Reduction scheme for semiclassical operator-valued Schrödinger type equation and application to scattering

Vania Sordoni[1]
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S.Donato,5 40127 Bologna, Italy

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Abstract

We obtain a general reduction scheme for the study of a selfadjoint semiclassical Schrödinger operator with operator-valued potential by the construction of almost invariant subspaces and we apply such results to scattering theory for matrix-valued operators.

1 Introduction and main results

In this paper, we give a general reduction scheme for the study of a selfadjoint semiclassical Schrödinger operator

$$P(h) = -h^2\Delta_x + V(x), \qquad x \in \mathbf{R}^n$$

with operator-valued potential $V(x)$, in the case where the spectrum of $V(x)$ is given by two disjoint components. Let $\Pi_0(x)$ denote the projection onto one of the two disjoint part of the spectrum of $V(x)$. Here we investigate, under suitable regularity assumptions on the potential $V(x)$, the existence of a projector $\Pi(h) \sim \Pi_0 + h\Pi_1 + ...$ (see [Ne, Sj, HeSj, BrNo, Ta, NeSo]) that commutes with $P(h)$ modulo remainder terms of order $\mathcal{O}(h^\infty)$ or modulo terms exponentially small as $h \to 0$. Even if almost invariant subspaces for $P(h)$ does not seem to exist according to the physical intuition saying that the adiabatic decoupling becomes poorer and poorer when the energy increases, here we show that it is possible to construct a projector Π such that, for any cutoff function χ, $[P,\Pi]\chi(P)$ are $\mathcal{O}(h^\infty)$ (or exponentially small) as $h \to 0$. However, these results permit to reduce (modulo these small error terms) the study of some spectral properties of P to the study of those of $P_1 = \Pi P\Pi$ and $P_2 = (1-\Pi)P(1-\Pi)$. In [MaSo] an application of our results to the time-dependent Born-Oppenheimer

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approximation is given. Here we apply our main results to scattering theory, generalizing to the Schrödinger case the results on scattering operator for Klein-Gordon matrix-valued operators given in [NeSo]. In particular, we recover, with a completely different method, some of the results of [MNS] (see also [BeMa]) on scattering matrix for matrix-valued operators. The plan of the paper is the following: in Section 2 we introduce our assumptions and we state our main results. The proof of this results are given in Section 3. In Section 4 we apply the reduction scheme obtained in the previous sections to scattering theory for matrix-valued Schrödinger operator.

2 Introduction and main results

Given an Hilbert space $\mathcal{H}$, we consider a family $V(x), x \in \mathbf{R}^n$, of selfadjoint linear operators on $\mathcal{H}$, bounded from below, with domain $\mathcal{D} = \mathcal{D}(V(x))$ independent of x. Denoting by $\mathcal{L}(\mathcal{D};\mathcal{H})$ the space of bounded operators from $\mathcal{D}$ to $\mathcal{H}$, we suppose that one of the following assumptions holds:

$(H1)$ the map $\mathbf{R}^n \ni x \to V(x) \in \mathcal{L}(\mathcal{D};\mathcal{H})$ is smooth

or

$(H1A)$ there exists $a > 0$ such that $\mathbf{R}^n \ni x \to V(x) \in \mathcal{L}(\mathcal{D};\mathcal{H})$ is analytic and admits an extension on the strip $\Gamma_a = \{x \in \mathbf{C}^n \ ; \ |\mathrm{Im}\, x| < a\}$

We also suppose that

$(H2)$ the spectrum $\sigma(V(x))$ of $V(x)$ is the union of two disjoint components $\sigma_j(x)$, $j = 0, 1$, and

$$\operatorname*{Inf}_{x\in\mathbf{R}^n} \mathrm{dist}(\sigma_0(x), \sigma_1(x)) \geq 2d > 0. \tag{2.1}$$

Moreover, we assume that, for any $x \in \mathbf{R}^n$ there exists a continuous loop $\gamma(x)$ depending smoothly on x, that surrounds $\sigma_0(x)$ and leaves $\sigma_1(x)$ in its exterior.

If (H1A) holds, one can suppose, by possibly shrinking a, that

$$\operatorname*{Inf}_{x\in\Gamma_a} \mathrm{dist}(\sigma_0(x), \sigma_1(x)) \geq d.$$

For our application to scattering, we need to consider the case where $V(x)$ admits a limit value as $|x| \to +\infty$. For this reason, we consider the following assumption:

$(H3)_\delta$ there exists a selfadjoint operator V_∞ such that:

$$\operatorname*{Sup}_{x\in\mathbf{R}^n} \langle x\rangle^\delta \|V(x) - V_\infty\|_{\mathcal{L}(\mathcal{D},\mathcal{H})} < \infty \tag{2.2}$$

with $\delta \geq 0$ and (in the case $(H1A)$ holds):

$(H3A)_\delta$

$$\operatorname*{Sup}_{x\in\Gamma_a} \langle x\rangle^\delta \|V(x) - V_\infty\|_{\mathcal{L}(\mathcal{D},\mathcal{H})} < \infty \tag{2.3}$$

Of course, if $\delta = 0$, a family $V(x)$ satisfying $(H1)$ (or $(H1A)$) and $(H2)$ whose L^2-norms are uniformly bounded with respect to $x \in \mathbf{R}^n$, automatically satisfies $(H3)_0$ (or $(H3A)_0$) with $V_\infty = 0$. In the following, we denote by $\Pi_0(x)$ the spectral projection of $V(x)$ corresponding to $\sigma_0(x)$. $\Pi_0(x)$ is a smooth (or analytic on Γ_a, if (H1A) holds) function of x with value in $\mathcal{L}(\mathcal{H};\mathcal{D})$. Let us denote by Π_0 the bounded operator from $L^2(\mathbf{R}^n;\mathcal{H})$ to $L^2(\mathbf{R}^n;\mathcal{D})$ acting as multiplication by $\Pi_0(x)$. Associated to $V(x)$ let us consider the Schrödinger Hamiltonian

$$P(h) = -h^2\Delta_x + V(x) \tag{2.4}$$

with domain $H^2(\mathbf{R}^n;\mathcal{D})$ and values in $L^2(\mathbf{R}^n;\mathcal{H})$. In the following, given a bounded operator A on $L^2(\mathbf{R}^n;\mathcal{H})$, we denote by $\|A\|_\delta$ the weighted norm

$$\|A\|_\delta = \operatorname*{Sup}_{t\in[0,1]} \|\langle x\rangle^{t\delta} A \langle x\rangle^{(1-t)\delta}\|_{\mathcal{L}(L^2(\mathbf{R}^n;\mathcal{H}))}$$

and simply by $\|A\|$ the one corresponding to $\delta = 0$. Our main results are the following

Theorem 2.1 *Assume $(H1), (H2)$ and $(H3)_\delta$ hold true for some $\delta \geq 0$. Then, for any $\Phi \in C_0^\infty(\mathbf{R})$, there exists a projection $\Pi(h)$ (depending on Φ) on $L^2(\mathbf{R}^n;\mathcal{H})$ such that*

$$\|\Pi - \Pi_0\|_\delta = \mathcal{O}(h) \tag{2.5}$$

and, for any $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi\Phi = \chi$, we have

$$\|\chi(P)[\Pi, P]\|_\delta + \|[\Pi, P]\chi(P)\|_\delta = \mathcal{O}(h^\infty) \tag{2.6}$$

uniformly with respect to $h > 0$ small enough. Moreover, $\Pi(h) = \Pi_0 + \sum_{j\geq 1} \Pi_j h^j$ modulo $\mathcal{O}(h^\infty)$ and, for any $j \geq 1$, $\Pi_j = \Phi(P)\hat{\Pi}_j + (1-\Phi(P))\hat{\Pi}_j\Phi(P)$ where $\hat{\Pi}_j$ is a differential operator of order j with smooth coefficients decaying as $\langle x\rangle^{-\delta}$ uniformly on $\mathbf{R}^n$.

Theorem 2.2 *Assume $(H1A), (H2)$ and $(H3A)_\delta$ hold true for some $\delta \geq 0$. Then, for any $\Phi \in C_0^\infty(\mathbf{R})$ and for any $B > 0$ sufficiently large, there exists an orthogonal projection $\Pi(h)$ on $L^2(\mathbf{R}^n;\mathcal{H})$ (depending on Φ and B) and $c > 0$ such that*

$$\|\Pi - \Pi_0\|_\delta = \mathcal{O}(h) \tag{2.7}$$

and

$$\|\chi(P)[\Pi, P]\|_\delta + \|[\Pi, P]\chi(P)\|_\delta = \mathcal{O}(e^{-c/h}) \tag{2.8}$$

uniformly with respect to $h > 0$ small enough. Moreover, $\Pi(h) = \Pi_0 + \sum_{j=1}^{[1/Bh]} \Pi_j h^j$ modulo $\mathcal{O}(e^{-c/h})$ and, for any $j \geq 1$, $\Pi_j = \Phi(P)\hat{\Pi}_j + (1-\Phi(P))\hat{\Pi}_j\Phi(P)$ where $\hat{\Pi}_j$ is a differential operator of order j with coefficients analytic on $\Gamma_{a'}, 0 < a' < a$, decaying as $\langle x\rangle^{-\delta}$ uniformly on $\Gamma_{a'}$.

3 Proof of the main results

The key point in the proof of the two Theorems is (as in [NeSo, Sj]) the construction of a formal symbol $\hat{\pi}(x,\xi;h) = \Pi_0(x) + \sum_{j\geq 1} \hat{\pi}_j(x,\xi)h^j \in C^\infty(\mathbf{R}^{2n};\mathcal{L}(\mathcal{H}))$ such that, denoting by $p(x,\xi) = \xi^2 + V(x)$ the symbol of $P(h)$, satisfy (at a formal series level)

$$\hat{\pi} = \hat{\pi}^* = \hat{\pi}\#\hat{\pi} \tag{3.1}$$

$$\hat{\pi}\#p - p\#\hat{\pi} = 0 \tag{3.2}$$

(here and in the following, # denotes the Weyl composition of symbols). Our first lemma is the following

Lemma 3.1 *Assume* $(H1), (H2)$ *and* $(H3)_\delta$ *hold true for some* $\delta \geq 0$. *Then, there exist functions*

$$\hat{\pi}_j(x,\xi) = \sum_{|\alpha|\leq j} a_{\alpha,j}(x)\xi^\alpha \tag{3.3}$$

polynomial of degree j *with respect to* ξ, *with smooth coefficient* $a_{\alpha,j}(x) \in \mathcal{L}(\mathcal{H})$ *such that the formal symbol*

$$\hat{\pi}(x,\xi;h) = \Pi_0(x) + \sum_{j\geq 1} \hat{\pi}_j(x,\xi)h^j, \tag{3.4}$$

satisfies, at a formal series level,

$$\hat{\pi}^* = \hat{\pi}, \quad \hat{\pi}\#\hat{\pi} = \hat{\pi}, \quad \hat{\pi}\#p - p\#\hat{\pi} = 0 \tag{3.5}$$

and, in addition, one has

$$a_{\alpha,j}(x) = \mathcal{O}(\langle x\rangle^{-\delta}), \quad \textit{for } j \geq 1 \tag{3.6}$$

uniformly on $\mathbf{R}^n$. *Moreover, if* $(H1A)$ *and* $(H3A)_\delta$ *hold for some* $\delta \geq 0$, *then* $a_{\alpha,j}(x)$ *are analytic on* Γ_a, *and (3.6) holds uniformly on* Γ'_a, *for any* $0 < a' < a$.

Proof: The construction of a symbol $\hat{\pi}$ satisfying (3.1) and (3.2) can be done following exactly the same scheme used in the proof of Lemma 3.2 of [NeSo] in the case of Klein-Gordon matrix-valued operators. For the sake of completeness we recall here the main arguments. Let $\gamma(x)$ be the continuous loop that surrounds $\sigma_0(x)$ and leaves $\sigma_1(x)$ in its exterior. In the case δ is strictly positive, we can also suppose that $\gamma(x)$ admits a limit loop as $|x| \to +\infty$. Let us consider $\Omega =: \{(x,\xi,z) \in \mathbf{R}^{2n} \times \mathbf{C}\, ; z - \xi^2 \in \gamma(x)\}$. For $(x,\xi;z) \in \Omega$, $p(x,\xi) - z$ is invertible and $q_0(x,\xi;z) = (p(x,\xi) - z)^{-1}$ is smooth and bounded. Let us define the symbol

$$r(x,\xi;h,z) = \sum_{j\geq 1} r_j(x,\xi;z)h^j \tag{3.7}$$

by

$$(p(x,\xi) - z)\#q_0(x,\xi;z) = 1 - r(x,\xi;h,z). \tag{3.8}$$

and $q(x,\xi;h,z)$ by

$$q = q_0 + q_0 \# \sum_{j\geq 1} r^{\# j} \tag{3.9}$$

One can easily prove (see e.g. the Appendix in [NeSo]), that for all $j = 1, 2, ...$, and for any multindex α, β there exist $K(\alpha,\beta,j) > 0$ such that,

$$\operatorname*{Sup}_{(x,\xi;z)\in\Omega} \langle x\rangle^{\delta}\langle\xi\rangle^{-|\alpha|-j}\|\partial_\xi^\alpha\partial_x^\beta q_j(x,\xi,z)\|_{\mathcal{L}(\mathcal{H})} \leq K(\alpha,\beta,j) \tag{3.10}$$

If $(H1A)$ and $(H3A)_\delta$ hold for some $\delta \geq 0$, one can take (see e.g. [BoKr]) $K(\alpha,\beta,j) = K_1^{j+|\alpha|+|\beta|+1}\alpha!\beta!j!$ for some constant $K_1 > 0$ independent of multiindex α,β and of j. Moreover, one can check that, for $j \geq 1$, $q_j(x,\xi,z)$ is given by (see e.g.[Ba])

$$q_j = q_0 \sum_{m=1}^{2j} \sum_{((\alpha_i),(\beta_i))\in E_m^j} C_m((\alpha)_i,(\beta)_i)\prod_{i=1}^{m}\left((\partial_x^{\alpha_i}\partial_\xi^{\beta_i}p)q_0\right) \tag{3.11}$$

where $E_m^j = \{((\alpha_i),(\beta_i)) \in (\mathbf{Z}_+^n)^m \times (\mathbf{Z}_+^n)^m \ ; |\sum_{i=1}^m \alpha_i| = |\sum_{i=1}^m \beta_i| = j\}$ and $C_m((\alpha)_i,(\beta)_i)$ are universal constants. Let us set

$$\Gamma(x,\xi) = \{z \in \mathbf{C} \ ; \ z - \xi^2 \in \gamma(x)\}$$

and let us define

$$\hat{\pi}(x,\xi;h) = \Pi_0(x) + \sum_{j\geq 1}\hat{\pi}_j(x,\xi)h^j$$

where

$$\begin{aligned} \Pi_0(x) &= \frac{i}{2\pi}\oint_{\Gamma(x,\xi)} q_o(x,\xi,z)\,dz = \frac{i}{2\pi}\oint_{\gamma(x)}(V(x)-z)^{-1}\,dz \\ \hat{\pi}_j(x,\xi) &= \frac{i}{2\pi}\oint_{\Gamma(x,\xi)} q_j(x,\xi,z)\,dz \end{aligned} \tag{3.12}$$

Then $\hat{\pi}$ satisfies (3.1) and (3.2) and, by making the change of variable $z \to z - \xi^2$ in the integral (3.12) and having in mind (3.11), one can easily check that $\hat{\pi}_j(x,\xi) = \frac{i}{2\pi}\oint_{\gamma(x)} q_j(x,\xi,z+\xi^2)\,dz = \sum_{|\alpha|\leq j} a_{\alpha,j}(x)\xi^\alpha$ are polynomials matrix valued functions of degree j, with smooth coefficients decaying as $\langle x\rangle^{-\delta}$ as $|x| \to \infty$, if $j \geq 1$. Moreover, if $(H1A)$ and $(H3A)_\delta$ hold for some $\delta \geq 0$, then the $a_{\alpha,j}$ are analytic on Γ_a and (3.6) holds uniformly on $\Gamma_{a'}$, for any $0 < a' < a$.
■

Proof of Theorem 2.1 Let us consider the h-differential operator given by the Weyl quantization of the symbols $\hat{\pi}_j$ i.e

$$\hat{\Pi}_j = \mathrm{Op}_h^w(\hat{\pi}_j(x,\xi)) \tag{3.13}$$

Now we choose $\Phi \in C_0^\infty(\mathbf{R})$ and we consider the family of bounded operator $\Phi(P)\hat{\Pi}_j$. Then we can resum in a standard way the formal series of operator $\Phi(P)\Pi_0 + \sum_{j\geq 1} \Phi(P)\hat{\Pi}_j h^j$ and $\Pi_0\Phi(P) + \sum_{j\geq 1} \hat{\Pi}_j \Phi(P) h^j$ and we denote by $\Phi(P)\hat{\Pi}$ and $\hat{\Pi}\Phi(P)$ the resulting operators. Let us define

$$\hat{\Pi}_\Phi := \Phi(P)\hat{\Pi} + (1-\Phi(P))\hat{\Pi}\Phi(P) + (1-\Phi(P))\Pi_0(1-\Phi(P)). \tag{3.14}$$

Then $\hat{\Pi}_\Phi$ satisfy

$$||\hat{\Pi}_\Phi - \Pi_0||_\delta + ||\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi||_\delta + ||[\hat{\Pi}_\Phi, P]||_\delta = \mathcal{O}(h).$$

In particular, the spectrum of $\hat{\Pi}_\Phi$ is close to $\{0,1\}$ up to $\mathcal{O}(h)$ and, by using the standard calculus for operator-valued pseudodifferential operators, (see also the direct proof of Theorem 2.2) it is easy to check that, for $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi\Phi = \chi$, then, for any $N \in \mathbf{N}$, we have

$$||\chi(P)[\hat{\Pi}_\Phi, P]||_\delta + ||[\hat{\Pi}_\Phi, P]\chi(P)||_\delta = \mathcal{O}(h^{N+1}) \tag{3.15}$$

$$||\chi(P)(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)||_\delta + ||(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)\chi(P)||_\delta = \mathcal{O}(h^{N+1}). \tag{3.16}$$

Taking

$$\Pi := \frac{i}{2\pi}\int_{|z-1|=1/2} \left(\hat{\Pi}_\Phi - z\right)^{-1} dz$$

and recalling (see [Ne] and [NeSo])) that

$$\Pi - \hat{\Pi}_\Phi = \frac{i}{2\pi}(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)\oint_{|z-1|=1/2} (\hat{\Pi}_\Phi - z)^{-1}(2\hat{\Pi}_\Phi - 1)(1-\hat{\Pi}_\Phi - z)^{-1}(1-z)^{-1} dz \tag{3.17}$$

it is easy to prove that, there exists $C > 0$ independent of Nsuch that

$$||\chi(P)(\Pi - \hat{\Pi}_\Phi)||_\delta \leq C||\chi(P)(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)||_\delta \tag{3.18}$$

Using (3.16) and (3.18), one easily obtain (2.5) and (2.6). ∎

Now, let us consider the analytic case. In such a case, in order to get exponentially small error terms, the idea is to take instead of a classical resummation of $\Phi(P)\Pi_0 + \sum_{j\geq 1} \Phi(P)\hat{\Pi}_j h^j$, an analytic type resummation. This means that we want to consider $\Phi(P)\Pi_0 + \sum_{j=1}^{N} \Phi(P)\hat{\Pi}_j h^j$ when N can be h dependent i.e. $N = \left[\frac{1}{Bh}\right]$ with B large enough (see [BoKr, Sj, Ne]). Due to the fact that we have introduced cutoff functions in the definition of $\Phi(P)\Pi_0 + \sum_{j\geq 1} \Phi(P)\hat{\Pi}_j h^j$, we cannot used the standard calculus for analytic pseudodifferential operators and we need to control directly the constants involved. Let us start with the following lemma:

Lemma 3.2 *Let*

$$q_j(x,\xi) = \sum_{|\alpha|\le j} q_{\alpha,j}(x)\xi^\alpha \tag{3.19}$$

be polynomial functions of degree j with respect to ξ, with coefficients $q_{\alpha,j}(x) \in \mathcal{L}(\mathcal{H})$ analytic on Γ_a such that

$$\|\partial_x^\gamma q_{\alpha,j}(x)\|_{\mathcal{L}(\mathcal{H})} \le C^{j+\gamma+1} j!\gamma!\langle x\rangle^{-\delta}$$

for some $C > 0$ and $\delta \ge 0$ uniformly on Γ_a and let us set $Q_j(x, hD_x) = \mathrm{Op}_h^w(q_j(x,\xi))$ the Weyl quantization of q_j. Then, for any $\Phi \in C_0^\infty(\mathbf{R})$, we have,

$$\|\Phi(P)Q_j\|_\delta + \|Q_j\Phi(P)\|_\delta \le C^{j+1}j! \tag{3.20}$$

for some constant $C > 0$ independent of h and j, as far as $j \le \big[1/h\big]$.

Proof: Taking $a \in \mathbf{R}$, $a >> 1$ such that $\hat{P} =: P + a$ is invertible, one can write

$$\langle x\rangle^\delta \Phi(P)Q_j = \Big(\langle x\rangle^\delta\Phi(P)\hat{P}^{j/2}\langle x\rangle^{-\delta}\Big)\Big(\langle x\rangle^\delta\hat{P}^{-j/2}\hat{P}_0^{j/2}\langle x\rangle^{-\delta}\Big)\Big(\langle x\rangle^\delta P_0^{-j/2}Q_j\Big).$$

If $\tilde{\Phi} \in C_0^\infty(\mathbf{C})$ is an almost analytic extension of Φ such that $|\overline{\partial}\tilde{\Phi}(z)| \le C_1|\mathrm{Im}z|^m$ for a fixed $m \ge 1$, we have (see [DiSj])

$$\Phi(P)\hat{P}^{j/2} = \frac{1}{\pi}\int \overline{\partial}\tilde{\Phi}(z)(z+a)^{j/2}(P-z)^{-1}dz\, d\bar{z}$$

and then

$$\|\langle x\rangle^\delta\Phi(P)\hat{P}^{j/2}\langle x\rangle^{-\delta}\| \le \frac{C_2^jC_1}{\pi}\int_{\mathrm{supp}\tilde{\Phi}} |\mathrm{Im}z|^m\|\langle x\rangle^\delta(P-z)^{-1}\langle x\rangle^{-\delta}\|dz\, d\bar{z} \le C_3C_2^j \tag{3.21}$$

for some $C_1, C_2, C_3 > 0$ independent of j. Moreover, let us prove:

$$\|\langle x\rangle^\delta\hat{P}^{-j/2}\hat{P}_0^{j/2}\langle x\rangle^{-\delta}\| \le C_4^{j+1} \tag{3.22}$$

for some $C_4 > 0$ independent of j as far as $j \le \big[1/h\big]$. To prove (3.22) it is sufficient to show by induction on j, that, if $j \le \big[1/h\big]$, then there exists $\tilde{C}_4 > 0$ independent of j such that

$$\sum_{|\alpha|=j}\|\langle x\rangle^\delta(hD_x)^\alpha\hat{P}^{-j/2}\langle x\rangle^{-\delta}\| \le \tilde{C}_4^{j+1} \tag{3.23}$$

Suppose that (3.23) holds for $j \le k-1$ and let us prove it for $j = k$. Setting $\tilde{K} = \sum_{|\beta|=2}\|\langle x\rangle^\delta\hat{P}^{-1}(hD_x)^\beta\langle x\rangle^{-\delta}\|$ and writing

$$\begin{aligned}\sum_{|\alpha|=k}\langle x\rangle^\delta(hD_x)^\alpha\hat{P}^{-\frac{k}{2}}\langle x\rangle^{-\delta} \quad &= \quad \sum_{|\beta|=2}\langle x\rangle^\delta\hat{P}^{-1}(hD_x)^\beta\langle x\rangle^{-\delta}\sum_{|\alpha|=k-2}\langle x\rangle^\delta(hD_x)^\alpha\hat{P}^{-\frac{k}{2}+1}\langle x\rangle^{-\delta}\\ &\quad + \sum_{|\alpha|=k}\langle x\rangle^\delta[\hat{P}^{-1},(hD_x)^\alpha]\hat{P}^{-\frac{k}{2}+1}\langle x\rangle^{-\delta}\end{aligned}$$

by inductive assumptions, one has

$$\begin{aligned}&\sum_{|\alpha|=k}\|\langle x\rangle^{\delta}(hD_x)^{\alpha}\hat{P}^{-k/2}\langle x\rangle^{-\delta}\|\\ &\qquad\le K\tilde{C}_4^{k-1}+\sum_{|\alpha|=k}\sum_{\beta\le\alpha,\beta\neq0}\frac{\alpha!}{(\alpha-\beta)!\beta!}h^{|\beta|}\|\langle x\rangle^{\delta}\hat{P}^{-1}(\partial_x^{\beta}V)(hD_x)^{\alpha-\beta}\hat{P}^{-\frac{k}{2}}\langle x\rangle^{-\delta}\|.\end{aligned}$$

Observe now that, since V is analytic on Γ_a, we have $\|\langle x\rangle^{\delta}\partial_x^{\beta}V\|\le K_1^{\beta}\beta!$, for some $K_1>0$. Moreover since $\alpha\le|\alpha|\le k\le\left[1/h\right]$, we have $\dfrac{\alpha!}{(\alpha-\beta)!}\le\alpha^{\beta}\le\left[1/h\right]^{|\beta|}\le h^{-|\beta|}$. Hence, setting $c^2=\|\langle x\rangle^{\delta}\hat{P}^{-1}\langle x\rangle^{-\delta}\|$, one has

$$\begin{aligned}\sum_{|\alpha|=k}\|\langle x\rangle^{\delta}(hD_x)^{\alpha}\hat{P}^{-k/2}\langle x\rangle^{-\delta}\| &\le K\tilde{C}_4^{k-1}\\ &+ c^2\sum_{|\alpha|=k}\sum_{0<|\beta|\le k}(cK_1)^{|\beta|}\,\|\langle x\rangle^{\delta}(hD_x)^{\alpha-\beta}\hat{P}^{\frac{-(k-|\beta|)}{2}}\langle x\rangle^{-\delta}\|\end{aligned}$$

and then, by inductive assumption

$$\begin{aligned}&\sum_{|\alpha|=k}\|\langle x\rangle^{\delta}(hD_x)^{\alpha}\hat{P}^{-k/2}\langle x\rangle^{-\delta}\|\le K\tilde{C}_4^{k-1}+\tilde{C}_4^{k+1}c^2\sum_{0<|\beta|\le k}\left(\frac{K_1c}{\tilde{C}_4}\right)^{|\beta|}\\ &\le\tilde{C}_4^{k+1}\left(\frac{K}{\tilde{C}_4^2}+c^2\left((1-\frac{K_1c}{\tilde{C}_4})^{-n}-1\right)\right)\le\tilde{C}_4^{k+1}\end{aligned}$$

if $\tilde{C}_4$ is chosen in such a way that $\dfrac{K}{\tilde{C}_4^2}+c\left((1-\dfrac{K_1c}{\tilde{C}_4})^{-n}-1\right)<1$. This proves (3.23) and then (3.22). Finally, since one can write

$$Q_j(x,hD_x)=\sum_{k=0}^{j}h^k\sum_{|\beta|\le j-k}(hD_x)^{\beta}q_{\beta,j,k}(x)$$

for some smooth functions $q_{\beta,j,k}$ analytic on Γ_a and satisfying

$$\|\partial_x^{\gamma}q_{\beta,j,k}(x)\|_{\mathcal{L}(\mathcal{H})}\le C^{j+k+|\gamma|+1}j!k!\gamma!\langle x\rangle^{-\delta}$$

for some $C>0$ and $\delta\ge0$ uniformly on Γ_a is easy to check that, there exist $c_1,c_2,C_5>0$ such that

$$\begin{aligned}&\|\langle x\rangle^{\delta}\hat{P}_0^{-j/2}Q_j\|\le\sum_{k=0}^{j}h^k\sum_{|\beta|\le j-k}\|\langle x\rangle^{\delta}\hat{P}_0^{-j/2}(hD_x)^{\beta}\langle x\rangle^{-\delta}\|\,\|\langle x\rangle^{\delta}q_{\beta,j,k}(x)\|\\ &\le c_1^{j+1}j!(\sum_{k=0}^{j}h^kc_2^kk!)\le C_5^{j+1}j! \qquad (3.24)\end{aligned}$$

if $j \le [1/h]$. Then by (3.22), (3.23) and (3.24) one has that that

$$\|\langle x\rangle^{\delta}\Phi(P)Q_j\| \le C^{j+1}j!$$

for some constant $C > 0$ independent of j. In the same way one can easily prove that also

$$\|\Phi(P)Q_j\langle x\rangle^{\delta}\| \le C^{j+1}j!$$

and then, by interpolation, one obtains (3.20). ∎

Proof of Theorem 2.1: Let us define $\hat{\Pi}_j$ as in (3.13) and, for $N = [\frac{1}{Bh}]$ with B sufficiently large, let us set

$$\hat{\Pi}_{\Phi}^{B} := \Pi_0 + \sum_{j=1}^{N}\Phi(H)\hat{\Pi}_j h^j + \sum_{j=1}^{N}(1-\Phi(H))\hat{\Pi}_j\Phi(H)h^j. \tag{3.25}$$

Observe that, by Lemma 3.2, there exists a constant $C > 0$ independent of j such that, we have that, for $1 \le j \le [1/Bh]$, $B \ge 1$

$$\|\Phi(P)\hat{\Pi}_j\|_{\delta} + \|\hat{\Pi}_j\Phi(P)\|_{\delta} \le C^{j+1}j!. \tag{3.26}$$

Denoting here $\Pi_0 = \hat{\Pi}_0$, we have

$$\chi(P)[\hat{\Pi}_{\Phi}^{B}, P] = \chi(P)[\sum_{j=0}^{N}\hat{\Pi}_j h^j, P] = \sum_{k=N+1}^{2N+2} h^k\chi(P)Q_k(x, hD_x) \tag{3.27}$$

where Q_k is a differential operator of order k satisfying for $(x,\xi) \in \Gamma_a \times \mathbf{C}^n$ and for any multi-index α, β

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}Q_k(x,\xi)| \le A^{k+|\alpha|+|\beta|+1}k!\alpha!\beta!\langle\xi\rangle^{k-|\beta|}\langle x\rangle^{-\delta} \tag{3.28}$$

for some constant $A > 0$ independent of h, α, β, k. Hence, by Lemma 3.2, one obtain that, if $N = [\frac{1}{Bh}]$ and B is sufficiently large,

$$\|\chi(P)[\hat{\Pi}_{\Phi}^{B}, P]\|_{\delta} + \|[\hat{\Pi}_{\Phi}^{B}, P]\chi(P)\|_{\delta} \le C(\sum_{s=N+1}^{2N+4} h^s C^s s!) \tag{3.29}$$

for some constant $C > 0$ independent of h and s. If $\Phi, \Phi_1 \in C_0^{\infty}(\mathbf{R})$, are such that $\chi\Phi = \chi$, $\Phi\Phi_1 = \Phi$ one has

$$\begin{aligned}
\chi(P)((\hat{\Pi}_{\Phi}^{B})^2 - \hat{\Pi}_{\Phi}^{B}) &= \chi(P)(\sum_{s=0}^{2N} h^s \sum_{k=0}^{s}\hat{\Pi}_k\hat{\Pi}_{s-k} - \sum_{j=0}^{N} h^j\hat{\Pi}_j)\Phi_1(P)\\
&+\chi(P)\sum_{k,s=0}^{N} h^{k+s}[\hat{\Pi}_k, \Phi(P)]\hat{\Pi}_s\Phi_1(P) - \chi(P)\sum_{k,s=0}^{N} h^{k+s}\hat{\Pi}_k\Phi(P)[\hat{\Pi}_s, \Phi_1(P)]\\
&-\chi(P)\sum_{s,k=0}^{N} h^{s+k}[\hat{\Pi}_k, \Phi(P)]\hat{\Pi}_s\Phi(P) - \chi(P)(\sum_{j=0}^{N} h^j[\hat{\Pi}_j, \Phi(P)]\hat{\Pi}_0(1-\Phi(P))\\
&+\chi(P)\sum_{j=0}^{N} h^j[\hat{\Pi}_j, \Phi_1(P)]
\end{aligned} \tag{3.30}$$

Observe that

$$\sum_{s=0}^{2N} h^s \sum_{k=0}^{s} \hat{\Pi}_k \hat{\Pi}_{s-k} - \sum_{j=0}^{N} h^j \hat{\Pi}_j = \sum_{k=N+1}^{4N} h^k \tilde{Q}_k$$

where $\tilde{Q}_k$ is a differential operator of order k satisfying (3.28). Hence, by Lemma 3.2,

$$\|\chi(P)(\sum_{s=0}^{2N} h^s \sum_{k=0}^{s} \hat{\Pi}_k \hat{\Pi}_{s-k} - \sum_{j=0}^{N} h^j \hat{\Pi}_j)\Phi_1(P)\|_\delta \leq C(\sum_{s=N+1}^{4N} h^s C^s s!)$$

for some constant $C > 0$ independent of h and s, if $N = [\frac{1}{Bh}]$, and B is sufficiently large. Moreover, writing

$$\chi(P)[\sum_{k=0}^{N} h^k \hat{\Pi}_k, \Phi(P)] = -\frac{1}{\pi}\int \overline{\partial}\tilde{\Phi}(z)(P-z)^{-1}\chi(P)[\sum_{k=0}^{N} h^k \hat{\Pi}_k, P](P-z)^{-1} dz\, d\bar{z}$$

and taking into account (3.29) one easily obtains

$$\begin{aligned} \|\chi(P)[\sum_{k=0}^{N} h^k \hat{\Pi}_k, \Phi(P)]\sum_{s=0}^{N} h^s \hat{\Pi}_s \Phi_1(P)\|_\delta &\leq C(\sum_{k=N+1}^{2N+4} h^k C^k k!)(\sum_{s=0}^{N} h^s C_1^s s!) \\ &\leq C_2(\sum_{k=N+1}^{2N+4} h^k C^k k!). \end{aligned} \tag{3.31}$$

for some new constants C, C_1, C_2 and $C > 0$ independent of h and s, for $N = [\frac{1}{Bh}]$, and $B \geq C_1$ sufficiently large. The other terms in (3.30) con be treated in the same way and one gets

$$\|\chi(P)((\hat{\Pi}_\Phi^B)^2 - \hat{\Pi}_\Phi^B)\|_\delta \leq C(\sum_{k=N+1}^{4N+4} h^k C^k k!) \tag{3.32}$$

for a new constant $C > 0$ independent of h and s and $N = [\frac{1}{Bh}]$,with B sufficiently large. By the same kind of arguments one can easily prove an estimates analogous to (3.32) for $((\hat{\Pi}_\Phi^B)^2 - \hat{\Pi}_\Phi^B)\chi(P)$. Hence, setting $\hat{\Pi}_\Phi = \hat{\Pi}_\Phi^B$ with B large enough and using Stirling formula, one gets

$$\begin{aligned} &\|\chi(P)[\hat{\Pi}_\Phi, P]\| + \|[\hat{\Pi}_\Phi, P]\chi(P)\| = \mathcal{O}(e^{-c/h}) \\ &\|\chi(P)(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)\| + \|(\hat{\Pi}_\Phi^2 - \hat{\Pi}_\Phi)\chi(P)\| = \mathcal{O}(e^{-c/h}) \end{aligned} \tag{3.33}$$

for some constant $c > 0$. Taking

$$\Pi := \frac{i}{2\pi}\int_{|z-1|=1/2} \left(\hat{\Pi}_\Phi - z\right)^{-1} dz$$

by (3.17) and (3.33) one obtains (2.7) and (2.8). ∎

4 An application to scattering theory for matrix-valued Schrödinger operators

In this section we give an application of the results obtained in the previous one to the scattering theory for two components Schrödinger systems. We consider the two-component Schrödinger Hamiltonian

$$P(h) = -h^2 \Delta_x \mathbf{1}_2 + V(x) \tag{4.1}$$

acting on $\mathcal{H} := L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n)$. Here $\mathbf{1}_2$ is the 2×2 identity matrix and, for $x \in \mathbf{R}^n$, $V(x)$ is a 2×2 symmetric matrix. We assume that
$(A1)$ the map $\mathbf{R}^n \ni x \to V(x)$ admits an analytic extension in some strip

$$\Gamma_a = \{x \in \mathbf{C}^n \ ; \ |\mathrm{Im} x| < a\}, \qquad a > 0. \tag{4.2}$$

Concerning the spectrum of $V(x)$, we assume that
(A2) the two real eigenvalues $\lambda_i(x)$ of the matrix $V(x)$ satisfy:

$$\operatorname*{Inf}_{x \in \mathbf{R}^n} (\lambda_2(x) - \lambda_1(x)) := 2d > 0. \tag{4.3}$$

Possibly by shrinking Γ_a, one can suppose that $\operatorname*{Inf}_{x \in \Gamma_a} (\lambda_2(x) - \lambda_1(x)) \geq d$. Moreover we assume that $V(x)$ is $''$ short range$''$ i.e.
$(A3)_\delta$ there exists a real symmetric 2×2 matrix V_∞ such that

$$|V(x) - V_\infty| = \mathcal{O}(\langle x \rangle^\delta) \tag{4.4}$$

with $\delta \geq 0$, uniformly for $x \in \Gamma_a$. Without loss of generality, we can assume that V_∞ is diagonal i.e.

$$V_\infty := \begin{pmatrix} \lambda_{1,\infty} & 0 \\ 0 & \lambda_{2,\infty} \end{pmatrix}. \tag{4.5}$$

Let $\gamma(x) = \{z \in \mathbf{C} \ ; \ |z - \lambda_1(x)| = d/2\}$ and let us define

$$\Pi_0(x) := \frac{i}{2\pi} \oint_{\gamma(x)} (V(x) - z)^{-1} \, dz$$

Due to (4.4) and (4.5), we have, setting $\tilde{\Pi}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$\operatorname*{Sup}_{x \in \Gamma_a} \langle x \rangle^\delta |\Pi_0(x) - \tilde{\Pi}_0| < \infty \tag{4.6}$$

Let us fix $\Phi \in C_0^\infty(\mathbf{R})$. Applying the results of the previous section to $P(h)$ and, in particular, Theorem 2.2, one can construct a projector $\Pi(h)$ (that depends on the choice of Φ) on $\mathcal{H}$ satisfying (2.7) and (2.8). Since $\|\Pi - \Pi_0\| = \mathcal{O}(h)$, one can define (see [Ka, NeSo]) the transformation matrix corresponding to the pair Π, Π_0:

$$U := \left(1 - (\Pi - \Pi_0)^2\right)^{-1/2} (\Pi\Pi_0 + (1 - \Pi)(1 - \Pi_0)) \tag{4.7}$$

having the properties

$$U^*U = UU^* = \mathbf{1}_2 \tag{4.8}$$

$$U\Pi_0 = \Pi U. \tag{4.9}$$

Furthermore, since $\mathbf{R}^n$ is contractible, and, by assumptions, $V(x)$ is real on the real axis, one can find a 2×2 matrix $Q(x)$, unitary for real x, satisfying

$$\Pi_0(x) = Q(x)\tilde{\Pi}_0 Q^*(x)$$

Hence, defining the unitary operator

$$\Omega := UQ$$

and setting

$$\tilde{P} := \Omega^* P \Omega$$

one has that, for any $\chi \in C_0^\infty(\mathbf{R})$ such that $\Phi\chi = \chi$, then $\tilde{P}\chi(\tilde{P})$ has an almost block diagonal form. More precisely we have (see [NeSo]):

Proposition 4.1 *Assume that* $(A1), (A2)$ *and* $(A3)_\delta$ *hold true for some* $\delta \geq 0$. *Then, for any* $\Phi \in C_0^\infty(\mathbf{R})$, *there exist a bounded operator* $\tilde{D}_j$ *on* $L^2(\mathbf{R}^n)$ *and a bounded operator* R *on* $\mathcal{H}$ *such that*

$$\tilde{P} = D + R \tag{4.10}$$

with

$$D := \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_j := -h^2\Delta_x + \lambda_j(x) + h\tilde{D}_j \tag{4.11}$$

and, for any $\chi \in C_0^\infty(\mathbf{R})$ *satisfying* $\chi\Phi = \Phi$, *one has*

$$\|R\chi(\tilde{P})\|_\delta + \|\chi(\tilde{P})R\|_\delta = \mathcal{O}(e^{-c/h}) \tag{4.12}$$

uniformly on h, *for* $h > 0$ *small enough.*

Let us apply the result of Proposition 4.1 to scattering theory. Let us consider the wave operator associated to the pairs (P, P_0), where

$$P_0 = \begin{pmatrix} P_{0,1} & 0 \\ 0 & P_{0,2} \end{pmatrix} := -h^2\Delta_x + \begin{pmatrix} \lambda_{1,\infty} & 0 \\ 0 & \lambda_{1,\infty} \end{pmatrix}$$

i.e.

$$W_\pm(P, P_0) \quad = \quad \mathrm{s} - \lim_{t \to \pm\infty} e^{itP} e^{-itP_0}$$

and, similarly, the wave operators $W_\pm(\tilde{P}, P_0)$, $W_\pm(D, P_0)$ associated to the pairs $(\tilde{P}, P_0)$ and (D, P_0) respectively. If $\delta > 1$, it is easy to check that this wave operators exist and are complete so we can define the scattering matrix

$$S(P, P_0) = W_+(P, P_0)^* W_-(P, P_0), \quad S(\tilde{P}, P_0) = W_+(\tilde{P}, P_0)^* W_-(\tilde{P}, P_0),$$

and

$$S(D,P_0)=\left(\begin{array}{cc} S(D_1,P_{0,1}) & 0 \\ 0 & S(D_2,P_{0,2}) \end{array}\right)=W_+(D,P_0)^*W_-(D,P_0).$$

Since

$$\|\Omega\|_\delta=\mathcal{O}(1) \tag{4.13}$$

one can easily prove (see Lemma 3.6 of [NeSo]) that

$$W_\pm(P,P_0)=\Omega^*W_\pm(\tilde{P},P_0)$$

and then, since Ω is an unitary operator,

$$S(P,P_0)=S(\tilde{P},P_0).$$

Using the same argument of the proof of Theorem 3.10 in [NeSo], one easily obtains

Theorem 4.2 *Assume that* $(A1),(A2)$ *and* $(A3)_\delta$ *hold true for some* $\delta>1$. *Then, for any* $\Phi\in C_0^\infty(\mathbf{R})$, *denoting by* D *the diagonal operator in (4.11), there exists* $c_\Phi>0$ *(independent of* h*) such that for any* $\phi_k\in\mathcal{H}$, $\hat{\phi}_k\in\mathcal{C}_0^\infty(\mathbf{R}^n\backslash\{0\})\oplus\mathcal{C}_0^\infty(\mathbf{R}^n\backslash\{0\}), k=1,2$,

$$|\langle(S(P,P_0)-S(D,P_0))\,\phi_1,\phi_2\rangle|\leq C_\Phi(\phi_1,\phi_2)e^{-c_\Phi/h}. \tag{4.14}$$

for some constant $C_\Phi(\phi_1,\phi_2)>0$ *depending on* Φ,ϕ_1,ϕ_2 *but not on* h, *for* $h>0$ *small enough.*

Due to the possible presence of resonances, it seems hard to prove an analogous result for the scattering matrix $S(P,P_0,\lambda)$ without any additional assumptions. On the contrary, if we assume that we stay close to a non-trapping energy level λ_0, we are able to prove that the scattering matrix $S(P,P_0,\lambda)$ is diagonal up to exponentially small terms, obtaining in particular (with complete different techniques) the results of [MNS]. Let us give the following definition:

Definition 4.3 *We say that* $\lambda_0>\lambda_{2,\infty}$ *is non-trapping if, denoting by* $p_j(x,\xi)=\xi^2+\lambda_j(x)$, $j=1,2$ *and by* H_{p_j} *the Hamiltonian flows generated by* p_j, *then for any* $(x,\xi)\in\mathbf{R}^{2n}$, *such that* $p_j(x,\xi)=\lambda_0$ *one has:*

$$|\exp(tH_{p_j})(x,\xi)|\to\infty,\quad \text{if } |t|\to\infty \tag{4.15}$$

for $j=1,2$.

In the following we denote by $S(P,P_0,\lambda)=S(\tilde{P},P_0,\lambda)=\left(\begin{array}{cc} S_{1,1}(\lambda) & S_{1,2}(\lambda) \\ S_{2,1}(\lambda) & S_{2,2}(\lambda) \end{array}\right)$ and $S(D,P_0,\lambda)=\left(\begin{array}{cc} S(D_1,P_0,\lambda) & 0 \\ 0 & S(D_2,P_0,\lambda) \end{array}\right)$ the scattering matrix associated to the scattering operator $S(P,P_0)$ and $S(D,P_0)$ respectively.

Theorem 4.4 *Assume that* $(A1), (A2)$ *and* $(A3)_\delta$ *hold true for some* $\delta \geq 1$ *and let us consider a non-trapping energy level if* $\lambda_0 > \lambda_{2,\infty}$. *Then, there exist a neighborhood* I *of* λ_0 *and constants* $C, c > 0$ *(independent of* h*) such that*

$$\|S(P, P_0, \lambda) - S(D, P_0, \lambda)\|_{\mathcal{L}(L^2(S^{n-1}) \oplus L^2(S^{n-1}))} \leq Ce^{-c/h}. \tag{4.16}$$

for $\lambda \in I$ *and for* $h \in (0, h_0]$ *small enough. In particular, for small* h

$$\|S_{i,j}(\lambda)\|_{\mathcal{L}(L^2(S^{n-1}))} \leq Ce^{-c/h}, \quad \lambda \in I \tag{4.17}$$

Proof: Let us consider a function $\Phi \in C_0^\infty(\mathbf{R})$ such that $\Phi(\lambda) = 1$ for λ in a neighborhood J of λ_0. Associated to Φ, let us consider $\tilde{P}$ (depending on Φ) as in Proposition 4.1 and let us write

$$S(P, P_0) = S(\tilde{P}, P_0) = S(D, P_0) - W_+^*(\tilde{P}, P_0)(S(D, \tilde{P}) - \mathbf{1}_2)W_-(\tilde{P}, P_0). \tag{4.18}$$

Denoting by $F_0(\lambda)$ the spectral representation of P_0 and by

$$F_\pm(\lambda) := F_0(\lambda)W_\pm^*(\tilde{P}, P_0), \quad \lambda \in I$$

we can write

$$S(\tilde{P}, P_0, \lambda) = S(D, P_0, \lambda) - F_+(\lambda)F_0(\lambda)^*(S(D, \tilde{P}, \lambda) - \mathbf{1}_2)F_0(\lambda)F_-(\lambda)^* \tag{4.19}$$

Observe that (see [Na] and [MNS]) $F_+(\lambda)$ is a spectral representation of $\tilde{P}$ on $\mathrm{Ran}E_J(\tilde{P})$ (here $E_J(\tilde{P})$ denotes the spectral projection of an operator $\tilde{P}$ on J). Moreover, denoting by

$$\mathcal{H}^\alpha := L^{2,\alpha}(\mathbf{R}^n) \oplus L^{2,\alpha}(\mathbf{R}^n)$$

with $L^{2,\alpha}(\mathbf{R}^n) := \langle x \rangle^{-\alpha} L^2(\mathbf{R}^n)$ and using a stationary representation of $W_+(\tilde{P}, P_0)$ and semiclassical resolvent estimates for $\tilde{P}$ as in [JeNa], one can show that $F_+(\lambda)$ is bounded from $\mathcal{H}^\alpha$ to $L^2(S^{n-1}) \oplus L^2(S^{n-1})$ for $\alpha > 1/2$ and its operator norm is bounded by ch^{-2} uniformly for $\lambda \in J$. Hence we can apply the representation formula for the scattering matrix

$$S(D, \tilde{P}, \lambda) = \mathbf{1}_2 + 2\pi i F_+(\lambda)(R + R(D - \lambda - i0)^{-1}R)F_+(\lambda)^* \tag{4.20}$$

with $R = R^* = \tilde{P} - D$. Observe, that, if $\chi \in C_0^\infty(\mathbf{R})$ is chosen in such a way that $\chi(\lambda) = 1$ for $\lambda \in I \subset J$ and $\chi\Phi = \chi$, then $\|R\chi(\tilde{P})\|_\delta = \mathcal{O}(e^{-c/h})$, for some constant $c > 0$ independent of h. As a consequence, by (4.12) for $\lambda \in I$, $\|\langle x \rangle^{\delta/2} R F_1(\lambda)^*\|_{\mathcal{L}(L^2(S^{n-1}) \oplus L^2(S^{n-1}), \mathcal{H})} + \|F_1(\lambda) R \langle x \rangle^{\delta/2}\|_{\mathcal{L}(\mathcal{H}, L^2(S^{n-1}) \oplus L^2(S^{n-1}))} = \mathcal{O}(e^{-c/h})$ and using the resolvent estimates

$$\|\langle x \rangle^{-\delta/2}(D - \lambda - i0)^{-1}\langle x \rangle^{-\delta/2}\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(h^{-1}), \quad \lambda \in I. \tag{4.21}$$

and (4.20) we can conclude that

$$\|S(D, \tilde{P}, \lambda) - \mathbf{1}_2\|_{\mathcal{L}(L^2(S^{n-1}) \oplus L^2(S^{n-1}))} = \mathcal{O}(e^{-c/h}) \tag{4.22}$$

uniformly with respect to h and then, by (4.19), Theorem 4.4 follows. ∎

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