Hydrodynamic Equation for a Deposition Model

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Abstract

We show that the two-component system of hyperbolic conservation laws $\partial_t \rho + \partial_x(\rho u) = 0 = \partial_t u + \partial_x \rho$ appears naturally in the formally computed hydrodynamic limit of some randomly growing interface models, and we study some properties of this system.

Key Words: Hyperbolic conservation laws, KPZ equation

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1 Introduction

The macroscopic behaviour of physical systems can often be described in terms of non-linear partial differential equations. In many cases, it had been shown that functionals of microscopic models from statistical physics converge in the hydrodynamic limit towards certain solutions of these partial differential equations.

Studying the partial differential equation (or the system of partial differential equations) can turn out to be a very hard challenge in itself: Appearance of singularities in finite time, shocks etc. The so-called hyperbolic conservation laws have in particular received a lot of interest. Even in one space dimension, these PDEs proved to be extremely interesting and challenging both mathematically and phenomenologically. These are partial differential equations of the type

$$\partial_t u + \partial_x J(u) = 0$$

where $u = u(t,x)$ takes its value in $\mathbb{R}^n$ and J is a non-linear function from $\mathbb{R}^n$ into $\mathbb{R}^n$.

The best known and most investigated examples are the following. (See e.g. [7, 14, 15] for a comprehensive introduction and survey of the subject.)

(1) Burgers' equation (with no viscosity): $n = 1$ and

$$\partial_t u + \partial_x (u^2/2) = 0.$$

(2) The isentropic gas dynamics equation in one space dimension: $n = 2$, the components are the density field $\rho(t, x)$ and momentum field $m(t, x)$

$$\begin{cases} \partial_t \rho + \partial_x m = 0 \\ \partial_t m + \partial_x \big(m^2/\rho + p(\rho)\big) = 0 \end{cases} \tag{1}$$

where $p(\rho)$ is the pressure, depending on density only.

(3) The so-called p-system, which is an alternative formulation of the dynamics of one-dimensional gas. The two components are the velocity field $u(t, x)$ and the specific volume (= inverse density) field $v(t, x)$:

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p(v) = 0. \end{cases} \tag{2}$$

Here $p(v)$ denotes the pressure, as a function of specific volume.

(4) The shallow water equation is another two component system: $h(t, x)$ denotes the height of the (shallow) layer of water, $u(t, x)$ is the velocity field:

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t u + \partial_x \big(u^2/2 + h\big) = 0. \end{cases} \tag{3}$$

Since Riemann, a considerable amount of knowledge and technology (more recently, for instance, entropy solutions, compensated compactness method) has been derived that give a better understanding of the physically relevant solutions to these equations.

In the present paper, we will be considering a particular two-component (i.e. $n = 2$) system of hyperbolic conservation laws that arises in the context of surface growth (or more precisely growing interfaces, since the surface is one-dimensional). In other words, at each time $t \geq 0$, one sees a landscape $x \mapsto h(t, x)$ where $x \in \mathbb{R}$. The function h is increasing in time. The rough phenomenological description of the phenomena we are interested in corresponds to the case where the surface is growing in the normal direction to its boundary, but there exists a 'tension' that tends to keep the surface together, in the sense that it will fill in holes quickly. In the physics literature, a famous equation has been proposed by Kardar, Parisi and Zhang (the KPZ equation) as a model for such situations, cf. [8]. It is (in the mathematical jargon) an ill-posed non-linear partial differential equation with a stochastic term:

$$\partial_t h = \Delta h - (\partial_x h)^2 + W$$

where $W = W(t, x)$ denotes a space-time white noise. We do not want to give a review of the huge physics literature on this equation, but we briefly stress two aspects. (See [2] for a state-of-the-art survey of the physics literature on the subject and an exhaustive list of references up to 1995.) First, there exists to our knowledge no completely satisfactory (see however [6]) derivation of this equation from a microscopic model. Second, it is predicted that 'the' solution to this equation has a special scaling behaviour at late times. More precisely, it is believed that when α, t, x are very large, $h^{(\alpha)}(t, x) = \alpha^{-1/3} h(\alpha t, \alpha^{2/3} x)$ is also a solution to the KPZ equation. The exponents $1/3$ and $2/3$ should be related to various conjectures and recent rigorous results concerning the fluctuations of highest eigenvalues of random matrices, of first passage percolation paths, of longest increasing sequences etc etc.

One way to define one-dimensional interfaces $h(t, x)$ in terms of particle systems goes as follows: Start with a (finite or infinite) system of particles that evolve randomly in the potential $h(t, x)$ (or in some potential defined in terms of h) and that all contribute to increase the potential in the sense that $h(t, x)$ corresponds to the joint local time (i.e. cummulated occupation time density) of the particles at time t and site x. In other words, $h(t, x)$ increases locally at x if there is a particle at x and time t. Note that this leads naturally to a two-component system in the (formally computed) hydrodynamical limit: the first component is the density of particles, and the second component is the gradient of the profile of the potential.

In [17], we constructed a continuous stochastic process, corresponding on a heuristic level to the case where there is exactly (and only) one particle (its location at time t is denoted by X_t) which is driven by

$$dX_t = -\partial_x h(t, X_t) dt$$

and $h(t, x)$ is the local time of X at x and time t, so that

$$\partial_t h(t, x) = \delta(X_t - x).$$

For details concerning the construction and primary properties of this process and a rigorous version of these equations, see [17]. Let us just emphasize a couple of features: The process $(X_t, t \geq 0)$ is a random process, even though the previous 'differential equations' look very deterministic. One reason is that (in the stationary regime), the function $x \mapsto h(t,x)$ is not regular; in fact, it is a Brownian motion in the space variable (for fixed t). Second, X_t is not a usual stochastic process (it is not solution of a stochastic differential equation for instance), it has the 2/3 scaling: $(\alpha^{-2/3} X_{\alpha t}, t \geq 0)$ has the same law as $(X_t, t \geq 0)$. In particular, $(\alpha^{-1/3} h(\alpha^{2/3} x, \alpha t), t \geq 0, x \in \mathbb{R})$ has the same law as $(h(x,t), t \geq 0, x \in \mathbb{R})$ so that h has the same scaling property as the asymptotic scaling conjectured for the KPZ equation.

The process $(X_t, t \geq 0)$ can be viewed as the scaling limit of a discrete negatively reinforced (i.e. self-repellent) random walk $(S_n, n \geq 0)$ on $\mathbb{Z}$ called the 'true self-avoiding walk' in the physics literature. This is a nearest-neighbour walk on $\mathbb{Z}$ that decides at each step to jump to the left or to the right according to a probability depending on how many times it has visited the neighbouring sites (or edges) before. Suppose for instance that after n steps $S_n = x$ and that the discrete walk $(S_i)_{i \leq n}$ has jumped already l (resp. r) times on the edge immediately to the left (resp. to the right) of x. Then, $S_{n+1} = x + 1$ with probability

$$\mathbf{P}\big(S_{n+1} = x+1 \big| l, r, S_n = x\big) = \frac{e^{-\beta l}}{e^{-\beta l} + e^{-\beta r}}$$

where $\beta > 0$ is some fixed constant. In other words, the walk will prefer to go along the edge it has visited less often in the past. Note also that the probability in fact depends only on the difference $l - r$ (which depends on all the past trajectory). The distribution of the rescaled position of the random walker, $S_n / n^{2/3}$, converges to (a multiple of) the one-dimensional marginal distribution of the continuous process X_t described above, [16].

It seems natural to consider the case where this one particle is replaced by many particles performing the same kind of self-repelling walk on $\mathbb{Z}$, with a *joint cumulated local time of all particles*. Or, in the continuous space-time setting: a continuously distributed cloud of particles (that all contribute to the same local time), which is the subject of the present paper. As we shall see, this leads in the (formally computed) hydrodynamic limit to the following system of hyperbolic conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x \rho = 0 \end{cases} \tag{4}$$

where ρ corresponds to the density of particles at x and time t, and $u(x,t) = -\partial_x h$ corresponds to the negative gradient of the interface. It seems, that although this system looks very natural, it has not been considered in the literature. We should emphasize that in spite of some formal similarities with the p-system (2) and the shallow water equation (3), the system (4) shows very different behaviour and describes a quite different phenomenon. We hope that its study may lead to improved understanding of some aspects of 'growing interfaces' in general. In particular, this equation could shed some light on some of the conjectured properties of the KPZ equation. *The goal of the present paper is not to present a complete treatment of this system of partial differential equation, but rather to initiate it as an alternative approach to 1-d domain growth and deposition phenomena.*

2 The PDE: phenomenological derivation

We define a deposition model in the following terms. The actual state of the system is described by two functions:

$$\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \quad \text{and} \quad h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}.$$

$\rho(t,x)$ is the density of the population performing the deposition, while $h(t,x)$ is the deposition height at time t and space coordinate x. The rules governing the time evolution of the system are the following

(1) The total population is conserved, so that the continuity equation

$$\partial_t \rho + \partial_x(\rho u) = 0$$

is valid, where $u(t,x)$ is the velocity field, to be specified by the dynamical rules.

(2) The deposition rate is proportional to the density of the population, i.e.

$$\partial_t h = c_1 \rho, \tag{5}$$

where c_1 is a positive constant.

(3) The population is driven by a velocity field proportional to the negative gradient of height

$$u = -c_2 \partial_x h, \tag{6}$$

where c_2 is another positive constant. This rule corresponds to the self-repellence mechanism described in the introductory section.

From (5) and (6) we readily get

$$\partial_t u + c_1 c_2 \partial_x \rho = 0$$

Without loss of generality, we can choose $c_1 c_2 = 1$ and get the two component system of hyperbolic conservation laws

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x \rho = 0 \end{array} \right. \tag{7}$$

This system of PDEs with initial conditions

$$\rho(0,x) = \rho^{(0)}(x), \qquad u(0,x) = u^{(0)}(x) \tag{8}$$

is the main object of the present paper.

As a first remark we mention here the scale invariance of (7). Let $\nu \in \mathbb{R}$ be fixed. Given the functions $(t,x) \mapsto \rho(t,x)$ and $(t,x) \mapsto u(t,x)$ and a positive fixed number α, define the rescaled functions

$$\begin{aligned} \rho^{(\alpha)}(t,x) &:= \alpha^{2(1-\nu)} \rho(\alpha t, \alpha^{\nu} x), \\ u^{(\alpha)}(t,x) &:= \alpha^{1-\nu} u(\alpha t, \alpha^{\nu} x). \end{aligned}$$

One can easily check that if (ρ, u) is solution of (7), then $(\rho^{(\alpha)}, u^{(\alpha)})$ is also a solution, for any $\alpha > 0$. The choice $\nu = 1$ yields the hyperbolic scale invariance valid for any hyperbolic conservation law. More interesting is for our purposes the choice $\nu = 2/3$. This is the physically relevant scale invariance, since the density changes covariantly under this scaling, i.e., the total mass $\int \rho^{(\alpha)} dx$ is unchanged.

With this choice of ν the following scale invariance of the deposition height follows:

$$h^{(\alpha)}(t,x) := \alpha^{-1/3} h(\alpha t, \alpha^{2/3} x).$$

Recall that this is exactly the *conjectured asymptotic scale invariance of the one-dimensional KPZ equation.*

3 Bricklayers

We define a system of interacting particles living on $\mathbb{Z}$, with *two conserved* quantities, whose hydrodynamic modes are governed by a two-component system of hyperbolic conservation laws which, after taking another limit (low density/late time), transforms into our system (7). The computations of the present section are somewhat formal. Working out all technical details (e.g. proving uniqueness of the equilibrium Gibbs measures or technical details of Yau's hydrodynamic limit) needs more effort. The present section serves as microscopic motivation of the PDE proposed above.

3.1 The particle system

The Great Wall of China is being built by a brigade of bricklayers. The wall consists of columns of unit-size bricks, piled above the edges of the lattice $\mathbb{Z}$. The height of the column piled above the edge $(j, j+1)$ (i.e., number of bricks in this column) is h_j. In the dynamics of the system the discrete negative gradients $z_j := h_{j-1} - h_j \in \mathbb{Z}$ will be relevant. The bricklayers occupy the sites of the lattice. At each site $j \in \mathbb{Z}$ there might be an unlimited number $n_j \in \mathbb{N}$ of bricklayers. Bricklayers jump to neighbouring sites and at each jump $j \to j \pm 1$ a brick is added to the respective column of bricks.

In more technical terms: particles (= bricklayers) perform continuous time nearest neighbour walk on the lattice $\mathbb{Z}$ and h_j measures the cumulated (discrete) local time on the lattice edge $(j, j+1)$.

About the dynamics: the jump rates are chosen so that the following conditions hold:

(1) the bricklayers' jumps are driven by the local shape of the wall so that they try to reduce the differences z_j (i.e. to keep the height of the wall even),

(2) conditionally on the actual shape of the wall the bricklayers jump independently.

This is done as follows. The instantaneous rate of jump from site j to site $j \pm 1$ (for each bricklayer sitting at site j) is equal to $r(\pm z_j)$, where $r : \mathbb{Z} \to (0, \infty)$ is a fixed monotone increasing function which defines the model. In order to be able to compute explicitely the stationary measures

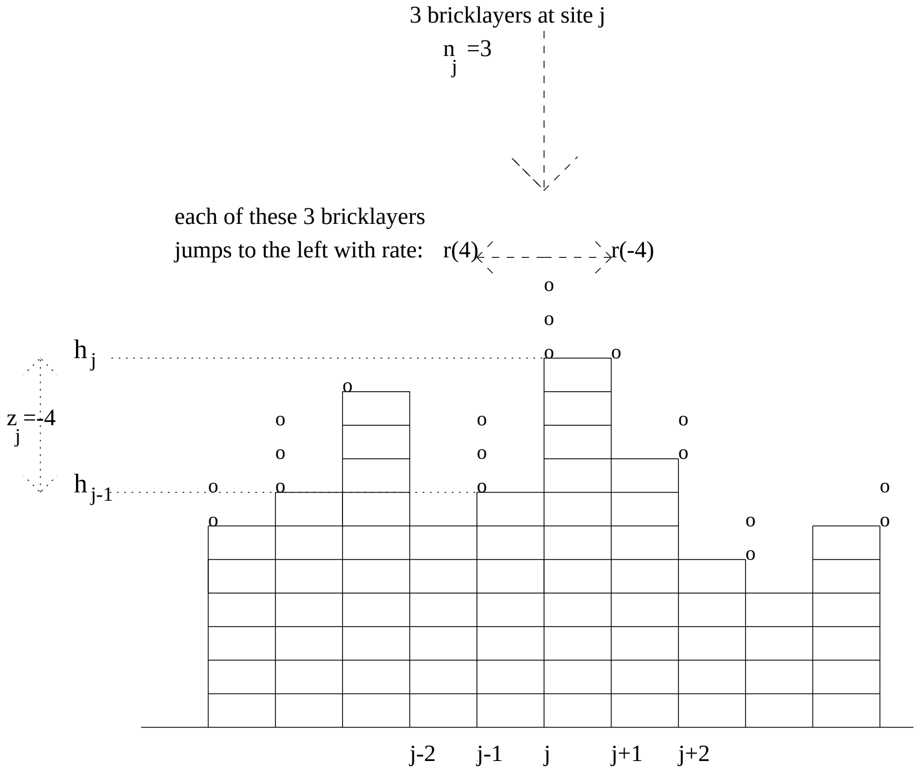

Figure 1: The great wall being built

(see subsection 3.2) we impose that $r(1-z)r(z)$ is a positive constant (this is for instance the case if $r(z) = \exp(\beta z)$), and multiplying time by a constant term, we can in fact restrict ourselves to the case where

$$r(z)r(-z+1) = 1, \quad \text{for all } z \in \mathbb{Z}. \tag{9}$$

Thus, the following changes of configuration may occur:

$$(n_j, z_j), (n_{j+1}, z_{j+1}) \to (n_j - 1, z_j - 1), (n_{j+1} + 1, z_{j+1} + 1)$$

with rate $n_j r(z_j)$, and

$$(n_j, z_j), (n_{j-1}, z_{j-1}) \to (n_j - 1, z_j + 1), (n_{j-1} + 1, z_{j-1} - 1)$$

with rate $n_j r(-z_j)$.

Clearly, $\sum_j n_j$ and $\sum_j z_j$ are formally conserved quantities of the dynamics. It is also clear that besides these globally conserved quantities the parity of $n_j + z_j$ is also conserved on each lattice site $j \in \mathbb{Z}$.

Now we give a more formal description of our interacting particle system. For $s \in \{0,1\}$ let

$$\big(\mathbb{N} \times \mathbb{Z}\big)_s := \{(n,z) \in \mathbb{N} \times \mathbb{Z} : n + z = s \bmod 2\}.$$

Given the sequence $\mathbf{s} = (s_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ we define the state space of our system as

$$\Omega_{\mathbf{s}} := \prod_{j\in\mathbb{Z}} \big(\mathbb{N} \times \mathbb{Z}\big)_{s_j}.$$

Elements of $\Omega_{\mathbf{s}}$ will be denoted by ω, i.e. $\omega = (\omega_j)_{j\in\mathbb{Z}}$ with $\omega_j = (n_j, z_j) \in \big(\mathbb{N} \times \mathbb{Z}\big)_{s_j}$. The (formal) infinitisimal generator of the Markov process described verbally in the first paragraph of this section, is:

$$Lf(\omega) = \sum_{j\in\mathbb{Z}} n_j r(z_j)\big(f(\Theta_{j+}\omega) - f(\omega)\big) + \sum_{j\in\mathbb{Z}} n_j r(-z_j)\big(f(\Theta_{j-}\omega) - f(\omega)\big),$$

where the maps Θ_{j+} and Θ_{j-} act on the subsets $\{\omega \in \Omega_{\mathbf{s}} : n_j \geq 1\}$ as

$$\big(\Theta_{j+}\omega\big)_i := \begin{cases} (n_i, z_i) & \text{if} \quad i \neq j, j+1 \\ (n_i - 1, z_i - 1) & \text{if} \quad i = j \\ (n_i + 1, z_i + 1) & \text{if} \quad i = j+1 \end{cases}$$

respectively

$$\big(\Theta_{j-}\omega\big)_i := \begin{cases} (n_i, z_i) & \text{if} \quad i \neq j, j-1 \\ (n_i - 1, z_i + 1) & \text{if} \quad i = j \\ (n_i + 1, z_i - 1) & \text{if} \quad i = j-1 \end{cases}$$

3.2 Equilibrium Gibbs measures

For $k \geq 0$ denote

$$R(z) := \prod_{k=1}^{|z|} r(k)$$

and

$$\theta^* := \lim_{k\to\infty} r(k) \in (1, \infty].$$

Note that (9) implies that for all $z \in \mathbb{Z}$,

$$R(-z) = R(z) = R(z-1)r(z) = R(z+1)r(-z). \tag{10}$$

Fix the parameters $s \in \{0,1\}$, $\lambda > 0$, $\theta \in (1/\theta^*, \theta^*)$ and define the probability measure $\mu_{s,\lambda,\theta}$ on $\big(\mathbb{N}\times\mathbb{Z}\big)_s$ as follows:

$$\mu_{s,\lambda,\theta}(n,z) := \frac{1}{Z_s(\lambda,\theta)}\frac{\lambda^n}{n!}\frac{\theta^z}{R(z)},$$

where

$$Z_s(\lambda,\theta) := \sum_{(n,z)\in(\mathbb{N}\times\mathbb{Z})_s} \frac{\lambda^n}{n!}\frac{\theta^z}{R(z)}$$

is the normalizing factor (partition function). The measure $\mu_{s,\lambda,\theta}$ is a product measure on $\mathbb{N}\times\mathbb{Z}$ restricted to the subset $n+z = s \bmod 2$. It is worth noting that

$$Z_s(\lambda,\theta) = Z_s(\lambda,\theta^{-1}) \qquad \text{and} \qquad \mu_{s,\lambda,\theta}(n,z) = \mu_{s,\lambda,\theta^{-1}}(n,-z). \tag{11}$$

For a fixed sequence $\mathbf{s} \in \{0,1\}^{\mathbb{Z}}$ and fixed parameters $\lambda > 0$, $\theta \in (1/\theta^*, \theta^*)$ we define on $\Omega_{\mathbf{s}}$ the probability measure

$$\mu_{\mathbf{s},\lambda,\theta} := \prod_{j\in\mathbb{Z}} \mu_{s_j,\lambda,\theta}.$$

By direct computations, one can check using (10) that for any function f that depends only on the value of finitely many ω_k's, for any fixed s_j, s_{j+1} and $(\omega_i)_{i\neq j,j+1}$,

$$\begin{aligned}&\sum_{\omega_j,\omega_{j+1}} n_j r(z_j)\mu_{s_j,\lambda,\theta}(\omega_j)\mu_{s_{j+1},\lambda,\theta}(\omega_{j+1}) f(\Theta_{j+}(\omega))\\ &\quad= \sum_{\omega_j,\omega_{j+1}} n_{j+1} r(z_{j+1})\mu_{s_j,\lambda,\theta}(\omega_j)\mu_{s_{j+1},\lambda,\theta}(\omega_{j+1}) f(\omega)\end{aligned}$$

and a similar dentity holds for the jumps to the left. It follows that given the local parities $n_j + z_j = s_j \bmod 2$, the probability measures $\mu_{\mathbf{s},\lambda,\theta}$ are stationary for the dynamics. These are the *equilibrium Gibbs measures* of our system. For a similar computation in the context of a simpler one-component domain growth model see also [1].

Invariance under spatial translations is unfortunately lost in this very general setup. In order to impose it, we restrict ourselves to one of the following two choices: either $\mathbf{s} = \mathbf{0}$ or $\mathbf{s} = \mathbf{1}$.

3.3 The hydrodynamic equations

For the rest of this section we fix either $\mathbf{s} = \mathbf{0}$ or $\mathbf{s} = \mathbf{1}$ and we do not denote any more the dependence on $\mathbf{s}$.

As we have mentioned already the globally conserved quantities of our system are $\sum_j n_j$ and $\sum_j z_j$. In the equilibrium regime $\mu_{\lambda,\theta}$ the averages of these quantities are

$$\rho := \langle n_j \rangle_{\lambda,\theta} = \lambda \frac{\partial \log Z(\lambda,\theta)}{\partial \lambda}, \quad u := \langle z_j \rangle_{\lambda,\theta} = \theta \frac{\partial \log Z(\lambda,\theta)}{\partial \theta}.$$

These are the particle density (per site) and the average slope of the height of the wall, in equilibrium. It is easy to see that the map $\mathbb{R}_+ \times (1/\theta^*, \theta^*) \ni (\lambda, \theta) \mapsto (\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$ is globally invertible. Indeed,

$$\begin{pmatrix} \partial\rho/\partial\lambda & \partial\rho/\partial\theta \\ \partial u/\partial\lambda & \partial u/\partial\theta \end{pmatrix} = \begin{pmatrix} \mathbf{Var}(n) & \mathbf{Cov}(n,z) \\ \mathbf{Cov}(n,z) & \mathbf{Var}(z) \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \theta^{-1} \end{pmatrix}. \tag{12}$$

So the gradient matrix on the left hand side of (12) is everywhere invertible and this implies global invertibility of the map $(\lambda, \theta) \mapsto (\rho, u)$. With slight abuse of notation we denote the components of the inverse function $\lambda = \lambda(\rho, u)$ and $\theta = \theta(\rho, u)$. From (11) it follows that

$$\lambda(\rho, -u) = \lambda(\rho, u) \quad \text{and} \quad \theta(\rho, -u) = 1/\theta(\rho, u). \tag{13}$$

In order to guess the system of hydrodynamic equations we have to see first how the infinitisimal generator acts on the conserved quantities. An easy computation shows:

$$\begin{aligned} Ln_j &= \big(n_{j-1}r(z_{j-1}) - n_j r(-z_j)\big) - \big(n_j r(z_j) - n_{j+1} r(-z_{j+1})\big) \\ Lz_j &= \big(n_{j-1}r(z_{j-1}) + n_j r(-z_j)\big) - \big(n_j r(z_j) + n_{j+1} r(-z_{j+1})\big) \end{aligned}$$

On the right hand side of these equations we see *discrete gradients of fluxes.* This fact helps us guessing the hydrodynamic equations. Applying the standard formal manipulations to our gradient system (see e.g. [5], [9]) and using the straightforward identities

$$\langle n_j r(\pm z_j) \rangle_{\lambda,\theta} = \lambda \theta^{\pm 1}$$

in the hydrodynamic limit taken with *hyperbolic (Eulerian) scaling* of space and time, we arrive at the system of PDEs

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \big(\lambda(\rho,u)(\theta(\rho,u) - \theta(\rho,u)^{-1})\big) = 0, \\ \partial_t u + \partial_x \big(\lambda(\rho,u)(\theta(\rho,u) + \theta(\rho,u)^{-1})\big) = 0. \end{array} \right. \tag{14}$$

Under growth conditions on the rate function $r(z)$, as $z \to \infty$, Yau's 'relative entropy method' (see e.g. [18], [5], [9]) in principle can be applied to our system of interacting particles, resulting in the validity of the above system of PDEs in the hydrodynamic limit, *as long as the solutions are smooth.*

From the system (14) we can derive the system (7) by taking a second limit: We replace $\rho(t,x)$ by $\alpha^{2/3}\rho(\alpha t, \alpha^{2/3}x)$ and $u(t,x)$ by $\alpha^{1/3}u(\alpha t, \alpha^{2/3}x)$ We note that for small values of the variables ρ and u,

$$\lambda(\rho,u) = \rho + o(\rho), \qquad \theta(\rho,u) = 1 + cu + o(u).$$

where

$$c = \left(\frac{\partial^2 Z}{\partial \theta^2}\bigg|_{\lambda=0,\theta=1} \right)^{-1} \in (0,\infty).$$

Letting now $\alpha \to 0$, we arrive at (7). We should emphasize here that this scaling limit does not depend much on the details of microscopic system. Also, from any conservation law of the form

$$\begin{cases} \partial_t \rho + \partial_x J(\rho,u) = 0 \\ \partial_t u + \partial_x K(\rho,u) = 0 \end{cases}$$

we would get (7) under the same limiting procedure, provided that

$$J(\rho,u) = \rho u + o(\rho u), \quad K(\rho,u) = \rho + o(\rho), \qquad \text{as } \rho, u \to 0.$$

This indicates that (7) is valid for a wider class of microscopic systems.

4 Analysis of the PDE

We are now going to see how the methods developed in the PDE literature (see [7, 14, 15]) can be applied to our system. In order to put things into perspective, we briefly recall general results and see how they can be applied in the context of our system (4).

4.1 Two-component systems of hyperbolic conservation laws

For a generic two-component system we shall use the notation $v = v(t,x) = (v_1(t,x), v_2(t,x))^T$. (The superscript ^{T} will denote transposition of vectors/matrices.) The generic two-component system is

$$\partial_t v + \partial_x J(v) = 0, \tag{15}$$

where $v \mapsto J(v) = (J_1(v), J_2(v))^T$ is a smooth vector field over $\mathbb{R} \times \mathbb{R}$. J is the flux of the flow of the conserved quantity v. The initial conditions are specified by

$$v(0,x) = v^{(0)}(x). \tag{16}$$

For a (possibly vector- or matrix valued) function $f = f(v)$ we denote the gradient with respect to the v-variables $\nabla f = (\partial f/\partial v_1, \partial f/\partial v_2)$. For classical *smooth* solutions $v(t,x)$, (15) is equivalent to

$$\partial_t v + \nabla J \cdot \partial_x v = 0 \tag{17}$$

(we use $\cdot$ to indicate products of matrices).

As a technical device one usually also considers the so-called viscous equations

$$\partial_t v + \nabla J \cdot \partial_x v = \varepsilon \partial_x^2 v. \tag{18}$$

Existence and unicity of smooth solution $v^{(\varepsilon)}(t,x)$ of (18), for any bounded and smooth initial conditions (16) is guaranteed by the smoothening effect of the *artificial viscosity term* on the right hand side. One hopes that physically acceptable (stable) solutions of the original system (15) can be obtained as a *strong* limit of the viscous solution $v^{(\varepsilon)}(t,x)$, as $\varepsilon \to 0$. The existence of this strong limit is a very difficult problem and is a main object of investigation in the context of hyperbolic conservation laws.

In our case (7) the two components are $v_1 = \rho$, $v_2 = u$, and the corresponding fluxes are $J_1(\rho,u) = \rho u$, $J_2(\rho,u) = \rho$. The inviscid system is (7). The (artificially) viscous system is

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_x^2 \rho \\ \partial_t u + \partial_x \rho = \varepsilon \partial_x^2 u. \end{array} \right. \tag{19}$$

The viscous solutions (which do exist and are unique) will be denoted by $\big(\rho^{(\varepsilon)}(t,x), u^{(\varepsilon)}(t,x)\big)$.

4.2 Hyperbolicity

One has to check that the matrix ∇J has two distinct real eigenvalues $\mu < \lambda$. The domain where this holds will be denoted

$$\mathcal{D}_{\text{hyp}} := \{v \in \mathbb{R} \times \mathbb{R} : \mu(v) < \lambda(v)\}.$$

The corresponding left (row) and right (column) eigenvectors will be denoted by l and r, respectively, m and s. That is:

$$l \cdot \nabla J = \lambda l, \qquad \nabla J \cdot r = \lambda r, \tag{20}$$

$$m \cdot \nabla J = \mu m, \qquad \nabla J \cdot s = \mu s. \tag{21}$$

For our system we find:

$$\nabla J = \begin{pmatrix} u & \rho \\ 1 & 0 \end{pmatrix},$$

and

$$\lambda = +\frac{1}{2}(\sqrt{u^2 + 4\rho} + u), \qquad l = (\lambda, \rho), \qquad r = (\lambda, 1)^T, \tag{22}$$

$$\mu = -\frac{1}{2}(\sqrt{u^2 + 4\rho} - u), \qquad m = (\mu, \rho), \qquad s = (\mu, 1)^T. \tag{23}$$

Note that $l \cdot s = m \cdot r = 0$, as it should be.

We conclude that for our system,

$$\mathcal{D}_{\text{hyp}} = \{(\rho, u) \in \mathbb{R} \times \mathbb{R} : u^2 + 4\rho > 0\}.$$

Note that in the physically relevant domain with non-negative densities

$$\mathcal{D}_{\text{ph}} := \{(\rho, u) \in \mathbb{R} \times \mathbb{R} : \rho \geq 0\},$$

there is one single point where strict hyperbolicity is lost, namely $(\rho, u) = (0, 0)$. On the other hand, we found that the system is still hyperbolic in the physically meaningless domain $\mathcal{D}_{\text{hyp}} \setminus \mathcal{D}_{\text{ph}} = \{(\rho, u) \in \mathcal{D}_{\text{hyp}} : \rho < 0\} \neq \emptyset$. At the moment nothing seems to prevent solutions to flow into this domain. Later we shall see that Lax's maximum principle (valid for stable entropy solutions) takes care of this problem.

4.3 Riemann invariants, characteristics

In the generic two-component case, we are looking for scalar functions $\mathcal{D}_{\text{hyp}} \ni v \mapsto w(v) \in \mathbb{R}$ and space-time trajectories $\mathbb{R}_+ \ni t \mapsto \xi(t) \in \mathbb{R}$ such that for smooth solutions of (15) (or, equivalently, of (17)) w is conserved along the trajectory $\xi(t)$, i.e.

$$\frac{d}{dt} w\big(v(t, \xi(t))\big) = 0.$$

Using (17) we find:

$$\frac{d\xi}{dt} = \frac{(\nabla w \cdot \nabla J) \cdot \partial_x v}{\nabla w \cdot \partial_x v}. \tag{24}$$

In order to solve (24), ∇w must be a left eigenvector of the matrix ∇J. It follows that this relation admits two solutions: one for each eigenvalue of ∇J. We denote the two solutions by w (corresponding to the eigenvalue λ), respectively, by z (corresponding to the eigenvalue μ). The gradients ∇w, respectively ∇z, are parallel to the row vectors l, respectively m, defined in (20), respectively (21). In other words,

$$\begin{aligned} \nabla w \cdot s = 0, \quad & \frac{d\xi}{dt} = \lambda, \\ \nabla z \cdot r = 0, \quad & \frac{d\xi}{dt} = \mu. \end{aligned}$$

These equations, of course, do not determine uniquely the functions $w(v)$ and $z(v)$. Given two smooth, monotone maps $f, g : \mathbb{R} \to \mathbb{R}$, the transformation $\hat{w} := f(w)$, $\hat{z} := g(z)$ leaves the above equations invariant. The functions w and z are called the *Riemann invariants*, or *characteristic coordinates* of the problem.

In our case the most convenient choice of the Riemann invariants w and z is the following: let

$$\begin{aligned} \mathcal{D}_w &:= \{(\rho, u) \in \mathcal{D}_{\text{hyp}} : \sqrt{u^2 + 4\rho} - u \geq 0\}, \\ \mathcal{D}_z &:= \{(\rho, u) \in \mathcal{D}_{\text{hyp}} : \sqrt{u^2 + 4\rho} + u \geq 0\}, \end{aligned}$$

and define $w : \mathcal{D}_w \to \mathbb{R}$, $z : \mathcal{D}_z \to \mathbb{R}$ by the formulas:

$$\begin{aligned} w(\rho, u) &= -\sqrt{\sqrt{u^2 + 4\rho} - u}\left(\sqrt{u^2 + 4\rho} + 2u\right), \\ z(\rho, u) &= -\sqrt{\sqrt{u^2 + 4\rho} + u}\left(\sqrt{u^2 + 4\rho} - 2u\right). \end{aligned}$$

Note that $\mathcal{D}_w \cap \mathcal{D}_z = \mathcal{D}_{\text{ph}}$, so that both Riemann invariants are defined in the physically relevant subdomain.

It is straightforward to check that both Riemann invariants w and z defined above are *convex* functions of the variables (ρ, u). This fact will have crucial importance in later analysis.

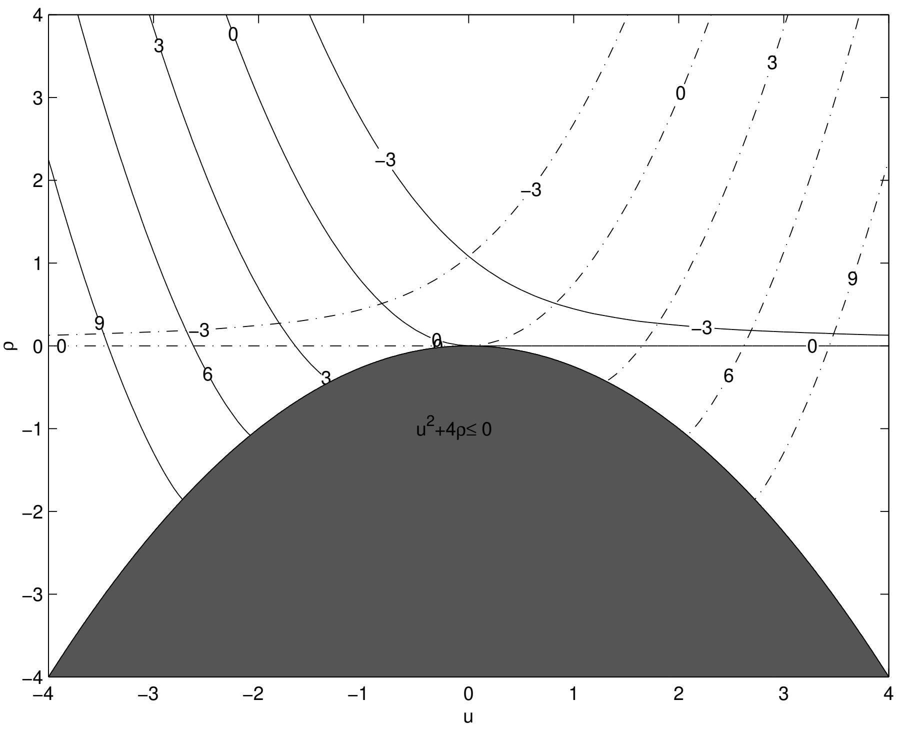

Figure 2: Level lines of the Riemann invariants: $z = cst$ and $w = cst$

4.4 Genuine nonlinearity

In plain words, genuine nonlinearity of a two-component system of hyperbolic conservation laws means that on the level curves $w(v) = \text{const.}$, respectively $z(v) = \text{const.}$, the characteristic speed μ, respectively λ, varies strictly monotonically. Formally:

$$\left.\frac{\partial \lambda}{\partial w}\right|_z \neq 0 \neq \left.\frac{\partial \mu}{\partial z}\right|_w .$$

Performing straightforward computations this turns out to be equivalent to

$$\nabla \lambda \cdot r \neq 0 \neq \nabla \mu \cdot s.$$

That is: the characteristic speeds λ and μ vary strictly monotonically in the direction of their corresponding right eigenvectors.

In our case, given the formulas (22) and (23) we easily get

$$\nabla \lambda \cdot r = \frac{2\lambda}{\lambda - \mu}, \qquad \nabla \mu \cdot s = \frac{2\mu}{\mu - \lambda}.$$

Recall from (22), (23) that on $\mathcal{D}_{\mathrm{ph}}$ we have $\mu \leq 0 \leq \lambda$, with strict inequalities for $\rho > 0$. We conclude that our system is genuinely nonlinear in the interior of the physically relevant domain $\mathcal{D}_{\mathrm{ph}}$. On the half lines $\rho = 0$, $u \leq 0$, respectively, $\rho = 0$, $u \geq 0$ (on the boundary of $\mathcal{D}_{\mathrm{ph}}$) genuine nonlinearity of the first, respectively, of the second, characteristic speed is lost.

4.5 Weak solutions, shocks, Rankine-Hugoniot conditions

As it is well-known, a nonlinear system of hyperbolic conservation laws (15) can develop singularities (e.g. discontinuities), irrespectively of the smoothness of the initial conditions. A *generalized* or *weak* solution of (15), (16) in a space-time domain is a bounded, measurable function $(t, x) \mapsto v(t, x)$ satisfying

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \big\{\partial_t\phi(t,x)\cdot v(t,x) + \partial_x\phi(t,x)\cdot J(v(t,x))\big\}dtdx + \tag{25}$$
$$+ \int_{-\infty}^{\infty} \phi(0,x)\cdot v^{(0)}(x)dx = 0$$

for any row vector valued test function $\phi = (\phi_1, \phi_2)$ with compact support in the respective space-time domain. This last equation is obtained by a formal integration by parts. It is easily seen that a strong (smooth) solution is also a weak solution.

Assuming a (locally) piecewise C^1 solution with a spatially isolated jump discontinuity at some space-time position $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, one derives the Rankine-Hugoniot conditions which relate the left- and right limits of the function $x \mapsto v(t, x)$ at the discontinuity and the propagation speed of the discontinuity:

$$\frac{J_1(v(t,x^+)) - J_1(v(t,x^-))}{v_1(t,x^+) - v_1(t,x^-)} = \sigma = \frac{J_2(v(t,x^+)) - J_2(v(t,x^-))}{v_2(t,x^+) - v_2(t,x^-)}, \tag{26}$$

where σ is the propagation speed of the discontinuity, i.e. the slope in space-time of the line of discontinuity. (26) is derived from (25) by an elementary local argument, using the divergence theorem (in space-time). Given the two independent relations in (26), any three of the five values $v_1(t, x^-)$, $v_1(t, x^+)$, $v_2(t, x^-)$, $v_2(t, x^+)$, σ determine the other two. This imposes a serious restriction on the possible jump discontinuities of weak solutions. Note that the conditions are left-right symmetric.

We turn now to our system (7). We denote by $(\rho^{\text{left}}, u^{\text{left}})$, respectively $(\rho^{\text{right}}, u^{\text{right}})$, the values of the component functions at the two sides of the presumed discontinuity. The Rankine-Hugoniot conditions are:

$$\frac{u^{\text{right}}\rho^{\text{right}} - u^{\text{left}}\rho^{\text{left}}}{\rho^{\text{right}} - \rho^{\text{left}}} = \sigma = \frac{\rho^{\text{right}} - \rho^{\text{left}}}{u^{\text{right}} - u^{\text{left}}}. \tag{27}$$

Given the value at one side of the discontinuity, the value at the other side as function of propagation speed is expressed as follows:

$$\rho^{\text{right}} = \sigma^2 - \sigma u^{\text{left}}, \qquad u^{\text{right}} = \sigma - \frac{\rho^{\text{left}}}{\sigma} \tag{28}$$

Note that ρ^{right}, respectively, u^{right}, is expressed as function of σ and u^{left}, respectively, as function of σ and ρ^{left}, only. (In principle, both should be expressed as functions of σ, ρ^{left} and u^{left}.) This is a special feature of our system.

The propagation speed, as function of the values of the components on both sides of the discontinuity, is expressed as:

$$\sigma_{\pm} = \pm\frac{1}{2}\left\{\sqrt{(u^{\text{right}})^2 + 4\rho^{\text{left}}} \pm u^{\text{right}}\right\} = \mp\frac{1}{2}\left\{\sqrt{(u^{\text{left}})^2 + 4\rho^{\text{right}}} \mp u^{\text{left}}\right\}.$$

Lax's condition of stability for Rankine-Hugoniot discontinuities, [10], specified for two-component systems reads as follows: Assume that the weak solution (25) of the two-component system (15) is piecewise smooth, with a spatially isolated discontinuity with values v^{left}, respectively, v^{right} on the two sides, propagating according to the Rankine-Hugoniot conditions (26). The discontinuity is a stable *back shock*, respectively, *front shock*, according whether

$$\mu(v^{\text{right}}) < \sigma < \min\{\mu(v^{\text{left}}), \lambda(v^{\text{right}})\}, \tag{29}$$

or

$$\max\{\lambda(v^{\text{right}}), \mu(v^{\text{left}})\} < \sigma < \lambda(v^{\text{left}}). \tag{30}$$

Rankine-Hugoniot discontinuities which do not obey either one of the conditions (29) or (30), are unstable, physically not realisable.

Tedious (but, in principle straightforward) computations show, that in the case of our system (4) the discontinuities propagating according to (27), or equivalently (28) are stable back shocks if $\sigma < 0$ and stable front shocks if $\sigma > 0$.

4.6 Entropies

Given the two-component system of conservation laws (15), we look for *additional* conserved quantities, i.e., for pairs of functions $\mathcal{D}_{\text{hyp}} \ni v \mapsto (S(v), F(v)) \in \mathbb{R} \times \mathbb{R}$ which satisfy

$$\partial_t S(v) + \partial_x F(v) = 0 \tag{31}$$

for *smooth solutions* of the original problem (15) (or, equivalently: for smooth solutions of (17)). Indeed, (31) means, that $S(v(t,x))$ is globally conserved quantity, with flux $F(v(t,x))$. The pair of functions (S, F) is called *entropy/flux* pair. Using the form (17), valid for smooth solutions of (15), one finds the system of PDEs defining an entropy/flux pair:

$$\nabla F = \nabla S \cdot \nabla J, \tag{32}$$

or, in extended form:

$$\frac{\partial F}{\partial v_k} = \sum_{l=1}^{2} \frac{\partial S}{\partial v_l} \frac{\partial J_l}{\partial v_k}, \qquad k = 1, 2.$$

This is a two-component linear hyperbolic system of PDEs for the two unknown functions S and F – just well determined. There are various alternative equivalent ways of writing it. E.g., eliminating the function F we get a second order hyperbolic PDE (a wave equation with variable coefficients) for S:

$$\frac{\partial J_1}{\partial v_2} \frac{\partial^2 S}{\partial v_1^2} + \left(\frac{\partial J_2}{\partial v_2} - \frac{\partial J_1}{\partial v_1} \right) \frac{\partial^2 S}{\partial v_1 \partial v_2} + \frac{\partial J_2}{\partial v_1} \frac{\partial^2 S}{\partial v_2^2} = 0.$$

Or, changing variables to the characteristic coordinates (w, z):

$$\frac{\partial F}{\partial w} = \lambda \frac{\partial S}{\partial z}, \qquad \frac{\partial F}{\partial z} = \nu \frac{\partial S}{\partial w}.$$

Or, eliminating F between these two equations:

$$\frac{\partial^2 S}{\partial w \partial z} = \frac{1}{\lambda - \mu} \left(\frac{\partial \mu}{\partial w} \frac{\partial S}{\partial z} - \frac{\partial \lambda}{\partial z} \frac{\partial S}{\partial w} \right).$$

These last two forms explicitly show the wave-character of the entropy equations (32). Of particular importance are those entropy/flux pairs for which the function $v \mapsto S(v)$ is *convex.* Such pairs will be simply called (with slight abuse of terminology) *convex entropy/flux pairs.*

In the case of our system (7) the entropy equations, written in terms of the physical variables ρ and u, are:

$$\frac{\partial F}{\partial \rho} = u\frac{\partial S}{\partial \rho} + \frac{\partial S}{\partial u}, \qquad \frac{\partial F}{\partial u} = \rho\frac{\partial S}{\partial \rho}$$

Or, eliminating F:

$$\rho\frac{\partial^2 S}{\partial \rho^2} - u\frac{\partial^2 S}{\partial \rho \partial u} - \frac{\partial^2 S}{\partial u^2} = 0 \tag{33}$$

The existence of a strictly convex entropy/flux pair, *globally defined* on $\mathcal{D}_{\text{ph}} = \{(\rho, u) : \rho \geq 0,\ u \in \mathbb{R}\}$ and with S bounded from below is very important, since the applicability of Lax's Maximum Principle cited in the next subsection relies on it. Here it is:

$$S(\rho, u) = \rho \log \rho + \frac{u^2}{2}, \qquad F(\rho, u) = u\rho(\log \rho + 1). \tag{34}$$

Lax's 'entropy wave construction' (cf. [11]) applies also to our system (4). Since these computations are rather involved, we do not reproduce them here. Let us just point out, that this robust method ensures the existence of a sufficiently rich family of convex entropy/flux pairs in any fixed subdomain compactly contained in $\mathcal{D}_{\text{ph}}$.

There are also other (more ad hoc) methods of constructing entropy/flux pairs. Following, e.g., the ideas of [13] we may try to find so called similarity solutions of the entropy equation (33) of the form:

$$S(\rho, u) = \rho^{\alpha}\phi(\rho^{\beta}u) \tag{35}$$

Elementary computations show that $\beta = -1/2$ is the only choice consistent with (33). Inserting (35), with $\beta = -1/2$ into (33) we find the following ordinary differential equation for the function $\phi : \mathbb{R} \to \mathbb{R}$:

$$3(y^2 - 4/3)\phi''(y) + (5 - 8\alpha)y\phi'(y) + 4\alpha(\alpha - 1)\phi(y) = 0. \tag{36}$$

Any solution of (36), with any $\alpha \in \mathbb{R}$ fixed provides an entropy of our system, via (35). So, we are able to construct a sufficiently rich family of entropy/flux pairs to our system (7).

4.7 Entropy solutions

A weak solution $(t,x) \mapsto v(t,x)$ of the generic system (15) is called *entropy solution* if for any convex entropy/flux pair (S,F) we have

$$\partial_t S + \partial_x F \leq 0 \tag{37}$$

in the sense of distributions, i.e., for any *positive* test function $(t,x) \mapsto \phi(t,x)$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \{\partial_t\phi(t,x)S(v(t,x)) + \partial_x\phi(t,x)F(v(t,x))\}\, dtdx \geq 0.$$

Entropy solutions are the only physically admissible, stable ones among the weak solutions. Strong limits of all convergent approximation schemes (such as vanishing viscosity or various convergent finite difference schemes) result in entropy solutions. It is also expected that convergent hydrodynamic limits of interacting particle systems result in entropy solution of the corresponding hyperbolic conservation laws. For piecewise smooth weak solutions, Lax's stability condition for the shocks mentioned in a previous paragraph is equivalent with the entropy conditions (37).

Of particular interest is the following Maximum Principle, due to P. Lax, see e.g. [11].

Maximum Principle for Entropy Solutions. *Assume that the following two conditions hold*

(i) The Riemann invariants $v \mapsto w(v)$ and $v \mapsto z(v)$ of the system of hyperbolic conservation laws (15) are (globally) convex functions of v.

(ii) There exists a globally defined convex entropy/flux pair, with entropy function bounded from below.

Then, starting with bounded initial data, $\sup_{-\infty<x<\infty}|v^{(0)}(x)| < \infty$, along entropy solutions $(t,x) \mapsto v(t,x)$ the maximum values of the Riemann invariants, $\sup_{-\infty<x<\infty} w(v(t,x))$ and $\sup_{-\infty<x<\infty} z(v(t,x))$ do not increase with t.

Remark: The same statement applies for solutions $v^{(\varepsilon)}(t,x)$ of the viscous system (18) — this follows from the classical maximum principle. If $v^{(\varepsilon)}$ converges strongly as $\varepsilon \to 0$, then the limiting v is in fact an entropy solution of the inviscid system (15) and forcibly it obeys Lax's Maximum Principle. It is not clear whether all entropy solutions arise as limits of viscous solutions, with vanishing viscosity. A general proof of the Maximum Principle for entropy solutions can be found in [11].

Applying this theorem to our system we find that if we start with bounded initial data $x \mapsto (\rho^{(0)}(x), u^{(0)}(x)) \in \mathcal{D}_{\mathrm{ph}}$ (that is: with non-negative initial density) then *entropy solutions will stay in the physical domain, i.e., for any* $t \geq 0$ $x \mapsto (\rho(t,x), u(t,x)) \in \mathcal{D}_{\mathrm{ph}}$. (See Fig. 2 for graphical representation of the level curves $w(\rho, u) =$ const. and $z(\rho, u) =$ const.) This is a very important consequence of the Maximum Principle: as we already mentioned, a priori we could not see any reason banning a (physically relevant) solution from flowing out into the physically meaningless domain with $\rho < 0$.

In the case of isentropic gas dynamics, (1), choosing convex versions of the Riemann invariants w and z, for any $w_{\max} \in \mathbb{R}$, $z_{\max} \in \mathbb{R}$, the domains

$$\{(\rho, m) \in \mathbb{R}_+ \times \mathbb{R} : w(\rho, m) \leq w_{\max}, z(\rho, m) \leq z_{\max}\}$$

are compact. So starting with bounded initial data global boundedness of (viscous and) entropy solutions is guaranteed by the Maximum Principle. This is unfortunately not the case for our system. The domains

$$\{(\rho, u) \in \mathcal{D}_{\mathrm{ph}} : w(\rho, u) \leq w_{\max}, z(\rho, u) \leq z_{\max}\}$$

are *not* compact, see Fig. 2. So here is an open question: *Is it the case, that if the initial data (8) are bounded then the solutions* $\left(\rho^{(\varepsilon)}(t,x), u^{(\varepsilon)}(t,x)\right)$ *of the viscous equation (19) stay bounded for ever? Similarly: is it the case that entropy solutions of (7) with bounded initial data stay bounded?* We guess that the answer to these questions are affirmative, but we could not prove this yet.

4.8 Vanishing viscosity, existence of entropy solutions

The *existence of entropy solutions* for a two-component syetem of hyperbolic conservationlaws (15) is a notoriously difficult question. The most powerful approach seems to be the program initiated by R. DiPerna in [3], completed for the case of isentropic gas dynamics (1) in [4], then refined and extended in Lions et al. [13] and in several other papers.

In [3], DiPerna proves the following result:

DiPerna's Theorem. *Consider the two-component system of hyperbolic conservation laws (15) and the corresponding viscous system (18). Assume that*

- *(i) The Riemann invariants $v \mapsto w(v)$ and $v \mapsto z(v)$ are convex. (More precisely: there are convex choices of the Riemann invariants. See subsection 4.3.)*
- *(ii) The system is genuinely nonlinear. (See subsection 4.4.)*

Let $\mathcal{C}$ be a domain compactly contained in $\mathcal{D}_{\text{hyp}}$ and assume that the sequence of solutions $v^{(\varepsilon)}(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}$, $\varepsilon \to 0$, of the viscous systems (18), (16) takes values from $\mathcal{C}$. Then there is a subsequence $v^{(\varepsilon')}(t,x)$ which converges strongly in $L^1_{\text{loc}}([0,T] \times \mathbb{R})$. The limit $v(t,x)$ is entropy solution of the system (15).

Some Remarks:

(1) The proof relies on the construction of Lax's 'entropy waves', hinted at in subsection 4.6 and essentially on the so-called compensated compactness method developed by Murat and Tartar. We do not have a chance to reproduce here any technical part of the proof.

(2) It is assumed that the viscous solutions stay in the domain $\mathcal{C}$. However, even in this form the theorem is technically very-very difficult. Extra difficulties arise by relaxing this condition and imposing conditions *only on the initial data*: in the isentropic gas dynamics and in our case too, the solution data will typically flow to the boundary of the domain of hyperbolicity and genuine nonlinearity, $\rho = 0$, where this theorem is not any more valid.

(3) For extensions, physically more satisfactory formulations and enormous further technical difficulties see e.g. [4], [13], etc.

This theorem can be applied in a straightforward way for domains $\mathcal{C}$, compactly contained in $\mathcal{D}_{\text{ph}}$. We can add to this that if initially

$$\max_x w(\rho^{(0)}(x), u^{(0)}(x)) < 0 \quad \text{or} \quad \max_x z(\rho^{(0)}(x), u^{(0)}(x)) < 0$$

then, due to the Maximum Principle, the viscous solutions $\rho^{(\varepsilon)}, u^{(\varepsilon)}$ are kept away from the 'dangerous' vacuum line $\rho = 0$, see Fig. 2. So, in this case one has to care only about the boundedness of the solutions.

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