Variational principle and almost quasilocality for renormalized measures.

Roberto Fernández,
Laboratoire de Mathématiques Raphël Salem,
Université de Rouen, UFR Sciences, site Colbert,
76821 Mont Saint-Aignan, France.
E-mail: Roberto.Fernandez@univ-rouen.fr
Arnaud Le Ny,
Laboratoire de Mathématiques Raphël Salem,
Université de Rouen, UFR Sciences, site Colbert,
76821 Mont Saint-Aignan, France.
E-mail: Arnaud.Leny@univ-rouen.fr
Frank Redig,
Faculteit Wiskunde en Informatica TU Eindhoven,
Postbus 513, 5600 MB Eindhoven, The Netherlands.
E-mail:f.h.j.redig@TUE.nl

July 11, 2001

Keywords: Gibbs vs non-Gibbs, almost quasilocality.
AMS classification: 60G60 (primary), 82B20, 82B30 (secondary).

Abstract

We restore part of the thermodynamic formalism for some renormalized measures that are known to be non-Gibbsian. We first point out that a recent theory due to Pfister implies that for block-transformed measures free energy and relative entropy densities exist and are conjugated convex functionals. We then determine a necessary and sufficient condition for consistency with a specification that is quasilocal in a fixed direction. As corollaries we obtain consistency results for models with FKG monotonicity and for models with appropriate "continuity rates". For (noisy) decimations or projections of the Ising model these results imply almost quasilocality of the decimated "+" and "−" measures.

1 Introduction

Non-Gibbsian measures were initially detected as "pathologies" of renormalization group transformations of low-temperature Gibbs measures [13, 15]. Initial efforts were directed towards the construction of a sufficiently rich catalogue of examples and mathematical mechanisms leading to non-Gibbsianness [7]. the present research is focused on two practical aspects of the phenomenon: (i) determination of weaker notions of Gibbsianness which are preserved by the transformations of interest, and (ii) extension of the thermodynamic formalism to these broader classes of measures. The first issue motivated the notions of weak Gibbsianness and almost quasilocality [3, 11, 2, 21, 22, 19, 20, 17, 18]. The second issue was treated in [19] and [22] which discuss: (a) Existence of thermodynamic quantities, and (b) relations between the properties "consistency with the same specification" (CSS) and "zero relative entropy" (ZRE). Block renormalizations are considered in [19] and projections to the line in [22]. In both cases thermodynamic functionals are shown to exist, and, under suitable hypothesis on measure supports, the implication "CSS $\Longrightarrow$ ZRE" is shown in [19] and the opposed implication in [22]. Work in both references is based on the weakly Gibbsian approach, that is on the determination of a set of configurations of full measure on which the conditional probabilities take a Gibbsian form, with an almost surely finite but unbounded Hamiltonian.

Our results are complementary to those of these references on two counts. First, we use recent results of Pfister [24] on asymptotically decoupled measures, to restore the specification-invariant part of the variational principle (Definition 2.17 below) for block-renormalized measures. The result (Theorem 3.2) is completely independent of any Gibbs-restoration approach. Second, we establish conditions for the validity of the implication "ZRE $\Longrightarrow$ CSS" (Theorem 3.3) that, as we show in Corollary 3.5, can be related to almost quasilocality. In particular almost quasilocality is proven for block decimations (Corollary 3.6), strengthening previous weakly Gibbsian results [2, 23] obtained by more laborious means. In Proposition 3.15 we present a further criterion that proves the implication "ZRE $\Longrightarrow$ CSS" for projections to a line without further support assumptions.

In general terms, our results illustrate the fact that important aspects of the variational principle can be directly related to properties of specifications, without having to rely on rather detailed descriptions of weakly Gibbsian potentials.

2 Basic definitions and notation

We start by summarizing some basic notions for the sake of completeness. As general reference we mention [12]. See also [7], Section 2, for a streamlined exposition.

2.1 Quasilocality, specifications, consistent measures

We consider configuration spaces $\Omega = \Omega_0^{\mathbb{L}}$ with Ω_0 finite and $\mathbb{L}$ countable (typically $\mathbb{L} = \mathbb{Z}^d$), equipped with the product discrete topology and the product Borel σ-algebra $\mathcal{F}$. More generally, for (finite or infinite) subsets Λ of $\mathbb{L}$ we consider the corresponding measurable spaces $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, where $\Omega_\Lambda = \{-1, 1\}^\Lambda$. For any $\omega \in \Omega$, ω_Λ denotes its projection on Ω_Λ. We denote by $\mathcal{S}$ the set of the finite subsets of $\mathbb{L}$. A function $f : \Omega \to \mathbb{R}$ is **local** if there exists a finite set Δ such that $\omega_\Delta = \sigma_\Delta$ implies $f(\omega) = f(\sigma)$. The set of local functions is denoted $\mathcal{F}_{\rm loc}$.

Definition 2.1 *Let* $f : \Omega \to \mathbb{R}$.

(i) f *is* **quasilocal** *if it is the uniform limit of local functions, that is, if*

$$\lim_{\Lambda \uparrow \mathbb{L}} \sup_{\substack{\sigma,\omega: \\ \sigma_\Lambda = \omega_\Lambda}} \Bigl| f(\sigma) - f(\omega) \Bigr| \;=\; 0\,. \tag{2.2}$$

[The notation $\Lambda \uparrow \mathbb{L}$ *means convergence along a net directed by inclusion.]*

(ii) f *is* **quasilocal in the direction** $\theta \in \Omega$ *if*

$$\lim_{\Lambda \uparrow \mathbb{L}} \Bigl| f(\omega_\Lambda \theta_{\Lambda^c}) - f(\omega) \Bigr| \;=\; 0 \tag{2.3}$$

for each $\omega \in \Omega$. *[No uniformity required.]*

Let us present some remarks.

Remark 2.4 In the present setting (product of finite single-spin spaces) quasilocality is equivalent to continuity and uniform continuity. This follows from Stone-Weierstrass plus the fact that local functions are continuous for discrete spaces.

Remark 2.5 The pointwise analogues of (2.2) and (2.3) are the following: (i) f is *quasilocal at* ω if $\lim_{\Lambda \uparrow \mathbb{L}} \sup_{\sigma,\eta} \Bigl| f(\omega_\Lambda \sigma_{\Lambda^c}) - f(\omega_\Lambda \eta_{\Lambda^c}) \Bigr| = 0$;

(ii) f is *quasilocal at ω in the direction θ* if (2.3) holds only for this ω.
We shall not resort to these notions, but let us point out that the previous remark is no longer valid at the pointwise level: A function can be quasilocal in every direction at a certain ω (that is, continuous at ω) and fail to be quasilocal at ω.

Here is an example illustrating the last remark. Let $d = 1$ and, for a fixed $\omega \in \Omega$, choose a countable family $\chi^{(m)}$ of configurations such that $\chi^{(m)}_{[-n,n]^c} \neq \chi^{(m')}_{[-n,n]^c}$ for all $n \in \mathbb{N}$, if $m \neq m'$, and such that $\chi_0^{(m)} \neq \omega_0$ for all m. Define

$$f(\eta) \;=\; \begin{cases} m/(n+m) \text{ if } \eta = \omega_{[-n,n]}\, \chi^{(m)}_{[-n,n]^c} \text{ for some } m, n \in \mathbb{N} \\ 0 \qquad\qquad\quad \text{otherwise .} \end{cases} \tag{2.6}$$

We see that, for all $\sigma \in \Omega$,

$$\lim_{\Lambda \uparrow \mathbb{L}} f(\omega_\Lambda \sigma_{\Lambda^c}) \to 0 = f(\omega). \tag{2.7}$$

Hence f is quasilocal at ω in every direction. However,

$$\sup_{\sigma,\eta} \Bigl| f(\omega_{[-n,n]}\sigma_{[-n,n]^c}) - f(\omega_{[-n,n]}\eta_{[-n,n]^c}) \Bigr| \;=\; \sup_m \frac{m}{n+m} \;=\; 1\;. \tag{2.8}$$

So f is not quasilocal at ω.

Definition 2.9 *A* **specification** *on $(\Omega, \mathcal{F})$ is a family $\gamma = \{\gamma_\Lambda, \Lambda \in \mathcal{S}\}$ of stochastic kernels on $(\Omega, \mathcal{F})$ that are*

(I) Proper*: $\forall B \in \mathcal{F}_{\Lambda^c}$, $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$.*

(II) Consistent*: If $\Lambda \subset \Lambda'$ are finite sets, then $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$.*

[We adopt the "conditional-probability" notation, that is, $\gamma_\Lambda(A|\,\cdot\,)$ is $\mathcal{F}_{\Lambda^c}$-measurable $\forall A \in \mathcal{F}$, and $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$ $\forall \omega \in \Omega$.] The notation $\gamma_{\Lambda'}\gamma_\Lambda$ refers to the natural composition of probability kernels: $(\gamma_{\Lambda'}\gamma_\Lambda)(A|\omega) = \int_\Omega \gamma_\Lambda(A|\omega')\gamma_{\Lambda'}(d\omega'|\omega)$. A specification is, in fact, a strengthening of the notion of system of proper regular conditional probabilities. Indeed, in the former, the consistency condition (II) is required to hold for *every* configuration $\omega \in \Omega$, and not only for almost every $\omega \in \Omega$. This is because the notion of specification is defined without any reference to a particular measure.

A probability measure μ on $(\Omega, \mathcal{F})$ is said to be *consistent* with a specification γ if the latter is a realization of its finite-volume conditional probabilities, that is, if $\mu[A|\mathcal{F}_{\Lambda^c}](\,\cdot\,) = \gamma_\Lambda(A|\,\cdot\,)$ μ-a.s. for all $A \in \mathcal{F}$ and $\Lambda \in \mathcal{S}$. Equivalently, μ is consistent with γ if it satisfies the *DLR equation* (for Dobrushin, Lanford and Ruelle):

$$\mu \;=\; \mu\,\gamma_\Lambda \tag{2.10}$$

for each $\Lambda \in \mathcal{S}$. The right-hand side is the composed measure: $(\mu\,\gamma_\Lambda)(f) = \int \gamma_\Lambda(f|\omega)\,\mu(d\omega)$ for f bounded measurable. We denote $\mathcal{G}(\gamma)$ the set of measures consistent with γ (measures *specified* by γ). The description of this set is, precisely, the central issue in equilibrium statistical mechanics.

A specification γ is *quasilocal* if for each $\Lambda \in \mathcal{S}$ and each f local, $\gamma_\Lambda f$ is a quasilocal function. Analogously, the specification is *quasilocal in the direction θ* if so are the functions $\gamma_\Lambda f$ for local f and finite Λ. A probability measure μ is *quasilocal* if it is consistent with some quasilocal specification. Gibbsian specifications —defined through interactions via Boltzmann's prescription— are the archetype of quasilocal specifications. Every Gibbs measure —i.e. every measure consistent with a Gibbsian specification— is quasilocal, and the converse requires only the additional property of non-nullness [16, 26, 17]. Reciprocally, a sufficient condition for non-Gibbsianness is the existence of an *essential non-quasilocality* (essential discontinuity), that is, of a configuration at which *every* realization of some finite-volume conditional probability of μ is discontinuous. These discontinuities are related to the existence of phase transitions in some constrained systems [13, 15, 7].

Two categories of measures have been defined in an effort to extend the Gibbsian formalism to non-quasilocal measures: *weakly Gibbsian* and *almost quasilocal* (see [21] and references therein). The latter is the object of the present paper and we include its definition for completeness. For a specification γ let Ω_γ be the set of configurations where $\gamma_\Lambda f$ is continuous for all $\Lambda \in \mathcal{S}$ and all f local, and Ω_γ^θ the set of configurations for which all the functions $\gamma_\Lambda f$ are quasilocal in the direction θ.

Definition 2.11

1. *A probability measure is* **almost quasilocal in the direction** θ *if it is consistent with a specification* γ *such that* $\mu(\Omega_\gamma^\theta) = 1$.

2. *A probability measure* μ *is* **almost quasilocal** *if it is consistent with a specification* γ *such that* $\mu(\Omega_\gamma) = 1$.

2.2 “Thermodynamic” functions and the variational principle

The variational principle links statistical mechanical and thermodynamical quantities. Rigorously speaking, the functions defined below (pressure and entropy density) do not quite correspond to standard thermodynamics. The corresponding notions in the latter depend only of a few parameters, while the objects below are functions on infinite dimensional spaces. These functions are, however, more informative from the probabilistic point of view, because, at least in the Gibbsian case, they are related to large-deviation principles.

Translation invariance plays an essential role in the thermodynamic formalism. That is, we assume that there is an action (“translations”) $\{\tau_i : i \in \mathbb{Z}^d\}$ on $\mathbb{L}$ which defines corresponding actions on configurations —$(\tau_i\omega)_x = \omega_{\tau_{-i}x}$—, on functions —$\tau_i f(\omega) = f(\tau_{-i}\omega)$—, on measures —$\tau_i\mu(f) = \mu(\tau_{-i}f)$— and on specifications —$(\tau_i\gamma)_\Lambda(f|\omega) = \gamma_{\tau_{-i}\Lambda}(\tau_{-i}f|\tau_{-i}\omega)$. [To simplify the notation we will write $\mu(f)$ instead of $E_\mu(f)$.] Translation invariance means invariance under all actions τ_i. We consider, in this section, only translation-invariant probability measures on Ω, whose space we denote by $\mathcal{M}^+_{1,\mathrm{inv}}(\Omega)$. We denote $\mathcal{G}_{\mathrm{inv}}(\gamma) = \mathcal{G}(\gamma) \cap \mathcal{M}^+_{1,inv}(\Omega)$. Furthermore, the convergence along subsets of $\mathbb{L}$ is restricted to sequences of cubes $\Lambda_n = \{\tau_{-i}(0) : i \in ([-n,n] \cap \mathbb{Z})^d\}$.

Definition 2.12 ([24]) *A measure $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ is* **asymptotically decoupled** *if there exist functions $g : \mathbb{N} \longrightarrow \mathbb{N}$ and $c : \mathbb{N} \longrightarrow [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{c(n)}{|\Lambda_n|} = 0\ , \tag{2.13}$$

such that for all $n \in \mathbb{N}$, $A \in \mathcal{F}_{\Lambda_n}$ and $B \in \mathcal{F}_{(\Lambda_{n+g(n)})^c}$,

$$e^{-c(n)}\,\nu(A)\,\nu(B) \;\le\; \nu(A\cap B) \;\le\; e^{c(n)}\,\nu(A)\,\nu(B)\ . \tag{2.14}$$

This class of measures strictly contains the set of all Gibbs measures. In particular, as we observe below, it includes measures obtained by block transformations of Gibbs measures, many of which are known to be non-Gibbsian.

For $\mu, \nu \in (\Omega, \mathcal{F})$, the *relative entropy* at volume $\Lambda \in \mathcal{S}$ of μ relative to ν is defined as

$$H_\Lambda(\mu|\nu) \;=\; \begin{cases} \displaystyle\int_\Omega \frac{d\mu_\Lambda}{d\nu_\Lambda} \log \frac{d\mu_\Lambda}{d\nu_\Lambda}\, d\nu & \text{if } \mu_\Lambda \ll \nu_\Lambda \\[2ex] +\infty & \text{otherwise.} \end{cases} \tag{2.15}$$

The notation μ_Λ refers to the projection (restriction) of μ to $(\Omega_\Lambda, \mathcal{F}_\Lambda)$. The *relative entropy density* of μ relative to ν is the limit

$$h(\mu|\nu) \;=\; \lim_{n\to\infty} \frac{H_{\Lambda_n}(\mu|\nu)}{|\Lambda_n|} \tag{2.16}$$

provided it exists. The limit is known to exist if $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ is a Gibbs measure (and $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$ arbitrary) and, more generally [24], if ν is asymptotically decoupled. In these cases $h(\,\cdot\,|\nu)$ is an affine non-negative function on $\mathcal{M}^+_{1,inv}(\Omega)$.

For $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ and f a bounded measurable function, the *pressure* (or minus free-energy density) for f relative to ν is defined as the limit

$$p(f|\nu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|^d} \log \int \exp\Bigl(\sum_{x\in\Lambda_n} \tau_x f\Bigr)\, d\nu$$

whenever it exists. This limit exists, for every quasilocal function f, if ν is Gibbsian or asymptotically decoupled [24], yielding a convex function $p(\,\cdot\,|\nu)$.

For our purposes, it is important to separate the different ingredients of the usual variational principle in statistical mechanics.

Definition 2.17 (Specification-independent variational principle) *A measure $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ satisfies a variational principle if the relative entropy $h(\mu|\nu)$ and the pressure $p(f|\nu)$ exist for all $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$ and all $f \in \mathcal{F}_{\rm loc}$, and they are conjugate convex functions in the sense that*

$$p(f|\nu) \;=\; \sup_{\mu\in\mathcal{M}^+_{1,\rm inv}(\Omega)} \Bigl[\mu(f) - h(\mu|\nu)\Bigr] \tag{2.18}$$

for all $f \in \mathcal{F}_{\rm loc}$, and

$$h(\mu|\nu) \;=\; \sup_{f\in\mathcal{F}_{\rm loc}} \Bigl[\mu(f) - p(f|\nu)\Bigr] \tag{2.19}$$

for all $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$.

Gibbs measures satisfy this specification-independent principle. Pfister [24, Section 3.1] has recently extended its validity to asymptotically decoupled measures. In these cases $h(\,\cdot\,|\nu)$ is the rate function for a (level 3) large-deviation principle for ν.

Definition 2.20 (Variational principle relative to a specification) *Let γ be a specification and $\nu \in \mathcal{G}_{\mathrm{inv}}(\gamma)$. We say that a variational principle occurs for ν and γ if for all $\mu \in \mathcal{M}^{+}_{1,inv}(\Omega)$*

$$h(\mu|\nu) = 0 \iff \mu \in \mathcal{G}_{\mathrm{inv}}(\gamma) \ . \tag{2.21}$$

The equivalence (2.21) holds for Gibbs measures ν, while the implication to the right is valid, more generally, for measures ν consistent with γ quasilocal (see [12], Chapter 10). In [19] the implication to the left was extended to block-transformed measures satisfying appropriate support hypothesis. Below we extend the implication to the right to some non-Gibbsian (non-quasilocal) measures.

2.3 Transformations of measures

Definition 2.22 *A* **renormalization transformation** *T from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ is a probability kernel $T(\,\cdot\,|\,\cdot\,)$ on $(\Omega, \mathcal{F}')$. That is, for each $\omega \in \Omega$ $T(\,\cdot\,|\omega)$ is a probability measure on $(\Omega', \mathcal{F}')$ and for each $A' \in \mathcal{F}'$ $T(A'|\,\cdot\,)$ is $\mathcal{F}$-measurable.*

The transformation is a **block-spin transformation** *if Ω' is of the form $(\Omega'_0)^{\mathbb{L}'}$ and there exists $\alpha > 0$ (compression factor) such that the following two properties hold*

(i) Strict locality*: For every n, $A' \in \mathcal{F}'_{\Lambda'_n}$ implies $T^{-1}(A') \in \mathcal{F}_{\Lambda'_{[\alpha n]}}$.*

(ii) Factorization*: There exists a distance* dist *in $\mathbb{L}'$ such that if $A' \in \mathcal{F}'_{D'}$ and $B' \in \mathcal{F}'_{E'}$ with $\mathrm{dist}(D', E') > \alpha$, then $T(A' \cap B'|\,\cdot\,) = T(A'|\,\cdot\,)\, T(B'|\,\cdot\,)$.*

A renormalization transformation is *deterministic* if it is of the form $T(\,\cdot\,|\omega) = \delta_{t(\omega)}(\,\cdot\,)$ for some $t : \Omega \to \Omega'$.

A renormalization transformation T induces a transformation $\mu \mapsto \mu T$ on measures, with $(\mu T)(f') = \mu[T(f')]$ for each $f' \in \mathcal{F}'$.

In most applications, block-spin transformations have a product form: $T(d\omega'|\omega) = \prod_{x'} T_{x'}(d\omega_{x'}|\omega)$, where $T_x(\{\omega_{x'}\}|\,\cdot\,) \in \mathcal{F}_{B_{x'}}$, for a family of sets $\{B_{x'} \subset \mathbb{L} : x' \in \mathbb{L}'\}$ —the *blocks*— with bounded diameter whose union covers $\mathbb{L}$. Transformations of this sort are called *real-space renormalization transformations* in physics. The transformations defining cellular automata (with local rules) fit also into this framework. The corresponding blocks overlap and the compression factor may be chosen arbitrarily close to one.

We briefly remind the reader of some of the transformations considered in the sequel:

- *Projections and decimations*: Given $D \subset \mathbb{L}$, this is the (product) deterministic transformation defined by $t(\omega) = (\omega_x)_{x \in D}$. The *decimation of spacing* $b \in \mathbb{N}$, for which $\mathbb{L} = \mathbb{Z}^d$, $D = b\mathbb{Z}^d$, is a block transformation, while *Schonmann's example* [25], corresponding to $D =$ hyperplane, is not because it fails to be strictly local.

- *Kadanoff*: This is a product block transformation defined by $T_x(d\omega_x|\omega) = \exp(p\,\omega'_x \sum_{y \in B_x} \omega_y)/\text{norm}$, for a given choice of parameter p and blocks B_x. In the limit $p \to \infty$ one obtains the *majority transformation* for the given blocks. If $B_x = bx$ this is a *noisy decimation*, that becomes the true decimation in the limit $p \to \infty$. More generally, one can define a *noisy projection* onto $D \subset \mathbb{L}$ through the transformation $\prod_{x \in D} \exp(p\,\omega'_x \omega_x)/\text{norm}$.

It is well known that renormalization transformations can destroy Gibbsianness (for reviews see [5, 6, 8, 9]). Most of the non-Gibbsian measures resulting from block transformations were shown to be weakly Gibbsian [2, 23]. In Corollaries 3.5 and 3.6 below, we show that in some instances they are, in fact, almost quasilocal.

2.4 Monotonicity-preserving specifications

Finally we review notions related to stochastic monotonicity. Let us choose an appropriate (total) order "$\leq$" for Ω_0 and, inspired by the case of the Ising model, let us call "plus" and "minus" the maximal and minimal elements. The choice induces a partial order on Ω: $\omega \leq \sigma \iff \omega_x \leq \sigma_x \ \forall x \in \mathbb{L}$. Its maximal and minimal elements are the configurations, denoted "$+$" and "$-$" in the sequel, respectively equal to "plus" and to "minus" at each site. For brevity, quasilocality in the "$+$", resp. "$-$", direction will be called *right continuity*, resp. *left continuity*. The partial order determines a notion of monotonicity for functions on Ω. A specification π is *monotonicity preserving* if for each finite $\Lambda \subset \mathbb{L}$, $\pi_\Lambda f$ is increasing whenever f is. These specifications have a number of useful properties. In the following lemma, we summarize the properties of monotonicity preserving specifications which we need in the sequel. Proofs and more details can be found in [11].

Lemma 2.23 *Let γ be a monotonicity-preserving specification*

(a) The limits $\gamma_\Lambda^{(\pm)}(\,\cdot\,|\omega) = \lim_{S \uparrow \mathbb{L}} \gamma_\Lambda(\,\cdot\,|\omega_S \pm_{S^c})$ exist and define two monotonicity-preserving specifications, $\gamma^{(+)}$ being right continuous and $\gamma^{(-)}$ left continuous. The specifications are translation-invariant if so is γ.

Furthermore, $\gamma^{(-)}(f) \leq \gamma(f) \leq \gamma^{(+)}(f)$ for any local increasing f, and the specifications $\gamma^{(+)}$, $\gamma^{(-)}$ and γ are continuous on the set

$$\Omega_{\pm} \;=\; \Big\{\omega \in \Omega : \gamma^{(+)}(f|\omega) = \gamma^{(-)}(f|\omega) \;\forall f \in \mathcal{F}_{\rm loc}, \Lambda \in \mathcal{S}\Big\} \;. \tag{2.24}$$

(b) *The limits $\mu^{\pm} = \lim_{\Lambda\uparrow\mathbb{L}} \gamma_\Lambda(\,\cdot\,|\pm)$ exist and define two extremal measures $\mu^{\pm} \in \mathcal{G}(\gamma^{(\pm)})$ [thus μ^{+} is right continuous and μ^{-} left continuous] which are translationn-invariant if so is γ. If f is local and increasing, $\mu^{-}(f) \leq \mu(f) \leq \mu^{+}(f)$ for any $\mu \in \mathcal{G}(\gamma)$.*

(c) *For each (finite or infinite) $D \subset \mathbb{L}$, the conditional expectations $\mu^{+}(f|\mathcal{F}_D)$ and $\mu^{-}(f|\mathcal{F}_D)$ can be given everywhere-defined monotonicity-preserving right, resp left, continuous versions. In fact, these expectations come, respectively, from* global specifications, *that is, from families of stochastic kernels satisfying Definition 2.9 also for infinite $\Lambda \subset \mathbb{L}$. Furthermore, $\mu^{-}(f|\mathcal{F}_D) \leq \mu^{+}(f|\mathcal{F}_D)$ for each f increasing.*

(d) *For each (infinite) $D \subset \mathbb{L}$ there exist monotonicity preserving specifications $\Gamma^{(D,\pm)}$ such that the projections $\mu^{\pm}_D \in \mathcal{G}(\Gamma^{(D,\pm)})$ and $\Gamma^{(D,-)}_\Lambda(f) \leq \Gamma^{(D,+)}_\Lambda(f)$ for each f increasing. [By (a) and (c) $\Gamma^{(D,+)}$ ($\Gamma^{(D,-)}$) can be chosen to be right (left) continuous and extended to a global specification on Ω_D with the same properties.]*

Models satisfying the FKG property [10] are the standard source of monotonicity-preserving specifications. This class of models includes the ferromagnets with two- and one-body interactions (eg. Ising). Item (d) of the lemma is potentially relevant for renormalized measures because of the fact that a transformed measure μT can be seen as the projection on the primed variables of the measure $\mu \times T$ on $\Omega \times \Omega'$ defined by

$$(\mu \times T)(d\omega, d\omega') \;=\; T(d\omega'|\omega)\, \mu(d\omega) \;. \tag{2.25}$$

To apply (d) of the lemma, however, one has to find a suitable specification for this measure $\mu \times T$. If $\mu \in \mathcal{G}(\gamma)$ and T is a product transformation, a natural candidate is the family $\gamma \otimes T$ of stochastic kernels

$$(\gamma \otimes T)_{\Lambda\times\Lambda'}(d\omega_\Lambda, d\omega'_{\Lambda'}|\omega_{\Lambda^c}, \omega'_{(\Lambda')^c}) \;=\; \prod_{\substack{x' : x' \in \Lambda' \\ \text{or } B_{x'}\cap\Lambda \neq \emptyset}} T_{x'}(d\omega_{x'}|\omega_{B_{x'}})\, \gamma_\Lambda(d\omega_\Lambda|\omega_{\Lambda^c}) \;. \tag{2.26}$$

Definition 2.27 *A pair* (γ, T)*, where* γ *is a specification and* T *a product renormalization transformation, is a* **monotonicity-preserving pair** *if the family* $\gamma \otimes T$ *is a monotonicity-preserving specification.*

It does not seem to be so simple to construct such monotonicity-preserving pairs. The only examples we know of are pairs for which $\gamma \otimes T$ is Gibbsian for a FKG interaction. This happens, for instance, for noisy projections (in particular noisy decimations) of the Ising measure.

3 Results

The following result follows immediately from Definitions 2.12 and 2.22.

Lemma 3.1 *If* $\mu \in \mathcal{M}_1^+(\Omega)$ *is asymptotically decoupled, then so is* μT *for every block-spin transformation* T*.*

From the results of Pfister, we can then conclude the following:

Theorem 3.2 *Let* $\mu \in \mathcal{M}_{1,inv}^+(\Omega)$ *be asymptotically decoupled and* T *be a block-spin transformation such that* μT *is translation-invariant. Then the renormalized measure* μT *satisfies the specification-independent variational principle of Definition 2.17.*

In [24, Section 3.4] it is showed that the relative entropy density $h(\,\cdot\,|\mu T)$ is the large deviation rate function of the empirical measure $L_\Lambda = \sum_{x\in\Lambda} \delta_{\tau_x \sigma}$.

The next theorem states the criterion used in this paper to prove the implication to the right in (2.21) for non-quasilocal measures ν.

Theorem 3.3 *Let* γ *be a specification that is quasilocal in the direction* $\theta \in \Omega$ *and* $\nu \in \mathcal{G}_{\rm inv}(\gamma)$*. For each* $\Lambda \in \mathcal{S}$*,* $M \in \mathbb{N}$*,* $\Lambda \subset \Lambda_M$ *and each local* f*, let* $\gamma_\Lambda^{M,\theta}(f)$ *denote the function* $\omega \to \gamma_\Lambda(f|\omega_{\Lambda_M}\,\theta_{\mathbb{L}\setminus\Lambda_M})$*. Then, if* $\mu \in \mathcal{M}_{1,\rm inv}^+(\Omega)$ *is such that* $h(\mu|\nu) = 0$*,*

$$\mu \in \mathcal{G}_{\rm inv}(\gamma) \iff \nu\Big[\,\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\Big(\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\Big)\Big] \underset{M\to\infty}{\longrightarrow} 0 \tag{3.4}$$

for all $\Lambda \in \mathcal{S}$ *and* $f \in \mathcal{F}_{\rm loc}$*.*

The right-hand-side of (3.4) shows that consistency requires the concentration properties of $d\mu_{\Lambda_M\setminus\Lambda}/d\nu_{\Lambda_M\setminus\Lambda}$ to beat asymptotic divergences due to the lack of continuity of γ_Λ. This imposes some conditions on μ which are reminiscent of what happens for unbounded spin-systems. This analogy between unbounded spin systems and non-Gibbsian measures is an early remark from Dobrushin. Within approaches based on potentials (weak Gibbsianness) these conditions are defined and handled by cluster-expansion methods [22, 19]. As we discuss below, in favorable cases montonicity arguments can be used instead.

We present two applications of the previous theorem. First we discuss systems with monotonicity-preserving specifications.

Corollary 3.5 *Consider a specification* γ *that is monotonicity preserving and translation invariant. Then, with the notation of Lemma 2.23,*

(a) $h(\mu^-|\mu^+) = 0$ *implies that* $\mu^- \in \mathcal{G}(\gamma^{(+)})$ *and* $\mu^-(\Omega_{\gamma^{(+)}}) = \mu^-(\Omega_{\gamma^{(-)}}) = 1$ *(hence* μ^- *is almost quasilocal).*

(b) For $\mu \in \mathcal{M}^+_{1,\mathrm{inv}}(\Omega)$*,* $h(\mu|\nu^+) = 0$ *and* $\mu(\Omega_\pm) = 1$ *implies* $\mu \in \mathcal{G}(\gamma^{(+)})$*, and thus* μ *almost quasilocal.*

Analogous results are valid interchanging "+" with "−".

By part (d) of Lemma 2.23, and the comments immediately thereafter, the preceding results apply when $\mu^\pm$ are the projections (possibly noisy) of the "plus" and "minus" phases of the Ising model. More generally, they can be the renormalized measures of the "plus" and "minus" measures of a monotonicity-preserving specification whenever the specification and the transformation form a monotonicity-preserving pair (Definition 2.27).

At low temperature, the decimations (possibly noisy) μ^+T and μ^-T of the "plus" and "minus" phases of the Ising model are non-Gibbsian [15, 7], that is, all specifications with which these measures are consistent show essential discontinuities. The preceding corollary shows that, nevertheless, in these cases the implication to the right of the variational principle (2.21) can be recovered up to a point.

If γ is a quasilocal translation-invariant specification and T a block-spin transformation, then $h(\mu T|\nu T) = 0$ for each $\mu, \nu \in \mathcal{G}_{\mathrm{inv}}(\gamma)$ such that μT and νT are translation invariant [7, formula (3.28)]. Hence, from part (a) of the previous corollary we conclude the following.

Corollary 3.6 *Let* γ *be a quasilocal, monotonicity-preserving, translation-invariant specification, and* T *a block-spin transformation that preserves translation invariance such that the pair* (γ, T) *is monotonicity-preserving.*

Let $\mu^{\pm}$ be the extremal measures for γ [part (b) of Lemma 2.23] and $\pi^{(\pm)}$ be the right(left)-continuous specifications such that $\mu^{\pm}T \in \mathcal{G}_{\rm inv}(\pi^{\pm})$.

Then

$$\mu^{-}T \in \mathcal{G}(\pi^{+}) \quad \textit{and} \quad \mu^{-}T(\Omega_{\pi^{+}}) = \mu^{-}T(\Omega_{\pi^{-}}) = 1 \tag{3.7}$$

(hence $\mu^{-}T$ is almost quasilocal). Analogous results are valid interchanging "+" with "−".

This corollary applies in particular for decimations (possibly noisy) of the Ising model. At low temperature, the renormalized measures $\mu^{+}T$ and $\mu^{-}T$ are in general non-Gibbsian [15, 7], that is, the specifications π^{+} and π^{-} show essential discontinuities. Nevertheless, the preceding corollary, together with part (b) of Corollary 3.5 shows that in these cases the implication to the right of the variational principle (2.21) can be recovered, together with almost quasilocality.

Several remarks are in order.

Remark 3.8 The preceding corollary strengthens, for (noisy) decimation transformations, the results of [2, 23] where only weak-Gibbsianness is proven. Our argument is apparently simpler than the renormalization and expansion-based procedures set up in these references, but, of course, it does not give such a complete description of the support of the decimated measures and it is only restricted to models with monotonicity properties.

Remark 3.9 For $d = 2$, the corollary implies that *all* the decimated measures of the Ising model are consistent with π^{+} and almost quasilocal. This follows from the results of Aizenman [1] and Higuchi [14] showing that μ^{+} and μ^{-} are the only extremal measures in $\mathcal{G}(\gamma)$.

Remark 3.10 Lefevere proves in [19] the implication to the left in (2.21), for ν a block-transformed measure and μ concentrated on an appropriate set $\widetilde{\Omega} \subset \Omega$ of ν-measure 1.

Our second application of Theorem 3.3 does not involve any monotonicity hypothesis. To formulate it we need some notation. For Λ a fixed finite volume, and f a local function, put

$$\delta^{\theta}_{\Lambda,M}(f) \;=\; \left|\gamma^{M,\theta}_{\Lambda}(f) - \gamma_{\Lambda}(f)\right| \tag{3.11}$$

and introduce for $\epsilon > 0$ the sets

$$A(\theta, \Lambda, f, \epsilon, M) \;=\; \{\eta \in \Omega : \delta^{\theta}_{\Lambda,M}(f) > \epsilon\} \,. \tag{3.12}$$

If γ is continuous in the direction θ, then $\delta^{\theta}_{\Lambda,M}$ tends to zero as M tends to infinity, and hence for any probability measure μ, $\mu[A(\theta,\Lambda,f,\epsilon,M)]$ tends to zero as M tends to infinity.

Definition 3.13 *Let $\alpha_M \uparrow \infty$ be an increasing sequence of positive numbers. Let μ be a probability measure on Ω. We say that the specification γ admits α_M as a μ-rate of continuity in the direction θ if for all $\epsilon > 0$, for all f local, and for all Λ:*

$$\limsup_{M\uparrow\infty} \frac{1}{\alpha_M} \log \mu[A(\theta,\Lambda,f,\epsilon,M)] \;<\; 0. \tag{3.14}$$

The following proposition shows that for a given rate of continuity, the condition of Theorem 3.3 will be satisfied if the relative entropies tend to zero at the same rate.

Proposition 3.15 *Let $\nu \in \mathcal{G}(\gamma)$ and suppose that α_M is a ν-rate of θ-continuity. Suppose furthermore that*

$$\lim_{M\uparrow\infty} \frac{1}{\alpha_M} H_{\Lambda_M}(\mu|\nu) \;=\; 0. \tag{3.16}$$

Then $\mu \in \mathcal{G}(\gamma)$.

This proposition applies, for instance, to Schonmann's example. Indeed, if ν^+ is the projection on a (one-dimensional) layer of the low-temperature plus-phase of the two-dimensional Ising model, then the estimates in [21] imply that the monotone right-continuous specification γ^+ (such that $\nu^+ \in \mathcal{G}(\gamma^+)$) admits $\alpha_M = M$ as ν^+-rate of right-continuity. Hence for this example we can conclude that for any other measure μ on the layer, $h(\mu|\nu^+) = 0$ implies $\mu \in \mathcal{G}(\gamma^+)$. This is a strengthening of part (b) of Corollary 3.5. We emphasize that such a μ can *not* be the projection ν^- of the minus Ising phase. Indeed, while at present the existence of $h(\nu^-|\nu^+)$ has not rigorously been established, Schonmann's original argument [25] implies that $h(\nu^-|\nu^+) > 0$ if it exists.

4 Proofs

4.1 Proof of Theorem 3.3

The hypothesis $h(\mu|\nu) = 0$ implies, by (2.15)–(2.16), that for n sufficiently large the $\mathcal{F}_{\Lambda_n}$-measurable function $d\mu_{\Lambda_n}/d\nu_{\Lambda_n}$ exists. Let's denote it g_{Λ_n}.

For f local and $\Lambda \in \mathcal{S}$, pick M such that $\Lambda_M \supset \Lambda$ and g_{Λ_M} exist and write

$$\mu(\gamma_\Lambda f - f) \;=\; A_M + B_M + C_M \tag{4.1}$$

with

$$A_M \;=\; \mu\Big[\gamma_{\Lambda_M}(f) - \gamma_{\Lambda_M}^{M,\theta}(f)\Big] \;,\; B_M \;=\; \nu\Big[(g_{\Lambda_M} - g_{\Lambda_M\setminus\Lambda})\, f\Big] \tag{4.2}$$

and C_M is the right-hand side in (3.4). We shall prove that A_M and B_M go to zero.

Indeed, $\lim_{M\to\infty} A_M = 0$ follows by dominated convergence, because γ is quasilocal in the direction θ and $|\gamma_\Lambda^M(f)| \le \|f\|_\infty$.

On the other hand, Csiszár's inequality [4]

$$H_\Delta(\mu|\nu) - H_{\Delta'}(\mu|\nu) \;\ge\; \frac{1}{2}\left[\int_\Omega \Big|g_\Delta(\omega) - g_{\Delta'}(\omega)\Big| d\nu(\omega)\right]^2 .$$

valid for $\Delta' \subset \Delta \in \mathcal{S}$, implies that

$$|B_M| \;\le\; \sqrt{2}\,\|f\|_\infty \Big[H_\Delta(\mu|\nu) - H_{\Delta\setminus\Lambda}(\mu|\nu)\Big]$$

for any $\Delta \supset \Lambda_M$. But the hypothesis $h(\mu|\nu) = 0$ implies that the difference in entropies in the right-hand side tends to zero as $\Delta \uparrow \mathbb{L}$, as shown in [12] or [24]. Hence $B(M) \to_M 0$.

4.2 Proof of Corollary 3.5

It is enough to verify the right-hand side of (3.4) for increasing local functions f since linear combinations of these are uniformly dense in the set of quasilocal functions.

Part (a) By Theorem 3.3 we only have to show that

$$C_M \;=\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\left(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)\right)\Big] \underset{M\to\infty}{\longrightarrow} 0\,, \tag{4.3}$$

where $g_D = d\mu_D^-/d\mu_D^+$ for $D \subset \mathbb{L}$. We first point out that

$$C_M \;\ge\; 0 \tag{4.4}$$

because γ is monotonicity preserving, while

$$\mu^+\left(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda^{M,+}(f)\right) \;=\; \mu^-\left(\gamma_\Lambda^{M,+}(f)\right) \tag{4.5}$$

because $\gamma_\Lambda^{M,+} f$ is $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurable. On the other hand,

$$\mu^+\Big(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda(f)\Big) \;=\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\mu^+(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big] \tag{4.6}$$

where $\mu^+(\,\cdot\,|\mathcal{F}_D)$ are the conditional expectations of part (c) of Lemma 2.23. By the last inequality there,

$$\mu^+\Big(g_{\Lambda_M\setminus\Lambda}\,\gamma_\Lambda(f)\Big) \;\geq\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\,\mu^-(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big] \;=\; \mu^-\Big(\gamma_\Lambda(f)\Big)\;. \tag{4.7}$$

because of the $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\mu^-(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})(\,\cdot\,)$. From (4.4), (4.5) and (4.7),

$$0 \;\leq\; C_M \;\leq\; \mu^-\Big(|\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)|\Big)$$

and (4.3) follows from the right-continuity of γ and dominated convergence.

Part (b) By monotonicity

$$\begin{aligned} 0 \;&\leq\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda(f)\Big)\Big] \\ &\leq\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big)\Big] \\ &=\; \mu\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big) \end{aligned}$$

where g_D indicates the Radon-Nikodym density of μ_D with respect to μ_D^+, and the last equality follows from the $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)$. The last line tends to zero with M by dominated convergence, because $\mu(\Omega_\pm) = 1$. This concludes the proof because of Theorem 3.3.

4.3 Proof of Proposition 3.15

Let us fix a local function f, a finite set Λ and some $\epsilon > 0$. We have

$$\nu\left[\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\Big(\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\Big)\right] \;\leq\; \epsilon + 2\|f\|_\infty\,\widetilde{\mu}_M(A_\epsilon^M) \tag{4.8}$$

where A_ϵ^M denotes the set (3.12) and we abbreviated

$$\widetilde{\mu}_M(A_\epsilon^M) \;=\; \nu\left(\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\,1_{A_\epsilon^M}\right)\;. \tag{4.9}$$

By (3.14) there exists $c > 0$ such that for M large enough,

$$\nu(A^M_\epsilon) \ \leq \ e^{-c\,\alpha_M} \ , \tag{4.10}$$

hence, for $0 < \ \delta < \ c$, and we can write the following inequalities:

$$\begin{aligned} \widetilde{\mu}_M(A^M_\epsilon) \quad &\leq \quad \frac{1}{\alpha_M\,\delta}\log\int \exp(\delta\,\alpha_M\,1_{A^M_\epsilon})\,d\nu \ + \ \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu) \\ &\leq \quad \frac{1}{\alpha_M\,\delta}\log\Big(1 + e^{\alpha_M\,\delta}\,\nu(A^M_\epsilon)\Big) \ + \ \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu) \\ &\leq \quad \frac{1}{\alpha_M\,\delta}\,e^{\alpha_M(\delta-c)} \ + \ \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu)\ . \end{aligned} \tag{4.11}$$

By (3.16), the last line tends to zero as $M \to \infty$. By (4.8), and the fact that $\epsilon > 0$ is arbitrary, we conclude that condition (3.4) of Theorem 3.3 is satisfied, which implies that $\mu \in \mathcal{G}(\gamma)$.

Acknowledgements

We thank Charles-Edouard Pfister for giving us an early version of [24] and Aernout van Enter for comments and suggestions. We are grateful to Eurandom (RF and ALN) and The University of Groningen (RF) for hospitality during the completion of this work.

References

[1] M. Aizenman. Translation invariance and instability of phase coexistence in the two dimensional Ising system. *Commun. Math. Phys.* **73**: 83–94, 1980.

[2] J. Bricmont, A. Kupiainen, R. Lefevere. Renormalization group pathologies and the definition of Gibbs states. *Comm.Math.Phys.* **194**:359–388, 1998.

[3] R.L. Dobrushin and S.B. Shlosman. Gibbsian description of 'non Gibbsian' field. *Russian Math Surveys*, **52**:285-297, 1997. Also 'Non Gibbsian' states and their description. *Comm.Math.Phys.* **200**:125–179, 1999.

[4] I. Csiszár. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.*, **2**:299–318, 1967.

[5] A. C. D. van Enter. On the possible failure of the Gibbs property for measures on lattice systems. *Markov Proc. Rel. Fields*, **2**:209–24, 1996.

[6] A. C. D. van Enter. The renormalization-group peculiarities of Griffiths and Pearce: What have we learned?, in *Mathematical Results in Statistical Mechanics* (Proceedings of the colloquium with the same name, Marseille-Luminy, July 1998), S. Miracle-Solé, J. Ruiz and V. Zagrebnov, World Scientific, Singapore, 1999.

[7] A.C.D. van Enter, R. Fernández, and A.D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Stat. Phys.* **72**:879–1167, 1993.

[8] R. Fernández. Random fields in lattices. The Gibbsianness issue. *Resenhas do IME-USP*, **3**: 391–421, 1998.

[9] R. Fernández. Measures for lattice systems. *Physica A*, **263**: 117–130, 1999.

[10] C. M. Fortuin, J. Ginibre and P. W. Kasteleyn. Correlation inequalities on some partially ordered sets. *Commun. Math. Phys.* **22**: 89–103, 1971.

[11] R. Fernández and C.-E. Pfister. Global specifications and nonquasilocality of projections of Gibbs measures. *Ann.Probab.***25**:1284–1315, 1997.

[12] H.O. Georgii. Gibbs Measures and Phase Transitions. Walter de Gruyter (de Gruyter Studies in Mathematics, Vol. 9), Berlin-New York, 1988.

[13] R.B. Griffiths and P.A. Pearce. Mathematicals properties of position-space renormalization-group transformations. *J.Stat.Phys.* **20**: 499–545, 1979.

[14] Y. Higuchi. On the absence of non-translationally invariant Gibbs states for the two-dimensional Ising system. In *Random Fields: Rigorous Results in Statistical Mechanics and Quantum Field Theory* (Esztergom 1979), J. Fritz, J. L. Lebowitz and D. Szasz (eds.), North Holland, Amsterdam, 1981.

[15] R.B. Israel. Banach algebras and Kadanoff transformations, in *Random Fields (Esztergom, 1979)* J.Fritz, J.L.Lebowitz, and D.Szász, eds (North-Holland, Amsterdam, 1981), Vol II, pp.593–608, 1981.

[16] O.K. Kozlov. Gibbs description of a system of random variables. *Probl. Inform. Transmission.***10**:258–265, 1974.

[17] A. Le Ny, Mesures de Gibbs sur un réseau et non-gibbsiannité: restauration du formalisme gibbsien. Thèse de doctorat de l'université de Rennes 1, juillet 2000: http://www.maths.univ-rennes1.fr/˜aleny.

[18] A. Le Ny, Fractal Failure of Quasilocality for a Majority Rule Transformation on a Tree, *Lett.Math.Phys.*,**54**:11–24, 2000.

[19] R. Lefevere. Variational principle for some renormalized measures. *J.Stat.Phys.*,**95**:785–803, 1999.

[20] R. Lefevere. Almost and weak Gibbsianness: A long-range pair-interaction example. *J.Stat.Phys.*,**96**:109–113, 1999.

[21] C. Maes, F. Redig and A. Van Moffaert. Almost Gibbsian versus Weakly Gibbsian. *Stoch.Proc.Appl.* **79**:1–15, 1999.

[22] C. Maes, F. Redig and A. Van Moffaert. The restriction of the Ising model to a Layer. *J.Stat.Phys* **96**:69–107, 1999.

[23] C. Maes, F. Redig, S. Shlosman and A. van Moffaert. Percolation, path large deviations and weak Gibbsianity. *Comm. Math. Phys.* **209**: 517–45, 2000.

[24] C.-E. Pfister. Thermodynamical aspects of classical lattice systems. To appear in the proceedings of the 4th brazilian school of probability, Mambucaba, RJ, 2000.

[25] R. H. Schonmann. Projections of Gibbs measures may be non-Gibbsian. *Commun. Math. Phys.*, **124**:1–7, 1989.

[26] W.G. Sullivan. Potentials for almost Markovian random fields. *Commun. Math. Phys.* **33**:61–74, 1973.