\| T_{\mathcal{T}}\, F_R\, e^{-itH_g}\, \Psi \right\|^2 , \tag{I.32}$$

In case that $\mathcal{T} = \mathbb{R}^3$, we write $Q^{\inf}(\Psi) := Q_{\mathcal{T}}^{\inf}(\Psi)$ and $Q^{\sup}(\Psi) := Q_{\mathcal{T}}^{\sup}(\Psi)$. Interpreting Ψ as an initial state for $t = 0$, the transported charge $Q^{\inf}(\Psi)$ measures the amount of its mass that is definitely transported away from the atom as time evolves, eventually. If $\mathcal{T} \subseteq \mathbb{R}^3$ is a small ball, then $Q_{\mathcal{T}}(\Psi)$ additionally filters out the part of the state with momentum in $\mathcal{T}$.

While it would be desirable, of course, to define only one type of transported charge, namely, $Q_{\mathcal{T}}(\Psi) := \lim_{R\to\infty} \lim_{t\to\infty} \|T_{\mathcal{T}}\, F_R\, e^{-itH_g}\, \Psi\|^2$, our methods described below do not allow us to prove the existence of such a limit – not even for the restricted class of initial states of the form (I.35). Nevertheless, $Q_{\mathcal{T}}^{\inf}(\Psi)$ and $Q_{\mathcal{T}}^{\sup}(\Psi)$ agree to leading order in the coupling constant g, as we demonstrate below. In fact, our main objective is the determination of the transported charges $Q_{\mathcal{T}}^{\inf}(\Psi)$ and $Q_{\mathcal{T}}^{\sup}(\Psi)$ to leading order in g.

Finally, we remark that it is important to observe the order of the two limits $R \to \infty$ and $t \to \infty$, as interchanging these limits would, indeed, yield the trivial result $\lim_{t\to\infty} \lim_{R\to\infty} \|F_R\, e^{-itH_g}\, \Psi\|^2 = 0$, for all $\Psi \in \mathcal{H}$. Likewise, we have

$$\lim_{R\to\infty} \sup_{t>0} \left\| F_R\, \mathbf{1}_{pp}(H_g)\, e^{-itH_g}\, \Psi \right\| \ = \ 0 \,, \tag{I.33}$$

$$\lim_{R\to\infty} \sup_{t>0} \left\| F_R\, \mathbf{1}_{\mathbb{R}_0^-}(H_g)\, e^{-itH_g}\, \Psi \right\| \ = \ 0 \,, \tag{I.34}$$

for all $\Psi \in \mathcal{H}$.

I.2.2 Ground State and Photon Cloud

Our choice for the initial state Ψ is of the form

$$\Psi \;:=\; A(\tau, \underline{f})\, \Phi_{\rm gs} \,, \tag{I.35}$$

where

$$A(0,\underline{f}) \;\equiv\; A(\underline{f}) \;:=\; \mathbf{1}_{el} \,\otimes\, a^*(f_1)\, a^*(f_2) \,\cdots\, a^*(f_N) \,, \tag{I.36}$$

and

$$A(\tau, \underline{f}) \;:=\; e^{-i\tau H_g}\, e^{i\tau H_0}\, A(\underline{f})\, e^{-i\tau H_0}\, e^{i\tau H_g} \,. \tag{I.37}$$

The operator $A(\tau, \underline{f})$ is called a *photon cloud* and represents N photons with corresponding smooth orbitals $\underline{f} = (f_1, f_2, \ldots, f_N)$, $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, of compact support away from zero momentum. These N photons are prepared at time $t = -\tau$ in such a way that they hit the atom at rest at time $t = 0$. (We thank S. De Bièvre, M. Griesemer, H. Spohn and especially S. Teufel for clarifying this point to us.) From Eq. (II.8) below we show that, in particular, $A(\infty, \underline{f})\Phi_{\rm gs} := \lim_{\tau\to\infty} A(\tau, \underline{f})\Phi_{\rm gs}$ exists, thus representing an incoming photon scattering state. In contrast, choosing Ψ according to (I.35) and with $\tau = 0$ amounts to adding at time $t = 0$ the photon cloud $A(\underline{f})$ to the (interacting) atom at rest.

We henceforth often leave out trivial tensor factors in our notation whenever it is clear from the context what is meant. For instance, we write $A(\underline{f}) = a^*(f_1)\, a^*(f_2) \,\cdots\, a^*(f_N)$, $H_0 = H_{el} + H_f$, etc.

I.2.3 The Photoelectric Effect

Having defined the transported charges with momentum in $\mathcal{T}$, we can now formulate a quantitative assertion about it. To this end, we fix $p \in \mathbb{R}^3$ and define two distributions $L_0^p, L_\infty^p \in \mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$ on smooth functions $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ of compact support away from zero by

$$\langle L_\infty^p \,,\, f\rangle \;\;:=\;\; 2\pi \int \delta\big(p^2 - E_0 - \omega(k)\big)\, \hat\rho(p,k)\, f(k)\, d^3k \,, \tag{I.38}$$

$$\langle L_0^p \,,\, f\rangle \;\;:=\;\; \int \frac{-i\, \hat\rho(p,k)\, f(k)\, d^3k}{p^2 - E_0 - \omega(k) + i0} \tag{I.39}$$

$$\begin{aligned} &= -\pi \int \delta\big(p^2 - E_0 - \omega(k)\big)\, \hat\rho(p,k)\, f(k)\, d^3k \\ &\quad - i\, \mathbb{P} \int \frac{\hat\rho(p,k)\, f(k)\, d^3k}{p^2 - E_0 - \omega(k)} \,, \end{aligned} \tag{I.40}$$

$\mathbb{P}\int$ denoting the Cauchy principal value and $\hat{\rho}(p,k)$ denoting the partial Fourier transform of $\rho(x,k)$ (as a square-integrable function) with respect to the particle position variable x.

Now we are in position to formulate our first main result.

Theorem I.1. *Assume Hypotheses 1–2, fix* $N, m \in \mathbb{N}$ *with* $m \leq N$, *and let* $\varphi_1, \varphi_2, \ldots \varphi_m \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ *be an orthonormal system,* $\langle \varphi_i | \varphi_j \rangle = \delta_{i,j}$. *Let* $A(\tau, \underline{\varphi})$ *be the corresponding photon cloud, i.e.,*

$$A(\tau, \underline{\varphi}) \; = \; e^{-i\tau H_g} \, e^{i\tau H_0} \, a^*(\varphi_1)^{n_1} \, a^*(\varphi_2)^{n_2} \; \cdots \; a^*(\varphi_m)^{n_m} \, e^{-i\tau H_0} \, e^{i\tau H_g} \, , \tag{I.41}$$

where $n_j \in \mathbb{N}$ *are such that* $n_1 + n_2 + \ldots + n_m = N$, *and fix a measurable set* $\mathcal{T} \subseteq \mathbb{R}^3$.

(i) If $\tau > g^{-1/K}$ *then the transported charges with momentum in* $\mathcal{T}$ *of the initial state* $A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}$ *satisfy*

$$\begin{aligned} Q_{\mathcal{T}}^{\inf}\big(A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}\big) \, + \, \mathcal{O}\big(g^{2+\mu}\big) \; &= \; Q_{\mathcal{T}}^{\sup}\big(A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}\big) \, + \, \mathcal{O}\big(g^{2+\mu}\big) \\ = \; g^2 \, Q_{\mathcal{T}}^{\tau}(\underline{\varphi}) \; &:= \; g^2 \, \big(n_1! \, n_2! \cdots n_m!\big) \sum_{j=1}^m n_j \int_{\mathcal{T}} |\langle L_\infty^p \, , \, \varphi_j \rangle|^2 d^3p \, . \end{aligned} \tag{I.42}$$

(ii) If $\tau < g^{\mu}$ *then the transported charge with momentum in* $\mathcal{T}$ *of the initial state* $A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}$ *is given by*

$$\begin{aligned} Q_{\mathcal{T}}^{\inf}\big(A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}\big) \, + \, \mathcal{O}\big(g^{2+\mu}\big) \; &= \; Q_{\mathcal{T}}^{\sup}\big(A(\tau, \underline{\varphi})\Phi_{\mathrm{gs}}\big) \, + \, \mathcal{O}\big(g^{2+\mu}\big) \\ = \; g^2 \, Q_{\mathcal{T}}^{\tau}(\underline{\varphi}) \; &:= \; g^2 \, \big(n_1! \, n_2! \cdots n_m!\big) \sum_{j=1}^m n_j \int_{\mathcal{T}} |\langle L_0^p \, , \, \varphi_j \rangle|^2 d^3p \, . \end{aligned} \tag{I.43}$$

Here, L_∞^p *and* L_0^p *are the distributions defined in Eqs. (I.38)–(I.40), and* $\mu \equiv \mu(K) := 1 - K^{-1} > 0$, *where* K *is the degree of differentiability in Hypothesis 1 and 2.*

Note that the coefficient $Q_{\mathcal{T}}^{\tau}(\underline{\varphi})$ of the leading order contribution to the photoelectric effect is *additive* (counting multiplicities) in the single photon states $\varphi_1, \ldots, \varphi_m$. This implies that the contribution of each φ_j is independent of the other single photon states φ_i, $i \neq j$, in the photon cloud, in contrast to the classical model for the photoelectric effect which would have predicted a dependence of $Q_{\mathcal{T}}^{\tau}(\underline{\varphi})$ only on the *total energy* of the photon cloud.

While, for large values of τ, i.e., the case (i) in Theorem I.1 above, the transported charge directly yields Einstein's predictions (I.1) and (I.2), there is still an inconsistency between (I.43) and Einstein's predictions (I.1) and (I.2), since $\langle L_0^p \, , \, \varphi_j \rangle$ may be non-vanishing, even if φ_j is supported in a region where $\omega(k) < p^2 - E_0$. It turns out that (I.43) is in accordance with (I.1) and (I.2) for *monochromatic light*, i.e., for a photon cloud with sharply localized momentum on (one or more) photon energy shell(s).

The photon cloud represented by $A(\infty, \underline{\varphi})$ may actually be physically closer to the experimental situation described by Einstein's theory than the one represented by $A(0, \underline{\varphi})$. Indeed, the photons in $A(\infty, \underline{\varphi})$ are localized at spatial infinity (and temporal negative infinity), when added to the system. Since the interaction is switched off at large distances from the origin, the sum of the free energies of the photon cloud and the atom coincides with the total energy of the interacting atom-photon system. In contrast, the photons in $A(0, \underline{\varphi})$ can be viewed as added to the system at $t = 0$, hence discontinuously changing the total energy of the system.

I.2.4 Monochromatic Light

In this subsection, we further analyze the leading order contribution to the transported charge given by g^2 times

$$Q_{\mathcal{T}}^0 \big(A(0, \underline{\varphi}) \Phi_{\rm gs} \big) \ = \ \big(n_1! \, n_2! \cdots n_m! \big) \sum_{j=1}^{m} n_j \int_{\mathcal{T}} | \langle L_0^p \, , \, \varphi_j \rangle |^2 d^3 p \qquad \text{(I.44)}$$

for the special case, in which each single photon wave function φ_j has support in a small neighborhood of an energy shell in momentum space. More precisely, for $N, m \in \mathbb{N}$, with $m \leq N$, we fix photon energies $\omega_1, \omega_2, \ldots, \omega_m \in \mathbb{R}^+$, $L^2(\mathbb{R})$-normalized functions $\chi_1, \chi_2, \ldots \chi_m \in C_0^\infty[(-\frac{1}{2}, \frac{1}{2})]$, and normalized functions on the two-sphere, $\kappa_1, \kappa_2, \ldots \kappa_m \in C^\infty[\mathbb{S}^2]$, for which we assume that $\langle \kappa_i | \kappa_j \rangle = 0$, whenever $\omega_i = \omega_j$. Then, for $0 < \delta < 1$, we define functions $\varphi_{1,\delta}, \varphi_{2,\delta}, \ldots, \varphi_{m,\delta} \in C_0^\infty[\mathbb{R}^3 \setminus \{0\}]$, in spherical coordinates $(r, \Omega_k) = (|k|, k/|k|)$ by

$$\varphi_{j,\delta}(r, \Omega_k) \ := \ \omega_j^{-1} \, \delta^{-1/2} \, \chi_j \Big(\frac{r - \omega_j}{\delta} \Big) \, \kappa_j(\Omega_k) \, . \qquad \text{(I.45)}$$

In order to ensure the disjointness of $\mathrm{supp} \varphi_{i,\delta}$ and $\mathrm{supp} \varphi_{j,\delta}$, whenever $\omega_i \neq \omega_j$, we additionally require that

$$\delta \ < \ \min \Big\{ \frac{\omega_i - \omega_j}{2} \ \Big| \ \omega_i \neq \omega_j \Big\} \, , \qquad \text{(I.46)}$$

and, furthermore,

$$\delta \ < \ \min\{ \omega_1, \omega_2, \ldots, \omega_m \} \, , \qquad \text{(I.47)}$$

to ensure that $\varphi_{j,\delta}$ is supported away from zero momentum. Note that the vectors $\varphi_{1,\delta}, \varphi_{2,\delta}, \ldots, \varphi_{m,\delta}$ are orthogonal and asymptotically orthonormal,

$$\lim_{\delta\to 0}\langle\varphi_{i,\delta}|\varphi_{j,\delta}\rangle \;=\; \delta_{i,j} \; . \tag{I.48}$$

Under these assumptions and choosing $n_1, n_2, \ldots, n_m \in \mathbb{N}$, with $n_1 + \ldots + n_m = N$, we define a photon cloud

$$A_\delta \;:=\; a^*(\varphi_{1,\delta})^{n_1}\, a^*(\varphi_{2,\delta})^{n_2} \;\cdots\; a^*(\varphi_{m,\delta})^{n_m} \tag{I.49}$$

of N asymptotically monochromatic photons, as $\delta \to 0$.

To formulate our result, we further define

$$\Theta_j(p,r) \;:=\; -i\int_{\mathbb{S}^2} \frac{r^2}{\omega_j}\, \hat\rho(p, r\sigma)\, \kappa_j(\sigma)\, d^2\sigma \; . \tag{I.50}$$

We require an additional condition on $\hat\rho$, the Fourier transform with respect to the electron variable x of ρ, defined in (I.17). More specifically, we require the existence of

$$J'''_{2,\gamma}(k) \;:=\; \max_{|\alpha|\le 2}\Big\{\int \big(|x|^{2\gamma}+1\big)\,\big|\partial_k^\alpha \rho(x,k)\big|^2\, d^3x\Big\}^{1/2} \;<\; \infty \, , \tag{I.51}$$

for all $k \in \mathbb{R}^3$, where our notation is similar to the one introduced in Subsect. I.1.3. Note that (I.51) implies that $\partial_k^\alpha \rho(x,k)$ is absolute integrable w. r. to x which, in turn, yields the following bound on its partial Fourier transform $\hat\rho$,

$$\sup_{p\in\mathbb{R}^3}\big\{|\hat\rho(p,k)| \;+\; |\nabla_k \hat\rho(p,k)|\big\} \;\le\; J'''_{2,\gamma}(k) \, , \tag{I.52}$$

for all $k \in \mathbb{R}^3$. We assume $J'''_{2,\gamma}$ to obey the following bound.

Hypothesis 3. *For some $\gamma > 3/2$ and all $\Lambda < \infty$,*

$$\sup\big\{J'''_{2,\gamma}(k)\;\big|\;|k|\le\Lambda\big\} \;<\; \infty \, . \tag{I.53}$$

This hypothesis implies the following bounds on Θ_j, for all $j = 1, \ldots, m$,

$$\sup\Big\{|\Theta_j(p,r)| + |\partial_r\Theta_j(p,r)|\;\Big|\;p\in\mathbb{R}^3 \;,\; r\in[0\,,\;\omega_j+1]\Big\} \;<\; \infty \, , \tag{I.54}$$

which we require for the proof of the following theorem.

Theorem I.2. *Assume Hypothesis 1–3 and Eqs. (I.45)–(I.49), and fix a measurable set $\mathcal{T} \subseteq \mathbb{R}^3$. Then in the limit of monochromatic light, the contribution of the photon $a^*(\varphi_{j,\delta})$ for the first order term of the transported charge is given by:*

$$\lim_{\delta \to 0} \int_{\mathcal{T}} |\langle L_0^p \, , \, \varphi_{j,\delta} \rangle|^2 \, d^3 p \;=\; \begin{cases} 0 & \textit{if} \;\; E_0 + \omega_j < 0 \, , \\ \Theta_{0,j} \, I_{0,j} & \textit{if} \;\; E_0 + \omega_j = 0 \, , \\ \Theta_{0,j} \, I_{+,j} & \textit{if} \;\; E_0 + \omega_j > 0 \, . \end{cases} \tag{I.55}$$

Here,

$$\Theta_{0,j} \;:=\; \int_{\mathbb{S}^2} \mathbf{1}_{\mathcal{T}}\big(\sqrt{E_0 + \omega_j}\sigma\big) \, \sqrt{E_0 + \omega_j} \, \big|\Theta(\sqrt{E_0 + \omega_j}\, \sigma \, , \, \omega_j)\big|^2 \, d^2\sigma \, , \tag{I.56}$$

and

$$I_{0,j} \;:=\; \int_0^\infty \left| \int_{-\infty}^\infty \frac{\chi_j(x) dx}{y - x + i0} \right|^2 dy \, , \tag{I.57}$$

$$I_{+,j} \;:=\; \int_{-\infty}^\infty \left| \int_{-\infty}^\infty \frac{\chi_j(x) dx}{y - x + i0} \right|^2 dy \, , \tag{I.58}$$

Einstein's predictions (I.1) and (I.2) are now manifest in the right hand side of (I.55). The coefficient $\lim_{\delta \to 0} Q^0_{\mathcal{T}}(A_\delta \Phi_{\mathrm{gs}})$ may still vanish,as the factor $\Theta_{0,j}$ may turn out to be zero, for all $j = 1, \ldots, m$. This case is in accordance with (I.1) and (I.2) and merely reflects the physical fact that some transitions (here: from a bound into a scattering state) are forbidden in lowest order perturbation theory.

II Dyson Series and Photoelectric Effect

In this section we study the asymptotics, as $t \to \infty$, for $e^{-itH_g} A(\tau, \underline{f})\Phi_{\rm gs}$, where $H_g = H_0 + gW$ is the Hamiltonian generating the dynamics of the interacting system, which is applied to an initial state $A(\tau, \underline{f})\Phi_{\rm gs}$ for $0 \le \tau \le g^{\mu}$ and for $\tau \ge g^{-\frac{1}{K}}$. The initial state is composed of a photon cloud

$$A \equiv A(\underline{f}) \; = \; a^*(f_1) a^*(f_2) \cdots a^*(f_N) \, , \quad \text{with } \; f_1, f_2, \ldots , f_N \, \in \, C_0^\infty(B_\Lambda \setminus \{0\}) \, , \tag{II.1}$$

of N photons and its time evolution

$$A(\tau) \, \Phi_{\rm gs} \; \equiv \; A(\tau, \underline{f})\Phi_{\rm gs} \; := \; e^{-i\tau H_g} \, e^{i\tau H_0} \, A(\underline{f}) \, e^{-i\tau H_0} \, e^{i\tau H_g} \, \Phi_{\rm gs} \tag{II.2}$$

applied to a ground state vector $\Phi_{\rm gs}$,

$$H_g \, \Phi_{\rm gs} \; = \; E_0 \, \Phi_{\rm gs} \, , \tag{II.3}$$

representing the interacting atom at rest. Note that each photon orbital f_j is assumed to be smooth and compactly supported away from zero momentum and in the ball of radius $\Lambda < \infty$ about the origin,

$$\operatorname{supp} f_j \; \subseteq \; B_\Lambda \setminus \{0\} \; := \; \big\{ k \in \mathbb{R}^3 \; \big| \; 0 < \omega(k) < \Lambda \big\} \, . \tag{II.4}$$

We now give a list of assumptions used to derive Theorem II.1, below. Given an operator X on $\mathcal{H}$ and $t \in \mathbb{R}$ we denote its *free* Heisenberg time evolution by

$$X_t \; := \; e^{-itH_0} \, X \, e^{itH_0} \, . \tag{II.5}$$

Next, we recall from (I.27) the estimate

$$\big\| (H_f^{(\Lambda)} + \mathbf{1})^\alpha \, P_c \, \Phi_{\rm gs} \big\| \; + \; \big\| (H_f^{(\Lambda)} + \mathbf{1})^\alpha \, P_\Omega^\perp \, \Phi_{\rm gs} \big\| \; \le \; \mathcal{O}(g) \, , \tag{II.6}$$

for any $\alpha \ge 1$ and any $\Lambda < \infty$, where $H_f^{(\Lambda)}$ is defined in (I.28). Eq. (II.6) is proved in Lemma IV.2, below. Furthermore, we recall from (I.22) that

$$P_d \, W \, P_d \; \equiv \; (P_d \otimes \mathbf{1}) \; W \; (P_d \otimes \mathbf{1}) \; = \; 0 \, . \tag{II.7}$$

In Lemma IV.3, we demonstrate that, for any $s \in \mathbb{R}$,

$$\big\| \, [W \, , \, A_s] \, (H_f^{(\Lambda)} + \mathbf{1})^{-1-(N/2)} \big\| \; \le \; \mathcal{O}\big((1 + |s|)^{-K} \big) \, , \tag{II.8}$$

where $K > 1$ is the degree of differentiability in Hypothesis 1. Since $\Phi_{\rm gs} \in \mathrm{dom}[(H_f + \mathbf{1})^{1+(N/2)}]$, for any $N \in \mathbb{N}_0$, the estimate (II.8) and the fact that H_f and P_d commute imply the norm-convergence of

$$\Phi_\infty\big(A(\underline{f}), \tau\big) \; := \; \lim_{T \to \infty} \Phi_T\big(A(\underline{f}), \tau\big) \, , \tag{II.9}$$

where

$$\Phi_T(X,\tau) \;:=\; \begin{cases} \displaystyle\int_0^T ds\, e^{is(H_0-E_0)}\,[W\,,\,X_s]\,P_d\,\Phi_{\rm gs} & \text{if} \quad 0\le\tau\le g^\mu \\ \displaystyle\int_{-T}^{T} ds\, e^{is(H_0-E_0)}\,[W\,,\,X_s]\,P_d\,\Phi_{\rm gs} & \text{if} \quad \tau\ge g^{-\frac{1}{K}} \end{cases} \tag{II.10}$$

provided $K>1$. Furthermore, we show in Lemma IV.4 that, for any $r\in\mathbb{R}$ and $s\in\mathbb{R}_0^+$,

$$\big\|\, W\, e^{-irH_0}\, P_c\,[W\,,\,A_s]\,(H_f+\mathbf{1})^{-2-(N/2)}\big\| \;\le\; \mathcal{O}\big((1+|r|)^{-3/2}\big)\,. \tag{II.11}$$

Under these assumptions, we prove the following Theorem.

Theorem II.1. *Assume Hypotheses 1–2 and (II.1)–(II.11). Then there exists a constant $C<\infty$ such that, for all $t\ge g^{-1/K}$,*

$$\big\|e^{-it(H_g-E_0)}\,A(\tau)\,\Phi_{\rm gs} \;-\; A_t\,\Phi_{\rm gs} \;+\; ige^{-it(H_0-E_0)}\,\Phi_\infty(A,\tau)\big\| \;\le\; C\,g^{1+\mu(K)}\,, \tag{II.12}$$

provided that either $0\le\tau\le g^\mu$ or $\tau\ge g^{-\frac{1}{K}}$, and where $\mu(K):=1-K^{-1}>0$ and $K>1$ is the degree of differentiability in Hypothesis 1.

Proof. First we compute the Heisenberg time evolution of the photon cloud $A(\tau)$ on $\mathrm{Ran}[(H_g-i)^{-1-N}]$,

$$\begin{aligned} e^{-itH_g}\,A(\tau)\,e^{itH_g} &= e^{-i(t+\tau)H_g}\,e^{i(t+\tau)H_0}\,A_t\,e^{-i(t+\tau)H_0}\,e^{i(t+\tau)H_g} \\ &= A_t \;+\; \Big[e^{-isH_g}\,e^{isH_0}\,A_t\,e^{-isH_0}\,e^{isH_g}\Big]_{s=0}^{s=t+\tau} \\ &= A_t \;-\; ig\int_0^{t+\tau} ds\; e^{-isH_g}\,\big[W\,,\,A_{t-s}\big]\,e^{isH_g} \\ &= A_t \;-\; ig\int_{-\tau}^{t} ds\; e^{-i(t-s)H_g}\,\big[W\,,\,A_s\big]\,e^{i(t-s)H_g}\,, \end{aligned} \tag{II.13}$$

using $A=A(0)=A(0,\underline{f})$ and a change of integration variables, $s\mapsto t-s$. Applying (II.13) to the ground state $\Phi_{\rm gs}$, we obtain

$$\begin{aligned} -ig\,\Psi(t,\tau) &:= e^{-it(H_g-E_0)}\,A(\tau)\,\Phi_{\rm gs} \;-\; A_t\,\Phi_{\rm gs} \\ &= -ig\int_{-\tau}^{t} ds\, e^{-i(t-s)(H_g-E_0)}\,[W\,,\,A_s]\,\Phi_{\rm gs}\,, \end{aligned} \tag{II.14}$$

and our goal is to show that

$$\big\|\Psi(t,\tau) \;-\; e^{-it(H_0-E_0)}\,\Phi_\infty(A(\tau),\tau))\big\| \;\le\; C\,g^\mu\,. \tag{II.15}$$

To this end, we first prove (II.15) for the special case $\tau = 0$. Given $T \geq 1$ and $K > 1$, and assuming that $t \geq T$, we observe that

$$\big\| \Phi_\infty(A,0) \ - \ \Phi_T(A,0) \big\| \ \leq \ \mathcal{O}\big(T^{1-K}\big) \, , \tag{II.16}$$

$$\Big\| \Psi(t,0) \ - \ \int_0^T ds \, e^{i(t-s)(H_g - E_0)} \, [W \, , \, A_s] \, \Phi_{\rm gs} \Big\| \ \leq \ \mathcal{O}\big(T^{1-K}\big) \, . \tag{II.17}$$

Next, we use that H_0 and A commute with P_d and that $P_d \, W \, P_d = 0$ which imply

$$[W \, , \, A_s] \, P_d \ = \ P_c \, [W \, , \, A_s] \, P_d \, . \tag{II.18}$$

Hence we have that

$$\begin{aligned} & \big\| e^{-itH_g} \, \Phi_T(A,0) \ - \ e^{-itH_0} \, \Phi_T(A,0) \big\| \qquad \qquad (\text{II.19}) \\ & = \ g \Big\| \int_0^t dr \, e^{-irH_g} \, W \, e^{-i(t-r)H_0} \, \Phi_T(A,0) \Big\| \\ & = \ g \Big\| \int_0^t dr \int_0^T ds \, e^{-isE_0} \, e^{-irH_g} \, W \, e^{-i(t-r-s)H_0} \, P_c [W \, , \, A_s] \, P_d \, \Phi_{\rm gs} \Big\| \\ & \leq \ g \big\| (H_f + \mathbf{1})^{2+(N/2)} \Phi_{\rm gs} \big\| \\ & \qquad \int_0^t dr \int_0^T ds \, \big\| W \, e^{-i(t-r-s)H_0} \, P_c [W \, , \, A_s] \, (H_f + \mathbf{1})^{-2-(N/2)} \big\| \, , \end{aligned}$$

and (II.11) and a change of variable $r \mapsto t - r$ now imply that

$$\begin{aligned} \big\| e^{-itH_g} \, \Phi_T(A,0) \ - \ e^{-itH_0} \, \Phi_T(A,0) \big\| \ & \leq \ C_1 \, g \int_0^T ds \int_0^t \frac{dr}{(1 + |r-s|)^{3/2}} \\ & \leq \ C_1 \, C_2 \, g \, T \, , \end{aligned} \tag{II.20}$$

where $C_1 < \infty$ and $C_2 := \int_{-\infty}^{\infty} (1 + |r|)^{-3/2} dr < \infty$. Applying the estimates (II.16)–(II.20) to the left side of (II.15), we hence obtain that

$$\begin{aligned} & \big\| \Psi(t,0) \ - \ e^{-it(H_0 - E_0)} \, \Phi_\infty(A,0) \big\| \qquad \qquad (\text{II.21}) \\ & \leq \ \Big\| \Psi(t,0) \ - \int_0^T ds \, e^{-i(t-s)(H_g - E_0)} \, [W \, , \, A_s] \, \Phi_{\rm gs} \Big\| \\ & \quad + \Big\| \int_0^T ds \, e^{-i(t-s)(H_g - E_0)} \, [W \, , \, A_s] \, \Phi_{\rm gs} \ - \ e^{-it(H_g - E_0)} \Phi_T(A,0) \Big\| \\ & \quad + \big\| \big(e^{-it(H_g - E_0)} \ - \ e^{-it(H_0 - E_0)} \big) \Phi_T(A,0) \big\| \ + \ \big\| \Phi_T(A,0) \ - \ \Phi_\infty(A,0) \big\| \\ & \leq \ \Big\| \int_0^T ds \, e^{is(H_g - E_0)} \, [W \, , \, A_s] \, \Phi_{\rm gs} \ - \ \Phi_T(A,0) \Big\| \ + \ \mathcal{O}\big(T^{1-K} + gT\big) \\ & \leq \ \Big\| \int_0^T ds \, \big(e^{is(H_g - E_0)} - e^{is(H_0 - E_0)} \big) \, [W \, , \, A_s] \, P_d \, \Phi_{\rm gs} \Big\| \\ & \quad + \Big\| \int_0^T ds \, e^{is(H_g - E_0)} \, [W \, , \, A_s] \, P_c \, \Phi_{\rm gs} \Big\| \ + \ \mathcal{O}\big(T^{1-K} + gT\big) \, , \end{aligned}$$

using that $P_d + P_c = \mathbf{1}$. It remains to bound the two integrals in the last line of (II.21). The second of these is bounded by

$$\begin{aligned} & \Big\| \int_0^T ds\, e^{is(H_g - E_0)}\, [W\,,\, A_s]\, P_c\, \Phi_{\rm gs} \Big\| \qquad\qquad \text{(II.22)} \\ & \leq \; \big\| (H_f^{(\Lambda)} + \mathbf{1})^{1+(N/2)}\, P_c\, \Phi_{\rm gs} \big\| \int_0^T ds \big\|\, [W\,,\, A_s]\, (H_f^{(\Lambda)} + \mathbf{1})^{-1-(N/2)} \big\| \\ & \leq \; \mathcal{O}(g) \;\leq\; \mathcal{O}(gT)\,, \end{aligned}$$

due to (II.6) and (II.8), since $K > 1$ and $T \geq 1$. To estimate the first integral, we use again the DuHamel formula as in (II.19). This yields

$$\begin{aligned} & \Big\| \int_0^T ds\, \big(e^{is(H_g - E_0)} - e^{is(H_0 - E_0)} \big)\, [W\,,\, A_s]\, P_d\, \Phi_{\rm gs} \Big\| \qquad\qquad \text{(II.23)} \\ & \leq \; g \Big\| \int_0^T ds\, e^{-isE_0} \int_0^s dr\, e^{irH_g}\, W\, e^{i(s-r)H_0}\, P_c [W\,,\, A_s]\, P_d\, \Phi_{\rm gs} \Big\| \\ & \leq \; g \big\| (H_f + \mathbf{1})^{2+(N/2)} \Phi_{\rm gs} \big\| \\ & \qquad \int_0^T ds \int_0^s dr\, \big\| e^{i(s-r)H_0}\, P_c [W\,,\, A_s]\, (H_f + \mathbf{1})^{-2-(N/2)} \big\| \\ & \leq \; C_1\, g \int_0^T ds \int_0^s \frac{dr}{(1+|r|)^{3/2}} \;\leq\; \mathcal{O}\big(gT\big)\,, \end{aligned}$$

changing again the integration variable $r \mapsto s - r$. Inserting (II.22) and (II.23) into (II.21), we arrive at

$$\big\| \Psi(t, 0) \;-\; e^{-it(H_0 - E_0)}\, \Phi_\infty(A, 0) \big\| \;\leq\; \mathcal{O}\big(T^{1-K} + gT \big)\,, \qquad \text{(II.24)}$$

which yields (II.15) upon choosing $T := g^{\mu(K)-1}$ and setting $\mu(K) := 1 - K^{-1} > 0$, for the special case $\tau = 0$.

Note that if more generally $0 \leq \tau < g^\mu$ then $\Phi_\infty(A, \tau) = \Phi_\infty(A, 0)$ and hence

$$\| A(\tau)\Phi_{\rm gs} \;-\; A\Phi_{\rm gs} \| \;=\; g \| \int_0^\tau ds\, e^{-is(H_g - E_0)} [W, A_{-s}] \Phi_{\rm gs} \|\,. \qquad \text{(II.25)}$$

Additionally using (II.8), we hence conclude $\| A(\tau)\Phi_{\rm gs} - A\Phi_{\rm gs} \| \leq \mathcal{O}(g^{1+\mu})$, proving the asymptotic expansion (II.12) in the case $\tau \in [0, g^\mu]$.

In order to deal with the case $\tau \in [g^{-1/K}, \infty)$, we observe that an application of (II.25), for $t \geq g^{-\frac{1}{K}}$, in combination with the decay property (II.8) establishes

$$\| e^{-itH_g} A(t) \Phi_{\rm gs} \;-\; e^{-itH_g} A(\tau) \Phi_{\rm gs} \| \;\leq\; \mathcal{O}(g^{1+\mu})\,, \qquad \text{(II.26)}$$

and we may replace the time-evolved state $e^{itH_g}A(\tau)\Phi_{\mathrm{gs}}$ by $e^{-itH_g}A(t)\Phi_{\mathrm{gs}}$ in this case. Using $\tau = t$ in Eq. (II.13), we furthermore obtain

$$e^{-it(H_g-E_0)}A(t)\Phi_{\mathrm{gs}} \;=\; A_t\Phi_{\mathrm{gs}} \;-\; ig\int_{-t}^{t} ds\, e^{-i(t-s)H_g}\,[W, A_s]\, e^{i(t-s)H_g}\Phi_{\mathrm{gs}}\,. \tag{II.27}$$

Using

$$\Psi(t,t) \;=\; \int_{-t}^{t} ds\, e^{-i(t-s)(H_g-E_0)}\,[W, A_s]\,\Phi_{\mathrm{gs}}\,, \tag{II.28}$$

computations running along the lines of (II.16)–(II.23) substituting $\Phi_\infty(A,\infty)$, $\Phi_T(A,\infty)$, $\Psi(t,\infty)$ for $\Phi_\infty(A,0)$, $\Phi_T(A,0)$, $\Psi(t,0)$ and replacing integrals of the form $\int\limits_0^T ds$ by those of the form $\int\limits_{-T}^{T} ds$ proves (II.12) for $\tau \geq g^{-\frac{1}{K}}$, as well. □

Theorem II.1 is the main tool for the analysis of the charges transported to infinity, $Q_{\mathcal{T}}^{\mathrm{inf}}(A(\tau)\Phi_{\mathrm{gs}})$ and $Q_{\mathcal{T}}^{\mathrm{sup}}(A(\tau)\Phi_{\mathrm{gs}})$, because it allows us to take the limits $t\to\infty$ and $R\to\infty$, as shown in the following lemma.

Lemma II.2. *Assume Hypotheses 1–2, and let $A = a^*(f_1)\, a^*(f_2)\cdots a^*(f_N)$ with $f_1, ..., f_N \in C_0^\infty(\mathbb{R}^3\setminus\{0\})$. Then there exists a constant $C<\infty$, such that the charges transported to infinity obey*

$$Q_{\mathcal{T}}^{\mathrm{sup}}(A(\tau)\Phi_{\mathrm{gs}}) \;-\; Cg^{2+\mu} \;\leq\; g^2\big\|T_{\mathcal{T}}\,\Phi_\infty(A,\tau)\big\|^2 \;\leq\; Q_{\mathcal{T}}^{\mathrm{inf}}(A(\tau)\Phi_{\mathrm{gs}}) \;+\; Cg^{2+\mu}\,, \tag{II.29}$$

for $0\leq\tau\leq g^\mu$ and $\tau \geq g^{-\frac{1}{K}}$, where $\mu(K) := 1 - K^{-1} > 0$ and $K>1$ is the degree of differentiability in Hypothesis 1.

Proof: First we observe that, since $F_R \equiv F_R\otimes \mathbf{1}_f$ commutes with $A_t \equiv \mathbf{1}_{el}\otimes A_t$ and $H_f = \mathbf{1}_{el}\otimes H_f$, we have

$$\begin{aligned}\big\|T_{\mathcal{T}}\,F_R\,A_t\,\Phi_{\mathrm{gs}}\big\| \;&\leq\; \big\|F_R\,A_t\,\Phi_{\mathrm{gs}}\big\| \;=\; \big\|A_t\,F_R\,\Phi_{\mathrm{gs}}\big\| \\ &\leq\; \big\|A_t\,(H_f+1)^{-N/2}\big\|\cdot\big\|F_R\,(H_f+1)^{N/2}\,\Phi_{\mathrm{gs}}\big\|\,.\end{aligned} \tag{II.30}$$

Applying Lemma IV.1 N-times, we observe that $\big\|A_t\,(H_f+1)^{-N/2}\big\|$ is bounded by $C_2<\infty$, uniformly in t,

$$\big\|A_t\,(H_f+1)^{-N/2}\big\| \;\leq\; \prod_{j=1}^{N}\Big\|(H_f+1)^{(j-1)/2}\,a^*(e^{-it\omega}f_j)\,(H_f+1)^{-j/2}\Big\| \;\leq\; C_2\,. \tag{II.31}$$

Moreover, $(H_f+1)^{N/2}\Phi_{\rm gs}\in\mathcal{H}$ and $F_R\to 0$ strongly, as $R\to\infty$, hence

$$\lim_{R\to\infty}\lim_{t\to\infty}\big\|T_{\mathcal{T}}\,F_R\,A_t\,\Phi_{\rm gs}\big\| \;=\; 0\,. \tag{II.32}$$

According to Theorem II.1 we have

$$e^{-it(H_g-E_0)}\,A(\tau)\,\Phi_{\rm gs} \;=\; A_t\,\Phi_{\rm gs} \;-\; ig\,e^{-it(H_0-E_0)}\,\Phi_\infty(A,\tau) \;+\; {\rm Rem}(t)\,, \tag{II.33}$$

where $\|{\rm Rem}(t)\|\leq C_1 g^{1+\mu}$, for some constant $C_1<\infty$ independent of $t\in\mathbb{R}^+$ and $\mu=1-1/K$. Inserting Eqs. (II.33) and (II.32) into the definitions (I.31)–(I.32) of the transported charges, we find that

$$Q_{\mathcal{T}}^{\sup}(A(\tau)\Phi_{\rm gs}) \;-\; C_1 g^{2+\mu} \;\leq\; g^2\widetilde{Q}_{\mathcal{T}}(A(\tau)\Phi_{\rm gs}) \;\leq\; Q_{\mathcal{T}}^{\inf}(A(\tau)\Phi_{\rm gs}) \;+\; C_1 g^{2+\mu}\,, \tag{II.34}$$

where

$$\widetilde{Q}_{\mathcal{T}}(A(\tau)\Phi_{\rm gs}) \;:=\; \lim_{R\to\infty}\lim_{t\to\infty}\big\|T_{\mathcal{T}}F_R e^{-it(H_0-E_0)}\Phi_\infty(A,\tau))\big\|^2\,, \tag{II.35}$$

and estimate (II.29) would follow from $\widetilde{Q}_{\mathcal{T}}(A(\tau)\Phi_{\rm gs})=\big\|T_{\mathcal{T}}\Phi_\infty(A,\tau)\big\|^2$.

Next, we turn to $T_{\mathcal{T}}F_R e^{-it(H_0-E_0)}\Phi_\infty(A,\tau)$. We note that $F_R=F_R\,P_c$ and hence

$$F_R\,e^{-it(H_0-E_0)} \;=\; e^{-it(H_f-E_0)}\,F_R\,P_c\,e^{-itH_{el}} \;=\; e^{-it(H_f-E_0)}\,F_R\,e^{-it(-\Delta)}\,P_c\,. \tag{II.36}$$

Furthermore, the absolute continuity of the spectrum of $-\Delta$ on $L^2(\mathbb{R}^3)$ implies that

$$(1-F_R)e^{-it(-\Delta)} \;=\; \mathbf{1}_{\{|x|<R\}}e^{-it(-\Delta)}\to 0 \tag{II.37}$$

strongly, as $t\to\infty$, for any $R<\infty$. Using (II.36), (II.37), and the fact that the operators $T_{\mathcal{T}}=T_{\mathcal{T}}P_c$, $e^{-it(-\Delta)}$, and $e^{-it(H_f-E_0)}$ commute, we thus obtain

$$\begin{aligned}&\lim_{R\to\infty}\lim_{t\to\infty}\big\|T_{\mathcal{T}}\,F_R\,e^{-it(H_0-E_0)}\,\Phi_\infty(A,\tau)\big\| = \\ &\quad= \lim_{R\to\infty}\lim_{t\to\infty}\big\|T_{\mathcal{T}}\,F_R\,e^{-it(-\Delta)}\,P_c\,\Phi_\infty(A,\tau)\big\| = \big\|T_{\mathcal{T}}\,\Phi_\infty(A,\tau)\big\|\,,\end{aligned} \tag{II.38}$$

which implies the claim. □

Next, we recall from Eq. (II.9) the definition of

$$\Phi_\infty(A,\tau) \;=\; \begin{cases} \displaystyle\int_0^\infty ds\,e^{is(H_0-E_0)}\,[W\,,\,A_s]\,P_d\,\Phi_{\rm gs} & \text{if } 0\leq\tau\leq g^\mu \\ \displaystyle\int_{-\infty}^\infty ds\,e^{is(H_0-E_0)}\,[W\,,\,A_s]\,P_d\,\Phi_{\rm gs} & \text{if } \tau\geq g^{-\frac{1}{K}} \end{cases} \tag{II.39}$$

which we complement by the definition of the vector

$$\Phi_\infty^\Omega(A,\tau) \;:=\; \begin{cases} \displaystyle\int_0^\infty ds\, e^{is(H_0-E_0)}\,[W\,,\,A_s]\,P_d\,(0\oplus\Omega) & \text{if } 0\le\tau\le g^\mu \\ \displaystyle\int_{-\infty}^\infty ds\, e^{is(H_0-E_0)}\,[W\,,\,A_s]\,P_d\,(0\oplus\Omega) & \text{if } \tau\ge g^{-\frac{1}{K}} \end{cases} \tag{II.40}$$

whose existence is guaranteed thanks to Lemma IV.3. Lemma II.3 below shows that these two vectors differ by at most $\mathcal{O}(g)$, i.e., a replacement of the ground state $\Phi_{\rm gs}$ of H_g by the ground state $0\oplus\Omega$ of H_0 in the definition of $\Phi_\infty(A,\tau)$ introduces only small errors which are negligible, as we shall see.

Lemma II.3. *Assume Hypotheses 1–2, and let* $A = a^*(f_1)\,a^*(f_2)\cdots a^*(f_N)$ *with* $f_1,...,f_N \in C_0^\infty(\mathbb{R}^3\setminus\{0\})$*. Then there exists a constant* $C<\infty$*, such that*

$$\big\|\Phi_\infty(A,\tau)\;-\;\Phi_\infty^\Omega(A,\tau)\big\|\;\le\; C\,g\,. \tag{II.41}$$

Proof: We first note that due to Lemma IV.3, we have the estimate

$$\begin{aligned} &\big\|\Phi_\infty(A,\tau)\;-\;\Phi_\infty^\Omega(A,\tau)\big\| \le \\ &\le\;\Big\|(H_f^{(\Lambda)}+1)^{1+\frac{N}{2}}\big(P_d\Phi_{\rm gs}\;-\;(0\oplus\Omega)\big)\Big\|\cdot\int_{\mathbb{R}} ds\big\|\,[W\,,\,A_s]\,(H_f^{(\Lambda)}+1)^{-1-\frac{N}{2}}\big\| \\ &\le\; C_1\,\Big\|(H_f^{(\Lambda)}+1)^{1+\frac{N}{2}}\big(P_d\Phi_{\rm gs}\;-\;(0\oplus\Omega)\big)\Big\|\,, \end{aligned} \tag{II.42}$$

for some constant $C_1<\infty$. Additionally using Lemma IV.2 and the decomposition $P_d\Phi_{\rm gs}-(0\oplus\Omega)=\big[P_dP_\Omega\Phi_{\rm gs}-(0\oplus\Omega)\big]+P_dP_\Omega^\perp\Phi_{\rm gs}$, we observe that

$$\Big\|(H_f^{(\Lambda)}+1)^{1+N/2}\big(P_d\Phi_{\rm gs}\;-\;(0\oplus\Omega)\big)\Big\|\;\le\;\mathcal{O}(g)\,. \qquad\square \tag{II.43}$$

The final ingredient for the proof of Theorem I.1 is Lemma II.4, below. Note that due to the definitions (II.40) we may restrict ourselves to the case $\tau=0$ and $\tau=\infty$.

Lemma II.4. *Assume Hypotheses 1–2, fix* $N,m\in\mathbb{N}$ *with* $m\le N$*, and let* $\varphi_1,\varphi_2,\dots\varphi_m\in C_0^\infty(\mathbb{R}^3\setminus\{0\})$ *be an orthonormal system,* $\langle\varphi_i|\varphi_j\rangle=\delta_{i,j}$*. Let* $A=a^*(\varphi_1)^{n_1}a^*(\varphi_2)^{n_2}\cdots a^*(\varphi_m)^{n_m}$*, where* $n_j\in\mathbb{N}$ *are such that* $n_1+n_2+\ldots+n_m=N$*, and fix a measurable set* $\mathcal{T}\subseteq\mathbb{R}^3$*. Then*

$$\begin{aligned} \big\|T_{\mathcal{T}}\,\Phi_\infty^\Omega(A,\tau)\big\|^2 \;&=\; Q_{\mathcal{T}}^\tau(\underline{\varphi}) = \\ &=\;\big(n_1!\,n_2!\cdots n_m!\big)\sum_{j=1}^m n_j\int_{\mathcal{T}}|\langle L_\tau^p\,,\,\varphi_j\rangle|^2 d^3p\,, \end{aligned} \tag{II.44}$$

with $L^p_\tau = L^p_0$*, for* $\tau \in [0, g^\mu]$*, and* $L^p_\tau = L^p_\infty$*, for* $\tau \in [g^{-1/K}, \infty)$*, where* L^p_0 *and* L^p_∞ *are defined in Eqs. (I.38)–(I.40).*

Proof: We first write $W = a^*(G) + a(G)$ and use the canonical commutation relations to obtain that

$$\begin{aligned} [W, A_s] &= \big[a^*(G) + a(G)\,,\ a^*(e^{-is\omega}\varphi_1)^{n_1} a^*(e^{-is\omega}\varphi_2)^{n_2} \cdots a^*(e^{-is\omega}\varphi_m)^{n_m}\big] \\ &= \big[a(G)\,,\ a^*(e^{-is\omega}\varphi_1)^{n_1} a^*(e^{-is\omega}\varphi_2)^{n_2} \cdots a^*(e^{-is\omega}\varphi_m)^{n_m}\big] \\ &= \sum_{j=1}^m n_j \left\{ \big\langle G \big| e^{-is\omega}\varphi_j \big\rangle \otimes \prod_{i=1}^m a^*(e^{-is\omega}\varphi_i)^{n_i - \delta_{i,j}} \right\}, \end{aligned} \tag{II.45}$$

on $\mathrm{dom}[H_f^{1+N/2}]$, where $\langle G | e^{-is\omega}\varphi_j \rangle \in \mathcal{B}(\mathcal{H}_{el})$ is a bounded operator acting on the electron variables given by

$$\big\langle G \big| e^{-is\omega}\varphi_j \big\rangle \ := \int d^3k\, e^{-is\omega(k)}\, \varphi_j(k)\, G^*(k)\,. \tag{II.46}$$

Moreover, for any $s \geq 0$,

$$\begin{aligned} & e^{is(H_0 - E_0)} \big\langle G \big| e^{-is\omega}\varphi_j \big\rangle \otimes \prod_{i=1}^m a^*(e^{-is\omega}\varphi_i)^{n_i - \delta_{i,j}}\ (0 \oplus \Omega) \\ &= e^{is(H_{el} - E_0)} \big\langle G \big| e^{-is\omega}\varphi_j \big\rangle \otimes e^{isH_f} \prod_{i=1}^m a^*(e^{-is\omega}\varphi_i)^{n_i - \delta_{i,j}}\ (0 \oplus \Omega) \\ &= e^{is(-\Delta - E_0)} \big\langle P_c\, G\, P_d \big| e^{-is\omega}\varphi_j \big\rangle \otimes \prod_{i=1}^m a^*(\varphi_i)^{n_i - \delta_{i,j}}\ (0 \oplus \Omega)\,. \\ &= \big(\psi_{j,s} \oplus 0\big) \otimes \prod_{i=1}^m a^*(\varphi_i)^{n_i - \delta_{i,j}}\, \Omega\,, \end{aligned} \tag{II.47}$$

where the Fourier transform of $\psi_j \in L^2(\mathbb{R}^3)$ with respect to the particle coordinate x is given by

$$\hat{\psi}_{j,s}(p) \ = \int d^3k\, e^{is(p^2 - E_0 - \omega(k))}\, \hat{\rho}(p,k)\, \varphi_j(k)\,. \tag{II.48}$$

Here, we used the fact that $P_d\, G(k)^* P_d = 0$ and hence $G(k)^* P_d = P_c\, G(k)^* P_d$, for all $k \in \mathbb{R}^3$. Moreover, $P_c\, G(k)^* P_d(\tilde{\psi} \oplus z) = z p_\downarrow(k)^* \oplus 0$, and this yields (II.48). Eqs. (II.45)–(II.48) imply that

$$\Phi^\Omega_\infty(A, \tau) \ = \ \sum_{j=1}^m n_j \left\{ \int_{-\tau}^\infty ds\, \big(\psi_{j,s} \oplus 0\big) \right\} \otimes \prod_{i=1}^m a^*(\varphi_i)^{n_i - \delta_{i,j}}\, \Omega\,, \tag{II.49}$$

Passing to the momentum representation for the particle variable and using that

$$\Big\langle \prod_{i=1}^{m} a^*(\varphi_i)^{n_i-\delta_{i,j}}\,\Omega \,\Big|\, \prod_{i=1}^{m} a^*(\varphi_i)^{n_i-\delta_{i,\ell}}\,\Omega \Big\rangle \;=\; \delta_{j,\ell}\cdot \prod_{i=1}^{m}(n_i-\delta_{i,j})!\,, \qquad \text{(II.50)}$$

we hence obtain

$$\begin{aligned} \big\| T_{\mathcal{T}}\,\Phi_\infty^\Omega(A,\tau)\big\|^2 \;&=\; \Big\| \sum_{j=1}^{m} n_j \Big\{ \int_{-\tau}^{\infty} ds\, \big(T_{\mathcal{T}}\,\psi_{j,s}\oplus 0\big)\Big\} \otimes \prod_{i=1}^{m} a^*(\varphi_i)^{n_i-\delta_{i,j}}\Omega \Big\|^2 \\ &=\; n_1! n_2! \cdots n_m! \sum_{j=1}^{m} n_j \int_{\mathcal{T}} d^3p \Big| \int_{-\tau}^{\infty} ds\, \hat\psi_{j,s}(p)\Big|^2 . \qquad \text{(II.51)} \end{aligned}$$

It remains to evaluate the integral $\int_{-\tau}^{\infty} ds\, \hat\psi_{j,s}(p)$ in (II.51), which is an exercise in distribution theory. To this end, we note that Hypothesis 3 implies that $\hat\rho(p,\,\cdot\,)\varphi_j$ is two times partially differentiable with respect to k, and that its derivatives of order ≤ 2 have bounded support away from $k=0$. Since the phase in $\hat\psi_{j,s}(p)$ is non-stationary, away from $k=0$, two times integration by parts back and forth yields

$$\begin{aligned} \int_1^\infty ds\, \hat\psi_{j,s}(p) \;&=\; \int_1^\infty ds \int d^3k\, e^{is(p^2-E_0-\omega(k))}\, \hat\rho(p,k)\,\varphi_j(k) \\ =\;& \int_1^\infty \frac{ds}{s^2} \int d^3k\, e^{is(p^2-E_0-\omega(k))} \Big(i\nabla\cdot \frac{k}{|k|}\Big)^2 \Big[\hat\rho(p,k)\,\varphi_j(k)\Big] \\ =\;& \lim_{\varepsilon\to 0} \int_1^\infty \frac{ds}{s^2} \int d^3k\, e^{is(p^2-E_0-\omega(k)+i\varepsilon)} \Big(i\nabla\cdot \frac{k}{|k|}\Big)^2 \Big[\hat\rho(p,k)\,\varphi_j(k)\Big] \\ =\;& \lim_{\varepsilon\to 0} \int d^3k \int_1^\infty ds\, e^{is(p^2-E_0-\omega(k)+i\varepsilon)}\, \hat\rho(p,k)\,\varphi_j(k) \\ =\;& \lim_{\varepsilon\to 0} \int d^3k\, \frac{-i\, e^{i(p^2-E_0-\omega(k))}\,\hat\rho(p,k)}{p^2-E_0-\omega(k)+i\varepsilon}\,\varphi_j(k)\,. \qquad \text{(II.52)} \end{aligned}$$

Thus in the case $\tau=0$:

$$\begin{aligned} \int_0^\infty ds\, \hat\psi_{j,s}(p) \;&=\; \int_0^1 ds\, \hat\psi_{j,s}(p) \;+\; \int_1^\infty ds\, \hat\psi_{j,s}(p) \qquad \text{(II.53)} \\ &=\; \lim_{\varepsilon\to 0} \int d^3k\, \frac{-i\,\hat\rho(p,k)}{p^2-E_0-\omega(k)+i\varepsilon}\,\varphi_j(k) = \big\langle L_0^p\,,\,\varphi_j\big\rangle\,. \end{aligned}$$

For the case $\tau = \infty$, we get

$$\begin{aligned}
\lim_{t\to\infty}\int_{-t}^{t} ds\, \hat\psi_{j,s}(p) &= \lim_{t\to\infty}\int_{-t}^{t} ds \int d^3k\, e^{is(p^2-E_0-\omega(k))}\,\hat\rho(p,k)\,\varphi_j(k) = \quad \text{(II.54)}\\
&= \lim_{t\to\infty}\int d^3k\, \frac{e^{it(p^2-E_0-\omega(k))}-e^{it(p^2-E_0-\omega(k))}}{i(p^2-E_0-\omega(k))}\,\hat\rho(p,k)\,\varphi_j(k)\\
&= \lim_{t\to\infty} 2\int d^3k\, \frac{\sin(t(p^2-E_0-\omega(k)))}{p^2-E_0-\omega(k)}\,\hat\rho(p,k)\,\varphi_j(k) =\\
&= 2\pi\int_{\mathbb{S}^2} d^2\Omega (p^2-E_0)^2\hat\rho(p;p^2-E_0,\Omega)\varphi_j(p^2-E_0,\Omega).\square
\end{aligned}$$

Proof of Theorem I.1: The proof of Theorem I.1 consists of the following chain of estimates,

$$\begin{aligned}
Q_{\mathcal{T}}^{\inf}(A(\tau)\Phi_{\rm gs}) \,+\, \mathcal{O}(g^{2+\mu}) &= Q_{\mathcal{T}}^{\sup}(A(\tau)\Phi_{\rm gs}) \,+\, \mathcal{O}(g^{2+\mu})\\
&= g^2\big\|T_{\mathcal{T}}\,\Phi_\infty(A,\tau)\big\|^2 \quad \text{(II.55)}\\
&= g^2\big\|T_{\mathcal{T}}\,\Phi_\infty^\Omega(A,\tau)\big\|^2 \,+\, \mathcal{O}(g^{2+\mu}) \quad \text{(II.56)}\\
&= Q_{\mathcal{T}}^{\tau}(\underline{\varphi}) \,+\, \mathcal{O}(g^{2+\mu})\,, \quad \text{(II.57)}
\end{aligned}$$

where Lemma II.2, II.3, and II.4 justify Eq. (II.55), (II.56), and (II.57), respectively. $\square$

III The Limit of Monochromatic Light

In this section we give a proof of Theorem I.2. We remark, that the functions

$$\varphi_{j,\delta}(r,\Omega_k) \;:=\; \omega_j^{-1}\,\delta^{-1/2}\,\chi_j\Big(\frac{r-\omega_j}{\delta}\Big)\,\kappa_j(\Omega_k) \tag{III.1}$$

defined in (I.45) form an orthonormal system under conditions (I.45)–(I.47) only in the limit $\delta \to 0$. For sufficiently small $\delta > 0$, these vectors are merely orthogonal. Hence in the previous calculations we have to replace (II.50) by

$$\begin{aligned}\Big\langle \prod_{i=1}^{m} a^*(\varphi_{i,\delta})^{n_i-\delta_{i,j}}\,\Omega \,\Big|\, \prod_{i=1}^{m} a^*(\varphi_{i,\delta})^{n_i-\delta_{i,\ell}}\,\Omega \Big\rangle & \qquad\qquad \text{(III.2)}\\ =\; \delta_{j,\ell}\cdot\prod_{i=1}^{m}(n_i-\delta_{i,j})!\,\|\varphi_i\|^{2(n_i-\delta_{i,j})}\,, &\end{aligned}$$

and we are interested in the limit $\delta \to 0$ of the quantity

$$\frac{Q^0_{\mathcal{T}}(A_\delta\Phi_{\rm gs})}{\|\varphi_{1,\delta}\|^{2n_1}\,n_1!\,\cdots\,\|\varphi_{m,\delta}\|^{2n_m}\,n_m!} \;=\; \int_{\mathcal{T}} d^3p \sum_{j=1}^{m}\frac{n_j}{\|\varphi_{j,\delta}\|^2}\,|\langle L_0^p\,,\,\varphi_{j,\delta}\rangle|^2 \tag{III.3}$$

Since the right side of (III.3) is additive in the functions $\varphi_{j,\delta}$, we may consider just a single photon with momentum distribution φ_δ of the form (III.1), centered about ω_0 and get

$$\langle L_0^p\,,\,\varphi_\delta\rangle \;=\; \int_{-\infty}^{\infty} F_{p,\delta}(p^2-E_0-x)\,\frac{dx}{x+i0} \;=\; \int_{-\infty}^{\infty} F_{p,\delta}(x)\,\frac{dx}{p^2-E_0-x+i0}\,, \tag{III.4}$$

where

$$F_{p,\delta}(r) \;=\; -i\int_{\mathbb{S}^2} r^2\,\varphi_\delta(r\sigma)\,\hat\rho(p,r\sigma)\,d^2\sigma \;=\; \delta^{-1/2}\,\chi\Big(\frac{r-\omega_0}{\delta}\Big)\,\Theta(p,r) \tag{III.5}$$

and

$$\Theta(p,r) \;=\; -i\int_{\mathbb{S}^2}\frac{r^2}{\omega_0}\,\hat\rho(p,r\sigma)\,\kappa(\sigma)\,d^2\sigma\,, \tag{III.6}$$

for $r \geq 0$, and $\Theta(p,r) \equiv F_{p,\delta} \equiv 0$ on $\mathbb{R}_-$. We now proceed to computing the limit $\lim\limits_{\delta\to 0^+}\int_{\mathcal{T}} d^3p\,|\langle L_p,\varphi_\delta\rangle|^2$. We write

$$\begin{aligned}\int_{\mathcal{T}} |\langle L_0^p\,,\,\varphi_\delta\rangle|^2\,d^3p \;&=\; \int_{\mathcal{T}}\Big|\int_{-\infty}^{\infty} F_{p,\delta}(x)\,\frac{dx}{p^2-E_0-x+i0}\Big|^2 d^3p \qquad \text{(III.7)}\\ &=\; \int_{\mathcal{T}}\Big|\int_{-\infty}^{\infty}\frac{\delta^{1/2}\,\chi(x)\,\Theta(p,\omega_0+\delta x)dx}{p^2-E_0-\omega_0-\delta x+i0}\Big|^2 d^3p\end{aligned}$$

Here, we substitute (III.4) in the integral (III.5) and perform a change of variables in x. We now change variables to polar coordinates in the p-integral to get

$$\begin{aligned} &\int_{\mathcal{T}} |\langle L_0^p \,,\, \varphi_\delta\rangle|^2 \, d^3p \qquad \text{(III.8)} \\ = \;& \int_{\mathbb{S}^2} d^2\sigma \int_0^\infty dp \, \mathbf{1}_{\mathcal{T}}(p\sigma) p^2 \bigg| \int_{-\infty}^\infty \frac{\delta^{1/2}\, \chi(x)\, \Theta(p\sigma,\, \omega_0 + \delta x) dx}{p^2 - E_0 - \omega_0 - \delta x + i0} \bigg|^2 \\ = \;& \frac{1}{2} \int_{\mathbb{S}^2} d^2\sigma \int_{-(E_0+\omega_0)/\delta}^\infty dy \, \mathbf{1}_{\mathcal{T}}\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma\big) \sqrt{\omega_0 + E_0 + \delta y} \\ & \qquad \bigg| \int_{-\infty}^\infty \chi(x)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta x\big) \, \frac{dx}{y - x + i0} \bigg|^2 . \end{aligned}$$

In the last step, we again change the variable, passing from p to $y := \delta^{-1}(p^2 - E_0 - \omega_0)$. As χ has support in $[-1, 1]$, we observe that the x-integral is non-singular, for $|y| > 1$, and we obtain the estimate

$$\begin{aligned} &\bigg| \int_{-\infty}^\infty \chi(x)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta x\big) \, \frac{dx}{y - x + i0} \bigg| \qquad \text{(III.9)} \\ & \qquad \leq \; \frac{\|\chi\|_{L^1}}{|y| - 1} \sup\Big\{ |\Theta(p, r)| \;\Big|\; p \in \mathbb{R}^3 \,,\; r \in [0 \,,\, \omega_0 + 1] \Big\} , \end{aligned}$$

for all $|y| > 1$. To estimate the integral in (III.9) in case that $|y| < 2$, we use the representation (I.40) of L_0^p and the identity

$$\int_{-\infty}^\infty \frac{f(x) - f(-x)}{2x} \, dx \;=\; - \int_{-\infty}^\infty f'(x) \, \ln|x| \, dx \,, \qquad \text{(III.10)}$$

which holds for all $f \in C_0^1(\mathbb{R})$. This yields

$$\begin{aligned} &\int_{-\infty}^\infty \chi(x)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta x\big) \, \frac{dx}{y - x + i0} \qquad \text{(III.11)} \\ = \;& - \int_{-\infty}^\infty \chi(y - z)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta(y - z)\big) \, \frac{dz}{z + i0} \\ = \;& i\pi\, \chi(y)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta y\big) \\ & - \int_{-\infty}^\infty \bigg\{ \frac{d}{dx} \Big[\chi(x)\, \Theta\big(\sqrt{E_0 + \omega_0 + \delta y}\,\sigma \,,\, \omega_0 + \delta x\big) \Big] \bigg\} \ln|y - x| \, dx \,. \end{aligned}$$

Thus we obtain the following estimate on the magnitude of this integral,

$$\begin{aligned}
&\Big| \int_{-\infty}^{\infty} \chi(x)\, \Theta\big(\sqrt{E_0+\omega_0+\delta y}\,\sigma\,,\ \omega_0+\delta x\big)\, \frac{dx}{y-x+i0} \Big| \\
&\leq\ \pi\, \|\chi\|_{L^\infty}\ \sup\Big\{ |\Theta(p,r)| \ \Big|\ p\in\mathbb{R}^3\,,\ r\in[0\,,\ \omega_0+2] \Big\} \\
&\qquad +\ \sup_{|y|\leq 2} \Big\{ \int_{-\infty}^{\infty} \big(|\chi'(x)| + |\chi(x)| \big)\, \big|\ln|y-x|\big|\, dx \Big\} \\
&\qquad\qquad \cdot \sup\Big\{ |\Theta(p,r)| + |\partial_r \Theta(p,r)| \ \Big|\ p\in\mathbb{R}^3\,,\ r\in[0\,,\ \omega_0+2] \Big\}\,,
\end{aligned} \tag{III.12}$$

for all $|y|<2$. Putting together the estimates (III.9) and (III.12), we conclude that

$$\Big| \int_{-\infty}^{\infty} \chi(x)\, \Theta\big(\sqrt{E_0+\omega_0+\delta y}\,\sigma\,,\ \omega_0+\delta x\big)\, \frac{dx}{y-x+i0} \Big| \ \leq\ \frac{C}{|y|+1}\,, \tag{III.13}$$

for some constant $C<\infty$ depending on χ, Θ, and their derivatives. Hence, we can apply Lebesgue's Dominated Convergence Theorem to (III.8) which gives

$$\lim_{\delta\to 0^+} \int_{\mathcal{T}} \langle L_0^p\,,\ \varphi_\delta\rangle|^2\, d^3p \ =\ \Theta_0 \int_{-\infty}^{\infty} h(y) \Big| \int_{-\infty}^{\infty} \frac{\chi(x)\, dx}{y-x+i0} \Big|^2 dy\,, \tag{III.14}$$

where Θ_0 is defined in (I.56), and

$$h(y) \ =\ \lim_{\delta\to+\infty} \mathbf{1}_{[-(E_0+\omega_0)/\delta\,,\,\infty)}(y)\,. \tag{III.15}$$

Hence, we obtain the claim (I.55),

$$\lim_{\delta\to 0} \int_{\mathcal{T}} |\langle L_0^p\,,\ \varphi_\delta\rangle|^2\, d^3p \ =\ \begin{cases} 0 & \text{if}\quad E_0+\omega_0<0\,, \\ \Theta_0\, I_0 & \text{if}\quad E_0+\omega_0=0\,, \\ \Theta_0\, I_+ & \text{if}\quad E_0+\omega_0>0\,, \end{cases} \tag{III.16}$$

with I_0 and I_+ as defined in (I.57) and (I.58). □

IV Technical Estimates

In this section we derive the estimates (II.6), (II.8), and (II.11), which is the basic input for the asymptotics of the time evolution asserted in Theorem II.1.

Before we turn to these bounds, we derive a preparatory lemma.

Lemma IV.1. *Suppose that* $\alpha \geq 1$, $\Lambda < \infty$, *and let* $G \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H})]$ *with* $C := \int d^3k \big(1 + \omega(k)^{-1}\big) \, \|G(k)\|^2 < \infty$. *Denote* $a^*(G) := \int d^3k \, G(k) \otimes a^*(k)$ *and* $a(G) := \int d^3k \, G(k)^* \otimes a(k)$. *Then*

$$
\begin{aligned}
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a(G) \, (H_f + 1)^{-\alpha-1/2}\big\| & \leq C \,, & \text{(IV.1)}\\
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a^*(G) \, (H_f + 1)^{-\alpha-1/2}\big\| & \leq C \, (2 + 4\Lambda)^\alpha \,, & \text{(IV.2)}\\
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a(G_\Lambda) \, (H_f^{(\Lambda)} + 1)^{-\alpha-1/2}\big\| & \leq C \,, & \text{(IV.3)}\\
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a^*(G_\Lambda) \, (H_f^{(\Lambda)} + 1)^{-\alpha-1/2}\big\| & \leq C \, (2 + 4\Lambda)^\alpha \,, & \text{(IV.4)}
\end{aligned}
$$

where $G_\Lambda(k) := \mathbf{1}_{\{\omega(k)<\Lambda\}} \, G(k)$, *and* $\mathbf{1}_{\{\omega(k)<\Lambda\}}$ *denotes the characteristic function of* $\{k \in \mathbb{R}^3 \mid \omega(k) < \Lambda\}$.

Proof. First, we introduce a more compact notation,

$$
\begin{array}{llllll}
G_1 := G, & & H_1 := H_f, & & \omega_1 := \omega, \\
G_2 := G_\Lambda, & & H_2 := H_f^{(\Lambda)}, & & \omega_2 := \omega_\Lambda,
\end{array}
\qquad \text{(IV.5)}
$$

and observe that Eqs. (IV.1)–(IV.4) are equivalent to

$$
\begin{aligned}
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a(G_j) \, (H_j + 1)^{-\alpha-1/2}\big\| & \leq C \,, & \text{(IV.6)}\\
\big\|(H_f^{(\Lambda)} + 1)^\alpha \, a^*(G_j) \, (H_j + 1)^{-\alpha-1/2}\big\| & \leq C \, (2 + 4\Lambda)^\alpha \,, & \text{(IV.7)}
\end{aligned}
$$

with $j = 1, 2$. We apply the operator on the left side of (IV.6) to a normalized vector $\psi \in \mathcal{H}$ and obtain the desired estimate by means of the pull-through formula, the Cauchy-Schwarz inequality, and $H_f^{(\Lambda)} + 1 \leq H_j + 1 + \omega_j$,

$$
\begin{aligned}
& \big\|(H_f^{(\Lambda)} + 1)^\alpha \, a(G_j) \, (H_j + 1)^{-\alpha-1/2} \, \psi\big\| \qquad \text{(IV.8)}\\
\leq \; & \int d^3k \, \|G_j(k)\| \left\| \frac{H_f^{(\Lambda)} + 1}{H_j + 1 + \omega_j(k)} \right\|^\alpha \left\| a(k) \, (H_j + 1)^{-1/2} \, \psi \right\| \\
\leq \; & \left(\int \frac{\|G_j(k)\|^2 \, d^3k}{\omega_j(k)} \right)^{1/2} \cdot \left\| \frac{H_j}{H_j + 1} \right\|^{1/2} \leq \left(\int \frac{\|G_j(k)\|^2 \, d^3k}{\omega_j(k)} \right)^{1/2} .
\end{aligned}
$$

for any $\psi \in \mathcal{H}$, $\|\psi\| = 1$. To derive (IV.7), we use the canonical commutation relations,

$$
\begin{aligned}
&\Big\|(H_f^{(\Lambda)}+1)^{\alpha}\, a^*(G_j)\,(H_j+1)^{-\alpha-1/2}\,\psi\Big\|^2 \qquad \text{(IV.9)}\\
&= \int d^3k\, d^3\tilde{k}\, \Big\langle \psi \,\Big|\, G_j(k)^* G_j(\tilde{k}) \otimes (H_j+1)^{-\alpha-1/2}\, a(k)\,\big(H_f^{(\Lambda)}+1\big)^{2\alpha}\, a^*(\tilde{k}) \\
&\qquad\qquad (H_j+1)^{-\alpha-1/2}\,\psi\Big\rangle \\
&= \int d^3k\, d^3\tilde{k}\, \Big\langle a(\tilde{k})\,(H_j+1)^{-1/2}\psi \,\Big|\, G_j(k)^* G_j(\tilde{k}) \\
&\qquad \otimes \Big[\frac{\big(H_f^{(\Lambda)}+1+\omega_\Lambda(k)+\omega_\Lambda(\tilde{k})\big)^2}{\big(H_j+1+\omega_j(k)\big)\big(H_j+1+\omega_j(\tilde{k})\big)}\Big]^{\alpha}\, a(k)\,(H_j+1)^{-1/2}\,\psi\Big\rangle \\
&\quad + \int d^3k\, \Big\langle \psi \,\Big|\, G_j(k)^* G_j(k) \otimes (H_j+1)^{-2\alpha-1}\,\big(H_f^{(\Lambda)}+1+\omega_\Lambda(k)\big)^{2\alpha}\,\psi\Big\rangle \\
&\le \Big\|\frac{H_f^{(\Lambda)}+1+2\Lambda}{H_j+1}\Big\|^{2\alpha} \Big(\int d^3k\, \|G_j(k)\|\, \Big\|a(k)\,(H_j+1)^{-1/2}\,\psi\Big\|\Big)^2 \\
&\quad + \Big\|(H_j+1)^{-2\alpha-1}\,\big(H_f^{(\Lambda)}+1+\omega_\Lambda(k)\big)^{2\alpha}\Big\| \Big(\int d^3k\, \|G_j(k)\|^2\Big) \\
&\le C\,(1+2\Lambda)^{2\alpha}\Big(1 + \int d^3k\, \omega_j(k)\, \Big\|a(k)\,(H_j+1)^{-1/2}\,\psi\Big\|^2\Big) \\
&\le C\,(1+2\Lambda)^{2\alpha}\Big(1 + \Big\|\frac{H_j}{H_j+1}\Big\|\Big) \le C\,(2+4\Lambda)^{2\alpha}\,.
\end{aligned}
$$

Note that in the last step we used $H_j = \int d^3k\, \omega_j(k)\, a^*(k)a(k)$. □

Now we are in position to establish Eq. (II.6) which shows that the interacting atom-photon ground state $\Phi_{\rm gs}$ is well-localized in energy, even for the noninteracting Hamiltonian.

Lemma IV.2. *For any $\alpha \ge 1$, $\Lambda < \infty$, and sufficiently small $g > 0$, we have*

$$\big\|(H_f^{(\Lambda)}+\mathbf{1})^{\alpha}\, P_c\, \Phi_{\rm gs}\big\| + \big\|(H_f^{(\Lambda)}+\mathbf{1})^{\alpha}\, P_\Omega^{\perp}\, \Phi_{\rm gs}\big\| \le \mathcal{O}(g)\,. \qquad \text{(IV.10)}$$

Proof. The proof is similar to the one for [2, Thm. X]. We first note that the asserted bound (IV.10) is implied by (I.25) and the following estimate,

$$\big\|(H_f^{(\Lambda)}+\mathbf{1})^{\alpha}\, \mathbf{1}_{\{H_0 \ge E_0/3\}}\, \Phi_{\rm gs}\big\| \le \mathcal{O}(g)\,, \qquad \text{(IV.11)}$$

because on $\mathrm{Ran}\,\mathbf{1}_{\{H_0 < E_0/3\}}$, we have $P_c = 0$ and $H_f^{(\Lambda)} < |e_0| - |E_0|/3$.

Let $\chi \in C_0^\infty(I;[0,1])$ be a real-valued, smooth function, compactly supported in $I := (\frac{4}{3}E_0\,,\,\frac{2}{3}E_0)$ and such that $\chi(E_0) = 1$. Clearly, $\mathbf{1}_{\{H_0 \ge E_0/3\}}\,\chi(H_0) = 0$, and $\chi(H_g)\,\Phi_{\rm gs} = \Phi_{\rm gs}$. Hence we observe that

$$
\begin{aligned}
(H_f^{(\Lambda)} + 1)^\alpha \, \mathbf{1}_{\{H_0 \geq E_0/3\}} \, \Phi_{\rm gs} \ = \qquad\qquad\qquad\qquad & \qquad\qquad\qquad (\mathrm{IV}.12)\\
\mathbf{1}_{\{H_0 \geq E_0/3\}} \, (H_f^{(\Lambda)} + 1)^\alpha \, \big[\chi(H_g) - \chi(H_0)\big] \, (H_g - E_0 + 1)^{-\alpha-1} \Phi_{\rm gs} \, . &
\end{aligned}
$$

Next, we represent $\chi(H_g)$ and $\chi(H_0)$ by the functional calculus based on almost analytic extensions of smooth functions of compact support [17]. Let $\tilde{\chi} \in C_0^\infty(\Delta;\mathbb{C})$ be an almost analytic extension of χ, supported in a small complex neighborhood $\Delta \subseteq \mathbb{C}$ of I and such that $\bar{\partial}\tilde{\chi}(z) = \mathcal{O}\big((\mathrm{Im}\, z)^2\big)$. By means of the measure $d\mu(z) := (2\pi i)^{-1} \bar{\partial}\tilde{\chi}(z)\, dz \wedge d\bar{z}$ and the second resolvent equation, we have the following identity

$$
\begin{aligned}
\chi(H_g) - \chi(H_0) \ &= \ \int \frac{d\mu(z)}{H_g - z} \ - \ \frac{d\mu(z)}{H_0 - z} \qquad (\mathrm{IV}.13)\\
&= \ -g \int d\mu(z) \big\{ (H_0 - z)^{-1} \, W \, (H_g - z)^{-1} \big\} \, .
\end{aligned}
$$

Inserting this identity into (IV.12) and applying Lemma IV.1 to $W = a^*(G) + a(G)$, we obtain the norm estimate

$$
\begin{aligned}
& \big\| (H_f^{(\Lambda)} + 1)^\alpha \, \mathbf{1}_{\{H_0 \geq E_0/3\}} \, \Phi_{\rm gs} \big\| \\
& \leq \ g \bigg\| \int d\mu(z) \big\{ (H_0 - z)^{-1} \, (H_f^{(\Lambda)} + 1)^\alpha \, W \\
& \qquad\qquad\qquad \cdot (H_g - E_0 + 1)^{-\alpha-1} \, (H_g - z)^{-1} \big\} \Phi_{\rm gs} \bigg\| \\
& \leq \ g \int |d\mu(z)| \Big\{ \big\| (H_0 - z)^{-1} \big\| \, \big\| (H_f^{(\Lambda)} + 1)^\alpha \, W \, (H_f + 1)^{-\alpha-1} \big\| \\
& \qquad\qquad \cdot \big\| (H_f + 1)^{\alpha+1} \, (H_g - E_0 + 1)^{-\alpha-1} \big\| \, \big\| (H_g - z)^{-1} \big\| \Big\} \\
& \leq \ \mathcal{O}(g) \, , \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{IV}.14)
\end{aligned}
$$

which finishes the proof. □

Our next goal is the derivation of (II.8). Actually, we prove a somewhat stronger estimate which is also used in the proof of Lemma IV.4, below. Recall that $A \equiv A(\underline{f}) = a^*(f_1) a^*(f_2) \cdots a^*(f_N)$ is a photon cloud of N photons $f_1, f_2, \ldots, f_N \in C_0^\infty(B_\Lambda \setminus \{0\})$, that $X_t := e^{-itH_0} X e^{itH_0}$ denotes the free time evolution of an observable X, and that $K > 1$ is the degree of differentiability in Hypothesis 1. Denoting

$$
(1 + |x|)^{3/2} \ := \ \begin{pmatrix} (1 + |x|)^{3/2} & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1} \, , \qquad (\mathrm{IV}.15)
$$

we prove the following lemma.

Lemma IV.3. *For any $N \in \mathbb{N}$ and $K > 1$, there exists a constant $C < \infty$ such that*

$$\big\|(1+|x|)^{3/2}\,[W\,,\,A_s]\,(H_f^{(\Lambda)} + \mathbf{1})^{-1-(N/2)}\big\| \;\le\; C\,(1+|s|)^{-K}\,, \tag{IV.16}$$

for all $s \in \mathbb{R}$.

Proof. First observe that on $\mathrm{dom}(H_f^{N/2})$, we have $e^{-isH_0}a^*(f)e^{isH_0} = a^*(e^{-is\omega}f)$ and hence

$$A_s \;=\; a^*(e^{-is\omega}f_1)\,a^*(e^{-is\omega}f_2)\cdots a^*(e^{-is\omega}f_N)\,. \tag{IV.17}$$

Writing $W = a^*(G) + a(G)$ and using the canonical commutation relations, we thus have

$$\begin{aligned}[W\,,\,A_s] \;&=\; \big[a^*(G)+a(G)\,,\;a^*(e^{-is\omega}f_1)\,a^*(e^{-is\omega}f_2)\cdots a^*(e^{-is\omega}f_N)\big]\\ &=\; \big[a(G)\,,\;a^*(e^{-is\omega}f_1)\,a^*(e^{-is\omega}f_2)\cdots a^*(e^{-is\omega}f_N)\big] \qquad\qquad \text{(IV.18)}\\ &=\; \sum_{j=1}^{N}\Big\{\big\langle G\big|e^{-is\omega}f_j\big\rangle \otimes \prod_{\substack{i=1,\\ i\neq j}}^{N} a^*(e^{-is\omega}f_i)\Big\}\,,\end{aligned}$$

where $\langle G|e^{-is\omega}f_j\rangle \in \mathcal{B}(\mathcal{H}_{el})$ is a bounded operator acting on the electron variables given by

$$\big\langle G\big|e^{-is\omega}f_j\big\rangle \;:=\; \int d^3k\;e^{-is\omega(k)}\,f_j(k)\,G^*(k)\,. \tag{IV.19}$$

Note that $k \mapsto f_j(k)G^*(k)$ is $K > 1$ times continuously differentiable, and, thanks to the support properties of f_j, there exists $r_j > 0$ such that $f_jG^* \in C^K[B_\Lambda \setminus B_{r_j}; \mathcal{B}(\mathcal{H}_{el})]$ has compact support away from zero. Thus, thanks to Hypothesis 1, K times ($\mathcal{B}(\mathcal{H}_{el})$-valued) integration by parts yields the standard estimate for oscillatory integrals,

$$\begin{aligned}&\Big\|(1+|x|)^{3/2}\,\big\langle G\big|e^{-is\omega}f_j\big\rangle\,\Big\| \qquad\qquad \text{(IV.20)}\\ &\le\; |s|^{-K}\int d^3k\,\Big\|(1+|x|)^{3/2}\,\big(\nabla_k\cdot(k/|k|)\big)^K\,f_j(k)\,G^*(k)\Big\|\\ &\le\; \mathcal{O}\big((1+|s|)^{-K}\big)\,, \qquad\qquad \text{(IV.21)}\end{aligned}$$

provided $|s| \ge 1$. For $|s| \le 1$, Estimate (IV.20) is trivial and hence holds for all $s \in \mathbb{R}$. Inserting (IV.20) into (IV.18) and undoing the free time evolution (which is possible because H_0 commutes with $H_f^{(\Lambda)}$), we obtain

$$\begin{aligned} & \big\| (1+|x|)^{3/2}\,[W\,,\,A_s]\,(H_f^{(\Lambda)}+\mathbf{1})^{-1-(N/2)}\big\| \qquad \text{(IV.22)} \\ \leq \;& \mathcal{O}\big((1+|s|)^{-K}\big)\sum_{j=1}^{N}\bigg\|\bigg(\prod_{\substack{i=1,\\ i\neq j}}^{N} a^*(e^{-is\omega}f_i)\bigg)\,(H_f^{(\Lambda)}+\mathbf{1})^{-1-(N/2)}\bigg\| \\ = \;& \mathcal{O}\big((1+|s|)^{-K}\big)\sum_{j=1}^{N}\bigg\|\bigg(\prod_{\substack{i=1,\\ i\neq j}}^{N} a^*(f_i)\bigg)\,(H_f^{(\Lambda)}+\mathbf{1})^{-1-(N/2)}\bigg\|\,. \end{aligned}$$

To estimate a product of $N-1$ creation operators $a^*(\tilde{f}_1)a^*(\tilde{f}_2)\cdots a^*(\tilde{f}_{N-1})$ with $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_N \in C_0^\infty(B_\Lambda \setminus \{0\})$, we use Estimate (IV.4) $N-1$ times and derive

$$\begin{aligned} & \bigg\|\bigg(\prod_{n=1}^{N-1} a^*(\tilde{f}_n)\bigg)\,(H_f^{(\Lambda)}+\mathbf{1})^{-(N-1)/2)}\bigg\| \qquad \text{(IV.23)} \\ = \;& \bigg\|\prod_{n=1}^{N-1}\Big((H_f^{(\Lambda)}+\mathbf{1})^{(n-1)/2}a^*(\tilde{f}_n)\,(H_f^{(\Lambda)}+\mathbf{1})^{-n/2}\Big)\bigg\| \\ \leq \;& \prod_{n=1}^{N-1}\Big\|(H_f^{(\Lambda)}+\mathbf{1})^{(n-1)/2}a^*(\tilde{f}_n)\,(H_f^{(\Lambda)}+\mathbf{1})^{-n/2}\Big\| \;\leq\; \mathcal{O}\big((1+\Lambda)^{(N^2)}\big)\,. \end{aligned}$$

Inserting this bound into (IV.22) yields (IV.16), since $\Lambda < \infty$ is bounded. $\square$

We finally turn to proving (II.11). Note that the following Lemma uses H_f rather than $H_f^{(\Lambda)}$.

Lemma IV.4. *There exists a constant $C < \infty$ such that, for any $r \in \mathbb{R}$ and $s \in \mathbb{R}_0^+$,*

$$\big\| W\,e^{-irH_0}\,P_c\,[W\,,\,A_s]\,(H_f+\mathbf{1})^{-2-(N/2)}\big\| \;\leq\; C\,\Big((1+|r|)^{-3/2}\Big)\,. \qquad \text{(IV.24)}$$

Proof. The proof is based on the decay of the Schwartz kernel $e^{-ir(-\Delta)}(x,y) = (4\pi i r)^{-3/2}\exp[i(x-y)^2(4r)^{-1}]$ of the propagator $e^{-ir(-\Delta)}$ of the free particle. More precisely, defining $C_1 := (4\pi)^{-1}\int(1+|x|)^{2\gamma}d^3x < \infty$, we observe that

$$\begin{aligned} & \big|\big\langle(1+|x|)^{-\gamma}\varphi\;\big|\;e^{-ir(-\Delta)}\,(1+|x|)^{-\gamma}\psi\big\rangle\big| \qquad \text{(IV.25)} \\ = \;& \frac{1}{(4\pi\,r)^{3/2}}\bigg|\int d^3x\,d^3y\,\frac{\overline{\varphi(x)}\,e^{i(x-y)^2/(4r)}\,\psi(y)}{(1+|x|)^\gamma\,(1+|y|)^\gamma}\bigg| \\ \leq \;& \frac{\|\varphi\|\,\|\psi\|}{(4\pi\,r)^{3/2}}\bigg(\int\frac{d^3x}{(1+|x|)^{2\gamma}}\bigg) \;\leq\; C_1\,\|\varphi\|\,\|\psi\|\,r^{-3/2}\,, \end{aligned}$$

for any $\varphi, \psi \in L^1 \cap L^2(\mathbb{R}^3)$, since $\gamma > 3/2$. This estimate yields

$$\begin{aligned} & \big\| (1+|x|)^{-\gamma} \, e^{-irH_0} \, P_c \, (1+|x|)^{-\gamma} \big\| && \text{(IV.26)} \\ & = \left\| \begin{pmatrix} (1+|x|)^{-\gamma} \, e^{-ir(-\Delta)} \, (1+|x|)^{-\gamma} & 0 \\ 0 & 0 \end{pmatrix} \otimes e^{-irH_f} \right\| \; \leq \; C_1 \, r^{-3/2} \, . \end{aligned}$$

Inserting (IV.26) into (IV.24) and using that H_f and H_0 commute, we obtain

$$\begin{aligned} & \big\| \, W \, e^{-irH_0} \, P_c \, [W \, , \, A_s] \, (H_f + \mathbf{1})^{-2-(N/2)} \big\| && \text{(IV.27)} \\ & \leq \; \big\| W \, (1+|x|)^{\gamma} \otimes (H_f+1)^{-1/2} \big\| \; \big\| (1+|x|)^{-\gamma} \, e^{-irH_0} \, P_c \, (1+|x|)^{-\gamma} \big\| \\ & \qquad \big\| (1+|x|)^{\gamma} \otimes (H_f+1)^{1/2} \, [W \, , \, A_s] \, (H_f+1)^{-2-(N/2)} \big\| \\ & \leq \; C_1 \, r^{-3/2} \, \big\| W \, (1+|x|)^{\gamma} \otimes (H_f+1)^{-1/2} \big\| \\ & \qquad \big\| (1+|x|)^{\gamma} \otimes (H_f+1)^{1/2} \, [W \, , \, A_s] \, (H_f+1)^{-2-(N/2)} \big\| \, . \end{aligned}$$

Next, we use (IV.18) and the pull-through formula to commute $(H_f+1)^{1/2}$ through $[W \, , \, A_s]$, observing that $\mathrm{supp} f_j \subseteq B_\Lambda(0)$. This yields

$$\begin{aligned} & \left\| (1+|x|)^{\gamma} \otimes (H_f+1)^{1/2} \, [W \, , \, A_s] \, (H_f+1)^{-2-(N/2)} \right\| . && \text{(IV.28)} \\ & \leq \; \left\| (1+|x|)^{\gamma} \, [W \, , \, A_s] \, (H_f+1)^{-2-(N/2)} \big(H_f + 1 + (N-1)\Lambda \big)^{1/2} \right\| \\ & \leq \; (1+N\Lambda)^{1/2} \left\| (1+|x|)^{\gamma} \, [W \, , \, A_s] \, (H_f+1)^{-3/2-(N/2)} \right\| \; \leq \; C_2 \, , \end{aligned}$$

for some constant $C_2 < \infty$, uniformly in $s \in \mathbb{R}$. Here we additionally inserted (IV.16) to derive the last inequality. Finally, there exists a constant $C_3 < \infty$, such that

$$\left\| W \, (1+|x|)^{\gamma} \otimes (H_f+1)^{-1/2} \right\| \; \leq \; C_3 \, , \qquad \text{(IV.29)}$$

as follows from writing $W \, (1+|x|)^{\gamma} \, = \, a^*[G \, (1+|x|)^{\gamma}] + a[G \, (1+|x|)^{\gamma}]$ and applying Lemma IV.1. $\square$

A Elimination of Matrix Elements by a Bogoliubov Transformation

Consider $H_{el} = -\Delta - V(x)$, a Schrödinger operator on $\mathbb{R}^3$ having a single, simple negative eigenvalue, $e_0 < 0$, associated to a normalized vector φ_0. This Hamiltonian represents an atom which we couple to the photonic field described in Subsect. I.1.2 with an interaction described in Subsect. I.1.3. More precisely, we assume the interaction W to be of the form (I.11), with $G \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{el})]$ being a square-integrable function with values in the bounded operators on $\mathcal{H}_{el}$, satisfying

$$\int d^3k \left\{ 1 + \omega(k)^{-2} \right\} \left\| G(k) \right\|^2 \leq 1 . \tag{A.1}$$

Note that this assumption is stronger than (I.15). We now show

Theorem A.1. *Assume (A.1). For $g > 0$ small enough, there exists a unitary transform $U_g \in \mathcal{B}(\mathcal{H}_{el} \otimes \mathcal{F})$ real analytic in g such that*

$$\widetilde{H}_g := U_g \, H_g \, U_g^* = \widetilde{H}_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + g\widetilde{W} \tag{A.2}$$

where

- $\widetilde{H}_{el} = H_{el} + g^2 \Delta V$*, and* ΔV *is a bounded self-adjoint operator;*
- *if W acts as a multiplication operator in the electron variable, ΔV is a potential, i.e., a multiplication operator;*
- $\widetilde{H}_{el}$ *has a single, simple negative eigenvalue, $e_g < 0$, associated to the one-dimensional eigenprojector Π_g;*
- *we have*

$$(\Pi_g \otimes \mathbf{1}_f) \, \widetilde{W} \, (\Pi_g \otimes \mathbf{1}_f) = 0 . \tag{A.3}$$

All the quantities introduced above are real-analytic in g.

Theorem A.1 proves that, starting from a Schrödinger operator on $\mathbb{R}^3$ with a single, simple negative bound state, using a unitary transform, one can always pass to an interaction fulfilling Assumption (I.12).

Proof. Pick $h \in L^2(\mathbb{R}^3)$, and consider the Bogoliubov transform

$$U(h) = \mathbf{1} \otimes e^{ig[a^*(h) + a(h)]} . \tag{A.4}$$

A standard computation gives

$$U(h)\, H_g\, U(h)^* \;=\; \widetilde{H}_{el} \otimes \mathbf{1}_f \;+\; \mathbf{1}_{el} \otimes H_f \;+\; g\widetilde{W}\,, \tag{A.5}$$

where

$$\begin{aligned} \widetilde{H}_{el} \;&=\; H_{el} \;+\; g^2\, \Delta V(h)\,, && \text{(A.6)}\\ \Delta V(h) \;&:=\; \int d^3k \Big\{ h(k) G(k) + \overline{h(k)} G^*(k) \Big\}\,, && \text{(A.7)}\\ \widetilde{W} \;&=\; W \;+\; \mathbf{1} \otimes \big\{ a^*(\omega h) + a(\omega h) \big\}\,. && \text{(A.8)} \end{aligned}$$

We remark that $\Delta V : L^2(\mathbb{R}^3) \to \mathcal{B}(\mathcal{H}_{el})$ is a bounded linear operator with norm bounded by one, thanks to (A.1). Hence, $\widetilde{H}_{el}$ has a single, simple eigenvalue $\tilde{e}(h) = e_0 + \mathcal{O}(g^2)$ in a vicinity of e_0, and we denote the corresponding normalized eigenvector by $\varphi_g(h)$.

To prove Theorem A.1, we only need to show that, for $g > 0$ sufficiently small, we can construct $h_g \in L^2(\mathbb{R}^3)$ such that

$$-\langle \varphi_g(h_g), \widehat{G}(k) \varphi_g(h_g) \rangle \;=\; h_g(k)\,, \qquad \text{where} \qquad \widehat{G}(k) \;:=\; \omega(k)^{-1}\, G(k)\,. \tag{A.9}$$

By Assumption (A.1), $\widehat{G}$ is a square-integrable function with values in the bounded operators on $\mathcal{H}_{el}$. Using standard perturbation theory, we construct $\varphi_g(h)$ simply by normalizing the vector

$$\hat{\varphi}_g(h) \;:=\; \frac{1}{2i\pi} \int_{\gamma_0} \big(z - H_{el} - g^2 \Delta V(h) \big)^{-1} \varphi_0 \, dz\,, \tag{A.10}$$

where γ_0 is a small circle of center e_0 and radius $|e_0/2|$. The function $\varphi_g(h)$ is real analytic in g and continuous in h, for h in the unit ball in $L^2(\mathbb{R}^3)$; real analytic perturbation theory and the bound $\|\Delta V(h)\| \leq \|h\|$ immediately give

$$\big\| \varphi_g(h) \;-\; \varphi_g(h') \big\| \;\leq\; C\, g^2\, \|h - h'\|\,, \tag{A.11}$$

for some constant $C < \infty$ and all $h, h' \in L^2(\mathbb{R}^3)$ with $\|h\|, \|h'\| \leq 1$. Set

$$h_0 \;=\; -\langle \varphi_0 |\, \widehat{G}(k)\, \varphi_0 \rangle\,. \tag{A.12}$$

For g fixed and small and $h \in L^2(\mathbb{R}^3)$, with $\|h\| \leq 1$, we define

$$T_g(h) \;=\; -\langle \varphi_g(h_0 + h) \,|\, \widehat{G}(\cdot)\, \varphi_g(h_0 + h) \rangle \;-\; h_0\,. \tag{A.13}$$

Note that T_0 has a trivial fixed point, $h = 0$. For $g > 0$ sufficiently small, T_g maps the unit ball of $L^2(\mathbb{R}^3)$ into itself. Moreover, one computes

$$\begin{aligned} \left\|T_g(h) - T_g(h')\right\| &\leq 4\|\widehat{G}(k)\|\,\|\varphi_g(h_0 + h) - \varphi_g(h_0 + h')\| \\ &\leq C\,g^2\,\|h - h'\|\,, \end{aligned} \tag{A.14}$$

for some $C < \infty$. Hence, T_g is contracting for g small enough. Therefore, the fixed point equation $h = T_g(h)$ has a unique solution in the unit ball of $L^2(\mathbb{R}^3)$. This fixed point is the desired function solving (A.9). We may construct this solution as the norm limit of the sequence $T_g^n(0)$. Each of these terms being real analytic in g and the convergence being uniform in g sufficiently small (the rate of convergence is given by g^2), the limit is real analytic in g. □

B Transported Charge for Bound and Negative Energy States

Theorem B.1. *Assume Hypothesis 1. Then*

$$\lim_{R\to\infty} \sup_{t\geq 0} \left\|F_R\, e^{-itH_g}\ \mathbf{1}_{pp}(H_g)\ \Psi\ \right\| \ = \ 0 \tag{B.1}$$

$$\lim_{R\to\infty} \sup_{t\geq 0} \left\|F_R\, e^{-itH_g}\ \mathbf{1}_{]-\infty,0]}(H_g)\ \Psi\right\| \ = \ 0 \tag{B.2}$$

for any $\Psi \in \mathcal{H}$*, where* $\mathbf{1}_{pp}(H_g)$ *and* $\mathbf{1}_{]-\infty,0]}(H_g)$ *are the spectral projections of* H_g *onto its point spectrum and onto* $]-\infty, 0]$*, respectively.*

Proof. To prove equality (B.1), we first assume that Ψ is an eigenvector of H_g with corresponding eigenvalue $E \in \mathbb{R}$. Since $F_R \to 0$ strongly, as $R \to \infty$, we then have

$$\lim_{R\to\infty} \sup_{t\geq 0} \left\|F_R\, e^{-itH_g}\ \Psi\ \right\| = \lim_{R\to\infty} \|F_R\Psi\| \ = \ 0\,. \tag{B.3}$$

This, of course, generalizes to any finite linear combination of eigenvectors of H_g. Since the norm closure of this set of vectors is the pure point subspace of H_g, Eq. (B.3) also generalizes to vectors in $\mathrm{Ran}\, \mathbf{1}_{pp}(H_g)$, which yields (B.1). As a consequence of (B.1) we may assume $\Psi = \mathbf{1}_c(H_g)\Psi$ in the proof of (B.2). These vectors out of the continuous subspace have the property that

$$\lim_{\delta\searrow 0} \mathbf{1}_{(E-\delta,E+\delta)}(H_g)\Psi \ = \ 0\,, \tag{B.4}$$

for any $E \in \mathbb{R}$. As H_g is bounded from below and in view of (B.4), it hence (choosing $E := 0$) suffices to prove that

$$\lim_{R\to\infty} \big\| F_R\, \chi[H_g] \,\big\| \; = \; 0 \, , \tag{B.5}$$

for any smooth function $\chi \in C_0^\infty[(-\infty, 0)]$, compactly supported on the negative half-axis and away from zero. The basic idea of our proof of (B.5) is essentially the same as [2, Thm. II.1] or Lemma IV.2 and uses a Combes-Thomas or Agmon Estimate. For $H = H_g$ or $H = H_0$, we use a representation

$$\chi[H] \; = \; \int_\Delta \frac{d\mu(z)}{z - H} \, , \tag{B.6}$$

based on an almost analytic extension $\tilde{\chi} \in C_0^\infty(\mathcal{M}; \mathbb{C})$ of χ, whose compact support $\mathcal{M} \subseteq \mathbb{C}$ can be chosen to include only $z \in \mathbb{C}$ with $\mathrm{Re} z \leq -\delta$, for some $\delta > 0$. Moreover, we can choose $\tilde{\chi}$ as to obey $\bar{\partial}\tilde{\chi}(z) = \mathcal{O}\big((\mathrm{Im}\, z)^2\big)$. The measure in (B.6) is then defined as $d\mu(z) := (2\pi i)^{-1}\bar{\partial}\tilde{\chi}(z)\, dz \wedge d\bar{z}$. By means of (B.6), the second resolvent equation, and the fact that $F_R\, \chi[H_0] = F_R\, P_c\, \chi[H_0] = 0$, since $H_0 \geq 0$ on $\mathrm{Ran}\, P_c$, we thus have

$$\begin{aligned} F_R\, \chi[H_g] \;&=\; F_R\, \Big(\chi[H_g] \; - \; \chi[H_0]\Big) \\ &=\; -g \int_{\mathcal{M}} d\mu(z) F_R\, \frac{P_c}{-\Delta + H_f - z}\, W\, \frac{1}{H_g - z} \, . \end{aligned} \tag{B.7}$$

Next, we pick $1 < \lambda < \infty$ and introduce $f_\lambda : \mathbb{R}^3 \to \mathbb{R}^+$ by $f_\lambda(x) := \sqrt{1 + (x/\lambda)^2}\,$, and we denote $f_\lambda(x)^{-1} := 1/f_\lambda(x)$. We note the following properties of f_λ,

$$\frac{|x|}{\lambda} \;\leq\; f_\lambda(x) \;\leq\; 1 + |x|^{3/2} \quad \text{and} \quad f_\lambda(x)^{-1}\, |\nabla f_\lambda(x)| \;\leq\; \lambda^{-1} \, , \tag{B.8}$$

which imply

$$\big\| F_R\, f_\lambda(x)^{-1} \big\| \;\leq\; C\, \lambda\, R^{-1} \, , \tag{B.9}$$

$$\begin{aligned} \mathrm{Re}\big\{ f_\lambda(x)\, (-\Delta)\, f_\lambda(x)^{-1} \big\} \;&=\; \\ -\Delta + 2\mathrm{Re}\big\{ f_\lambda(x)^{-1}\, \nabla f_\lambda(x) \cdot i\nabla \big\} \;&\geq\; -\lambda^{-2} \, , \end{aligned} \tag{B.10}$$

$$\big\| f_\lambda(x)\, W\, (H_g + i)^{-1} \big\| \;\leq\; C \, , \tag{B.11}$$

for some constant $C < \infty$. Here, (B.9) is trivial, (B.10) is meant in the quadratic form sense on smooth functions of compact support, and (B.11) uses Hypothesis 1 and Lemma IV.1. Choosing $\lambda \geq 2/\sqrt{\delta}$ and inserting the bounds (B.9)–(B.11) into (B.7), we arrive at (B.5). □

References

[1] W. Amrein, A. Boutet de Monvel, and V. Georgescu. *C_0-Groups, Commutator Methods, and Spectral Theory of N-Body Hamiltonians*. Birkhäuser, 1996.

[2] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.* , 137:299–395, 1998.

[3] V. Bach, J. Fröhlich, and I. M. Sigal. Renormalization group analysis of spectral problems in quantum field theory. *Adv. in Math.* , 137:205–298, 1998.

[4] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Commun. Math. Phys.*, 207(2):249–290, 1999.

[5] V. Bach, J. Fröhlich, and I. M. Sigal. Return to equilibrium. *J. Math. Phys.*, 41(6):3985–4060, June 2000.

[6] J. Derezinski and C. Gérard. *Scattering theory of classical and quantum N-particle systems*. Text and Monographs in Physics. Springer, 1997.

[7] D. Eastham. *Atomic Physics of Lasers*. Taylor and Francis, London, 1986.

[8] A. Einstein. Über einen die Erzeugung und Verwandlung des Lichts betreffenden heuristischen Gesichtspunkt. *Annalen der Physik*, 17:132–148, 1905.

[9] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic completeness for Rayleigh scattering. *Preprint*, 2001.

[10] Ch. Gérard. Asymptotic completeness for the spin-boson model with a particle number cutoff. *Rev. Math. Phys.*, 8:549–589, 1996.

[11] M. Griesemer, E. Lieb, and M. Loss. Ground states in nonrelativistic quantum electrodynamics. *Invent. math.*, June 2001. DOI 10.1007/s002220100159.

[12] H. Haken. *Light*, volume 1. North-Holland, 1981.

[13] D. Haliday and R. Resnick. *Physics (Part I and II)*. Wiley, 1978.

[14] W. Hallwachs. Ueber den Einfluss des Lichtes auf electrostatisch geladene Körper. *Annalen der Physik*, 33:301, 1888.

[15] H. Hertz. Ueber einen Einfluss des ultravioletten Lichtes auf die electrische Entladung. *Annalen der Physik*, 31:983, 1887.

[16] P. Lenard. Über die lichtelektrische Wirkung. *Annalen der Physik*, 8:149–198, 1902.

[17] A. Melin and J. Sjöstrand. *Fourier Integral Operators with Complex-valued Phase Functions*, volume 459 of *L. N. M.* . Springer, Heidelberg, 1976.

[18] R. Millikan. A direct photoelectric determination of Planck's h. *Phys. Rev.*, 7:355–388, 1916.

[19] R. Millikan. Einstein's photoelectric equation and contact electromotive force. *Phys. Rev.*, 7:18–32, 1916.

[20] B. Schlein. Nichtrelativistische Elektronensysteme in einem quantisierten elektromagnetischen Feld. *diploma thesis, ETH-Zürich*, August 1999.