FROM GLOBAL TO
LOCAL FLUCTUATION THEOREMS

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Abstract: The Gallavotti-Cohen fluctuation theorem suggests a general symmetry in the fluctuations of the entropy production, a basic concept in the theory of irreversible processes, based on results in the theory of strongly chaotic maps. We study this symmetry for some standard models of nonequilibrium steady states. We give a general strategy to derive a *local* fluctuation theorem exploiting the Gibbsian features of the stationary space-time distribution. This is applied to spin flip processes and to the asymmetric exclusion process.

Dedicated in honor of Robert Minlos on the occasion of his 70th birthday.

1 Introduction

A basic feature of equilibrium systems is that the restriction to a subsystem is again in equilibrium and with respect to the same microscopic interaction, for the same temperature, pressure and chemical potential. We can imagine cutting out a much smaller but still macroscopic region from our system and we will still find the same equilibrium state apart from possible boundary effects. Mathematically, this is expressed via the DLR-equations stating that the local conditional probabilities of a Gibbs measure coincide with the corresponding finite volume Gibbs measures. This really amounts to the fact that, for Gibbs measures, the ratio of probabilities for two different microscopic configurations that are identical outside a finite volume, is given by the Boltzmann factor $\sim \exp \beta \Delta H$ for relative Hamiltonian ΔH depending continuously on the configuration far out. In other words, relative energies make sense and they can be written as a sum of sufficiently local interaction potentials.

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Important consequences are found in the theory of equilibrium fluctuations and in the framework of the theory of large deviations; Bob Minlos was among the very first to develop a mathematical theory of this Gibbs formalism. In particular in [1], the description of the thermodynamic limit of Gibbs measures is given. We indeed usually have in mind here (very large) spatially extended systems which are monitored locally.

The idea that time does not enter this equilibrium description is further stimulated from the fact that at least in classical statistical mechanics, the momenta (entering only in the kinetic energy part of the Hamiltonian) can be integrated out at once from the partition function. Time enters already more explicitly in equilibrium dissipative dynamics such as via Langevin equations (Markov diffusion processes) or Glauber dynamics (for spin relaxation). Yet, under the condition of detailed balance, the stationary dynamics is microscopically reversible and the past cannot be distinguished from the future. The equilibrium steady state probability distribution on the space-time histories still has a Gibbsian structure with as extra *bonus* that the restriction to a spatial layer (at fixed time) is still explicitly Gibbsian and in fact, the restriction of the dynamics to a subregion still satisfies the detailed balance condition with respect to it. It was again Bob who was among the pioneers in the subject of space-time Gibbs measures and who, with his great experience in cluster expansion techniques and his love for field theory, saw the advantage of the space-time approach in the construction of solutions to infinite dimensional Markov diffusion processes, see e.g. [2].

While the condition of detailed balance reflects a symmetry (the space-time distribution is time-reversal invariant), the fact that the space-time distribution is Gibbsian (at least in some sense) does however not at all depend on it. In other words, the fact itself that the spatio-temporal probability distribution enjoys Gibbsianness is much more general and has nothing to do with microscopic reversibility. This can be checked readily for probabilistic cellular automata, [10, 3], but, more generally, it is the locality of the space-time interaction that does the job. In the present paper, we will exploit this fact in going from a global to a local fluctuation theorem for the entropy production in some models of interacting particle systems. At this moment, we need a second introduction to write about the statistical mechanics of steady state entropy production and how its fluctuations can give interesting information about the response of the system to perturbations. As we will see and as introduced in [3], once the conceptual framework and the Gibbsian basis of the fluctuation theorem is understood, the transition from a global to a local fluctuation theorem will be merely a technical matter. Physically speaking however, it is much better for the obvious reason that global fluctuations are far too improbable to be observed, [3, 17, 18].

1.1 Fluctuation theorem

First observed in [12] and later derived in [13, 14, 22], the Gallavotti-Cohen fluctuation theorem proves a symmetry in the fluctuations in time of the phase space contraction rate for a class of dynamical systems. The dynamics must obey certain conditions; it is a reversible smooth dynamical system $\xi \mapsto \phi(\xi), \xi \in K$ on a phase space K that is in some sense bounded carrying only a finite number of degrees of freedom (a compact and connected manifold). The transformation ϕ is a diffeomorphism on K. The resulting (discrete) time evolution is obtained by iteration and the reversibility means that there is a diffeomorphism θ on K with $\theta^2 = 1$ and $\theta \circ \phi \circ \theta = \phi^{-1}$. It is assumed that the dynamical system satisfies some technical (ergodic) condition: it is a transitive Anosov system. This ensures that the system allows a Markov partition (and the representation via some symbolic dynamics) and the existence of the SRB measure ρ, an invariant measure with expectations

$$\rho(f) = \lim_N \frac{1}{N} \sum_0^N f(\phi^n \xi) \tag{1.1}$$

corresponding to time-averages for almost every randomly chosen initial point $\xi \in K$ (i.e., for an absolutely continuous measure with respect to the Riemann volume element $d\xi$ on K). The change of variables implied by the dynamics defines the Jacobian determinant J and one writes $\dot{S} \equiv -\ln J$. This is the phase space contraction rate which is identified with the entropy production rate via the following argument: Define the (Shannon) entropy of a probability distribution $m(d\xi) = m(\xi)d\xi$ on K as

$$S(m) = -\int d\xi m(\xi) \ln m(\xi) \tag{1.2}$$

With m_n as density at time n, under the dynamics, the density at time $n+1$ is

$$m_{n+1}(v) = \frac{m_n(\phi^{-1}v)}{J(\phi^{-1}v)} \tag{1.3}$$

and the change in entropy is therefore

$$S(m_N) - S(m_0) = \int d\xi m_0(\xi) \sum_0^{N-1} \ln J(\phi^n \xi) \tag{1.4}$$

Dividing by N and taking N to infinity, the empirical probability distribution approaches the SRB distribution ρ, as in (1.1). Therefore, the time-averaged change in the entropy of the imagined reservoir is $\rho(\dot{S})$, see also [20, 21, 22]. One further assumes (and sometimes proves) dissipativity:

$$\rho(\dot{S}) > 0. \tag{1.5}$$

One is interested in the fluctuations of

$$\bar{s}_N(\xi) \equiv \frac{1}{\rho(\dot{S})N} \sum_{-N/2}^{N/2} \dot{S}(\phi^n(\xi)), \tag{1.6}$$

in the state ρ. Informally, the fluctuation theorem then states that $\bar{s}_N(\xi)$ has a distribution ρ_N with respect to ρ such that

$$\lim_N \frac{1}{N\rho(\dot{S})a} \ln \frac{\rho_N(a)}{\rho_N(-a)} = 1 \tag{1.7}$$

always. In other words, the distribution of the time-averaged $\dot{S}$ over long time intervals satisfies some general symmetry property. A more precise phrasing can be obtained via large deviation theory. For a continuous time version (Anosov flows) we refer to [19].

The reason why we are interested in the fluctuation theorem is because the established symmetry in fluctuations is very general and it may be important for the construction of nonequilibrium statistical mechanics beyond linear order perturbation theory. Now there are various proposals, ideas and results but, at any rate, whatever the point of view, it is rather natural asking how to establish a local version of the fluctuation theorem. The title of the present paper refers to that problem with the understanding that local refers to a space-time window within a much larger spatially and temporally extended nonequilibrium system. This was already the subject of [3, 17]. We want to understand how general the local fluctuation theorem can be and what form it takes for some standard nonequilibrium models. It turns out that, as was explicitly pointed out in [3], it is the Gibbsian structure of the space-time distribution that allows a local fluctuation theorem. This was already apparent from [17, 3] but here we add further systematization and explain and illustrate this local version of the fluctuation theorem.

One further question concerns the physical identification of the quantity for which we are investigating the symmetry in the fluctuations. We will call it entropy production. This name already exists for a physical quantity that appears in close to equilibrium thermodynamics, and indeed we believe that our choice of words reflects a generalization. The basic idea is that nonequilibrium steady states are not time-reversal invariant and that the mean entropy production should give a measure of discriminating between the original space-time distribution and its time-reversal. That is the relative space-time entropy density. For the variable entropy production, we must look up the source of the time-reversal symmetry breaking in the space-time interaction. It turns out that once it is recognized that the entropy production is the antisymmetric part of the space-time interaction under time-reversal, the symmetry in its local fluctuations (as expressed in the local fluctuation theorem (LFT)) is almost an immediate consequence of the Gibbsian structure. This we will show.

Of course, the question remains how we wish to use the local fluctuation theorem.

That is not the subject of the present paper but we refer to [15, 16, 22, 3, 11, 9] for some ideas.

1.2 Example

We sketch here the nature of a local versus global fluctuation theorem via a simple model. We have in mind a (1 + 1)-dimensional Ising spin system with formal Hamiltonian

$$H(\sigma) = \sum_{x,t} \sigma_t(x)[\sigma_{t+1}(x) + b\sigma_{t+1}(x+1)] \tag{1.8}$$

where we think $x \in \mathbb{Z}$ as the spatial coordinate and $t \in \mathbb{Z}$ as the (discrete) time; $\sigma_t(x) = \pm 1,\ b \neq 0$.
Look at the function

$$\bar{S}_{n,t}(\sigma) = b \sum_{t=-T}^{T} \sum_{x=-n}^{n} \sigma_t(x)[\sigma_{t-1}(x+1) - \sigma_{t+1}(x+1)]$$

of the spins in a space-time window parametrized by $n, T > 0$. We are interested in its fluctuations under the probability laws

- P_n, the Gibbs measure on $\{-1,1\}^{\{-n,\dots,n+1\}\times\mathbb{Z}}$ with respect to the Hamiltonian

  $$H_n(\sigma) = \sum_t \sum_{x=-n}^{n} \sigma_t(x)[\sigma_{t+1}(x) + b\sigma_{t+1}(x+1)]$$

  and

- P, any infinite volume Gibbs measure on $\{-1,1\}^{\mathbb{Z}^2}$ for the Hamiltonian (1.8).

In both cases we take the counting measure as reference and set the inverse temperature $\beta = 1$.
The difference is that P_n is an Ising model on a one-dimensional strip (finite spatial volume with infinite time-extension) and P is the corresponding model for infinite space-time volume.
We start with the statement of a global fluctuation theorem; that concerns the law P_n. A simple calculation shows that for every function g

$$\int dP_n(\sigma) g(\bar{S}_{n,T}(\sigma)) = \int dP_n(\sigma) g(-\bar{S}_{n,T}(\sigma)) e^{-\bar{S}_{n,T}(\sigma) + B_{n,T}(\sigma)} \tag{1.9}$$

with

$$\begin{aligned} B_{n,T}(\sigma) = \sum_{-n}^{n} \quad & \{-b\sigma_{-T-1}(x)\sigma_{-T}(x+1) + b\sigma_{-T}(x)\sigma_{-T-1}(x+1) \\ & + b\sigma_T(x)\sigma_{T-1}(x+1) - b\sigma_{-T}(x)\sigma_{T+1}(x+1) \\ & + \sigma_{T-1}(x)\sigma_T(x) - \sigma_{-T}(x)\sigma_{T+1}(x) \\ & - \sigma_{-T-1}(x)\sigma_{-T}(x) + \sigma_{-T-1}(x)\sigma_{-T}(x)\} \end{aligned}$$

Clearly, $|B_{n,T}(\sigma)| \leq cn$.
As a result, for fixed n, for all functions f,

$$\lim_T \frac{1}{T} \ln |\frac{\int dP_n(\sigma) f(\bar{S}_{n,T}(\sigma)/T)}{\int dP_n(\sigma) f(-\bar{S}_{n,T}(\sigma)/T) e^{-\bar{S}_{n,T}(\sigma)}}| = 0 \tag{1.10}$$

which implies the symmetry expressed in (1.7) with $N = T$.
Now to a local fluctuation theorem; that concerns the law P. A similar calculation shows that

$$\int dP(\sigma) g(\bar{S}_{n,T}(\sigma)) = \int dP(\sigma) g(-\bar{S}_{n,T}(\sigma)) e^{-\bar{S}_{n,T}(\sigma)+B_{n,T}(\sigma)+F_{n,T}(\sigma)}$$

with

$$|F_{n,T}(\sigma)| \leq cT$$

and we conclude that in both order of limits,

$$\lim_{n,T} \frac{1}{nT} \ln |\frac{\int dP(\sigma) f(\bar{S}_{n,T}(\sigma)/(nT))}{\int dP(\sigma) f(-\bar{S}_{n,T}(\sigma)/(nT)) e^{-\bar{S}_{n,T}(\sigma)}}| = 0 \tag{1.11}$$

This is the same symmetry as in (1.7) but for the local fluctuations in a spatially extended system. Of course, (1.11) involves limits but the basic fact behind (1.11) is that there is a local function $R_{n,T}$ with $|R_{n,T}(\sigma) - \bar{S}_{n,T}(\sigma)| \leq c_1 n + c_2 T$ for which

$$\int dP(\sigma) g(R_{n,T}) = \int dP(\sigma) g(-R_{n,T}) e^{-R_{n,T}(\sigma)}$$

which is an exact local fluctuation symmetry. Various things are lacking from this example. Mathematically, things will be more complicated when the $B_{n,T}$ or $F_{n,T}$ are not uniformly bounded or when time is not discrete or when the space-time Hamiltonian (1.8) is not local or contains hard-core interactions. Physically, the example above carries no interpretation of $\bar{S}_{n,T}$ as entropy production.

1.3 Local fluctuation theorem

The main theme of the present paper is a general strategy to find a local fluctuation theorem for the entropy production in a nonequilibrium steady state, in the context of stochastic interacting particle systems. To get the idea we present the result informally for a typical application. The details and mathematically precise statements about this model are given in Section 4. The model is a microscopic version of a reaction-diffusion system where the reaction consists of the birth and death of particles on the sites of a regular lattice and the diffusion part lets these particles hop to nearest neighbor vacancies subject to an external field.

Consider the square lattice $\mathbb{Z}^2$ to each site i of which we assign a variable $\eta(i) = 0, 1$, meaning that site is empty or occupied by a particle. The configuration η can change in two ways: first, a particle can be created or destroyed at lattice site i: $\eta \to \eta^i$ where η^i is identical to η except that the occupation at the site i is flipped. Secondly, a particle at i can hop to one of the four nearest neighbor sites j under the condition that j is empty: $\eta \to \eta^{ij}$ where η^{ij} is the new configuration obtained by exchanging the occupations at sites i and j. We make a nonequilibrium dynamics by adding an external field $E > 0$ which introduces a bias for particle hopping in a certain direction.

In formula, first, a particle is destroyed or created at any given site at fixed rates. The transition from a configuration η to the new η^i takes place at rate

$$c(i, \eta) = \gamma_+(1 - \eta(i)) + \gamma_-\eta(i)$$

where γ_+ is the rate for the transition $0 \to 1$ and γ_- is the rate for $1 \to 0$. Secondly, the particles on the lattice undergo a diffusive motion. To be specific, we choose a large square V centered around the origin with periodic boundary conditions and we first introduce hopping rates over a nearest neighbor pair $\langle ij \rangle$ in the horizontal direction, $i = (i_1, i_2), j = (i_1 + 1, i_2)$:

$$c(i, j, \eta) = e^{E/2}\eta(i)(1 - \eta(j)) + e^{-E/2}\eta(j)(1 - \eta(i))$$

The hopping rate in the vertical direction is constant (put $E = 0$ in the above if $j = (i_1, i_2 \pm 1)$). Taking E large, we expect to see many more jumps of particles to the right than to the left. In the absence of reaction rates, that is for $\gamma_\pm = 0$, we recover the so called asymmetric exclusion process and particle number is strictly conserved. More generally, the Master Equation is

$$\frac{d\rho_t(\eta)}{dt} = \sum_i [\rho_t(\eta^i)c(i, \eta^i) - \rho_t(\eta)c(i, \eta)] + \sum_{\langle ij \rangle} [\rho_t(\eta^{ij})c(i, j, \eta^{ij}) - \rho_t(\eta)c(i, j, \eta)]$$

For this model, the stationary measure ρ is the product measure with uniform density equal to $\gamma_+/(\gamma_- + \gamma_+)$ corresponding to a chemical potential $\ln \gamma_+/\gamma_-$ of the particle reservoir.

For a fixed nearest neighbor pair $\langle ij \rangle$, with $j = (i_1 + 1, i_2)$ to the right of i, the time-integrated microscopic current over an interval $[-T, T]$ is

$$J_T^i \equiv N_T^{i\to j} - N_T^{j\to i}$$

with $N_T^{i\to j}$ the total number of particles that have passed from site i to site j. We have the convention to take this current positive when the net number of particles jumping to the right (i.e., in the direction of the external field) is positive. Multiplying the sum of all the current contributions in V with the field E we get

$$W_{V,T}(\eta_s, s \in [-T, T]) \equiv E \sum_{i\in V} J_T^i(\eta_s, s \in [-T, T])$$

which is a random variable representing the work done on our system over the time-interval $[-T, T]$. Its expectation in the stationary state equals (up to a temperature factor) the expected heat dissipated in the environment and is given by

$$\langle W_{V,T}\rangle = 2T|V|\, E\sinh(E/2)\,\frac{\gamma_+\gamma_-}{(\gamma_+ + \gamma_-)^2}$$

If we now fix another square $\Lambda \subset V$ inside our large system, then

$$W_{\Lambda,T} \equiv E \sum_{i\in\Lambda:(i_1+1,i_2)\in\Lambda} J^i_T$$

is the random variable "work done on the system in Λ over the time-interval $[-T, T]$". That constitutes the main contribution to the local random variable "entropy production in the space-time window $\Lambda \times [-T, T]$". Yet, this is only its bulk contribution. We have indeed only included in $W_{\Lambda,T}$ the microscopic currents between the sites strictly inside Λ while particles will of course also hop in and out of Λ via its boundary. In other words, the region $V \setminus \Lambda$ acts as a particle reservoir from which particles can enter or leave Λ. That also contributes to the entropy production as, quite generally, the change in entropy in the particle reservoir equals the number of particles transferred to it, multiplied by its chemical potential. Now usually, this chemical potential is fixed and constant, i.e., not depending on whatever happens in the system itself. Here this is not the case. It suffices to imagine that almost all particles are in fact inside Λ with therefore a low density of particles in $V \setminus \Lambda$. As a result, the *effective* chemical potential for creating or destroying particles at the boundaries of Λ will depend on time and on whatever happened inside Λ before that time. Moreover this will contribute to the nonequilibrium condition only for $E \neq 0$ because only then will there be a different rate of leaving/entering Λ at the right versus the left vertical boundaries of Λ. This is not the case for the upper versus the lower boundaries but also there, even when there would not be a field strictly inside Λ, the dynamics inside will be influenced by the field outside. This is summarized in the form of the second contribution to the time-integrated entropy production and it is a boundary term:

$$\mathcal{J}_{\partial\Lambda,T} \equiv R_\ell + R_r + R_u + R_d$$

where the various terms correspond to the reactions taking place at the left, right, upper and lower boundaries of the square Λ. We will not write all of them down explicitly but here is for example

$$R_r \equiv \sum_{i\in\Lambda:j\in V\setminus\Lambda}\ \sum_{-T\leq t\leq T} [\eta_t(i)\ln\frac{\gamma_- + q_{\Lambda,t}(j,\eta,E)}{\gamma_- + q_{\Lambda,t}(j,\eta,-E)} + (1-\eta_t(i))\ln\frac{\gamma_+ + p_{\Lambda,t}(j,\eta,E)}{\gamma_+ + p_{\Lambda,t}(j,\eta,-E)}$$

where $j = (i_1+1, i_2)$, the sum over times t is over the times when a particle is created or destroyed at i, and the rates p and q are given by

$$q_{\Lambda,t}(j,\eta,E) \equiv e^{E/2}\mathrm{Prob}[\eta_t(j) = 0|\eta_s(k), k\in\Lambda, s\in[-T,t]]$$

and

$$p_{\Lambda,t}(j,\eta,E) \equiv e^{-E/2}\mathrm{Prob}[\eta_t(j) = 1|\eta_s(k), k \in \Lambda, s \in [-T,t]]$$

where the probabilities refer to the steady state in V. In other words, the external field does not only work on the particles in Λ it also creates a gradient in chemical potential (large at the left boundary and smaller at the right) in Λ. The total random variable "entropy production in Λ" now reads

$$\bar{S}_{\Lambda,T} \equiv W_{\Lambda,T} + \mathcal{J}_{\partial\Lambda,T}$$

The result proved in Section 4 is the fluctuation theorem symmetry for $\bar{S}_{\Lambda,T}$:

$$\lim_{\Lambda,T}\lim_{V}\frac{1}{|\Lambda|T}\ln\frac{\mathrm{Prob}[\frac{\bar{S}_{\Lambda,T}}{|\Lambda|T} = a]}{\mathrm{Prob}[\frac{\bar{S}_{\Lambda,T}}{|\Lambda|T} = -a]} = a \tag{1.12}$$

uniformly in the $\gamma_\pm$.

One may wonder whether the work $W_{\Lambda,T}$ satisfies a similar fluctuation symmetry. That is (1.12) with $W_{\Lambda,T}$ replacing $\bar{S}_{\Lambda,T}$. It remains uncertain however whether that is true uniformly in the values $\gamma_\pm \downarrow 0$ but, as we will show, it remains true whenever $\gamma_\pm \neq 0$.

The rest of our paper is organized as follows: in Section 2 we give a general strategy to obtain LFT, which we apply in Section 3 for spinflip processes and in Section 4 for the asymmetric exclusion process.

2 Abstract setting

We identify the essential mathematical structure, needed to pass from a global to a local fluctuation theorem. Our later specific illustrations will then just be applications of the same theme.

We consider a measurable space $(\Omega,\mathcal{F})$ on which two sequences of probability measures P_n and P_n^r. Suppose that Θ_n is an involution on Ω such that P_n and $P_n \circ \Theta_n$ are mutually absolutely continuous and the same for the pairs P_n^r and $P_n^r \circ \Theta_n$. We write

$$R_n \equiv \ln\frac{dP_n}{dP_n\Theta_n}, F_n \equiv R_n + \ln\frac{dP_n^r\Theta_n}{dP_n^r}$$

then, by definition, for all functions f,

$$\int dP_n f(R_n) = \int dP_n e^{-R_n} f(-R_n) \tag{2.13}$$

and

$$\int dP_n^r f(R_n) = \int dP_n^r e^{-R_n+F_n} f(-R_n) \tag{2.14}$$

The identity (2.13) expresses an exact symmetry in the fluctuations of R_n but should be compared with the global symmetry (1.9, 1.10). The next equality (2.14) is very similar but there is the correction term F_n. To get rid of it (at least asymptotically in n) we need extra assumptions. This will then yield the local fluctuation theorem. Before we give a general way of expressing these assumptions, the reader may appreciate some more explication concerning our choice of 'global' versus 'local' as there is of course no natural interpretation of this within the proposed abstraction.
As we will see in the next sections, we really start from two measures P and P_n on Ω where Ω will be the pathspace of an (infinite volume) interacting particle process on the $d-$dimensional regular lattice $\mathbb{Z}^d$; P will be an infinite volume steady state measure (i.e., the path-space measure of a stationary process over some time interval $[-T,T]$); n will refer to a finite space-time volume (corresponding to a sequence of cubes Λ_n centered around the origin times the interval $[-T,T]$) and Θ_n will be time-reversal on the space-time volume $\Lambda_n \times [-T,T]$. The process P_n will be the path-space measure of the stationary interacting particle process on this finite $\Lambda_n \times [-T,T]$. P_n^r is the marginal distribution of the trajectories restricted to the space-time window $\Lambda_n \times [-T,T]$ under P. In the context of interacting particle systems, P and P_n will be path-space measures of a Markovian process, whereas P_n^r will be non-Markovian. In the local fluctuation theorem it is attempted to recover the global symmetry of R_n under P_n also in the restrictions P_n^r of P to finite volumes Λ_n. Clearly then, what we need is that the difference between P_n and P_n^r is a boundary term but this is more or less implied by having our interacting particle systems enjoy Gibbsianness on space-time. Finally, the meaning of R_n is that it gives, at least up to space-time boundary terms, a statistical mechanical representation of the thermodynamic steady-state entropy production. We wish however to refer to [3, 4, 5, 6, 7] for explaining this. Still, it should be kept in mind that the B_n introduced in the following proposition will measure the difference between the *true* entropy production (denoted there by $\bar{S}_n$) and R_n.
There are in fact various strategies; we present two of them.
Proposition 2.1: Let B_n be a measurable function so that $B_n \circ \Theta_n = -B_n$. Define $\bar{S}_n \equiv R_n + B_n$ and let (a_n) be a sequence of positive numbers tending to infinity with n. Assume that P_n and P_n^r are mutually absolutely continuous and so that

$$\lim_n \frac{1}{a_n} \ln \int dP_n^r \left(\frac{dP_n}{dP_n^r}\right)^{\lambda_1} e^{\lambda_2 B_n} = 0 \tag{2.15}$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$. Suppose that for all $z \in \mathbb{R}$

$$p(z) = \lim_n \frac{1}{a_n} \ln \int e^{-z\bar{S}_n} dP_n \tag{2.16}$$

exist and is finite. Then, whenever

$$q(z) = \lim_n \frac{1}{a_n} \ln \int e^{-z\bar{S}_n} dP_n^r \tag{2.17}$$

exists, then $p(z) = q(z)$ and $q(z) = q(1-z)$.

Remarks:

1. The symmetry $q(z) = q(1-z)$ is dual to the symmetry as expressed in (1.7). Its Legendre transform $i(a) = \sup_z(-q(z) - za)$ satisfies $i(a) - i(-a) = -a$. If $\bar{S}_n$ satisfies a large deviation principle under P_n, respectively P_n^r, then $i(a)$ is the corresponding rate function, and the symmetry $q(z) = q(1-z)$ is equivalent with the large deviation symmetry $i(a) - i(-a) = -a$.

2. We will apply the strategy of Proposition 2.1 for obtaining a local fluctuation theorem for spinflip processes in the next Section.

Proof of Proposition 2.1: Since $B_n \circ \Theta_n = -B_n$, in the same way as for (2.13), we deduce that

$$\int dP_n f(\bar{S}_n) = \int dP_n e^{-\bar{S}_n + B_n} f(-\bar{S}_n) \tag{2.18}$$

Starting with the left hand side, for $f(s) = e^{-zs}$, by the Hölder inequality, for $1/a + 1/b = 1 = 1/v + 1/w$,

$$\begin{aligned}
&\ln \int dP_n e^{-z\bar{S}_n} \leq \frac{1}{a} \ln \int dP_n^r (\frac{dP_n}{dP_n^r})^a + \frac{1}{b} \ln \int dP_n^r e^{-bz\bar{S}_n} \\
&\leq \frac{1}{a} \ln \int dP_n^r (\frac{dP_n}{dP_n^r})^a + \frac{1}{bv} \ln \int dP_n e^{-bvz\bar{S}_n} + \\
&\frac{1}{bw} \ln \int dP_n^r (\frac{dP_n^r}{dP_n})^{w-1}
\end{aligned} \tag{2.19}$$

Dividing this by a_n and taking limits, we can use condition (2.15) with $\lambda_2 = 0$ to get

$$p(z) \leq \frac{q(bz)}{b} \leq \frac{p(bvz)}{bv}$$

Again by the Hölder inequality, both functions p and q are convex, and hence continuous. Therefore we can take the limit for $b, v \to 1$ to conclude that $p(z) = q(z)$. The right hand side of (2.18) can be treated in the same way:

$$\begin{aligned}
&\ln \int dP_n e^{-(1-z)S_n + B_n} \leq \frac{1}{a} \ln \int dP_n^r (\frac{dP_n}{dP_n^r})^a e^{aB_n} + \frac{1}{b} \ln \int dP_n^r e^{-b(1-z)\bar{S}_n} \\
&\leq \frac{1}{a} \ln \int dP_n^r (\frac{dP_n}{dP_n^r})^a e^{aB_n} + \frac{1}{bv} \ln \int dP_n e^{-bv(1-z)\bar{S}_n + B_n} + \\
&\frac{1}{bw} \ln \int dP_n^r (\frac{dP_n^r}{dP_n})^{w-1} e^{-wB_n/v}
\end{aligned} \tag{2.20}$$

which, again after taking limits $n \uparrow +\infty$, and using $B_n = \bar{S}_n - R_n$, gives

$$p(z) = q(z) \leq \frac{q(b(1-z))}{b} \leq \frac{q(-bv(1-z)+1)}{bv}$$

and we can take the limits $b, v \to 1$ to get the desired $q(z) = q(1-z)$. ■

Proposition 2.2: Let B_n be a measurable function such that $B_n \circ \Theta_n = -B_n$ and define $\bar{S}_n = R_n + B_n$. Let (a_n) be a sequence of positive numbers tending to infinity with n so that for all $\lambda \in \mathbb{R}$

$$\lim_n \frac{1}{a_n} \ln \int dP_n^r e^{\lambda(B_n+F_n)} = 0 \tag{2.21}$$

Suppose that for all $z \in \mathbb{R}$

$$q(z) = \lim_n \frac{1}{a_n} \ln \int e^{-z\bar{S}_n} dP_n^r \tag{2.22}$$

exists and is finite. Then, $q(z) = q(1-z)$.
Proof of Proposition 2.2: By definition of F_n, we have

$$\int dP_n^r f(\bar{S}_n) = \int dP_n^r e^{-\bar{S}_n + F_n + B_n} f(-\bar{S}_n)$$

We thus leave the left hand side and apply a similar chain of inequalities to the right hand side as was used in the proof of Proposition 2.1:

$$\begin{aligned}
&\ln \int dP_n^r e^{-(1-z)S_n+F_n+B_n} \leq \frac{1}{a} \ln \int dP_n^r e^{aF_n+aB_n} + \frac{1}{b} \ln \int dP_n^r e^{-b(1-z)\bar{S}_n} \\
&\leq \frac{1}{a} \ln \int dP_n^r e^{a(F_n+B_n)} + \frac{1}{bv} \ln \int dP_n^r e^{-bv(1-z)\bar{S}_n+F_n+B_n} + \\
&\frac{1}{bw} \ln \int dP_n^r e^{-w(F_n-B_n)/v}
\end{aligned} \tag{2.23}$$

We may thus again divide by a_n and take limits first $n \uparrow +\infty$ to reach

$$q(z) \leq \frac{q(b(1-z))}{b} \leq \frac{q(-bv(1-z)+1)}{bv}$$

By convexity we can take the limits $b, v \downarrow 1$ to obtain the desired conclusion. ■

Remarks:
1. Of course, if it happens that $|F_n + B_n|/a_n \to 0$ uniformly, then, for all positive functions f,

$$\lim_n \frac{1}{a_n} \ln \frac{\int dP_n^r f(\bar{S}_n)}{\int dP_n^r e^{-\bar{S}_n} f(-\bar{S}_n)} = 0$$

without further ado.
2. The difference between Proposition 2.1 and Proposition 2.2 is that in the first we suppose that P_n and P_n^r are mutually absolutely continuous while in the latter, we need that P_n^r and $P_n^r\Theta_n$ are mutually absolutely continuous. We will follow the second strategy in Section 4 for the asymmetric exclusion process.
3. The condition that the limits defining $p(z)$ and $q(z)$ exist is natural in the context where we have a large deviation principle for $\bar{S}_n$ under P_n and P_n^r resp. However if we define p^+, p^-, q^+, q^- by the corresponding limsup, resp. liminf, then we still have convexity of p^+, q^+ (the limsups), but not necessarily of p^-, q^-. We can still conclude however the equality $p^+(z) = q^+(z)$, and $q^+(z) = q^+(1-z)$.

3 LFT for spinflip processes

We start our study with the, for physical applications, less interesting case of pure spinflip processes. For details on the construction of spinflip processes, we refer to [24].

The configuration space is $K = \{+1, -1\}^{\mathbb{Z}^d}$ (spins on the $d-$dimensional regular lattice) and the path space is $\Omega = D(K, [-T, T])$ the set of right-continuous trajectories having left limits, parametrized by time $t \in [-T, T], T > 0$ and having values $\omega_t \in K$. Our processes are specified in terms of spinflip rates $c(x, \sigma), x \in \mathbb{Z}^d, \sigma \in K$ for which our first most important assumption is that they are positive and bounded: there are constants $b_1 > 0, b_2 < +\infty$ so that $b_1 < c(x, \sigma) < b_2$ for all x, σ. For convenience we assume that $c(x, \sigma)$ only depends on the neighboring spins $\sigma(y)$ with $|y - x| \leq 1$. Thirdly, we assume the rates to be translation invariant: $c(x, \sigma) = c(0, \tau_x \sigma)$. Here and afterwards we put $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$ Θ_n denotes time-reversal on Λ_n defined by $(\Theta_n \omega)_t(x) \equiv \omega_{-t}(x)$ if $x \in \Lambda_n$, and $(\Theta_n \omega)_t(x) \equiv \omega_t(x)$ if $x \notin \Lambda_n$. On the jump-times we adapt $\Theta_n \omega$ so that it becomes right-continuous, and thus obtain Θ_n as an involution on Ω

We define $\Lambda_n^\star \equiv \{x \in \Lambda_n, c(x, \sigma) = c(x, \sigma')$ for all $\sigma, \sigma' \in K$ with $\sigma(y) = \sigma'(y), y \in \Lambda_n\}$ for the subset of sites where the spinflip rates do not depend on the configuration outside Λ_n.

We first describe the sequence of processes P_n corresponding to P_n in the previous abstract setting. For this we fix a boundary condition $\eta \in K$ and we define spinflip rates

$$c_n(x, \sigma) \equiv I[x \in \Lambda_n] c(x, \sigma_{\Lambda_n} \eta_{\Lambda_n^c}), x \in \mathbb{Z}^d, \sigma \in K \tag{3.24}$$

where $I[\cdot]$ is the indicator function and $\sigma_{\Lambda_n} \eta_{\Lambda_n^c} \in K$ coincides with σ on Λ_n and equals η on the complement $\Lambda_n^c \equiv \mathbb{Z}^d \setminus \Lambda_n$. P_n is the stationary process on Ω with generator

$$L_n f(\sigma) \equiv \sum_x c_n(x, \sigma)[f(\sigma^x) - f(\sigma)]$$

corresponding to a spinflip process in Λ_n with rates $c(x,\sigma)$ and boundary condition η. We call the (unique) stationary measure $\rho_n : \int d\rho_n L_n f = 0$. We always assume that for all $\sigma \in \{-1,+1\}^{\Lambda_n}, \rho_n(\sigma) \geq b_1 \exp[-b_2|\Lambda_n|]$. We can compute the density of P_n with respect to $P_n\Theta_n$ via a Girsanov formula for point processes, e.g. in [8, 25]. For given $\omega \in \Omega$ we let $N_s^x(\omega), s \in [-T,T], x \in \mathbb{Z}^d$ denote the number of spinflips at x up to time s; that is, $N_s^x(\omega) \equiv |\{t \in [-T,s], \omega_{t^-}(x) = -\omega_t(x)\}|$; then,

$$R_n \equiv \ln \frac{dP_n}{dP_n\Theta_n} = \sum_{x\in\Lambda_n} \int_{-T}^{T} dN_s^x(\omega) \ln \frac{c_n(x,\omega_{s-})}{c_n(x,\omega_s)} + \ln \frac{\rho_n(\omega_{-T})}{\rho_n(\omega_T)}$$

As a consequence, the distribution of R_n as induced from P_n satisfies immediately the global fluctuation symmetry (2.13).
We can also consider the observable

$$\bar{S}_n \equiv \sum_{x\in\Lambda_n^\star} \int_{-T}^{T} dN_s^x(\omega) \ln \frac{c(x,\omega_{s-})}{c(x,\omega_s)}$$

which is measurable inside Λ_n. This is not an arbitrary choice but there is too little physics here to call R_n or $\bar{S}_n$ the entropy production; we will not elaborate on this. The difference $B_n \equiv \bar{S}_n - R_n$ is mainly a sum over $x \in \Lambda_n \setminus \Lambda_n^\star$.
The other process P_n^r we need to look at is very similar but it is in general not Markovian. To define it, we take a stationary process P on Ω and we take its restriction to Λ_n. We write ρ for the corresponding stationary measure and we let $\bar{\rho}_n$ be its restriction to Λ_n. We always assume that for all $\sigma \in \{-1,+1\}^{\Lambda_n}, \bar{\rho}_n(\sigma) \geq a_1 \exp[-a_2|\Lambda_n|]$. Being more explicit, we let P be an infinite volume stationary process with formal generator

$$Lf(\sigma) \equiv \sum_x c(x,\sigma)[f(\sigma^x) - f(\sigma)]$$

and put P_n^r the unique path-space measure such that

1. The distribution of $\{\omega_t(x) : x \in \Lambda_n, t \in [-T,T]\}$ under P_n^r and P coincide.
2. Under P_n^r, $\omega_t(x) = \eta(x)$ for all $x \notin \Lambda_n$, $t \geq 0$.

Theorem 3.1 [LFT for spinflip processes] For all $z \in \mathbb{R}$,

$$\lim_{n,T} \frac{1}{n^d T} \ln \frac{\int dP e^{-z\bar{S}_n}}{\int dP e^{-(1-z)\bar{S}_n}} = 0$$

Proof of Theorem 3.1:
Even though P_n^r is not Markovian (in general), it remains a jump process and the jump-intensities can be computed from the original spinflip rates. In order to have a Gibbsian structure these intensities must be the same in the bulk of Λ_n as they were

for the infinite volume process P. As the rates are local, the process P_n^r restricted to Λ_n indeed has the same intensities as the process P_n except at the sites of the boundary $\Lambda_n \setminus \Lambda_n^\star$. This is a consequence of the following generally stated

Lemma 3.2:
Suppose N_t is a point process with intensity c_s, i.e., $M_t = N_t - \int_0^t c_s \, ds$ is a martingale for the filtration $\mathcal{F}_t$. Suppose that $\mathcal{F}'_t \subset \mathcal{F}_t$ is a subfiltration of $\mathcal{F}_t$, and define

$$N'_t = \mathbb{E}[N_t | \mathcal{F}'_t] \tag{3.25}$$

Then N'_t is a point process with intensity

$$c'_s = \mathbb{E}[c_s | \mathcal{F}'_s] \tag{3.26}$$

Proof of Lemma 3.2:
It is easy to see that $M'_t = N'_t - \mathbb{E}(\int_0^t c_s ds | \mathcal{F}'_t) = \mathbb{E}[M_t | \mathcal{F}'_t]$ is a $\mathcal{F}'_t$ martingale. Hence, it suffices to show that

$$B_t = \mathbb{E}[\int_0^t c_s ds | \mathcal{F}'_t] - \int_0^t \mathbb{E}[c_s | \mathcal{F}'_s] ds \tag{3.27}$$

is a $\mathcal{F}'_t$-martingale. This is a consequence of the following equalities:

$$\begin{aligned} &\mathbb{E}[B_t | \mathcal{F}'_s] \\ &= B_s + \mathbb{E}\left[\int_s^t (c_r - \mathbb{E}[c_r | \mathcal{F}'_r]) dr | \mathcal{F}'_s\right] \\ &= B_s + \mathbb{E}[\int_s^t c_r dr | \mathcal{F}'_s] - \mathbb{E}\left[\int_s^t \mathbb{E}[c_r | \mathcal{F}'_r] | \mathcal{F}'_s\right] \\ &= B_s \end{aligned} \tag{3.28}$$

■

Therefore, the rates of the restricted process on Λ_n are given by

$$\tilde{c}_s(x, \omega) = \mathbb{E}[c(x, \sigma_s) | \sigma_\tau(y) = \omega_\tau(y), -T \leq \tau \leq s, y \in \Lambda_n]$$

where the expectation is with respect to P.
Or, for all $x \in \Lambda_n$, $N_t^x(\omega) - \int_0^t \tilde{c}_s(x, \omega)$ is a martingale under P_n^r. As a consequence, $\tilde{c}_s(x, \omega) = c(x, \sigma)$ when $x \in \Lambda_n^\star$ and $\omega_s = \sigma$.
Just as for the pair $P_n, P_n \Theta_n$, the absolutely continuity of $P_n^r \theta_n$ with respect to P_n^r and vice versa is guaranteed by the positivity of the spinflip rates inside Λ_n. We are thus ready to apply Proposition 2.1. We must first verify the corresponding assumption (2.15). We find

$$B_n = \bar{S}_n - R_n = - \sum_{x \in \Lambda_n \setminus \Lambda_n^\star} \int dN_s^y(\omega) \ln \frac{c_n(x, \omega_{s^-})}{c_n(x, \omega_s)} + \ln \frac{\rho_n(\omega_T)}{\rho_n(\omega_{-T})} \tag{3.29}$$

and

$$\ln \frac{dP^r_n}{dP_n} = \ln \frac{\bar\rho(\omega_{-T})}{\rho_n(\omega_{-T})} + \sum_{x\in\Lambda_n\setminus\Lambda^\star_n} \int dN^x_s(\omega) \ln \frac{\tilde c_s(x,\omega)}{c_n(x,\omega_s)} - \int_{-T}^{T} ds[\tilde c_s(x,\omega) - c_n(x,\omega_s)] \tag{3.30}$$

Clearly, both $|B_n|$ and $|\ln dP^r_n/dP_n|$ are bounded by $c_1 N([-T,T],\Lambda_n\setminus\Lambda^\star_n) + c_2|\Lambda_n| + c_3 T|\Lambda_n\setminus\Lambda^\star_n|$ for some constants $c_1, c_2, c_3 < \infty$, where $N([-T,T],\Lambda_n\setminus\Lambda^\star_n)$ is the number of spinflips that have occurred in the space-time window $[-T,T]\times(\Lambda_n\setminus\Lambda^\star_n)$. It remains thus to show for all λ

$$\begin{aligned}&\lim_{n\uparrow\infty}\frac{1}{n^d}\ln\int dP^r_n e^{\lambda N([-T,T],\Lambda_n\setminus\Lambda^*_n)}\\ &= \lim_{n\uparrow\infty}\frac{1}{n^d}\ln\int dP^0\frac{dP^r_n}{dP^0_n} e^{\lambda N([-T,T],\Lambda_n\setminus\Lambda^*_n)} = 0\end{aligned}$$

where we inserted the reference process P^0 (and its restriction P^0_n to Λ_n) corresponding to the product process of independent spinflips (rate 1). In particular,

$$\lim_{n\uparrow\infty}\frac{1}{n^d}\ln\int dP^0 e^{\lambda N([-T,T],\Lambda_n\setminus\Lambda^*_n)} = \lim_{n\uparrow\infty}\frac{|\Lambda_n\setminus\Lambda^*_n|}{n^d} 2T(e^\lambda - 1) = 0$$

and we can apply the same argument as in Proposition 2.1. Finally, the condition (2.16) of Proposition 2.1 is a consequence of the large deviation results of [23]. ■

4 LFT for the asymmetric exclusion process

The configuration space is now $K = \{0,1\}^{\mathbb{Z}^2}$ (occupation variables on the 2-dimensional regular lattice) and the pathspace $\Omega = D(K,[-T,T])$ is essentially unchanged from that in the previous section. For $\eta\in K$, $\eta(x) = 1, 0$ indicates the presence, respectively absence of a particle at the site $x\in\mathbb{Z}^d$. This hopping dynamics will be modeled by an asymmetric exclusion process. This is a bulk driven diffusive lattice gas. The hopping rates for vertical (v) and horizontal bonds (h) depend on the direction in the following way:

$$\begin{aligned} c^v(x,\eta) &\equiv \frac{1}{2}[\eta(x)(1-\eta(x+e_2)) + \eta(x+e_2)(1-\eta(x))]\\ c^{h,E}(x,\eta) &\equiv \frac{1}{2}[e^{E/2}\eta(x)(1-\eta(x+e_1)) + e^{-E/2}\eta(x+e_1)(1-\eta(x))]\end{aligned}$$

where e_1, e_2 are the unit vectors in the positive horizontal and vertical direction.
In addition, for the moment, we allow for the possibility of particle creation and destruction. We put the birth/death rate $c(x,\eta)\equiv\epsilon$ independent of the configuration

η and the site x.
The formal Markov generator L to the infinite volume process is then found as the sum

$$Lf(\eta) \equiv \epsilon \sum_x [f(\eta^x) - f(\eta)] + \sum_{\langle xy \rangle} c(x,y,\eta)[f(\eta^{xy}) - f(\eta)]$$

where η^x is the new configuration after changing the occupation at x, η^{xy} is the new configuration after exchanging the occupations at x and y and $c(x,y,\eta)$ is given by (4.31) for nearest neighbors $x, y = x \pm e_i$. We can allow more general reaction-diffusion processes (e.g. with extra interaction, speed change, etc.) but we will stick here to this choice. What is simpler here is that the Bernoulli measure ρ with density $1/2$ is a non-reversible stationary measure. The corresponding pathspace measure over the time-interval $[-T,T]$ is $P = P^E$ and we put P_n^E the process restricted to the finite square Λ_n This P_n^E will now play the role of P_n^r of Section 2. For a given trajectory $\omega \in \Omega$ we let $N_s^{xy}(\omega), s \in [-T,T]$ be the number of hopping times where the occupation at the nearest neighbor sites $\langle xy \rangle$ was exchanged. Since this model has a clear physical interpretation we can define the variable entropy production in Λ_n.

The first contribution comes from the work done by the external field

$$\bar{W}_n \equiv E \sum_{\langle xy \rangle, y = x + e_1 \in \Lambda_n} \int_{-T}^{T} dN_s^{xy}(\omega)[\omega_{s^-}(x)(1 - \omega_{s^-}(y)) - \omega_{s^-}(y)(1 - \omega_{s^-}(x))] \tag{4.31}$$

This is of the form field (E) times current. There is a second contribution from differences in the reaction rates at the boundary: particles enter or leave at different rates at the various boundaries; this contribution is present even in the case where no external field is applied inside Λ_n:

$$\begin{aligned} \mathcal{J}_n(\omega) \equiv & \sum_{x \in \partial\Lambda} \sum_{y = x \pm e_1 \in \Lambda_n^c} \int_{-T}^{T} dN_s^x(\omega)[\omega_{s^-}(x) \ln \frac{2\epsilon + \lambda_y^E(\omega,s) e^{E_j/2}}{2\epsilon + \lambda_y^{-E}(\omega,s) e^{-E_j/2}} \\ & + (1 - \omega_{s^-}(x)) \ln \frac{2\epsilon + \kappa_y^E(\omega,s) e^{-E_j/2}}{2\epsilon + \kappa_y^{-E}(\omega,s) e^{E_j/2}}] \\ & + \sum_{x \in \partial\Lambda} \sum_{y = x \pm e_2 \in \Lambda_n^c} \int_{-T}^{T} dN_s^x(\omega)[\omega_{s^-}(x) \ln \frac{2\epsilon + \lambda_y^E(\omega,s)}{2\epsilon + \lambda_y^{-E}(\omega,s)} \\ & + (1 - \omega_{s^-}(x)) \ln \frac{2\epsilon + \kappa_y^E(\omega,s)}{2\epsilon + \kappa_y^{-E}(\omega,s)}] \end{aligned} \tag{4.32}$$

where the second sum is over all (external) neighbors y of x and $\partial\Lambda_n$ is the interior boundary of Λ_n. Here, the additional rates are

$$\lambda_y^E(\eta,t) \equiv \mathbb{E}^E[1 - \eta_t(y) | \eta_s(z), z \in \Lambda_n, s \in [-T,t]]$$

and

$$\kappa_y^E(\eta,t) \equiv \mathbb{E}^E[\eta_t(y)|\eta_s(z), z \in \Lambda_n, s \in [-T,t]]$$

for $E_j = \pm E$ if $y = x \pm e_1, E_j = 0$ if $y = x \pm e_2$ and the expectations are in the process $P = P^E$. The variable entropy production is put

$$\bar{S}_n = \bar{W}_n + \mathcal{J}_n$$

The symmetry in the fluctuations of $\bar{S}_n$ is given by
Theorem 4.1 [LFT for the aymmetric exclusion process] For all ϵ (including $\epsilon = 0$), for all $z \in R$,

$$\lim_{n,T} \frac{1}{n^2 T} \ln \frac{\int dP e^{-z\bar{S}_n}}{\int dP e^{-(1-z)\bar{S}_n}} = 0$$

Proof of Theorem 4.1 We start by noting that for the time-reversal Θ_n,

$$P_n^E \Theta_n = P_n^{-E}$$

Obviously then, for a function f measurable in $\Lambda_n \times [-T,T]$,

$$\int dP f(\omega) = \int dP_n^E f(\omega) = \int dP_n^E \frac{dP_n^{-E}}{dP_n^E} f(\Theta_n \omega)$$

and we must investigate express the density dP_n^E/dP_n^{-E} via a Girsanov formula. This is the strategy of Proposition 2.2. The Girsanov formula gives

$$\ln \frac{dP_n^E}{dP_n^{-E}} = \bar{S}_n + F_n \tag{4.33}$$

with the following correction term:

$$\begin{aligned} F_n(\omega) \equiv \sinh(E/2) \sum_{x,y=x+e_1 \in \Lambda_n} \int_{-T}^{T} ds[\omega_s(y)(1-\omega_s(x)) - \omega_s(x)(1-\omega_s(y))] \\ + \sum_{x \in \partial\Lambda} \sum_{y=x\pm e_1, x\pm e_2 \in \Lambda_n^c} \int_{-T}^{T} ds[\omega_s(x)[\lambda_y^E(\omega,s)) - \lambda_y^{-E}(\omega,s)] \\ +(1-\omega_s(x))[\kappa_y^E(\omega,s)) - \kappa_y^{-E}(\omega,s)] \end{aligned} \tag{4.34}$$

Now, $|F_n| \leq c|\partial\Lambda|T$ because the (first) bulk term in (4.34) telescopes to a boundary term. We can thus apply Remark 1 after Proposition 2.2 to finish the proof. ■

Next, we investigate whether the variable work W_n of (4.31) itself satisfies the same local symmetry as the entropy production.

Theorem 4.2 [LFT for the work done] For all $\epsilon > 0$, for all $z \in \mathbb{R}$,

$$\lim_{n,T} \frac{1}{n^2 T} \ln \frac{\int dP e^{-z\bar{W}_n}}{\int dP e^{-(1-z)\bar{W}_n}} = 0$$

Proof of Theorem 4.2 Clearly, since $\epsilon > 0$, $|\mathcal{J}_n| \leq cN([-T,T], \Lambda_n \setminus \Lambda_n^*)$, that is bounded, up to a constant, by the number of flips in the trajectory on sites $x \in \Lambda_n \setminus \Lambda_n^*$, for times $t \in [-T,T]$. We can therefore verify condition (2.21) in the same way as we did for Theorem 3.1. Finally, the large deviation results of [23] remain valid for $\epsilon > 0$, so that we can finish the proof along the lines of Proposition 2.2. ■

5 Remarks

1. It is clear from the preceding analysis that the reasons for having a global or local fluctuation theorem do not in any way depend on the Θ_n being time-reversal. Thus, the same results will be reproduced in exactly the same form for any other involution. Of course, the symmetry breaking part in the pathspace action functional will be the variable for which the fluctuation symmetry holds (replacing entropy production corresponding to time-reversal symmetry breaking). As an example, if for a spinflip process, the rates are not even under a global spinflip (by the presence of a bias or magnetic field), then a local fluctuation theorem will be established for the variable magnetization. Furthermore, we may consider the composition of two or more involutions — in this way, we could e.g. obtain a local fluctuation theorem for the odd part (under spinflip) of the variable entropy production. Finally, we can even go beyond the case of involutions and consider instead the generators of the symmetry group for the unperturbed dynamics. In this case, the precise form of the fluctuation symmetry is not preserved but its modification presents no real problem.

2. We restricted our discussion to interacting particle systems where the evolution is Markovian. Within the Gibbsian space-time picture, this means that the interaction is "nearest neighbor" in the time direction (the jump intensity at time t depends only on the configuration at time t^-). However, this restriction is not at all necessary. If the jump intensities are local in space and bounded from above and from below, then we can still apply the Girsanov formula for point processes to obtain the local fluctuation theorem from the global fluctuation theorem.

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