LONG DISTANCE CORRELATION STRUCTURE OF INTERMITTENCY IN PARABOLIC ANDERSON MODELS

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Abstract. We study the *long distance* correlation structure of intermittency in parabolic Anderson models. We use the Green's function formulation and the analytical frame work established in [W1-4]. Contrary to the case in [W1-4], this is the regime of large fluctuations. We prove that for a suitable class of probability distributions, the higher moments of the Green's functions *do not* decay faster than the first moment. The Green's functions therefore exhibit an intermittent structure. This result on long distance asymptotics is complementary to other known results on intermittency, which pertain to *long time* asymptotics.

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I. INTRODUCTION

In this paper we use the analytical framework established in [W1-4] to study parabolic Anderson models, which describe diffusion in random potentials. Let

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d), \tag{1.1}$$

where the matrix elements

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell_1} = 1 \\ &= 0 \quad \text{otherwise}; \end{aligned} \tag{1.2}$$

γ is a positive parameter, the potential function V is a diagonal matrix: $V = \text{diag}(v_j)$, $j \in \mathbf{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution dg. From now on, we write $|\,|_1$ for $|\,|_{\ell^1}$. ℓ^2 norms will be denoted by $\|\,\|$. In this paper we will be mostly using ℓ^1 norms. The probability measure is taken to be $P = \prod_{j \in \mathbf{Z}^d} dg(v_j)$. We use the notation $\langle\,\rangle$ to denote the expectation value with respect to P.

We are concerned with the following Cauchy problem in the *large disorder* regime: $\gamma >> 1$,

$$\begin{cases} \dfrac{\partial}{\partial t} u(j,t) = (\Delta + \gamma V) u(j,t) \qquad (j \in \mathbf{Z}^d,\, t \geq 0) \\ u(j,0) = \delta_0(j). \end{cases} \tag{1.3}$$

In particular, we are interested in the behavior as $t \to \infty$ of $u(j(t),t)$, where $j(t) \to \infty$ as $t \to \infty$ for $\gamma >> 1$. This is a regime which is qualitatively different from that studied in [W1-4].

Since V is stationary in time, the problem in (1.3) can be related to the spectral study of H. As is well known, the spectrum $\sigma(\Delta)$ is $[-2d, 2d]$. Let supp dg be the support of dg, then we have the well established fact (See e.g., [CFKS,PF].) that

$$\sigma(H) = [-2d, 2d] + \gamma \,\text{supp}\, dg \qquad a.s. \tag{1.4}$$

For $E \notin \sigma(H)$, we define

$$G(E) = (E - H)^{-1}, \tag{1.5}$$

and its kernel

$$G(E,i,j) = (\delta_i, (E-H)^{-1}\delta_j), \qquad i,\, j \in \mathbf{Z}^d, \tag{1.6}$$

to be the Green's function of H at energy E. For a given realization of V, $G(E,\mu,\nu)$ is the fundamental solution to the equation

$$(E - H)G = \delta_\mu. \tag{1.7}$$

We assume dg has semi-bounded support. Without loss of generality, we assume

(H1) supp $dg \subseteq (-\infty, 0]$ or supp $dg \subseteq [0, +\infty)$.

The behavior of $u(j(t),t)$, as $t \to \infty$, $j(t) \to \infty$ can then be deduced from that of $G(E,i,j)$ as $|i-j| \to \infty$ for $E = 2d$ if supp $dg \subseteq [0,+\infty)$ or $-2d$ if supp $dg \subseteq (-\infty, 0]$. (For more precise statements, see [sect. 7.2, Sz].) Henceforth, we will only address the asymptotic behavior of $G(E,i,j)$ as $|i-j| \to \infty$.

The appropriate mathematical notion for studying long distance behavior of Green's functions is the notion of Lyapunov exponents as introduced by Sznitman in [Sz]. By using the FKG inequality [FKG, Li] as stated in Remark 3.2 on p. 241-242 of [Sz] and subadditivity ergodic theorem, it can be shown, similar to [Sz, Z] that for all $E \geq 2d$ or $E \leq -2d$, the limit

$$\beta_E(j) = \lim_{n\to\infty} -\frac{\log\langle G(E,0,nj)\rangle}{n}, \tag{1.8}$$

exists. Moreover by patching of limits:

$$\lim_{j\to\infty} \frac{|\beta_E(j) + \log\langle G(E,0,j)\rangle|}{|j|_1} = 0. \tag{1.9}$$

The $\beta_E(j)$ are called the annealed Lyapunov exponents. For a fixed E, β_E defines a norm on $\mathbf{R}^d$ [Sz, Z].

Similarly it is shown in [Z] that for all $E \geq 2d$ or $E \leq -2d$, the limit

$$\alpha_E(j) = \lim_{n\to\infty} -\frac{\log G(E,0,nj)}{n}, \tag{1.10}$$

exists a.s. and that

$$\lim_{j\to\infty} \frac{|\alpha_E(j) + \log G(E,0,j)|}{|j|_1} = 0, \qquad a.s. \tag{1.11}$$

The $\alpha_E(j)$ are called the quenched Lyapunov exponents. Like β_E, for a fixed E, α_E defines a norm on $\mathbf{R}^d$ [Sz, Z].

In order to describe the moments of G, we extend the above notions as follows. We define

$$\begin{aligned} \bar{b}_E^{(p)}(j) &= \limsup_{n\to\infty} -\frac{\log\langle [G(E,0,nj)]^p\rangle}{n}, \qquad p = 1,\, 2, \ldots \\ \underline{b}_E^{(p)}(j) &= \liminf_{n\to\infty} -\frac{\log\langle [G(E,0,nj)]^p\rangle}{n}, \qquad p = 1,\, 2, \ldots \end{aligned} \tag{1.12}$$

to be the p^{th} generalized "annealed Lyapunov exponents". We note that when $p=1$,

$$\bar{b}_E^{(1)}(j) = \underline{b}_E^{(1)}(j) = \beta_E(j),$$

where $\beta_E(j)$ is the Lyapunov exponent defined in (1.8) satisfying (1.9).

Remark. It would be of interest to determine whether $\bar{b}_E^{(p)}(j) = \underline{b}_E^{(p)}(j)$ for $p > 1$ and that we have the p^{th} generalized Lyapunov exponent in the sense of (1.9).

In analogy with the definition of intermittency for long time asymptotics, see e.g., [Mo], we say that there is intermittency at long distance if for all $m = 1, 2, ...,$ all $j \in \mathbf{Z}^d$ $(j \neq 0)$, the following inequality holds:

$$\beta_E(j) > \frac{\bar{b}_E^{(2)}(j)}{2} \geq \frac{\underline{b}_E^{(2)}(j)}{2} > \frac{\bar{b}_E^{(3)}(j)}{3} \geq \frac{\underline{b}_E^{(3)}(j)}{3} \cdots > \frac{\bar{b}_E^{(m)}(j)}{m} \geq \frac{\underline{b}_E^{(m)}(j)}{m}. \tag{1.13}$$

Note that the inequalities are *strict* among the generalized Lyapunov exponents for different p. (Due to the minus signs in the definition of the Lyapunov exponents in (1.8, 1.12), the inequality signs in (1.13) are reversed compared to the definition in [Mo].)

To prove our results, aside from (H1), we need one more condition on dg. Define

$$g(a) = \int_{-a}^{0^+} dg(v), \qquad (a \geq 0) \tag{1.14}$$

in the first case of (H1) and

$$g(a) = \int_{0^-}^{a} dg(v), \qquad (a \geq 0) \tag{1.15}$$

in the second case of (H1). We assume further

(H2) dg is such that g is of regular variation at 0 with exponent $1 - \delta$ $(0 < \delta < 1)$, i.e.,

$$\frac{g(Ca)}{g(a)} \to f(C) = C^{1-\delta} \quad (0 < \delta < 1) \tag{1.16}$$

as $a \to 0$ for all $C > 0$.

Remarks.

(i) Under the assumption (H1), $\sigma(H) \subseteq (-\infty, 2d]$ or $\sigma(H) \subseteq [-2d, \infty)$.

(ii) Since dg is a probability measure, g is a positive monotone function on $[0, \infty)$. Once we assume the limit of $g(Ca)/g(a)$ exists and is finite as $a \to 0$ for a dense set of C's in $\mathbf{R}^+$, then the limit $f(C)$ is necessarily of the form C^ρ for some $0 \leq \rho < \infty$.(For more details on this, see p. 268 of [F].) Here we further specify that $0 < \rho = 1 - \delta < 1$.

Assumption (H2) is quite natural, as we are studying the case where the potential is "dominant": $\gamma >> 1$ and the Green's function $G(E) = (E - H)^{-1} = (E - \Delta - \gamma V)^{-1}$. So the behavior of the distribution g around 0 is crucial.

The Bernoulli distribution:

$$\begin{aligned} dg(v) &= \frac{\delta(v) + \delta(v+1)}{2} \\ &= \frac{dh(v) + dh(v+1)}{2} \end{aligned} \tag{1.17}$$

where δ is the Dirac distribution at 0 and h is the Heaviside function, is of regular variation with exponent 0 $(\delta = 1)$. The uniform distribution:

$$\begin{aligned} dg(v) &= 1 \qquad 0 < v < 1 \\ &= 0 \qquad v = 0, 1 \end{aligned} \tag{1.18}$$

is of regular variation at 0 with exponent 1 $(\delta = 0)$. So the class of dg in (H2) is "in between" the Bernoulli distribution and the uniform distribution.

Our main result is :

Theorem 1.1. *Assume dg satisfies (H1,2) with $0<\delta<1$. There exist $\gamma_0>0$, c_m, $C_m>0$ ($m\in\mathbf{N}$) such that for all $\gamma>\gamma_0$, all m, all μ, $\nu\in\mathbf{Z}^d$,*

$$(1.19)\quad \left(\frac{c_m}{\gamma}\right)^{[(1-\delta)|\mu-\nu|_1+1]}\le\langle[(E-\Delta-\gamma V)^{-1}(\mu,\nu)]^m\rangle\le\left(\frac{C_m}{\gamma}\right)^{[(1-\delta)|\mu-\nu|_1+1]},$$

where $E=2d$ if $\operatorname{supp}dg\subseteq(-\infty,0]$; $E=-2d$ if $\operatorname{supp}dg\subseteq[0,\infty)$.

Remark. The proof of Theorem 1.1 is applicable to dg of regular variation with exponents ≥ 1. We opt to write the theorem for dg satisfying (H2) only, because of the interesting application in Theorem 1.2 below in this case. When the exponent is 0, e.g., for the Bernoulli distribution, the estimates here do not apply directly. But since

$$(E-\Delta-\gamma V)^{-1}(\mu,\nu)\le(E-\Delta-\gamma' V)^{-1}(\mu,\nu)$$

for $\gamma>\gamma'$, we still have exponential decay by applying the results in [W3]. It can be shown however that the rate is only of order 1.

Using Theorem 1.1 in (1.9), the following is immediate:

Corollary. *Assume dg satisfies (H1,2). Let $\hat{j}=j/|j|_1\in\ell^1$-unit ball. There exists $\gamma_0>0$ such that for $\gamma>\gamma_0$,*

$$(1.20)\qquad \beta_E(\hat{j})=(1-\delta)\log\gamma+\mathcal{O}(1)\qquad(0<\delta<1)$$

where $E=2d$ if $\operatorname{supp}dg\subseteq(-\infty,0]$; $E=-2d$ if $\operatorname{supp}dg\subseteq[0,\infty)$.

The main consequence of Theorem 1.1 is

Theorem 1.2. *Assume dg satisfies (H1,2) with $0<\delta<1$. Let $E=2d$ if $\operatorname{supp}dg\subseteq(-\infty,0]$; $E=-2d$ if $\operatorname{supp}dg\subseteq[0,\infty)$. For all m, there exist $\gamma_0>0$, such that for all $\gamma>\gamma_0$, the Green's function $G(E,i,j)$ defined in (1.6) is intermittent. More precisely, the generalized Lyapunov exponents defined in (1.8, 1.12) satisfy (1.13).*

Geometric interpretation of Theorem 1.2.

Similar to the long time asymptotics (see e.g., [Mo]), Theorem 1.2 implies the following hierarchical structure of $G(E,\mu,\nu)$. Without loss of generality, we take $\mu=0$. For $\nu\in\mathbf{Z}^d$ ($\nu\neq0$), let

$$(1.21)\qquad a^{(1)}(\nu)\in(\frac{\bar{b}_E^{(2)}(\nu)}{2},\beta_E(\nu))$$

In what follows, ν is held fixed. So generally, we will not make the dependence on ν explicit in $a^{(1)}$, $\bar{b}_E^{(2)}$, β_E.

Let $n\in\mathbf{N}$ and consider the set

$$(1.22)\qquad B_{n,a^{(1)}}=\{j\in\mathbf{Z}^d|G(E,j,j+n\nu)>e^{-na^{(1)}}\}.$$

Let $\Lambda = [-L, L]^d$ $(L > 0)$. The volume density of the set:

$$\rho_{n,a^{(1)}} = \lim_{L\to\infty} \frac{\mathrm{vol}[B_{n,a^{(1)}} \cap \Lambda]}{(2L+1)^d}$$

exists almost surely by the ergodic theorem and

$$\rho_{n,a^{(1)}} = P\{G(E, 0, n\nu) > e^{-na^{(1)}}\}.$$

Using the Tchebychev inequality, we have further

$$\begin{aligned}\rho_{n,a^{(1)}} &= P\{G(E, 0, n\nu) > e^{-na^{(1)}}\}\\ &\leq \langle G(E, 0, n\nu)\rangle e^{na^{(1)}}\\ &\leq C_1 e^{-n(\beta_E - a^{(1)})},\end{aligned}$$

for all $a^{(1)}$ satisfying (1.21). So for large n, the density $\rho_{n,a^{(1)}}$ of the set $B_{n,a^{(1)}}$ is exponentially small in n. As $n \to \infty$, $B_{n,a^{(1)}}$ typically consists of exponentially rare islands or points in $\mathbf{Z}^d$.

Using (1.13, 1.22) and the ergodic theorem, as $n \to \infty$, the second moment

$$\langle [G(E, 0, n\nu)]^2\rangle \sim \lim_{L\to\infty} \frac{1}{(2L+1)^d} \sum_{j\in B_{n,a^{(1)}}} [G(E, j, j + n\nu)]^2$$

i.e., the second moment is almost entirely generated by the large fluctuations (large deviation from the mean) of the Green's function concentrated in $B_{n,a^{(1)}}$.

By the same reasoning, for any

$$a^{(2)} \in (\frac{\bar{b}_E^{(3)}}{3}, \frac{b_E^{(2)}}{2}),$$

the third moment $\langle [G(E, 0, n\nu)]^3\rangle$ is concentrated in the set:

$$B_{n,a^{(2)}} = \{j \in \mathbf{Z}^d | G(E, j, j + n\nu) > e^{-na^{(2)}}\}.$$

Since $a^{(2)} < a^{(1)}$, $B_{n,a^{(2)}} \subset B_{n,a^{(1)}}$.

More generally, Theorem 1.2 says that for all $m \in \mathbf{N}$ and all sufficiently large γ, for any

$$a^{(p)} \in (\frac{\bar{b}_E^{(p+1)}}{p+1}, \frac{b_E^{(p)}}{p}), \qquad p = 1, 2, \ldots m,$$

there exist a corresponding hierarchical sequence of sets in $\mathbf{Z}^d$:

$$B_{n,a^{(1)}} \supset B_{n,a^{(2)}} \supset \ldots \supset B_{n,a^{(m)}},$$

such that for large n, the p^{th} moment $\langle [G(E, 0, n\nu)]^p\rangle$ is concentrated in $B_{n,a^{(p)}}$.

The attraction centers.

The so called attraction centers for the Green's function are the islands of sites $i \in \mathbf{Z}^d$, where v_i are near 0. These are the islands where the local Green's function is large. For the class of probability measures that satisfy (H2), in dimensions $d \geq 2$, there is percolation of sites i such that $v_i \leq a$ for some appropriate $a > 0$. (See e.g., [Gr].) And yet for $0 < \delta < 1$ and sufficiently large γ, the averaged Green's function decays at a rate: $\mathcal{O}((1-\delta)\log\gamma)) >> \mathcal{O}(1)$. This is much larger than the rate indicated by a. When $\delta = 0$, the class of probability measure in (H2) includes the Bernoulli distribution. Taking $a = 0$, we have the standard percolation scenario. There is percolation in $d \geq 2$. But as mentioned in the remark after Theorem 1.1, the averaged Green's function still decays exponentially, even though the rate is now only of order 1.

The attraction centers are also manifested at the level of the correponding field theory. We will not go into the details here except mentioning that by using the Gaussian-Grassmann representation initiated in [W1-4], the averaged Green's functions are mapped to correlation functions of an *attractive* field theory, (see sect. II). The measure here is the effective measure after integrating over the random potentials. The precise expression of the measure is obtained by using (2.22) of sect. II and integrating over the Grassmann (anti-commuting) variables.

Connections with existing results.

So far, the majority of the literature on intermittency in the Anderson model is concerned with the long time asymptotics. (See e.g., [CM,GM,Mo].) There one considers the Cauchy problem:

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t} v(j,t) = (\Delta + \gamma V) v(j,t) & (j \in \mathbf{Z}^d,\ t \geq 0) \\ v(j,0) = 1. & \end{array} \right. \tag{1.23}$$

One obtains asymptotics for $\langle v(j,t)^p \rangle$, $p = 1, 2, ...$ and also $v(j,t)$ *a.s.*, for *fixed* j and $t \to \infty$, (see e.g., [GM]). In [GH], the authors obtained the asymptotics for

$$\lim_{t\to\infty} \frac{\langle v(i,t) v(j,t) \rangle}{\langle v^2(0,t) \rangle}.$$

But again i, j are held *fixed.*

The main difference between the problem in (1.3) and that in (1.23) is the *initial* condition. In (1.3), the initial condition corresponds to a δ source:

$$u(j,0) = \delta_0(j). \tag{1.24}$$

Note that knowing u implies knowing v, since by symmetry:

$$v(0,t) = \sum_{j\in\mathbf{Z}^d} u(j,t),$$

where $u(j,t)$ is the solution to the Cauchy problem in (1.3).

A sketch of the proof of Theorem 1.1.

The proof of Theorem 1.1 consists of (a) a finite volume Green's function estimate; (b) an iteration procedure. In (a), we use the supersymmetric representation, i.e., the combination of a Gaussian and a Grassmann integral as in [W1-4]. Due to the presence of the large parameter γ, the elementary matrix using which we do perturbation theory is $\mathcal{O}(1) \times \mathcal{O}(1)$. (In the case $\gamma << 1$ in [W2-4], the elementary matrix is of size $\mathcal{O}(1/\sqrt{\gamma}) \times \mathcal{O}(1/\sqrt{\gamma})$.)

The main novelty here is that by manipulating Grassmann integrals, we are able to identify all the terms contributing to the determinant of this $\mathcal{O}(1)\times\mathcal{O}(1)$ elementary matrix. Hence our estimates are optimal in powers of $1/\gamma$.

The iteration procedures in (b) are the same as in [W4]. The upper bound part was first devised in [W3]. The iteration for the upper bound uses the resolvent expansion and a priori estimates for the free Green's function; while the lower bound uses the resolvent expansion and the FKG inequalities [FKG, Li]. The FKG inequalities are applicable here as $[G(E, \mu, \nu)]^m$ are monotone in V. Due to the presence of the parameter $\gamma >> 1$, the precisions afforded by these iteration procedures are sufficient to obtain the first term in the asymptotics for the annealed Lyapunov exponent β in the Corollary.

II. THE SUPERSYMMETRIC FORMALISM AND EXPRESSIONS FOR $\langle \prod G_\Lambda \rangle$

Without loss of generality, we may assume $\operatorname{supp} dg \subseteq (-\infty, 0]$. This is because the replacement of V by $-V$ followed by the unitary map $u(n) \to (-1)^{|n|}u(n)$ takes H into $-H$. For definitiveness, we henceforth assume $\operatorname{supp} dg \subseteq (-\infty, 0]$.

Let $E = 2d$. Let Λ be the ℓ^1-ball in $\mathbf{Z}^d$ of radius L ($L \in \mathrm{N}$), i.e., $\Lambda = \{i \in \mathbf{Z}^d | |i|_1 \leq L\}$, we define H_Λ as the operator:

$$H_\Lambda = \Delta_\Lambda + \gamma V, \text{ on } \ell^2(\Lambda). \tag{2.1}$$

The spectrum

$$\sigma(H_\Lambda) \subset [-2d, 2d] + \gamma \operatorname{supp} dg.$$

We define the finite volume Green's function as

$$G_\Lambda(E, i, j) = (E - H_\Lambda)^{-1}(i, j), \qquad E \notin \sigma(H_\Lambda). \tag{2.2}$$

It is well known that for all ℓ^1-balls Λ in $\mathbf{Z}^d$,

$$E - \Delta_\Lambda = 2d - \Delta_\Lambda > 0$$

as an operator. Hence

$$E - \Delta_\Lambda - \gamma V > 0,$$

when $\operatorname{supp} dg \subseteq (-\infty, 0]$. The finite volume Green's function

$$G_\Lambda = (E - \Delta_\Lambda - \gamma V)^{-1}$$

is well defined at $E = 2d$.

Our aim in this section is to recall the convenient representation for $\langle \prod_{\alpha=1}^m G_\Lambda(E, \mu_\alpha, \nu_\alpha)\rangle = \langle \prod_{\alpha=1}^m (E - \Delta_\Lambda - \gamma V)^{-1}_{\mu_\alpha \nu_\alpha}\rangle$, μ_α, $\nu_\alpha \in \Lambda$, derived in [W4]. We start by presenting a brief introduction to the supersymmetric formalism. For a general proper formulation, see e.g., [Be,V]. (For earlier uses of supersymmetry in the context of localization, see e.g., [BCKP, K, KS].)

For any symmetric $n \times n$ matrix A with $\Re A > 0$, we recall the following well-known formulae:

$$(\det A)^{-1} = \int e^{-\sum_{i,j=1}^n A_{ij} x_i \cdot x_j} \prod_{i=1}^n \frac{d^2 x_i}{\pi}, \tag{2.3}$$

and

$$A^{-1}_{\mu\nu} = \int x_\mu \cdot x_\nu \det A e^{-\sum_{i,j=1}^n A_{ij} x_i \cdot x_j} \prod_{i=1}^n \frac{d^2 x_i}{\pi}, \tag{2.4}$$

where $x_i \in \mathbf{R}^2$ and $x_i \cdot x_j$ is the usual scalar product in $\mathbf{R}^2$. Substituting $E - H_\Lambda$ for A in (2.4), and drop the subscript Λ for simplicity, we then have

$$G(E, \mu, \nu) = \int x_\mu \cdot x_\nu \det(E - H) e^{-\sum_{i,j \in \Lambda} (E-H)_{ij} x_i \cdot x_j} \prod_{i \in \Lambda} \frac{d^2 x_i}{\pi}. \tag{2.5}$$

(In this section, since we only work in Λ, if it is convenient, we will continue to drop the subscript Λ without further notice.)

Let $|\Lambda|$ be the number of points in Λ. We use the Grassmann algebra of $2|\Lambda|$ generators to express $\det[E - H]$. This algebra is generated by $2|\Lambda|$ anticommuting variables ξ_i, η_i, $i \in \Lambda$ satisfying the relations:

$$\begin{aligned} [\xi_i, \eta_j] &= \xi_i \eta_j + \eta_j \xi_i = 0, \\ [\xi_i, \xi_j] &= \xi_i \xi_j + \xi_j \xi_i = 0, \\ [\eta_i, \eta_j] &= \eta_i \eta_j + \eta_j \eta_i = 0, \end{aligned} \tag{2.6}$$

where we write $[a, b] = ab + ba$ for the anti-commutator. It is denoted by $A[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$ (if we identify Λ with $\{1, .., |\Lambda|\}$). From (2.4), we see in particular that $\xi_i^2 = \eta_i^2 = 0$. We write these anticommutative variables collectively as ξ, η.

To make connections with better known objects, ξ_j, η_j can be thought of as the two 1-forms on the j^{th} copy of $\mathbf{R}^2$. "C^∞ functions" $F(\xi_i, \eta_j)$ of these anticommuting variables are defined by Taylor's formula at $(0, 0)$ which contains a finite number of terms because of nilpotency. In this way $F(\xi, \eta)$ becomes an element of the Grassmann algebra. For example if

$$F(\xi, \eta) = e^{A_{ij} \xi_i \eta_j}, \tag{2.7}$$

then

$$F(\xi, \eta) = 1 + A_{ij} \xi_i \eta_j. \tag{2.8}$$

This is the function that we need to write the determinant. We also need to define the notions of differentiation and integration. Define:

$$\begin{aligned}\frac{\partial}{\partial\xi_i}(\xi_i) &= 1,\\ \frac{\partial}{\partial\eta_i}(\eta_i) &= 1.\end{aligned} \tag{2.9}$$

We require further that these differentiations be linear operators and that Leibnitz' rule hold. We can then define integrals (with respect to ∂) as follows:

$$\int 1 d\xi_i = 0, \quad \int \xi_i d\xi_i = 1, \quad \int 1 d\eta_i = 0, \quad \int \eta_i d\eta_i = 1. \tag{2.10}$$

A multiple integral is defined to be a repeated integral. For example,

$$\int \xi_i\eta_j d\xi_i d\eta_j = -\int \eta_j\xi_i d\xi_i d\eta_j = -\int \eta_j d\eta_j = -1. \tag{2.11}$$

Using (2.7,2.8, 2.10,2.11), we obtain

$$\det(E-H) = \int e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.12}$$

$$(E-H)^{-1}_{\mu\nu}\det(E-H) = \int \xi_\mu\eta_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.13}$$

We illustrate (2.12) in the case where $(E-H) = M$ is a 2×2 matrix. The integrand in the RHS of (2.12) is

$$e^{-\sum_{j,k=1,2} M_{jk}\eta_j\xi_k} = \prod_{j,k=1,2}(1 - M_{jk}\eta_j\xi_k), \tag{2.14}$$

where we used the commutativity of $e^{-M_{jk}\eta_j\xi_k}$ and (2.7, 2.8). Doing the integral in the RHS of (2.14) upon using (2.10) and (2.11), we obtain

$$\text{RHS} = M_{11}M_{22} - M_{12}M_{21} = \det M, \tag{2.15}$$

as expected. Similarly, we obtain for the RHS of (2.13):

$$\text{RHS} = (-1)^{|\mu|+|\nu|}\det(M^{\mu\nu}) = M^{-1}_{\mu\nu}\det M, \qquad \mu,\ \nu = 1,\ 2, \tag{2.16}$$

where $M^{\mu\nu}$ is the co-matrix with the μth row and νth column removed. (In the 2×2 case it is just the one matrix element that is left.)

Combining (2.5) with (2.12) and (2.13) with (2.3) (replacing A by $E-H$ in (2.3)), we obtain the following expression:

$$\begin{aligned}G(E,\mu,\nu) &= \int x_\mu\cdot x_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}X_j\cdot X_k}\prod_{j\in\Lambda} d^2X_j,\\ &= \int \xi_\mu\eta_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}X_j\cdot X_k}\prod_{j\in\Lambda} d^2X_j,\end{aligned} \tag{2.17}$$

where

$$\begin{aligned} x_j &\in \mathbf{R}^2, \\ X_j &\overset{\text{def}}{=} (x_j, \xi_j, \eta_j), \\ X_j \cdot X_k &\overset{\text{def}}{=} x_j \cdot x_k + \frac{1}{2}(\eta_j \xi_k + \eta_k \xi_j), \\ d^2 X_j &\overset{\text{def}}{=} \frac{d^2 x_j}{\pi} d\eta_j d\xi_j, \end{aligned} \tag{2.18}$$

and we used that $(E - H)$ is a symmetric matrix.

Let $\hat{g}$ be the Laplace transform of dg:

$$\hat{g}(s) = \int e^{sv} dg(v), \qquad s \geq 0 \tag{2.19}$$

For example, for the Bernoulli distribution:

$$dg(v) = \frac{\delta(v+1) + \delta(v)}{2}, \tag{2.20}$$

$\hat{g}(s) = e^{-s/2} \cosh(s/2)$; while for the uniform distribution:

$$\begin{aligned} dg(v) &= 1 \qquad 0 < v < 1 \\ &= 0 \qquad v = 0, 1 \end{aligned} \tag{2.21}$$

$$\hat{g}(s) = \frac{1 - e^{-s}}{s}.$$

Using (2.17) and taking the expectation value with respect to dg, we obtain two equivalent expressions:

$$\begin{aligned} \langle G(E, \mu, \nu) \rangle &= \int x_\mu \cdot x_\nu e^{-(\sum_{j \in \Lambda} E X_j \cdot X_j - \sum_{j,k \in \Lambda, |j-k|_1 = 1} X_j \cdot X_k - \gamma \sum_{j \in \Lambda} v_j X_j \cdot X_j)} \\ &\qquad \prod_{j \in \Lambda} dg(v_j) \prod_{j \in \Lambda} d^2 X_j \\ &= \int x_\mu \cdot x_\nu e^{-(\sum_{j \in \Lambda} E X_j \cdot X_j - \sum_{j,k \in \Lambda, |j-k|_1 = 1} X_j \cdot X_k)} \prod_j \hat{g}(\gamma X_j \cdot X_j) \prod_{j \in \Lambda} d^2 X_j \\ &\overset{\text{def}}{=} \int x_\mu \cdot x_\nu e^{-2\Phi_0(X)} \prod_j \hat{g}(\gamma X_j \cdot X_j) \prod_{j \in \Lambda} d^2 X_j, \end{aligned} \tag{2.22}$$

and

$$\langle G(E, \mu, \nu) \rangle = \int \xi_\mu \eta_\nu e^{-2\Phi_0(X)} \prod_j \hat{g}(\gamma X_j \cdot X_j) \prod_{j \in \Lambda} d^2 X_j, \tag{2.23}$$

where

$$\begin{aligned} \hat{g}(\gamma X_j \cdot X_j) &= \hat{g}(\gamma x_j \cdot x_j + \gamma \eta_j \xi_j) \\ &= \hat{g}(\gamma x_j \cdot x_j) + \gamma \hat{g}'(x_j \cdot x_j) \eta_j \xi_j, \end{aligned} \tag{2.24}$$

by using (2.8) to expand $e^{\gamma v_j \xi_j \eta_j}$ and direct computations, and

$$(2.25) \qquad 2\Phi_0(X) \overset{\text{def}}{=} \sum_{j\in\Lambda} EX_j \cdot X_j - \sum_{i,j\in\Lambda, |i-j|_1=1} X_i \cdot X_j.$$

We notice in (2.22), (2.25) that apart from the factor $x_\mu \cdot x_\nu$, the integrand is only a "function" of $X_j \cdot X_k$. More precisely, we consider the algebra $C^\infty(\mathbf{R}^{2|\Lambda|}) \otimes \Lambda[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$. We call elements of this algebra, (super)functions. Clearly,

$$\begin{aligned} D(X) = X_i \cdot X_j &= x_i \cdot x_j + \frac{1}{2}(\eta_i\xi_j + \eta_j\xi_i) \\ &\in C^\infty(\mathbf{R}^{2|\Lambda|}) \otimes \Lambda[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}], \end{aligned}$$

and is hence a (super)function. We call the scalars (which are functions in $C^\infty(\mathbf{R}^{2|\Lambda|})$) in front of the Grassmann generators ξ, η, coefficients. In the case of $X_i \cdot X_j$, the coefficients are $x_i \cdot x_j$ and $1/2$. Those (super)functions which are only functions of $X_i \cdot X_j$ $(i,\ j \in \Lambda)$ are called supersymmetric functions. More precisely, supersymmetries are defined to be the set of transformations (in the alegebraic sense) that leave the dot products $X_i \cdot X_j$ $(i,\ j \in \Lambda)$ invariant. Two obvious transformations that leave the dot product invariant are the usual rotations O in $\mathbf{R}^2$,

$$x_i = x_i' O \qquad (i = 1, \cdots, n)$$

and the transformations $A \in Sp(2)$ acting on ξ_i, η_i $(i = 1, \cdots, n)$ such that

$$(\xi_i, \eta_i) = (\xi_i', \eta_i')A,$$

where $\{x_i', \xi_i', \eta_i'\}_{i=1}^n$ is another set of coordinates, x_i' being the even ones and ξ_i', η_i' the odd ones. We put $X_i' = (x_i', \xi_i', \eta_i')$ $\qquad (i = 1, \cdots, n)$. Aside from these two linear transformations, supersymmetries also include transformations generated by (super)vector fields of the type:

$$\begin{aligned} V &= \sum_i V_i \\ &= \sum_i (\xi_i a + \eta_i b)\frac{\partial}{\partial x_i} + 2(b \cdot x_i)\frac{\partial}{\partial \xi_i} - 2(a \cdot x_i)\frac{\partial}{\partial \eta_i}, \end{aligned}$$

where a, $b \in \mathbf{R}^2$, and

$$\begin{aligned} a\frac{\partial}{\partial x_i} &\overset{\text{def}}{=} a^{(1)}\frac{\partial}{\partial x_i^{(1)}} + a^{(2)}\frac{\partial}{\partial x_i^{(2)}}, \\ b\frac{\partial}{\partial x_i} &\overset{\text{def}}{=} b^{(1)}\frac{\partial}{\partial x_i^{(1)}} + b^{(2)}\frac{\partial}{\partial x_i^{(2)}}. \end{aligned}$$

(Note that it is the same transformation in all the X_i.) The above transformation is to be understood in the algebraic sense. We check that $VD(X_i, X_j) = 0$. Analogous to the usual change of variables, the (super) change of variables also generates a

Jacobian, which is called a Berezinian. We will not go into the details, except mentioning that the Berezinian here is 1.

More generally, we would also like to obtain an expression for the average of products of G_Λ: $\langle\prod_{\alpha=1}^m G(E,\mu_\alpha,\nu_\alpha)\rangle$ for all $m\in\mathbf{N}^+$, even though in this paper we will only use it for $m=1$, 2. For that purpose, for each $j\in\Lambda$, we introduce $x_j^{(\alpha)}\in\mathbf{R}^2$, $\alpha=1,\cdots,m$, and $\xi_i^{(\alpha)}$, $\eta_i^{(\alpha)}$, $i\in\Lambda$, $\alpha=1,\cdots,m$ satisfying:

$$\begin{aligned}&[\xi_i^{(\alpha)},\eta_j^{(\beta)}]=0,\\ &[\xi_i^{(\alpha)},\xi_j^{(\beta)}]=0,\\ &[\eta_i^{(\alpha)},\eta_j^{(\beta)}]=0, \qquad\qquad (\alpha,\,\beta=1,\cdots,m)\end{aligned}$$

i.e., the algebra

$$C^\infty((\mathbf{R}^{2m})^\Lambda)\otimes\Lambda[\xi_1^{(1)},\eta_1^{(1)},\cdots,\xi_{|\Lambda|}^{(1)},\eta_{|\Lambda|}^{(1)},\cdots,\xi_1^{(m)},\eta_1^{(m)},\cdots,\xi_{|\Lambda|}^{(m)},\eta_{|\Lambda|}^{(m)}].$$

We obtain similar to (2.22) (2.23), two equivalent expressions for $\langle\prod_{\alpha=1}^m G(E,\mu_\alpha,\nu_\alpha)\rangle$:
(2.26)

$$\begin{aligned}&\langle\prod_{\alpha=1}^m G(E,\mu_\alpha,\nu_\alpha)\rangle\\ &=\int\prod_{\alpha=1}^m x_{\mu_\alpha}^{(\alpha)}\cdot x_{\nu_\alpha}^{(\alpha)}e^{-(\sum_{j\in\Lambda}(\sum_{\alpha=1}^m EX_j^{(\alpha)}\cdot X_j^{(\alpha)}-\sum_{j,k\in\Lambda,|j-k|_1=1}X_j^{(\alpha)}\cdot X_k^{(\alpha)})}\prod_{j\in\Lambda}\hat{g}(\gamma\sum_{\alpha=1}^m X_j^{(\alpha)}\cdot X_j^{(\alpha)})\\ &\qquad\qquad\prod_{\alpha=1}^m\prod_{j\in\Lambda}d^2X_j^{(\alpha)}\\ &\overset{\text{def}}{=}\int\prod_{\alpha=1}^m x_{\mu_\alpha}^{(\alpha)}\cdot x_{\nu_\alpha}^{(\alpha)}e^{-2\Phi_0^{(m)}(X)}\prod_{j\in\Lambda}\hat{g}(\gamma\sum_{\alpha=1}^m X_j^{(\alpha)}\cdot X_j^{(\alpha)})\prod_{\alpha=1}^m\prod_{j\in\Lambda}d^2X_j^{(\alpha)},\end{aligned}$$

and
(2.27)

$$\langle\prod_{\alpha=1}^m G(E,\mu_\alpha,\nu_\alpha)\rangle=\int\prod_{\alpha=1}^m\xi_{\mu_\alpha}^{(\alpha)}\eta_{\nu_\alpha}^{(\alpha)}e^{-2\Phi_0^{(m)}(X)}\prod_{j\in\Lambda}\hat{g}(\gamma\sum_{\alpha=1}^m X_j^{(\alpha)}\cdot X_j^{(\alpha)})\prod_{\alpha=1}^m\prod_{j\in\Lambda}d^2X_j^{(\alpha)},$$

for all $m\in\mathbf{N}^+$, where $\hat{g}(\gamma\sum_{\alpha=1}^m X_j^{(\alpha)}\cdot X_j^{(\alpha)})$ is defined by formal Taylor series similar to (2.24), which terminates after a finite number of terms due to nilpotency, e.g., when $m=2$, writing X_j for $X_j^{(1)}$, $Y_j=(y_j,\tau_j,\omega_j)$ for $X_j^{(2)}$,

$$\begin{aligned}&\hat{g}(\gamma X_j\cdot X_j+\gamma Y_j\cdot Y_j)\\ &=\hat{g}(\gamma x_j\cdot x_j+\gamma y_j\cdot y_j)+\gamma\hat{g}'(\gamma x_j\cdot x_j+\gamma y_j\cdot y_j)(\eta_i\xi_j+\omega_j\tau_j)\\ &\quad+\gamma^2\hat{g}''(\gamma x_j\cdot x_j+\gamma y_j\cdot y_j)\eta_j\xi_j\omega_j\tau_j;\end{aligned}\tag{2.28}$$

and

$$2\Phi_0^{(m)}(X)=\sum_{j\in\Lambda}\sum_{\alpha=1}^m EX_j^{(\alpha)}\cdot X_j^{(\alpha)}-\sum_{j,k\in\Lambda,|j-k|_1=1}\sum_{\alpha=1}^m X_j^{(\alpha)}\cdot X_k^{(\alpha)}.\tag{2.29}$$

Note that $2\Phi_0^{(1)}(X)$ is the same as $2\Phi_0(X)$ defined in (2.25). For simplicity we write $\Phi_0(X)$ for $\Phi_0^{(1)}(X)$ from now on.

Let τ be a supersymmetric transformation. Let $X_i=\tau X_i'$ $(i=1,\cdots,n)$.

Definition. *A superfunction F is supersymmetric if it is invariant under all supersymmetries:*

$$F(X_1, \cdots, X_n) = F(X'_1, \cdots, X'_n),$$

for all τ supersymmetric transformations.

Clearly, supersymmetric functions belong to a rather restricted class of functions. For example, in $\mathbf{R}^2$, with two generators ξ, η, F is supersymmetric if and only if there exists f: $[0, \infty) \mapsto \mathbf{R}$ of class C^∞, such that

$$F(X) = f(X \cdot X) = f(x \cdot x) + f'(x \cdot x)\eta\xi. \tag{2.30}$$

For the general case, see *e.g.*, [KS].

Let $d^2X_i = (d^2x_i/\pi)d\eta_i d\xi_i$ $(i = 1, \cdots, n)$ as in (2.18). One of the most useful properties of the supersymmetric functions is the following:

Proposition 2.1. *If F is supersymmetric with all of its coefficients in $\mathcal{S}(\mathbf{R}^{2n})$, then*

$$\int F(X_1, \cdots, X_n) dX_1 \cdots dX_n = F(0, \cdots 0). \tag{2.31}$$

For a proof, see e.g., [K,KS]. In this paper we only need to use (2.31) for $n = 1$. The proof of which is obvious from (2.30). Proposition 2.1 is related to the so called "localization formula" in geometry, see e.g., [BGV].

In order to prove Proposition 2.2 and also for later use, we summarize below (without proof) some basic estimates on the Green's functions. See e.g., [La]. Let Λ be the ℓ^1-ball in $\mathbf{Z}^d$ of radius L, $(L \in \mathrm{N})$, as before.

Lemma 2.4. *For $E \geq 2d$,*

$$0 < (E - \Delta_\Lambda)^{-1}_{ij} \leq \begin{cases} \mathcal{O}(L), & d = 1 \\ \mathcal{O}(\log(\dfrac{L}{|i-j|+1})) & d = 2 \\ \dfrac{\mathcal{O}_d(1)}{(|i-j|+1)^{d-2}} & d \geq 3, \end{cases} \tag{2.32}$$

for all i, $j \in \Lambda$ and $\mathcal{O}_d(1)$ is uniform in L.

Using Lemmas 2.4 and 2.3, we have the following Corollary, proved in [W2,3].

Corollary 2.5. *(maximum principle) Let $S \subseteq \Lambda$, be any subset of $\Lambda = [-L, L]^d$. Define*

$$\tilde{\Delta}_\Lambda(i,j) = \begin{cases} \Delta_\Lambda(i,j), & i, j \in \Lambda \backslash S, \\ 0 & \textit{otherwise}; \end{cases} \tag{2.33}$$

$$\tilde{M} = E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k' \qquad (E \geq 2d).$$

Then

$$0 < \tilde{M}_{ij}^{-1} \le \begin{cases} \mathcal{O}(L), & d=1 \\ \mathcal{O}(\log(\dfrac{L}{|i-j|+1})) & d=2 \\ \dfrac{\mathcal{O}_d(1)}{(|i-j|+1)^{d-2}} & d \ge 3, \end{cases} \tag{2.34}$$

for all i, $j \in \Lambda\backslash S$ and $\mathcal{O}_d(1)$ is uniform in L.

III. Computations of finite volume Green's functions

Using the representation in (2.27) with $\mu_\alpha = \mu$, $\nu_\alpha = \nu$, we arrive at our main estimate of the section:

Proposition 3.1. *Assume dg satisfies (H1,2) with $0 < \delta < 1$. For all Λ, ℓ^1-balls in $\mathbf{Z}^d$, there exist $\gamma_0 > 0$, $c_{m,\Lambda}$, $C_{m,\Lambda} > 0$ ($m \in \mathbf{N}$) such that for all $\gamma > \gamma_0$, all m, all μ, $\nu \in \Lambda$,*

(3.1)

$$\left(\frac{c_{m,\Lambda}}{\gamma}\right)^{[(1-\delta)|\mu-\nu|_1+1]} \le \langle[(E-\Delta_\Lambda-\gamma V)^{-1}(\mu,\nu)]^m\rangle \le \left(\frac{C_{m,\Lambda}}{\gamma}\right)^{[(1-\delta)|\mu-\nu|_1+1]}, \quad m = 1, 2, \ldots$$

where $E \ge 2d$ if $\operatorname{supp} dg \subseteq (-\infty, 0]$*; $E \le -2d$ if* $\operatorname{supp} dg \subseteq [0,\infty)$.

Remark. (3.1) shows that the average of the higher order moments of the Green's functions *do not* "decay" faster. This is characterictic of systems with *large* fluctuations. In the next section, we show that this phenomenon persists as $\Lambda \nearrow \mathbf{Z}^d$.

The proofs of (3.1) for different $m \ge 1$ are similar. We need the following Tauberian theorem. (See Theorems 2, 3 on p. 418-422 of [F].) Recall also the notion of regularly varying functions introduced earlier, immediately proceding (H2).

Let dU be a measure defined on $[0,\infty)$ and such that its Laplace transform

$$\omega(\lambda) = \int_0^\infty e^{-\lambda x} dU(x) \tag{3.2}$$

exists for $\lambda > 0$.

Theorem 3.2. *Let $\rho \in [0,\infty)$. If $L \ge 0$ defined on $[0,\infty)$ is regularly varying at 0 with exponent 0, then each of the relations*

$$\omega(\lambda) \sim \lambda^{-\rho} L(\frac{1}{\lambda}), \qquad \lambda \to \infty, \tag{3.3}$$

and

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^\rho L(t), \qquad t \to 0 \tag{3.4}$$

implies the other.

We first write in detail the proof for $m = 1$.

Proof of Proposition 3.1 for $m = 1$. Let $t = 1/\gamma$ ($\gamma > 0$). It is more convenient to look at

$$\tilde{G}_\Lambda(E, \mu, \nu) = \gamma G_\Lambda(E, \mu, \nu) = (t(E - \Delta_\Lambda) - V)^{-1}_{\mu\nu}. \tag{3.5}$$

When $m = 1$, analogous to (2.23), we have

(3.6)

$$\begin{aligned}
\langle \tilde{G}_\Lambda(E, \mu, \nu)\rangle &= \langle (t(E - \Delta_\Lambda) - V)^{-1}_{\mu\nu}\rangle \\
&= \int \xi_\mu \eta_\nu e^{-t\sum_{j,\,k\in\Lambda}(E-\Delta)_{j,\,k} X_j\cdot X_k} \prod_j \hat{g}(X_j \cdot X_j) \prod_{j\in\Lambda} d^2X_j \\
&= \int \xi_\mu \eta_\nu e^{-t\sum_{j,\,k\in\Lambda}(E-\Delta)_{j,\,k} x_j\cdot x_k} \prod_j (\hat{g}(x_j \cdot x_j) + \hat{g}'(x_j \cdot x_j)\eta_j\xi_j) \\
&\qquad \prod_j (1 - tE\eta_j\xi_j) \prod_{k\in\Lambda,\, |k-j|_1=1} (1 + (t/2)(\eta_j\xi_k + \eta_k\xi_j) + (t^2/4)\eta_j\xi_k\eta_k\xi_j) \prod_{j\in\Lambda} \frac{d^2x_j}{\pi} d\eta_j d\xi_j.
\end{aligned}$$

We first do the $d\eta\, d\xi$ integrals. Due to anticommutivity (2.6), only selective terms contribute. It is natural to classify these terms with a graph. For a given subset $\Lambda \subset \mathbf{Z}^d$, we consider the *oriented* graph $\tilde{\Lambda}$ defined by:

(i) the set of vertices in Λ;

(ii) the edges are pairs $(m,\, n) \in \Lambda^2$, $|m - n|_1 = 0,\, 1$.

A directed path is $\tilde{\gamma}$, a set $(m_0, ..., m_p)$ ($p \geq 1$) with $m_i \in \Lambda$ and $(m_i,\, m_{i+1})$ edge of $\tilde{\Lambda}$. A loop is a directed path such that $m_p = m_0$. The vertices of the path $(m_0, ..., m_p)$ is the set $\{m_0, ..., m_p\}$. It is written γ.

Let $\mu,\, \nu \in \Lambda$, $\tilde{\gamma}_{\mu\nu}$ be a directed path from μ to ν and $\mathcal{D}_{\mu\nu}$ the set of such paths. A partition π of $\tilde{\Lambda}\backslash\tilde{\gamma}$ is a set of loops such that the union of the sets of vertices of loops in π is $\Lambda\backslash\gamma$. For a given π, let $\pi_1 \subseteq \pi$ be the subset of loops of 1 vertex in π. Let n_π be the number of loops in π. Let the set of partitions of $\tilde{\Lambda}\backslash\tilde{\gamma}$ be $\mathcal{P}(\tilde{\Lambda}\backslash\tilde{\gamma})$.

In the integrand in the RHS of (3.6), the η, ξ terms in the parenthesis under the product signs are of 4 types:

$$\eta_j\xi_j,\ \eta_j\xi_k,\ \eta_k\xi_j,\ \eta_j\xi_k\eta_k\xi_j \quad (j,\, k \in \Lambda,\ |j - k|_1 = 1).$$

Using the graph notations introduced earlier, they correspond to the edges $(j,\, j)$, $(j,\, k)$, $(k,\, j)$ and the loop of two vertices i, j. Note that the edge $(j,\, j)$ is also the loop of 1 vertex j.

In order to lighten up the notation, we now drop the tilde on γ and Λ when it does not cause confusion, i.e., γ and Λ can stand both for the sets of vertices and the associated graphs. By the same reasoning, π also stands for the set of vertices of π. It is easy to see that after multiplying the terms in the parenthesis under the product signs, all the resulting terms that contribute to the integral consist of a directed path from μ to ν: $\gamma_{\mu\nu}$ and a partition of $\Lambda\backslash\gamma_{\mu\nu}$.

Let $L_{\mu\nu}$ be the length of $\gamma_{\mu\nu}$. Let $\mathcal{D}_{\mu\nu}$ be the set of all possible $\gamma_{\mu\nu}$. We have therefore,

(3.7)
$$\langle \tilde{G}_\Lambda(E,\mu,\nu)\rangle = \int \prod \frac{d^2 x_j}{\pi} e^{-t\sum_{j,\,k\in\Lambda}(E-\Delta)_{j,\,k} x_j\cdot x_k} \sum_{\mathcal{D}_{\mu\nu}} \Big(\prod_{j\in\gamma_{\mu\nu}} (t/2)^{L_{\mu\nu}} \hat{g}(x_j\cdot x_j)\Big)$$
$$\Big(\sum_{\pi\in\mathcal{P}} (-1)^{n_\pi} \prod_{k\in\pi_1} (\hat{g}'(x_k\cdot x_k) - tE\hat{g}(x_k\cdot x_k)) \prod_{\ell\in\pi\setminus\pi_1} (t/2)\hat{g}(x_\ell\cdot x_\ell)\Big).$$

The factor $(-1)^{n_\pi}$ comes from the fact that any loop contributes a minus sign after the $d\eta d\xi$ integration, i.e., for all n

$$\int \eta_{j_1}\xi_{j_2}\eta_{j_2}\xi_{j_3}\cdots\eta_{j_n}\xi_{j_1}\prod_{i=1}^{n} d\eta_{j_i}d\xi_{j_i} = -1.$$

Remark. The RHS of (3.7) is positive, as once the directed path $\gamma_{\mu\nu}$ is chosen, the dominant term is the term where $\pi = \pi_1$. Since $\hat{g}' < 0$ from (2.19), the dominant terms are positive. Of course, this must be the case, as we know a priori that the LHS of (3.7) is positive.

We now estimate the terms under the integral sign in the RHS of (3.7). Since for all $\Lambda \in \mathbf{Z}^d$, ℓ^1-unit balls, there exist $0 < a_\Lambda < b_\Lambda < 4d$ such that

$$\sigma(E-\Delta_\Lambda) \subseteq [a_\Lambda,\, b_\Lambda] \subset [0, 4d],$$

we have the following estimate:

$$e^{-a_\Lambda t\sum_{i\in\Lambda} x_i\cdot x_i} \le e^{-t\sum_{i,\,j\in\Lambda}(E-\Delta_\Lambda)_{ij}x_i\cdot x_j} \le e^{-b_\Lambda t\sum_{i\in\Lambda} x_i\cdot x_i}. \tag{3.8}$$

Using (3.8) in (3.7), the RHS can be estimated by a sum of products of integrals of one variable only. Moreover all such integrals are either of the form

$$\hat{\hat{g}}(\tilde{t}) = \int_0^\infty \hat{g}(s)e^{-\tilde{t}s}ds \qquad \text{or} \qquad \hat{\hat{g}}'(\tilde{t}) = \int_0^\infty \hat{g}'(s)e^{-\tilde{t}s}ds \qquad (\tilde{t}>0), \tag{3.9}$$

where $\tilde{t} = a_\Lambda t$ or $b_\Lambda t$.

Using (H2) and the Tauberian Theorem 3.2 twice, we have

$$\begin{aligned} &\hat{\hat{g}}(\tilde{t}) \sim \frac{1}{\tilde{t}^\delta} \sim \frac{1}{t^\delta} \qquad \text{as} \quad t\to 0 \quad (\delta>0) \\ &\hat{\hat{g}}'(\tilde{t}) \sim \mathcal{O}(1) \qquad \text{as} \quad t\to 0 \quad (\delta>0) \end{aligned} \tag{3.10}$$

where $\tilde{t} = a_\Lambda t$ or $b_\Lambda t$.

Using (3.10) in (3.7) and noticing that each factor of $\hat{g}$ comes with a corresponding factor t, we obtain immediately (3.1) of Proposition 3.1 for $m = 1$. (Recall that $t = 1/\gamma > 0$.) □

Proof of Proposition 3.1 (general m). We first write the proof for $m = 2$.

(3.11)

$$\langle[\tilde{G}(E,\mu,\nu)]^2\rangle = \langle[(t(E-\Delta_\Lambda)-V)^{-1}_{\mu\nu}]^2\rangle$$

$$= \int \xi_\mu\eta_\nu\tau_\mu\omega_\nu e^{-[t\sum_{i,j}(E-\Delta)_{ij}(X_i\cdot X_j+Y_i\cdot Y_j)]} \prod_{j\in\Lambda}\hat{g}(X_j\cdot X_j+Y_j\cdot Y_j)]\prod_{j\in\Lambda}d^2X_j\prod_{j\in\Lambda}d^2Y_j$$

$$= \int \xi_\mu\eta_\nu\tau_\mu\omega_\nu e^{-[t\sum_{i,j}(E-\Delta)_{ij}(x_i\cdot x_j+y_i\cdot y_j)]}$$

$$\prod_j[\hat{g}(x_j\cdot x_j+y_j\cdot y_j)+\hat{g}'(x_j\cdot x_j+y_j\cdot y_j)(\eta_j\xi_j+\omega_j\tau_j)+\hat{g}''(x_j\cdot x_j+y_j\cdot y_j)\eta_j\xi_j\omega_j\tau_j]$$

$$\prod_j[(1-tE\eta_j\xi_j)\prod_{k\in\Lambda,\,|k-j|_1=1}(1+(t/2)(\eta_j\xi_k+\eta_k\xi_j)+(t^2/4)\eta_j\xi_k\eta_k\xi_j)]$$

$$\prod_j[(1-tE\omega_j\tau_j)\prod_{k\in\Lambda,\,|k-j|_1=1}(1+(t/2)(\omega_j\tau_k+\omega_k\tau_j)+(t^2/4)\omega_j\tau_k\omega_k\tau_j)]$$

$$\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\frac{d^2y_j}{\pi}\prod_j d\eta_j d\xi_j d\omega_j d\tau_j$$

$$= \int \xi_\mu\eta_\nu\tau_\mu\omega_\nu e^{-[t\sum_{i,j}(E-\Delta)_{ij}(x_i\cdot x_j+y_i\cdot y_j)]}$$

$$\prod_j\{(\hat{g}(x_j\cdot x_j+y_j\cdot y_j)+[\hat{g}'(x_j\cdot x_j+y_j\cdot y_j)-tE\hat{g}(x_j\cdot x_j+y_j\cdot y_j)](\eta_j\xi_j+\omega_j\tau_j)$$

$$+[(\hat{g}''(x_j\cdot x_j+y_j\cdot y_j)-2tE\hat{g}'(x_j\cdot x_j+y_j\cdot y_j)+t^2E^2\hat{g}(x_j\cdot x_j+y_j\cdot y_j)]\eta_j\xi_j\omega_j\tau_j\}$$

$$\prod_{k\in\Lambda,\,|k-j|_1=1}(1+(t/2)(\eta_j\xi_k+\eta_k\xi_j)+(t^2/4)\eta_j\xi_k\eta_k\xi_j)$$

$$\prod_{k\in\Lambda,\,|k-j|_1=1}(1+(t/2)(\omega_j\tau_k+\omega_k\tau_j)+(t^2/4)\omega_j\tau_k\omega_k\tau_j)\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\frac{d^2y_j}{\pi}\prod_j d\eta_j d\xi_j d\omega_j d\tau_j.$$

Compared to (3.6), we see that (3.11) has double the amount of variables. Like the evaluation of the RHS of (3.6), we do the $d\eta\, d\xi\, d\omega\, d\tau$ integration first. We use the same rules to classify the terms. We work on $\tilde{\Lambda}\times\tilde{\Lambda}$.

Analogous to (3.7), we have after $d\eta\, d\xi\, d\omega\, d\tau$ integrations

(3.12)

$$\langle[\tilde{G}(E,\mu,\nu)]^2\rangle = \langle[(t(E-\Delta_\Lambda)-V)^{-1}_{\mu\nu}]^2\rangle$$

$$= \int\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\frac{d^2y_j}{\pi}e^{-[t\sum_{i,j}(E-\Delta)_{ij}(x_i\cdot x_j+y_i\cdot y_j)]}$$

$$\sum_{\mathcal{D}^{(1)}_{\mu\nu},\,\mathcal{D}^{(2)}_{\mu\nu}}\ \sum_{\mathcal{P}^{(1)},\,\mathcal{P}^{(2)}}\ \prod_{\alpha,\,\beta=1,2;\,\alpha\neq\beta}(-1)^{n_{\pi^{(\alpha)}}}\Big(\frac{t}{2}\Big)^{L^{(\alpha)}_{\mu\nu}}\prod_{j\in\gamma^{(\alpha)}_{\mu\nu},\,j\notin\pi^{(\beta)}_1}\hat{g}^{1/2}\prod_{j\in\gamma^{(\alpha)}_{\mu\nu},\,j\in\pi^{(\beta)}_1}(\hat{g}'-tE\hat{g})^{1/2}$$

$$\prod_{k\in\pi^{(\alpha)}_1,\,k\notin\pi^{(\beta)}_1}(\hat{g}'-tE\hat{g})^{1/2}\prod_{k\in\pi^{(\alpha)}_1,\,k\in\pi^{(\beta)}_1}(\hat{g}''-2tE\hat{g}'+t^2E^2\hat{g})^{1/2}$$

$$\prod_{\ell\in\pi^{(\alpha)}\setminus\pi^{(\alpha)}_1,\,\ell\in\pi^{(\beta)}_1}\Big(\frac{t}{2}\Big)(\hat{g}'-tE\hat{g})^{1/2}\prod_{\ell\in\pi^{(\alpha)}\setminus\pi^{(\alpha)}_1,\,\ell\notin\pi^{(\beta)}_1}\Big(\frac{t}{2}\Big)\hat{g}^{1/2},$$

where $\mathcal{D}^{(\alpha)}_{\mu\nu}$ $(\alpha = 1, 2)$ is the α^{th} copy of $\mathcal{D}_{\mu\nu}$,... etc., and to shorten the expression, we have omitted the arguments for $\hat{g}$, $\hat{g}'$ and $\hat{g}''$, e.g.,

$$\prod_{j\in\gamma^{(\alpha)}_{\mu\nu},\, j\in\pi^{(\beta)}_1} (\hat{g}' - tE\hat{g})^{1/2}$$

stands for

$$\prod_{j\in\gamma^{(\alpha)}_{\mu\nu},\, j\in\pi^{(\beta)}_1} (\hat{g}'(x_j \cdot x_j + y_j \cdot y_j) - tE\hat{g}(x_j \cdot x_j + y_j \cdot y_j))^{1/2}$$

etc. The functions under the exponent 1/2 are not necessarily positive. But this is immaterial, as after expanding the product $\prod_{\alpha,\,\beta=1,2\,\alpha\neq\beta}$, only the functions themselves appear. The exponent 1/2 is only there to help keeping track of the terms. Recall that we work on $\tilde{\Lambda} \times \tilde{\Lambda}$, while the underlying vertices form the set Λ.

The largest terms in the RHS of (3.12) are the ones where $\gamma^{(1)}_{\mu\nu} = \gamma^{(2)}_{\mu\nu} \overset{\text{def}}{=} \gamma_{\mu\nu}$ with $L^{(1)}_{\mu\nu} = L^{(2)}_{\mu\nu} = |\mu - \nu|_1$ and $\pi^{(1)} = \pi^{(2)} = \pi^{(1)}_1 = \pi^{(2)}_1$. To leading order in t, they correspond to the integral:

(3.13)

$$I_2 = \int \prod_{j\in\Lambda} \frac{d^2x_j}{\pi}\frac{d^2y_j}{\pi} e^{-[t\sum_{i,j}(E-\Delta)_{ij}(x_i\cdot x_j + y_i\cdot y_j)]}$$

$$\sum_{\{\gamma_{\mu\nu}|L_{\mu\nu}=|\mu-\nu|_1\}} (\frac{t}{2})^{2|\mu-\nu|_1} \prod_{j\in\gamma_{\mu\nu}} \hat{g}(x_j \cdot x_j + y_j \cdot y_j) \prod_{k\in\Lambda\setminus\gamma_{\mu\nu}} \hat{g}''(x_j \cdot x_j + y_j \cdot y_j).$$

Using (3.8), the RHS of (3.13) is reduced to estimate integrals of the form:

$$\hat{\hat{g}}(\tilde{t}) = \int_0^\infty \hat{g}(s)e^{-\tilde{t}s}ds \qquad \text{or} \qquad \widehat{\hat{g}''}(\tilde{t}) = \int_0^\infty \hat{g}''(s)e^{-\tilde{t}s}ds \qquad (\tilde{t} > 0),$$

where $\tilde{t} = \mathcal{O}(1)t$ as in (3.10).

Using (H2) and the Tauberian Theorem 3.2 twice as before, we have

$$\frac{1}{\mathcal{O}_\Lambda(1)} t^{[(1-\delta)|\mu-\nu|_1-(1+\delta)]} \le I_2 \le \mathcal{O}_\Lambda(1)t^{[(1-\delta)|\mu-\nu|_1-(1+\delta)]}. \tag{3.14}$$

The rest of the terms in (3.12) are of higher orders in t and can be estimated in the same way. We have thus proved (3.1) for $m = 2$.

The proofs for $m > 2$ are similar, we work on m copies of $\tilde{\Lambda}$, i.e., $(\tilde{\Lambda})^m$. Unfortunately they become increasingly cumbersome to write. So we do not present all the details here. We remark however, that for each m, the largest terms are always of the forms:

(3.15)

$$I_m = \int \prod_{j\in\Lambda}\prod_{\alpha=1}^m \frac{d^2x^{(\alpha)}_j}{\pi} e^{-t\sum_{i,j}\sum_{\alpha=1}^m(E-\Delta)_{ij}(x^{(\alpha)}_i\cdot x^{(\alpha)}_j)}$$

$$\sum_{\{\gamma_{\mu\nu}|L_{\mu\nu}=|\mu-\nu|_1\}} (\frac{t}{2})^{m|\mu-\nu|_1} \prod_{j\in\gamma_{\mu\nu}} \hat{g}(\sum_{\alpha=1}^m(x^{(\alpha)}_j\cdot x^{(\alpha)}_j)) \prod_{k\in\Lambda\setminus\gamma_{\mu\nu}} \hat{g}^{(m)}(\sum_{\alpha=1}^m(x^{(\alpha)}_k\cdot x^{(\alpha)}_k))$$

$$=\mathcal{O}_{\Lambda,\,m}(1)t^{[(1-\delta)|\mu-\nu|_1-(m-1+\delta)]},$$

where

$$\begin{aligned}&\gamma_{\mu\nu} \overset{\text{def}}{=} \gamma^{(1)}_{\mu\nu} = \cdots = \gamma^{(m)}_{\mu\nu} \qquad \text{with}\\ &L^{(1)}_{\mu\nu} = L^{(2)}_{\mu\nu} = \cdots = L^{(m)}_{\mu\nu} = |\mu - \nu|_1\\ &\pi^{(1)} = \pi^{(2)} = \cdots = \pi^{(m)}\\ &= \pi^{(1)}_1 = \pi^{(2)}_1 = \cdots = \pi^{(m)}_1.\end{aligned}$$

□

IV. UPPER AND LOWER BOUNDS ON THE INFINITE VOLUME GREEN'S FUNCTIONS

We now iterate (3.1) to obtain the corresponding upper and lower bounds for $\langle[(E - \Delta - \gamma V)^{-1}(\mu, \nu)]^m\rangle$, the infinite volume Green's functions. This is the content of Theorem 1.1.

We use the induction schemes devised in [W3, 4] to prove (1.19). The only difference is that we replace ℓ^∞ balls by ℓ^1 balls in $\mathbf{Z}^d$. The induction mechanism itself, however, remains unchanged.

Let L_0, $L \in \mathbf{N}$, Λ_0 be the ℓ^1 ball of radius L_0, i.e., $\Lambda_0 = \{i \in \mathbf{Z}^d \,||i|_1 \le L_0\}$. Let $\tilde\Lambda_0 = \{i \in \mathbf{Z}^d \,||i|_1 \le L_0 + 1\}$, $\Lambda = \{i \in \mathbf{Z}^d \,||i|_1 \le L\}$. Define $\partial\Lambda_0 = \{j \,|\, |j|_1 = L_0\}$ and similarly for $\partial\tilde\Lambda_0$, We also extend the notation Δ_S to denote the operator with

$$\begin{cases} \Delta_{S;ij} = \Delta_{ij}, & i,\, j \in S,\ \text{or}\ i,\, j \in \mathbf{Z}^d \backslash S \\ \qquad 0 & \qquad \text{otherwise.} \end{cases} \tag{4.1}$$

(Assume $L >> L_0$.) We have the following decoupling lemma:

Lemma 4.2. *Assume $\mu \in \Lambda_0$, $\nu \notin \tilde\Lambda_0$. For all $m \in \mathbf{N}^+$,*

$$\begin{aligned}&\langle[(E - \Delta_\Lambda - \gamma V)^{-1}_{\mu\nu}]^m\rangle\\ \le&[\bar L L_0^{2(d-1)}]^m \sup_{j \in \partial\Lambda_0\, j' \in \partial\tilde\Lambda_0} \langle[(E - \Delta_{\Lambda_0} - \gamma V)^{-1}_{\mu j}]^m\rangle\langle[(E - \Delta_\Lambda - \gamma V)^{-1}_{j'\nu}]^m\rangle,\end{aligned} \tag{4.2}$$

where $E \ge 2d$ and

$$\bar L = \begin{cases} \mathcal{O}(1)L, & d = 1 \\ \mathcal{O}(1)\log L & d = 2 \\ \mathcal{O}_d(1) & d \ge 3, \end{cases} \tag{4.3}$$

The above lemma is proved in detail in [W3, 4], so we do not repeat it here. (The proof in [W3, 4] is independent of the norm that one uses for unit balls. In [W3, 4], $\tilde\Lambda_0$ was taken to be $\tilde\Lambda_0 = \{i \in \mathbf{Z}^d \,||i|_1 \le L_0 + 2\}$. But this does not alter the proof in an essential way. See (4.6) below.) Nonetheless, we recall the main idea,, which is the same for all m. We illustrate it for $m = 1$.

Since $\partial\Lambda_0$ separates μ and ν, by the resolvent equation we have

$$(E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu}=\sum_{<jj''>}(E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}(E-\Delta_\Lambda-\gamma V)^{-1}_{j''\nu}, \tag{4.4}$$

where $< jj'' >$ denotes the set

$$\{j\in\partial\Lambda_0,\, j''\notin\partial\Lambda_0 |\, |j''-j|_1=1\}. \tag{4.5}$$

Note that $j''\in\partial\tilde{\Lambda}_0$. Using the resolvent equation on the second factor in the sum in (4.4), we have

(4.6)

$$\begin{aligned}&(E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu}\\ =&\sum_{<jj''>}(E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}[(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j''\nu}+\sum_{<j'j'''>}(E-\Delta_\Lambda-\gamma V)^{-1}_{j''j'''}(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu}\\ \le&\bar{L}\sum_{j\in\partial\Lambda_0,\, j'\in\partial\tilde{\Lambda}_0}(E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu},\end{aligned}$$

where $\bar{L}$ is as defined in (4.3), $< j'j''' >$ denotes the set $\{j'\in\partial\tilde{\Lambda}_0,\, j'''\notin\partial\tilde{\Lambda}_0 |\, |j'''-j'|_1=1\}$ and we used Lemma 3.3 to bound $(E-\Delta_\Lambda-\gamma V)^{-1}_{j''j'''}$. Summing over the boundary, we obtain (4.2) for $m=1$. The proof for general m follows the same line of arguments supplemented with some elementary inequalities, see [W4].

Proof of Theorem 4.1.

The Upper Bound.

We use Proposition 3.1 and iterate Lemma 4.2 at increasing scale as in [W3, 4]. We start with $m=1$. Let $L_0\in\mathbf{N}$ and take $L_1=L_0^2$. Let $\Lambda_0(\mu)=\{i\in\mathbf{Z}^d| |i-\mu|_1\le L_0\}$, $\tilde{\Lambda}_0(\mu)=\{i\in\mathbf{Z}^d| |i-\mu|_1\le L_0+1\}$, $\Lambda_1(\mu)=\{i\in\mathbf{Z}^d| |i-\mu|_1\le L_1\}$ be the ℓ^1 balls in $\mathbf{Z}^d$ centered at μ with radius L_0, L_0+1, L_1. Let $\nu\in\Lambda_1(\mu)$ be such that $|\mu-\nu|_1\ge L_1/3$. Using Lemma 4.2 with $\Lambda_0(\mu)$ replacing Λ_0, $\tilde{\Lambda}_0(\mu)$ replacing $\tilde{\Lambda}_0$, $\Lambda_1(\mu)$ replacing Λ, we have,

(4.7)

$$\begin{aligned}&\langle(E-\Delta_{\Lambda_1(\mu)}-\gamma V)^{-1}_{\mu\nu}\rangle\\ \le&\bar{L}_1L_0^{2(d-1)}\sup_{j_1\in\partial\Lambda_0(\mu),\, j_1'\in\partial\tilde{\Lambda}_0(\mu)}\langle(E-\Delta_{\Lambda_0(\mu)}-\gamma V)^{-1}_{\mu j_1}\rangle\langle(E-\Delta_{\Lambda_1(\mu)}-\gamma V)^{-1}_{j_1'\nu}\rangle\\ \le&(C_{d,\,L_0}L_0)^{2d}\Big(\frac{1}{\gamma}\Big)^{[(1-\delta)(L_0+1)]}\sup_{j_1'\in\partial\tilde{\Lambda}_0(\mu)}\langle(E-\Delta_{\Lambda_1(\mu)}-\gamma V)^{-1}_{j_1'\nu}\rangle,\end{aligned}$$

where $C_{d,\,L_0}$ is some constant which only depends on L_0 and the dimension d.

We now use Lemma 4.2 on $\langle(E-\Delta_{\Lambda_1(\mu)}-\gamma V)^{-1}_{j_1'\nu}\rangle$ with $\Lambda_0(j_1')$ in place of $\Lambda_0(\mu)$... etc., so that we “pave” a path from μ toward ν using ℓ^1 balls congruent to $\Lambda_0(\mu)$, centered at j_1' ... etc. Let [] denote the integer part. This iteration process can be repeated at least

$$n=\left[\frac{|\mu-\nu|_1}{L_0+1}\right] \tag{4.8}$$

times before either (a) the $\ell^1-\operatorname{dist}(j'_n, \partial\Lambda_1(\mu)) \le L_0$ or (b) $|j'_n - \nu|_1 \le L_0$.

Let

$$r = \min(L_0,\, \operatorname{dist}(j'_n, \partial\Lambda_1(\mu)),\, |j'_n - \nu|_1). \tag{4.9}$$

Let $\bar\Lambda(j'_n, r)$ be the ℓ^1-ball centered at j'_n with radius r. Clearly $\bar\Lambda(j'_n, r) \subseteq \Lambda_1(\mu)$. We obtain after $(n+1)$ iterations, the last one using $\bar\Lambda(j'_n, r)$:

(4.10)

$$\begin{aligned}
&\langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{\mu\nu}\rangle \\
\le&[(C_{d,\,L_0}L_0)^{2d}(\frac{1}{\gamma})^{[(1-\delta)(L_0+1)]}]^n \sup_{j'_n\in\partial\tilde\Lambda_0(j_{n-1})} \langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j'_n\nu}\rangle \\
\le&[(C_{d,\,L_0}L_0)^{2d}(\frac{1}{\gamma})^{[(1-\delta)(L_0+1)]}]^n \\
&\sup_{j'_n\in\partial\tilde\Lambda_0(j_{n-1})} \{\langle (E - \Delta_{\bar\Lambda(j'_n,r)} - \gamma V)^{-1}_{j'_n\nu}\rangle \\
&\qquad + \langle \sum_{<j_{n+1}j'_{n+1}>} (E - \Delta_{\bar\Lambda(j'_n,r)} - \gamma V)^{-1}_{j'_n j_{n+1}} (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j'_{n+1}\nu}\rangle\} \\
\le&[(C_{d,\,L_0}L_0)^{2d}(\frac{1}{\gamma})^{[(1-\delta)(L_0+1)]}]^n (C_{d,\,L_0}L_0)^{2d}(\frac{1}{\gamma})^{[(1-\delta)(r+1)]} \\
\le&(C_{d,\,L_0}L_0)^{2d(n+1)}(\frac{1}{\gamma})^{[(1-\delta)(n(L_0+1)+(r+1))]},
\end{aligned}$$

To determine r, we notice that if it is case (a), then $\nu \in \partial\Lambda_1(\mu)$, since otherwise (b) must occur before (a). So

$$|\mu - \nu|_1 - \left[\frac{|\mu - \nu|_1}{L_0 + 1}\right](L_0 + 1) \le r \le L_0. \tag{4.11}$$

Using (4.8), (4.9), we have

$$n(L_0 + 1) + (r + 1) \ge |\mu - \nu|_1 + 1. \tag{4.12}$$

Using (4.12), (4.8) in (4.10), we have at scale $L_1 > L_0$, for $\Lambda_1(\mu) = \{i \in \mathbf{Z}^d \,||i - \mu|_1 \le L_1\}$, $\nu \in \Lambda_1(\mu)$ with $|\mu - \nu|_1 \ge L_1/3$, the following estimate:

$$\begin{aligned}
&\langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{\mu\nu}\rangle \\
&\le \left[\frac{(C_{d,\,L_0}L_0)^{pd/L_0}}{\gamma}\right]^{[(1-\delta)(|\mu-\nu|_1+1)]} \qquad (|\mu - \nu|_1 \ge L_1/3),
\end{aligned} \tag{4.13}$$

for some $p > 0$ depending only on δ, L_0 and $C_{d,\,L_0}$ is as before, a constant depending only on L_0 and the dimension d.

Let $L_2 = L_1^2$. Repeating the above procedure with L_2 replacing L_1, L_1 replacing L_0, we obtain for $\Lambda_2(\mu) = \{i \in \mathbf{Z}^d \,||i - \mu|_1 \le L_2\}$, $\nu \in \Lambda_2(\mu)$ with $|\mu - \nu|_1 \ge L_2/3$, the estimate:

$$\begin{aligned}
&\langle (E - \Delta_{\Lambda_2(\mu)} - \gamma V)^{-1}_{\mu\nu}\rangle \\
&\le \left[\frac{(C_{d,\,L_0}L_0)^{(pd/L_0)}L_1^{(pd/L_1)}}{\gamma}\right]^{[(1-\delta)(|\mu-\nu|_1+1)]} \qquad (|\mu - \nu|_1 \ge L_1/3).
\end{aligned} \tag{4.14}$$

More generally, let $L_n = L_{n-1}^2$, Λ_n the corresponding ℓ^1 ball of radius L_n. Repeating the above iteration procedure with the scale L_{n-1} replacing L_0, and the scale L_n replacing L_1, we obtain after n iterations:

$$\begin{aligned}(4.15)\qquad &\langle (E-\Delta_{\Lambda_n(\mu)}-\gamma V)^{-1}_{\mu\nu}\rangle \\ &\leq \left[\frac{(C_{d,\,L_0}^{(pd/L_0)}\prod_{i=0}^{n-1}(L_i)^{(pd/L_i)}}{\gamma}\right]^{[(1-\delta)(|\mu-\nu|_1+1)]} \qquad (|\mu-\nu|_1\geq L_n/3).\end{aligned}$$

Since $L_n = L_{n-1}^2$, the product in (4.15) can be estimated uniformly in n, once $L_0 > 1$ is specified. We obtain that for all $\delta > 0$, there exist $0 < C_\delta < \infty$, such that

$$(4.16)\qquad \langle (E-\Delta_{\Lambda_n(\mu)}-\gamma V)^{-1}_{\mu\nu}\rangle \leq (\frac{C_\delta}{\gamma})^{[(1-\delta)(|\mu-\nu|_1+1)]} \qquad (|\mu-\nu|_n\geq L_n/3),$$

for $\gamma >> 1$, *uniformly* in n.

To finish proving the upper bound in (1.19) for $m = 1$, assume $j \in \Lambda_n$ for some n, $|j|_1 \geq L_n/3$ and $\text{dist}(j, \partial\Lambda_n) \geq |j|_1$. Let $G_n(E, i, j) = (E - \Delta_{\Lambda_n} - \gamma V)^{-1}_{ij}$. We proceed as in [W3]. We write

$$(4.17)\qquad G_{n+1}(E,0,j) = G_n(E,0,j) + \sum_{<j'j''>} G_n(E,0,j')G_{n+1}(E,j'',j),$$

where $< j'j'' >$ is the set $\{j' \in \partial\Lambda_n,\ j'' \notin \partial\Lambda_n|\,|j''-j'|_1 = 1\}$. From (3.1),

$$(4.18)\qquad \langle G_n(E,0,j')\rangle \leq (\frac{C_\delta}{\gamma})^{[(1-\delta)(L_n+1)]} \qquad (\gamma >> 1).$$

Using (4.18) in (4.17),

$$\begin{aligned}(4.19)\qquad &\langle G_{n+1}(E,0,j)\rangle - \langle G_n(E,0,j)\rangle \\ \leq&\mathcal{O}(1)(\frac{C_\delta}{\gamma})^{[(1-\delta)(L_n+1)]}L_{n+1}(C_dL_n)^{(d-1)} \\ \leq&\mathcal{O}(1)(\frac{C_\delta}{\gamma})^{[(1-\delta')(L_n+1)]}\end{aligned}$$

for $\gamma >> 1$ and some $\delta' \in (\delta,\ 1)$.

So $\langle G_n(E,0,j)\rangle$ form a Cauchy sequence and for a C_δ slightly larger than that in (4.18),

$$(4.20)\qquad \langle G(E,0,j)\rangle = \lim_{n\to\infty}\langle G_n(E,0,j)\rangle \leq (\frac{C_\delta}{\gamma})^{[(1-\delta)(|\mu-\nu|_1+1)]}$$

The proof for $m > 1$ are similar, upon using Lemma 4.2 for $m > 1$ and iteration, so we do not repeat all the details. In the proof of the existence of the infinite volume limit, we do the following. We illustrate it for $m = 2$. In this case, squaring both sides of (4.17), we have

$$\begin{aligned}[G_{n+1}(E,0,j)]^2 =&[G_n(E,0,j)]^2 + [\sum_{<j'j''>} G_n(E,0,j')G_{n+1}(E,j'',j)]^2 \\ &+ 2G_n(E,0,j)[\sum_{<j'j''>} G_n(E,0,j')G_{n+1}(E,j'',j)].\end{aligned}$$

So

$$\begin{aligned}&\langle[G_{n+1}(E,0,j)]^2\rangle - \langle[G_n(E,0,j)]^2\rangle \\ =&\langle[\sum_{<j'j''>} G_n(E,0,j')G_{n+1}(E,j'',j)]^2\rangle \\ &+ 2\langle G_n(E,0,j)[\sum_{<j'j''>} G_n(E,0,j')G_{n+1}(E,j'',j)]\rangle.\end{aligned}$$

We estimate all the terms except $G_n(E,0,j')$ (as functions of V) by their sup-norm. We obtain

$$\begin{aligned}&\langle[G_{n+1}(E,0,j)]^2\rangle - \langle[G_n(E,0,j)]^2\rangle \\ &\le \mathcal{O}(1)L_{n+1}^2 L_n^{2(d-1)} \sup_{j'\in<j'j''>} \langle[G_n(E,0,j)]^2\rangle \\ &\quad + \mathcal{O}(1)L_n L_{n+1} L_n^{d-1} \sup_{j'\in<j'j''>} \langle G_n(E,0,j)\rangle \\ &\le \mathcal{O}(1)(\frac{C_\delta}{\gamma})^{[(1-\delta')(L_n+1)]}\end{aligned} \tag{4.21}$$

for $\gamma >> 1$, some $\delta' \in (\delta, 1)$. We can now proceed to prove the existence of limit for $m = 2$ as for $m = 1$. Generalizing the estimate in (4.21) to all m, we prove the upper bound in (4.1).

The Lower Bound.

To prove the lower bound in (1.19), we use the resolvent expansion and the FKG inequality [FKG, Li]. Let $j \in \mathbf{Z}^d$, $j \neq 0$. Let τ_{0j} be a directed path from the origin to j, such that the length of the path $L_{0j} = |j|_1$. Let ℓ_i ($i = 0, 1, 2...|j|_1$) with $\ell_0 = 0$, $\ell_{|j|_1} = j$ be the vertices in τ_{0j}. As before, we also use τ_{0j} to denote this set of vertices.

Fix $L_0 \in \mathbf{N}$ and assume $j > L_0$. Let $\Lambda_0(0)$ be the ℓ^1-ball centered at 0 with radius L_0. We start with $m = 1$. Using the resolvent equation and taking $E = 2d^+$, we have

$$G(E,0,j) = \sum_{<ii'>} G_{\Lambda_0(0)}(E,0,i)G(E,i',j), \tag{4.22}$$

where $< ii' >$ is the set defined in (4.5). Since $\{\ell_{L_0}, \ell_{L_0+1}\} \in < ii' >$ and all terms in the sum in (4.22) are positive, we have

$$G(E,0,j) \ge G_{\Lambda_0(0)}(E,0,\ell_{L_0})G(E,\ell_{L_0+1},j). \tag{4.23}$$

Taking the expectation value and using the FKG inequality, as G, $G_{\Lambda_0(0)}$ are monotone decreasing functions of V, we obtain

$$\langle G(E,0,j)\rangle \geq \langle G_{\Lambda_0(0)}(E,0,\ell_{L_0})\rangle\langle G(E,\ell_{L_0+1},j)\rangle. \tag{4.24}$$

Since after taking the average, the limit $E \to 2d$ exists for both the LHS and the RHS of (4.24) following the previous arguments from (4.7) to (4.20), we have after taking the limit

$$\langle G(E,0,j)\rangle \geq \big(\frac{c_1}{\gamma}\big)^{[(1-\delta)(L_0+1)]}\langle G(E,\ell_{L_0+1},j)\rangle, \tag{4.25}$$

where $E = 2d$ and we used (3.1).

We now repeat the procedure with $\Lambda_0(\ell_{L_0+1})$ the ℓ^1-ball centered at ℓ_{L_0+1} with radius L_0. Clearly $\ell_{2L_0+1} \in \partial\Lambda_0(\ell_{L_0+1}) \cap \tau_{0j}$. So we have

$$\langle G(E,0,j)\rangle \geq \big(\frac{c_1}{\gamma}\big)^{[2(1-\delta)(L_0+1)]}\langle G(E,\ell_{2L_0+1},j)\rangle. \tag{4.26}$$

This iteration can be repeated

$$n = [\frac{|j|_1}{L_0+1}]$$

times, before $|j-\ell_{(nL_0+n)}|_1 \leq L_0$, at which time, we have

$$\begin{aligned}\langle G(E,0,j)\rangle &\geq \big(\frac{c_1}{\gamma}\big)^{[n(1-\delta)(L_0+1)]}\langle G(E,\ell_{(nL_0+n)},j)\rangle\\ &\geq \big(\frac{c_1}{\gamma}\big)^{[n(1-\delta)(L_0+1)]}\langle G_{\Lambda_0(\ell_{(nL_0+n)})}(E,\ell_{(nL_0+n)},j)\rangle\\ &\geq \big(\frac{c_1}{\gamma}\big)^{[(1-\delta)(|j|_1+1)]},\end{aligned}$$

where we used the a priori bound $G \geq G_{\Lambda_0}$.

For $m = 2$, we write

$$\begin{aligned}[G(E,0,j)]^m &= [\sum_{<ii'>} G_{\Lambda_0(0)}(E,0,i)G(E,i',j)]^m\\ &\geq \sum_{<ii'>}[G_{\Lambda_0(0)}(E,0,i)]^m[G(E,i',j)]^m\end{aligned} \tag{4.27}$$

So

$$\begin{aligned}\langle[G(E,0,j)]^m\rangle &\geq \langle[G_{\Lambda_0(0)}(E,0,\ell_{L_0})]^m\rangle\langle[G(E,\ell_{L_0+1},j)]^m\\ &\geq \cdots\\ &\quad\vdots\\ &\geq \big(\frac{c_m}{\gamma}\big)^{[(1-\delta)(|j|_1+1)]},\end{aligned}$$

by exactly the same manipulation as for $m = 1$. This concludes the proof of the lower bound and hence Theorem 1.1. □

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