# ON A CLASS OF ANISOTROPIC SCHRÖDINGER OPERATORS

Marius Măntoiu

Institute of Mathematics of the Romanian Academy,
P.O.Box 1-764, Bucharest; RO-70700, ROMANIA

Abstract. We introduce a C*-algebra containing the resolvents of a certain class of Schrödinger operators. Their potentials are anisotropic; they tend in radial directions to (different) almost periodic functions. The technique consists in defining a suitable compactification of the configuration space; a rather straightforward procedure leads then to the relevant C*-algebra. It is shown that the quotient of this C*-algebra by the ideal of compact operators has a remarcable structure. As an application, we calculate the essential spectrum of the affiliated anisotropic operators in terms of the spectra of a family of "asymptotic Hamiltonians".

## 1. Introduction

We shall call *(generalized) Schrödinger operator* in the Hilbert space $\mathcal{H} = L^2(X)$, $X = \mathbb{R}^n$ a self-adjoint operator of the form

$$H := h(P) + V. \tag{1-1}$$

We denoted by $P := -i\nabla$ the momentum, $h : X \to \mathbb{R}$ is a continuous function diverging at infinity and $V : X \to \mathbb{R}$ is (for simplicity) continuous and bounded.

We say that $V$ *has radial almost periodic limits* if there exists a continuous function $W : S \times X \to \mathbb{R}$ such that for any $s \in S$ =the unit sphere in $X$, the function $W_s(\cdot) := W(s, \cdot)$ is almost periodic on $X$ and

$$\lim_{r\to\infty} \mid V(rs) - W_s(rs) \mid = 0, \text{ uniformly in } s \in S. \tag{1-2}$$

We shall say that $W$ is *the asymptotic function* of $V$.

The main aim of this paper is to prove

**Theorem 1.1.** *Assume that $V$ has radial almost periodic limits and let $W$ be its asymptotic function. For any $s \in S$ set $H_s := h(P) + W_s$ (it is a self-adjoint, almost periodic operator in $\mathcal{H}$).*

---

*Key words and phrases.* Schrödinger operator, anisotropic, almost periodic, compactification, C*-algebra.

*Acknowledgement:* The author take this opportunity to express his gratitude to Vladimir Georgescu and Radu Purice for many helpful discussions .

*Then the essential spectrum of $H$ is the closure of the union of the family of spectra of the operators $H_s$:*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \overline{\bigcup_{s\in S} \boldsymbol{\sigma}(H_s)}. \tag{1-3}$$

The result is still interesting when the functions $W_s$ are purely periodic, with a $s-$ depending period; the spectrum of $H_s$ will then have a band structure. But there is essentially no extra effort to prove the general case.

A simple potential to which Theorem 1.1 applies is

$$V(x) = A(x)\cos\left[< B(x), x > +C(x)\right],$$

where $A, B_j, C : X \to \mathbb{R}$ are continuous and for some continuous functions $a, b_j, c : S \to \mathbb{R}$, one has

$$\lim_{r\to\infty}\left[\mid A(rs) - a(s) \mid +r\sum_{j=1}^{n} \mid B_j(rs) - b_j(s) \mid + \mid C(rs) - c(s) \mid\right] = 0$$

uniformly in $s \in S$. The asymptotic function is, of course

$$W(s,x) = a(s)\cos\left[< b(s), x > +c(s)\right].$$

It seems not an easy task to obtain Theorem 1.1 by traditional methods (Weyl sequences, partition of unity). In fact we shall use the algebraic approach of Vladimir Georgescu. We send to [1] for a systematic presentation and to [7] for further informations and applications to other types of anisotropic Schrödinger operators.

The main ideea is to show that the resolvent of $H$ belongs to a C*-algebra $\mathfrak{C}$ containing the ideal $\mathfrak{K}$ of compact operators. If a good choice for $\mathfrak{C}$ is made, the quotient $\mathfrak{C}/\mathfrak{K}$ can be embedded in some other C*-algebra with a special structure and this gives efficient formulae for the essential spectrum of $H$.

One way to perform this is to associate first a suitable abelian C*-algebra $\mathcal{A}$ to the class of potentials which are to be treated. A very effective way to do this is to think in terms of compactifications $\hat{X}$ of $X$: the functions in the algebra are exactly those which can be extended continuously on $\hat{X}$.

In a second step one also takes into account the "free energy". This is done by constructing the C*-algebra generated by $\mathcal{A}$ and the resolvents of the operators $h(P)$ for all continuous, divergent functions $h$. This works well if $\mathcal{A}$ is stable with respect to the natural action of the group X by translations. The Neumann series shows that this is the right procedure. Fortunately, one gets a C*-algebra which is also a crossed product and this is very helpful in the proofs. Crossed products have wonderful functorial properties and this permits essentially to concentrate on the abelian parts.

At this abelian level, one is led to study the dynamical system $(\hat{X}, \hat{\alpha}, X)$, where $\hat{\alpha}$ is the natural action of $X$ by homeomorphisms of $\hat{X}$ which extends the group low $\alpha$ of $X$. Performing the quotient by the ideal of compact operators may be seen as substracting $X$ from $\hat{X}$. One may hope that the remaining dynamical system $(\hat{X} \setminus X, \hat{\alpha}, X)$ is transparent enough to lead to some comprehensive formula for the essential spectrum.

A brief survey of the algebraic formalism, crossed products and topological dynamical systems may be found in Section 2.

The compactification which is relevant for our problem is as follows: Let $AP(X)$ the C*-algebra of continuous, almost periodic functions on $X$. We shall denote by $T$ its spectrum; it is a quite famous object in harmonic analysis, called *the Bohr group*. Essentially, one must compactify $X$ by "adding at infinity" the space $S \times T$.

Very often, the compactifications of quite simple-looking spaces are very complicated. In Section 3 we introduce a class of simple compactifications $\hat{X}$. We also describe in concrete terms the C*-algebra of continuous functions on $\hat{X}$ as well as the way in which $\hat{X}$ becomes the space of a natural dynamical system.

Section 4 is essentially dedicated to the proof of Theorem 1.1. The general construction of Section 3 is particularized to produce a dynamical system $(\hat{X}, \hat{\alpha}, X)$, where $\hat{X} \setminus X = S \times T$, as promised before. We shall still be placed at a level general enough to admit the alternative case $X = \mathbb{Z}^n$. So there will also be a result analogous to Theorem 1.1 for finite-difference operators.

We felt that invoking the theory of extensions of C*-algebras would be illuminating. It shows how a general theory works to produce rather concrete results, it was a source of inspiration for the author and also motivates the way the construction of $\hat{X}$ is performed in [**3.1**]. But since the theory of extensions is not a standard tool in the spectral theory of self-adjoint operators, this will only be the topic of an appendix. The reader may skip it with no essential loss.

We end the Introduction by specifying some notations and conventions. A closed, self-adjoint, two-sided ideal in a C*-algebra will be called simply *ideal*. For any Hilbert space $\mathcal{H}$ we denote by $B(\mathcal{H})$ the C*-algebra of all linear bounded operators in $\mathcal{H}$ and by $K(\mathcal{H})$ the ideal of compact operators. If $\mathfrak{B}$ is a C*-algebra and $Y$ a locally compact space, the set of all continuous functions $\varphi : Y \to \mathfrak{B}$ which tend to 0 at infinity will be denoted by $C_0(Y; \mathfrak{B})$. It is a C*-algebra in a standard way. When we are sure that $Y$ is compact, we drop the index 0. When $\mathfrak{B} = \mathbb{C}$, we write simply $C_0(Y)$. $BC(Y)$ consists of continuous, bounded functions on $Y$. Those which are also uniformly continuous are elements of $BC_u(Y)$. The sign $\cong$ will be used for canonical isomorphisms of C*-algebras. We write $\mathfrak{B} \hookrightarrow \mathfrak{C}$ when a monomorphism (=injective morphism) of C*-algebras is given.

Just proving Theorem 1.1 would have led to a shorter article. We tried to situate each construction at a suitable level of generality, hoping that the mathematics involved would present some intrinsic interest and might also serve other purposis. In fact, by similar techniques, a certain type of non-propagation results at suitable energies is available for anisotropic Schrödinger operators (see [2]).

## 2. An algebraic framework for the essential spectrum

This Section has a preparatory nature. We begin in [**2.1**] by recalling the notion of observable affiliated to a C*-algebra. The basic reference is [1]. Crossed products constitutes a major topic of modern C*-algebra theory (see for example [9]). We shall not give the most general definitions. Since it is difficult to send to the general literature for the few facts we need, we refer to [7], where the information relevant to spectral analysis is given in a rather self-contained way. So, the paragraph [**2.2**] is essentially a short abstract of Section 3 from [7], to which we send for proofs and further details. The abelian case is reviewed in [**2.3**].

**[2.1] Observables affiliated to C*-algebras.**
To define affiliation in an abstract setting, we think in terms of functional calculus.

**Definition 2.1.** *By* (self-adjoint) observable affiliated to the C*-algebra $\mathfrak{C}$ *we mean a *-morphism* $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.

Observables affiliated to $B(\mathcal{H})$ are just couples $(\mathcal{H}_0, H)$, where $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$ and $H$ is a densely defined, self-adjoint operator in $\mathcal{H}_0$. The case $\mathcal{H}_0 \neq \mathcal{H}$ effectively appears in a natural way in some applications, but this will not be the case in the present article. We shall prefer the notation $f(H)$ instead of $\Phi(f)$ for $f \in C_0(\mathbb{R})$ and we shall call $H$ the affiliated observable, even if the C*-algebra $\mathfrak{C}$ is abstract. It is useful to note that, if $\mathfrak{C}$ is a C*-subalgebra of $B(\mathcal{H})$, then a self-adjoint operator $H$ in $\mathcal{H}$ is affiliated to $\mathfrak{C}$ iff the resolvent $(H-z)^{-1}$ is in $\mathfrak{C}$ for at least one point $z$ not belonging to the spectrum of $H$.
One may transport observables by *-morphisms:

**Definition 2.2.** *Let* $\rho : \mathfrak{C} \to \mathfrak{C}'$ *a *-morphism and* $H$ *an observable affiliated to* $\mathfrak{C}$. *Then* $\rho[H] : C_0(\mathbb{R}) \to \mathfrak{C}'$ *given by* $f(\rho[H]) := \rho[f(H)]$ *is an observable affiliated to* $\mathfrak{C}'$, *called* the image of $H$ through $\rho$.

We note that *-morphisms do not increase the spectrum:

$$\boldsymbol{\sigma}(\rho[H]) \subset \boldsymbol{\sigma}(H) := \{\lambda \in \mathbb{R} \mid \text{ if } \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0, \text{ then } \varphi(H) \neq 0\}.$$

in fact, if $\rho$ is a monomorphism, we have equality.
Two special instances will systematically appear in the sequel: quotients and direct products. Let $\mathfrak{K}$ be an ideal of the C*-algebra $\mathfrak{C}$ and $\pi$ the canonical surjection $: \mathfrak{C} \to \mathfrak{C}/\mathfrak{K}$. Then the spectrum of the image $\pi[H]$ of an observable $H$ affiliated to $\mathfrak{C}$ will be denoted by $\boldsymbol{\sigma}_{\mathfrak{K}}(H)$. The guiding case is $\mathfrak{K} = K(\mathcal{H}) \subset \mathfrak{C} \subset B(\mathcal{H})$. In this situation, as it is well-known, one has $\boldsymbol{\sigma}_{\mathfrak{K}}(H) = \boldsymbol{\sigma}_{\text{ess}}(H)$, the essential spectrum of the self-adjoint operator $H$. By extension, we may call $\boldsymbol{\sigma}_{\mathfrak{K}}(H)$ *the* $\mathfrak{K}$*-essential spectrum of* $H$. Now assume that $\mathfrak{C} = \prod_{a \in A} \mathfrak{C}_a$ is a direct product of C*-algebras and set $p_a : \mathfrak{C} \to \mathfrak{C}_a$ for the $a$'th projection. Then one has $\boldsymbol{\sigma}(H) = \overline{\cup_{a \in A} \boldsymbol{\sigma}(p_a[H])}$ for any observable $H$ affiliated to $\mathfrak{C}$.

**[2.2] Crossed products.**

**Definition 2.3.** *A* C*-dynamical system $(\mathcal{C}, \tau, X)$ *consists of a C*-algebra* $\mathcal{C}$, *an abelian locally compact group* $(X, +)$ *with Haar measure* $dx$ *and a continuous action* $\tau$ *of* $X$ *in* $\mathcal{C}$ *by *-automorphisms.*

We assign to $(\mathcal{C}, \tau, X)$ a Banach *-algebra, generalizing the well-known (convolution) group algebra $L^1(X)$, which is recovered if one takes $\mathcal{C} = \mathbb{C}$. For this, we consider the Banach space $L^1(X; \mathcal{C})$ of Bochner integrable functions $S : X \to \mathcal{C}$ and we endow it with an involution

$$S^*(x) := \tau_x[S(-x)^*]$$

and a multiplication

$$(S \cdot T)(x) := \int_X dy S(y)\tau_y[T(x-y)].$$

**Definition 2.4.** *The enveloping C*-algebra of the Banach *-algebra defined above is denoted by $\mathcal{C} \rtimes_\tau X \equiv \mathcal{C} \rtimes X$ and is called* the crossed product of $\mathcal{C}$ by the action $\tau$ of $X$.

We recall that there exist C*-norms on $L^1(X;\mathcal{C})$ and the enveloping C*-algebra is-by definition-the completion with respect to the largest one. Thus, $\mathcal{C} \rtimes X$ is the smallest C*-algebra (in the sense of surjections) in which $L^1(X;\mathcal{A})$ is a densa *-subalgebra.

**Definition 2.5.**
*(a) A $X$-*morphism *between two C*-dynamical systems $(\mathcal{C}_k,\tau_k,X)$, $k=1,2$ is a *-morphism $\rho:\mathcal{C}_1\to\mathcal{C}_2$ satisfying $\rho[\tau_{1,x}(C)]=\tau_{2,x}(\rho[C])$ for all $x\in X$ and $C\in\mathcal{C}_1$.*
*(b) In the framework of (a) we set $\rho_0 : L^1(X;\mathcal{C}_1)\to L^1(X;\mathcal{C}_2)$ by $(\rho_0[S])(x) = \rho[S(x)]$. By extending canonically $\rho_0$ to the C*-enveloping algebra one gets a *-morphism $\rho_\rtimes : \mathcal{C}_1 \rtimes_{\tau_1} X \to \mathcal{C}_2 \rtimes_{\tau_2} X$.*

**Proposition 2.6.** *Let $(\mathcal{C}_k,\tau_k,X)$, $k=1,2,3$ be C*-dynamical systems, $\rho_{12}:\mathcal{C}_1\to\mathcal{C}_2$ and $\rho_{23}:\mathcal{C}_2\to\mathcal{C}_3$ be $X$-morphisms. Set $\mathfrak{C}_k := \mathcal{C}_k \rtimes_{\tau_k} X$. If*

$$0 \to \mathcal{C}_1 \xrightarrow{\rho_{12}} \mathcal{C}_2 \xrightarrow{\rho_{23}} \mathcal{C}_3 \to 0$$

*is an exact sequence, then*

$$0 \to \mathfrak{C}_1 \xrightarrow{(\rho_{12})_\rtimes} \mathfrak{C}_2 \xrightarrow{(\rho_{23})_\rtimes} \mathfrak{C}_3 \to 0$$

*is also an exact sequence.*

In particular, Proposition 2.6 gives immediately

**Corollary 2.7.** *(a) Let $(\mathcal{C},\tau,X)$ be a C*-dynamical system and $\mathcal{K}$* a stable ideal *in $\mathcal{C}$ (i.e. $\tau_x\mathcal{K}\subset\mathcal{K}$ for all $x\in X$). Then $X$ acts in obvious ways by *-automorphisms in $\mathcal{K}$ and in $\mathcal{C}/\mathcal{K}$ and the C*-algebras $(\mathcal{C}\rtimes X)/(\mathcal{K}\rtimes X)$ and $(\mathcal{C}/\mathcal{K})\rtimes X$ are canonically isomorphic.*
*(b) Let $(\mathcal{B},\sigma,X)$, $(\mathcal{C},\tau,X)$ two C*-dynamical systems and $\rho:\mathcal{B}\to\mathcal{C}$ a $X$-morphism. Then $Ker(\rho)$ is a stable ideal and $Ker(\rho_\rtimes)\cong[Ker(\rho)\rtimes X]$.*

We also have

**Proposition 2.8.** *Let $(\mathcal{C}_a,\tau_a,X)_{a\in A}$ a family of C*-dynamical systems. Let us form the natural direct product C*-dynamical system $\left(\prod_{a\in A}\mathcal{C}_a,\prod\tau_a,X\right)$, where $\left(\prod\tau\right)_x[(S_a)_{a\in A}] := (\tau_{a,x}[S_a])_{a\in A}$. Then the C*-algebras $\left(\prod_{a\in A}\mathcal{C}_a\right)\rtimes_{\prod\tau_a}X$ and $\prod_{a\in A}(\mathcal{C}_a\rtimes_{\tau_a}X)$ are canonically isomorphic.*

**[2.3] Topological dynamical systems.**

In the case of abelian C*-algebras there is a bijection between C*-dynamical systems and the notion defined (in a generality suited to our purposes) in

**Definition 2.9.** *A* topological dynamical system $(Y, \alpha, X)$ *is composed of a locally compact space* $Y$*, a locally compact group* $X$ *and an action of* $X$ *on* $Y$ *by homeomorphisms such that the application* $\alpha : X \times Y \to Y$ *is continuous.*

A theorem of Ellis (see [6]) asserts that if $\alpha$ is separately continuous it is also (jointly) continuous. By viewing the abelian C*-algebra as the set of functions of class $C_0$ on its spectrum (Gelfand theory), the correspondence stated above becomes obvious.

A particular case will be of special interest. Assume that the locally compact group $X$ is abelian and consider the action of $X$ on itself by left translations. Let $\mathcal{A}$ be a C*-subalgebra of $BC_u(X)$ containing $C_0(X)$, which is stable under translations, i.e. if $\varphi \in \mathcal{A}$, then $\tau_x(\varphi) := \varphi(\cdot + x) \in \mathcal{A}$ for all $x \in X$. Then $(\mathcal{A}, \tau, X)$ is a C*-dynamical system. The spectrum $\hat{X}$ of $\mathcal{A}$ is a *compactification* of $X$ (a compact space in which $X$ is dense), the action of $X$ on itself extends continuously to an action $\hat{\alpha}$ on $\hat{X}$ and $(\hat{X}, \hat{\alpha}, X)$ is the underlying topological dynamical system.

Very often it is better to have a represented version of the crossed product $\mathcal{A} \rtimes_\tau X$. Assume that $X$ is abelian. We endow its dual $X^\sharp$ (the space of all continuous morphisms : $X \to \mathbb{T} := \{z \in \mathbb{C} \mid \mid z \mid = 1\}$) with the standard structure; it will also be an abelian, locally compact group. Consider the Hilbert space $\mathcal{H} := L^2(X)$. Any element $\varphi \in \mathcal{A}$ may be viewed as a multiplication operator in $\mathcal{H}$, denoted $\varphi(Q)$. For every $\psi \in C_0(X^\sharp)$ one defines $\psi(Q^\sharp)$ the corresponding multiplication operator in $\mathcal{H}^\sharp := L^2(X^\sharp)$. By setting $\mathcal{F} : \mathcal{H} \to \mathcal{H}^\sharp$ the Fourier transform, we can define $\psi(P) := \mathcal{F}^* \psi(Q^\sharp)\mathcal{F}$. Both $\varphi(Q)$ and $\psi(P)$ are bounded, normal operators in $\mathcal{H}$. A quite natural object is $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$, the closed vector space generated by products $\varphi(Q)\psi(P)$, where $\varphi \in \mathcal{A}$ and $\psi \in C_0(X^\sharp)$. It can be shown that it coincides with $\langle C_0(X^\sharp) \cdot \mathcal{A} \rangle$ (obvious notation) and that it is a C*-subalgebra of $B(\mathcal{H})$. Now the really useful fact is that the crossed product $\mathcal{A} \rtimes X$ admits a faithful representation in $\mathcal{H}$, whose image is exactly $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$. More precisely, consider an element $\varphi$ in $\mathcal{A}$ and an element $\psi$ in $C_0(X^\sharp)$ with Fourier transform $\psi^\sharp$ in $L^1(X)$. The function given by $(\varphi \odot \psi^\sharp)(x) := \psi^\sharp(x)\varphi$ is an element of $L^1(X; \mathcal{A})$, hence of $\mathcal{A} \rtimes X$. Then the image of $\varphi \odot \psi^\sharp$ under the representation is $\varphi(Q)\psi(P)$. There will be a certain tendancy to identify the C*-algebras $\mathcal{A} \rtimes X$ and $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$. Both points of view are important; the crossed product has excelent functorial properties, while the amalgamated form $\langle \mathcal{A} \cdot C_0(X^\sharp) \rangle$ is more expressive with respect to affiliated observables.

Let us recall, finally, that $K(\mathcal{H}) = \langle C_0(X) \cdot C_0(X^\sharp) \rangle$. This entitles us to conclude that $K(\mathcal{H}) \cong C_0(X) \rtimes X$.

## 3. A class of compactifications

We shall introduce here a certain class of compactifications $\hat{X}$ of a space $X$, defined by adding a compact piece $Z$ at infinity, much in the spirit of the Alexandrov compactification, which is recovered when $Z$ is a singleton. The compactification will have the form $\hat{X} = X \sqcup Z$, with a topology defined suitably in terms of the topologies on $X$ and $Z$. The way we define $\hat{X}$ in [**3.1**] will be as simple as possible, quite close to an approach of [10]. In particular, we avoid using ultrafilters. The elements of $C(\hat{X})$ will admit a transparent interpretation as functions on $X$, in terms of their asymptotic behaviour; see [**3.2**]. Until now we assumed only that $X$ is a locally compact, non-compact space. In [**3.3**] we also assume it to be an abelian group and extend to $\hat{X}$ the action of $X$ on itself.

**[3.1] The construction of the compactification.**
Let us consider a locally compact, non-compact space $X$. Motivated by the Appendix, we shall define (in a direct manner) a particular type of compactification $\hat{X}$ of $X$, by "adding the compact piece $Z$ at infinity". One cand find in [10] a similar construction for the case $X = [0, \infty)$.

**Definition 3.1.** *A triplet* $(X_\infty, \zeta, Z)$ *will be called* a compactifying triplet *for* $X$ *if*

*(i)* $X_\infty$ *is a closed subset of* $X$ *with relatively compact complement,*

*(ii)* $Z$ *is a compact space,*

*(iii)* $\zeta : X_\infty \to Z$ *is a continuous function such that the following condition is satisfied:*

*[C]* $\zeta^{-1}(\mathcal{Z})$ *is not relatively compact in* $X_\infty$ *for any non-void open set* $\mathcal{Z} \subset Z$.

Let us call $\Omega \subset X_\infty$ *a large (sub)set* if it has a compact complement. Condition [C] says exactly that inverse images by $\zeta$ of open, non-void sets $\mathcal{Z}$ meet any large subset of $X_\infty$ (however "small" would it be). The algebraic interpretation of [C] is given by

**Lemma 3.2.** *Assume that* $\zeta(X_\infty)$ *is dense in* $Z$. *Let us set* $C(Z) \circ \zeta := \{\phi \circ \zeta \mid \phi \in C(Z)\}$ *(it is a C*-subalgebra of* $BC(X_\infty)$*). Then condition [C] is equivalent to* $C_0(X_\infty) \cap [C(Z) \circ \zeta] = \{0\}$.

*Proof.* If the function $\phi \circ \zeta$ is in $C_0(X_\infty)$, it takes arbitrary small values on sufficiently remote large sets $\Omega$. By [C], $\zeta(\Omega)$ is dense in $Z$ and this implies that $\phi$ is null.

Reciprocally, let $\mathcal{Z} \subset Z$ a non-void, open set such that $\zeta^{-1}(\mathcal{Z})$ is relatively compact. By Urysohn's lemma, there exists a non-null function $\phi$ in $C(Z)$ such that $\phi|_{\mathcal{Z}^c} = 0$. Then $\phi \circ \zeta$ is a non-trivial function taking the value 0 on $\zeta^{-1}(\mathcal{Z})^c$, i.e. a continuous function with compact support. ☐

We shall turn $\hat{X} := X \sqcup Z$ into a compactification of $X$ by using the topology of $X$, the inverse image by $\zeta$ of the topology of $Z$ and the large sets. For $\mathcal{Z} \subset Z$ open and $\Omega \subset X_\infty$ large, we set

$$\mathcal{O}_{\mathcal{Z},\Omega} := \zeta^{-1}(\mathcal{Z}) \cap \Omega$$

and

$$\hat{\mathcal{O}}_{\mathcal{Z},\Omega} := \mathcal{O}_{\mathcal{Z},\Omega} \sqcup \mathcal{Z}.$$

If $\mathcal{Z} \neq \emptyset$ then $\mathcal{O}_{\mathcal{Z},\Omega} \neq \emptyset$; this is equivalent to [C]. Note that

$$\hat{\mathcal{O}}_{\mathcal{Z},\Omega} \cap \hat{\mathcal{O}}_{\mathcal{Z}',\Omega'} = \hat{\mathcal{O}}_{\mathcal{Z}\cap\mathcal{Z}',\Omega\cap\Omega'}. \tag{3-1}$$

Then, obviously, the open sets of $X$ together with the family $\{\hat{\mathcal{O}}_{\mathcal{Z},\Omega} \mid \mathcal{Z},\Omega\}$ form a basis for a topology on $\hat{X}$. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to $z\in Z$ iff $x_\lambda\to\infty$ (i.e. $x_\lambda$ leaves any compact set of $X$ for $\lambda$ large enough) and $\zeta(x_\lambda)\to z$ in $Z$. It follows that two compactifying triplets $(X^1_\infty,\zeta^1,Z)$ and $(X^2_\infty,\zeta^2,Z)$ will define the same topology on $\hat{X}=X\sqcup Z$ if and only if for every net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $X^1_\infty\cap X^2_\infty$ such that $x_\lambda\to\infty$ one has $\zeta^1(x_\lambda)\to z$ iff $\zeta^2(x_\lambda)\to z$.

**Proposition 3.3.** *$\hat{X}$ is a compactification of $X$.*

*Proof.* (i) *$\hat{X}$ is Hausdorff.* Two points of $X$ are separated by usual open sets of $X$. A point of $X$ is separated of any point of $Z$ by including it in an open subset of $X$ with compact closure. Two points of $Z$ are separated by $\hat{\mathcal{O}}$-sets because of (3-1).

(ii) $X$ *is dense in* $\hat{X}$ since any $\hat{\mathcal{O}}$-set has a non-trivial intersection with $X$: $\hat{\mathcal{O}}_{\mathcal{Z},\Omega}\cap X=\mathcal{O}_{\mathcal{Z},\Omega}\neq\emptyset$ if $\mathcal{Z}\neq\emptyset$, because of condition [C] (which is critical).

(iii) $\hat{X}$ *is compact.* Let $\mathfrak{c}$ be an open covering of the space $\hat{X}$. It is enough to consider the case $\mathfrak{c}=\{(\mathcal{D}_j)_{j\in J},(\hat{\mathcal{O}}_{\mathcal{Z}_k,\Omega_k})_{k\in K}\}$, where the $\mathcal{D}_j$'s are open sets in $X$. Then $\{\mathcal{Z}_k\}_{k\in K}$ is a covering of the compact space $Z$, so we can restrict to a finite one $\{\mathcal{Z}_k\}_{k\in K_0}$. Set $\Omega=\cap_{k\in K_0}\Omega_k$; it is a large set. Choose $\Omega'$ large, with closure contained in $\Omega$. Then $X\setminus\Omega'$ is compact, hence covered by $\{\mathcal{D}_j\}_{j\in J_0}$, where $J_0$ is finite. We show that $\{\mathcal{O}_{\mathcal{Z}_k,\Omega_k}\}_{k\in K_0}$ covers $\Omega$, hence $\{\hat{\mathcal{O}}_{\mathcal{Z}_k,,\Omega_k}\}_{k\in K_0}$ covers $\Omega\sqcup Z$. Let $x\in\Omega$ and set $z=\zeta(x)$. We choose $k\in K_0$ such that $z\in\mathcal{Z}_k$, i.e. $x\in\mathcal{O}_{\mathcal{Z}_k,\Omega_k}$. The proof is finished. $\square$

**[3.2] The C*-algebra $C(\hat{X})$.**

Let us preserve the notations above. We would like to present the C*-algebra of complex, continuous functions on $\hat{X}$ as an algebra of functions acting on $X$ only. For any $\hat{\varphi}\in C(\hat{X})$ we set $\varphi:=\hat{\varphi}|_X$ and $\phi:=\hat{\varphi}|_Z$. One also writes $\hat{\varphi}=\varphi\sqcup\phi$. Since $\hat{X}$ induces on $X$ and on $Z$ the usual topologies, $\varphi$ and $\phi$ are continuous. More generally, if $\varphi$ is a function on $X$ and $\phi$ is a function on $Z$, we denote by $\varphi\sqcup\phi$ the function on $\hat{X}$ which coincides with $\varphi$ on $X$ and with $\phi$ on $Z$.

**Proposition 3.4.** *Let $(\varphi,\phi)\in BC(X)\times C(Z)$. A necessary and sufficient condition for $\hat{\varphi}=\varphi\sqcup\phi$ to belong to $C(\hat{X})$ is that $\varphi|_{X_\infty}-\phi\circ\zeta\in C_0(X_\infty)$.*

*Proof.* Since $\varphi$ and $\phi$ are continuous, the condition $\varphi|_{X_\infty}-\phi\circ\zeta\in C_0(X_\infty)$ translates into

$$\forall\varepsilon>0,\ \exists\Omega\subset X_\infty \text{ large, s.t. if } x\in\Omega,\ \mid\varphi(x)-\phi[\zeta(x)]\mid<\varepsilon. \tag{3-2}$$

We take once again into account the continuity of $\varphi$ and $\phi$ as well as the fact that neighbourhood bases in $\hat{X}$ for the points in $Z$ may be constructed only with $\hat{\mathcal{O}}$-sets. Then the continuity of $\hat{\varphi}=\varphi\sqcup\phi$ in $z\in Z$ may be written:

$$\forall\delta>0,\ \exists\hat{\mathcal{O}}_{\mathcal{Z},\Omega} \text{ neighbourhood of } z \text{ in } \hat{X} \text{ s.t. if } x\in\mathcal{O}_{\mathcal{Z},\Omega},\ \mid\varphi(x)-\phi(z)\mid<\delta. \tag{3-3}$$

But one has

$$\mid\ |\varphi(x)-\phi[\zeta(x)]|-|\varphi(x)-\phi(z)|\ \mid\ \leq\ \mid\phi[\zeta(x)]-\phi(z)\mid.$$

The right-hand side can be made small if $\zeta(x)\in\mathcal{Z}$ and $\mathcal{Z}$ is chosen suitably, so we get easily that (3-2) is true iff (3-3) is true for any $z\in Z$. $\square$

An obvious consequence is

**Corollary 3.5.** *$C(\hat{X})$ is isomorphic (by restriction to $X$) with the C*-algebra $C^{\mathrm{asy}}_{C(Z)}(X)$ of all bounded, continuous functions $\varphi : X \to \mathbb{C}$ such that there exists $\phi \in C(Z)$ with*

$$\lim_{x\to\infty} \mid \varphi(x) - \phi[\zeta(x)] \mid = 0. \tag{3-4}$$

One might say that the functions in $C^{\mathrm{asy}}_{C(Z)}(X)$ *are asymptotically of class* $C(Z)$. When (3-4) is verified, we shall say that $\phi$ *is the asymptotic function for* $\varphi$. Its unicity follows already from the isomorphism above, since the continuous function $\hat{\varphi}$ is uniquely determined by the values taken on the dense set $X$ and $\phi = \hat{\varphi} \mid_Z$. For a direct proof one may use Lemma 3.2. It is obvious that any $\varphi \in C(Z)$ is the asymptotic function for some element of $C^{\mathrm{asy}}_{C(Z)}(X)$.

To summarize, the compactifying triplet $(X_\infty, \zeta, Z)$ led to a compactification $\hat{X}$ of the space $X$ and to two (related) extensions of $C(Z)$ by $C_0(X)$, i.e. two (related) short exact sequences:

$$0 \to C_0(X) \xrightarrow{i} C(\hat{X}) \xrightarrow{r} C(Z) \to 0$$

and

$$0 \to C_0(X) \xrightarrow{i'} C^{\mathrm{asy}}_{C(Z)}(X) \xrightarrow{r'} C(Z) \to 0.$$

Here $i$ is "extension by null values" (i.e. $i(\varphi) = \varphi \sqcup 0$), $r$ is "restriction to $Z$", $i'$ is the canonical injection and $r'(\varphi) \equiv \phi$ is the asymptotic function assigned to $\varphi$. Let us also denote by $\gamma : C(\hat{X}) \to C^{\mathrm{asy}}_{C(Z)}(X)$ the isomorphism of Corollary 3.5. Then plainly $\gamma \circ i = i'$ and $r' \circ \gamma = r$. This means exactly that the two extensions are isomorphic. When we can take $X_\infty = X$ they are split, since "composition with $\zeta$" becomes a morphism : $C(Z) \to C^{\mathrm{asy}}_{C(Z)}(X)$ and a right inverse for $r'$. The C*-subalgebra $C(Z) \circ \zeta$ is then a (non-canonical) supplement for the ideal $C_0(X)$. This case leads to "two-body type" algebras (and Hamiltonians) and it is rather trivial. This is how we may treat for instance Schrödinger operators with potentials approaching at infinity some (almost) periodic function.

**[3.3] A dynamical system.**

We tackle now the problem of defining a dynamical system on $\hat{X}$. We preserve the notations above, but also assume that $X$ is an abelian locally compact group. The action of $X$ on itself by left translations will be denoted by $\alpha$. There is also given a continuous action $\beta$ of $X$ on $Z$ by homeomorphisms. One defines the action $\hat{\alpha} \equiv \alpha \diamond \beta$ of $X$ on $\hat{X}$ by $\hat{\alpha}_x = \alpha_x \diamond \beta_x : \hat{X} \to \hat{X}$ for all $x \in X$, where $\alpha_x \diamond \beta_x$ coincides with $\alpha_x$ on $X$ and with $\beta_x$ on $Z$.

We would like to write down an explicit condition on $\alpha$, $\beta$ and $\zeta$ equivalent to the continuity of the map $\hat{\alpha} : X \times \hat{X} \to \hat{X}$. Let us recall that by Ellis's theorem one need only check separate continuity. We also take account of the continuity of the functions $\alpha$ and $\beta$ and the way we constructed the topology on $\hat{X}$. It follows easily that $\hat{\alpha}$ is continuous iff for any $x \in X$ and any net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $X$ such that $x_\lambda \to \infty$ and $\zeta(x_\lambda) \to z \in Z$ one has $\zeta(x + x_\lambda) \to \beta_x(z)$.

Let us denote by the same letter the actions on the C*-algebras. For example, $\alpha_x(\varphi) = \varphi \circ \alpha_{-x}$. We are in the framework of Proposition 2.6, since for all $x \in X$ the next diagram commutes:

$$\begin{array}{ccccc} C_0(X) & \xrightarrow{i} & C(\hat{X}) & \xrightarrow{r} & C(Z) \\ {\scriptstyle \alpha_x}\downarrow & & {\scriptstyle \hat{\alpha}_x}\downarrow & & {\scriptstyle \beta_x}\downarrow \\ C_0(X) & \xrightarrow{i} & C(\hat{X}) & \xrightarrow{r} & C(Z). \end{array}$$

(note that $\hat{\alpha}_x(\phi \sqcup \varphi) = \alpha_x(\phi) \sqcup \beta_x(\varphi)$). It follows by Corollary 2.7.(a) that

$$\left[C(\hat{X}) \rtimes X\right] \Big/ [C_0(X) \rtimes X] \cong C(Z) \rtimes X. \tag{3-5}$$

In the same way we get

$$\left[C^{\rm asy}_{C(Z)}(X) \rtimes X\right] \Big/ [C_0(X) \rtimes X] \cong C(Z) \rtimes X. \tag{3-6}$$

For brievty we used similar notations, although the $X$-actions and the isomorphisms are different (but connected). This should not cause confusion.

## 4. Potentials with anisotropic almost periodic behaviour

Results as (3-5) and (3-6) constitute only a first step towards an efficient formula for the essential spectrum. As explained in [**2.1**], if $H$ is an observable affiliated to $C^{\rm asy}_{C(Z)}(X) \rtimes X$, then its essential spectrum equals the spectrum of its image in $C(Z) \rtimes X$, which is usually a complicated object. We shall put in evidence a situation in which it can be "decomposed". This decomposition comes from writting $Z$ as a cartesian product $S \times T$. $S$ will be a sphere situated at infinity and this is explained in [**4.1**]. We denote by $T$ the Bohr group, coinciding with the spectrum of the algebra of continuous, almost periodic functions on $X$; this is the topic of [**4.2**]. In [**4.3**] we put all these together and construct a rather rich dynamical system. Its C*-algebra is decomposed in [**4.4**]. As a direct consequence, we derive a formula for the essential spectrum in [**4.5**], which has Theorem 1.1 as a particular case.

### [4.1] Radial-type compactifications.

**Definition 4.1.** *Assume that $X$ is a locally compact, non-compact abelian group and $(X_\infty, \sigma, S)$ is a compactifying triplet as in Definition 3.1. We call $(X_\infty, \sigma, S)$ a* radial-type compactifying triplet *(=*RTCT*) if for any $x \in X$ and any net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $X$ such that $x_\lambda \to \infty$ and $\sigma(x_\lambda) \to s \in S$ one has $\sigma(x + x_\lambda) \to s$.*

Let us set $\tilde{X} := X \sqcup S$, structured as a compactification of $X$ as in [**3.1**]. We shall say that $\tilde{X}$ is *a radial-type compactification.* Obviously, Definition 4.1. was given exactly such that the usual action $\alpha$ of $X$ on itself extends in a trivial way to $\tilde{X}$. All the points of $S$ will be fixed points for the action $\beta = id$ and $\alpha \diamond \beta$ is continuous.

**Remark.** It is clear that the Alexandrov (one-point) compactification is radial-type by our definition. Unfortunately, the results obtained in [**4.3**] by using it would be modest. For a general abelian, locally compact group I don't know if there are others or when there is a maximal one. But at least if $X$ is compactly generated, there is a convenient construction, given below. We remark that in fact there exists a largest compactification of an arbitrary abelian, locally compact, non-compact group $X$ such that all its points not belonging to $X$ are fixed by translations. It is the spectrum of the C*-algebra consisting of continuous functions on $X$ with vanishing asymptotic oscillations. Unfortunately, it seems that it can't be obtained by means of a compactifying triplet, hence the present formalism doesn't apply. We intend to come back to this topic in a future publication.

Recall that a locally compact, abelian group is called *compactly generated* if it posseses a compact subset $K$ such that the smallest closed subgroup containing $K$ is $X$ itself. A well-known structure result (see [8], Theorem 9.8.) asserts that such a group is isomorphic to $\mathbb{R}^k \times \mathbb{Z}^l \times G$, where $k, l \in \mathbb{N}$ and $G$ is a compact, abelian group. Planely, it is enough to find a RTCT for $\mathbb{R}^k \times \mathbb{Z}^l \times G$. Let us set $S := S^{k+l}$ (the unit sphere in $\mathbb{R}^{k+l}$) and $\sigma(\xi, n, g) := \frac{(\xi,n)}{|(\xi,n)|}$ for $\xi \in \mathbb{R}^k$, $n \in \mathbb{Z}^l$, $g \in G$ and (let's say) $|\ (\xi, n)\ | \geq 1$.

**Proposition 4.2.** *We defined a RTCT.*

*Proof.* (a) The set $X_\infty := \left\{(\xi, n, g) \in \mathbb{R}^k \times \mathbb{Z}^l \times G \mid |\ (\xi, n)\ | \geq 1\right\}$ is closed, with relatively compact complement and $\sigma : X_\infty \to S$ is continuous.

(b) Let $\mathcal{S}$ be a non-void, open subset of $S$. Then $\sigma^{-1}(\mathcal{S}) = \left[(\mathbb{R}^k \times \mathbb{Z}^l) \cap C_{\mathcal{S}}\right] \times G$, where $C_{\mathcal{S}}$ is a truncated, non-void, open cone in $\mathbb{R}^{k+l}$. Obviously the set $\sigma^{-1}(\mathcal{S})$ is non-void and stable under multiplication in the first two variables by any strictly positive integer. Terefore it is not relatively compact and the condition $[C]$ of Definition 3.1 is verified.

(c) We consider a point $(\eta, g)$ and a net $\{(\eta_\lambda, g_\lambda)\}_{\lambda\in\Lambda}$ in $X_\infty$ such that $|\ \eta_\lambda\ | \to \infty$ in $\mathbb{R}^k \times \mathbb{Z}^l$ and $\frac{\eta_\lambda}{|\eta_\lambda|} \to s \in S$. Then one will also have $\frac{\eta_\lambda+\eta}{|\eta_\lambda+\eta|} \to s$, because of the simple inequality $\left|\frac{\eta_\lambda+\eta}{|\eta_\lambda+\eta|} - \frac{\eta_\lambda}{|\eta_\lambda|}\right| \leq \frac{2|\eta|}{|\eta_\lambda+\eta|}$. The compactifying triplet is radial-type. □

**[4.2] The Bohr group and the C*-algebra of almost periodic functions.**

We recall briefly the construction of the Bohr group of $X$, valid for any locally compact, abelian group $X$. For details we send to [8].

On the dual $X^\sharp$ of $X$ we consider the discrete topology. One gets a discrete, abelian group, denoted by $X^\sharp_d$. The dual group $(X^\sharp_d)^\sharp$ with the usual (dual) topology is then a compact group, denoted by $T$ and called *the Bohr group* of $X$. By standard duality arguments one can identify $X$ with a dense subgroup of $T$. More precisely, there exists a continuous morphism $\tau$ from $X$ to $T$ with dense image. We avoid to call $T$ the Bohr compactification of $X$ (as it is done occasionally), since the topology induced by $T$ on $X$ (called *the Bohr topology*) is much weaker than the initial one (i.e. the morphism $\tau : X \to T$ is not open). In particular, no neighbourhood of the origin in the Bohr topology is relatively compact with respect to the initial topology on $X$. In fact this follows as a special case of the point (a) of our Proposition 4.4.

So $X$ will be identified with a dense subgroup of $T$. We denote by $\gamma$ the action of $X$ on $T$ obtained by restricting to $X \times T$ the group low of $T$; it extends the action

$\alpha$ of $X$ on itself. It is plain that $(T,\gamma,X)$ is a *minimal dynamical system*, i.e. the orbit $\{\gamma_x(t) \mid x \in X\}$ is dense in $T$ for all $t \in T$. In fact it can be shown that it is the largest minimal dynamical system with group $X$ for which the space is also a group.

The next classical result is a particular case of [5] 16.2.1.

**Proposition 4.3.** *For a continuous function $\psi : X \to \mathbb{C}$, the next assertions are equivalent:*

*(a) $\psi$ extends to a continuous function on $T$.*

*(b) The set $\{\psi(\cdot + y) \mid y \in X\}$ is relatively compact in $BC(X)$.*

*(c) For any $\varepsilon > 0$, there exists a trigonometric polynomial $p$ on $X$ such that $\| \psi - p \| < \varepsilon$.*

We recall that a trigonometric polynomial is just a finite linear combination of characters of $X$. A function $\psi$ satisfying the equivalent conditions of Proposition 4.1. is called *almost periodic*. The set of all almost periodic functions on $X$ is a C*-subalgebra of $BC_u(X)$, denoted $AP(X)$, canonically isomorphic to $C(T)$. All the continuous functions : $X \to \mathbb{C}$ which are periodic with respect to some closed subgroup $\Gamma$ of $X$ with compact quotient $X/\Gamma$ lie in $AP(X)$, but there are many others.

**[4.3] A dynamical system constructed on $Z = S \times T$.**

We combine now the two preceeding paragraphs to construct a rather simple dynamical system. We fix an abelian, locally compact, non-compact group $X$ and we denote by $\alpha$ the canonical action on itself by translations. $(X_\infty,\sigma,S)$ will be a RTCT as in [**4.1**] and $\tau : X \to T$ the canonical morphism of $X$ into the Bohr group $T$, described in [**4.2**]. We recall that $X$ acts trivially on $S$ and that $(T,\gamma,X)$ is a minimal dynamical system ($\gamma$ is induced by the group law of $T$). The fact that $\tau$ is a group morphism implies that for all $x \in X$

$$\tau \circ \alpha_x = \gamma_x \circ \tau. \tag{4-1}$$

We set $Z := S \times T$ and $\zeta := (\sigma,\tau) : X_\infty \to Z$, $\zeta(x) := (\sigma(x),\tau(x))$. Finally, setting $\hat{X} := X \sqcup Z$ we define $\hat{\alpha} : X \times \hat{X} \to \hat{X}$ as $\hat{\alpha}_x := \alpha_x \diamond \beta_x$ for $x \in X$; by a previous notation $\alpha_x \diamond \beta_x$ coincides with $\alpha_x$ on $X$ and with $\beta_x := id_S \times \gamma_x$ on $Z$.

**Proposition 4.4.** *(a) $(X_\infty,\zeta,Z)$ is a compactifying triplet for $X$.*

*(b) Let us consider on $\hat{X}$ the topological structure induced by $(X_\infty,\zeta,Z)$ as in* [**3.1**]. *Then $(\hat{X},\hat{\alpha},X)$ is a topological dynamical system.*

*Proof.*

(a) Obviously $Z$ is a compact set and $\zeta$ is a continuous function. Let us verify condition [C] from Definition 3.1. A moment of reflection shows that this is not a trivial matter.

It is enough to show that for $\mathcal{S} \subset S$ and $\mathcal{T} \subset T$ open, non-void sets, $\zeta^{-1}(\mathcal{S}\times\mathcal{T}) = \sigma^{-1}(\mathcal{S}) \cap \tau^{-1}(\mathcal{T})$ is not relatively compact.

We note first that the minimality of the action $\gamma$ implies that $T = \cup_{x\in X}\gamma_x^{-1}(\mathcal{T})$. Since $T$ is compact, there exist $x_1,...,x_m \in X$ such that $T = \cup_{j=1}^m \gamma_{x_j}^{-1}(\mathcal{T})$. Since $(X_\infty,\sigma,S)$ is a RTCT, the action $\tilde{\alpha} : X \times \tilde{X} \to \tilde{X} := X \sqcup S$ obtained by extending

$\alpha$ trivially on $S$ is continuous. By taking into account the way we defined the topology on $\tilde{X}$, it follows that there is a non-void, open set $\mathcal{S}' \subset S$ and a large set $\Omega \subset X_\infty$ such that for $j = 1, ..., m$:

$$\Omega \cap \sigma^{-1}(\mathcal{S}') \subset \Omega \cap \alpha_{x_j}^{-1}\left[\sigma^{-1}(\mathcal{S})\right]. \tag{4-2}$$

We have

$$\sigma^{-1}(\mathcal{S}') = \sigma^{-1}(\mathcal{S}') \cap \tau^{-1}(T) = \bigcup_{j=1}^{m}\left(\sigma^{-1}(\mathcal{S}') \cap \tau^{-1}\left[\gamma_{x_j}^{-1}(\mathcal{T})\right]\right).$$

But (4-1) implies that $\tau^{-1}\left[\gamma_{x_j}^{-1}(\mathcal{T})\right] = \alpha_{x_j}^{-1}\left[\tau^{-1}(\mathcal{T})\right]$, so we get

$$\sigma^{-1}(\mathcal{S}') = \bigcup_{j=1}^{m}\left(\sigma^{-1}(\mathcal{S}') \cap \alpha_{x_j}^{-1}\left[\tau^{-1}(\mathcal{T})\right]\right). \tag{4-3}$$

By combining (4-2) and (4-3) we obtain

$$\Omega \cap \sigma^{-1}(\mathcal{S}') \subset \Omega \cap \left(\bigcup_{j=1}^{m} \alpha_{x_j}^{-1}\left[\zeta^{-1}(\mathcal{S} \times \mathcal{T})\right]\right). \tag{4-4}$$

Suppose now that $\zeta^{-1}(\mathcal{S} \times \mathcal{T})$ is relatively compact. Then the right-hand side in (4-4) will also be relatively compact, hence the same will be true for $\sigma^{-1}(\mathcal{S}') = \left[\Omega \cap \sigma^{-1}(\mathcal{S}')\right] \cup \left[\Omega^c \cap \sigma^{-1}(\mathcal{S}')\right]$. This is forbidden by condition [C] written for the compactifying triplet $(X_\infty, \sigma, S)$.

(b) The conditions $\hat{\alpha}_{x+y} = \hat{\alpha}_x \circ \hat{\alpha}_y$ and $\hat{\alpha}_0 = id_{\hat{X}}$ are readily verified. By Ellis' theorem, to get the continuity of $\hat{\alpha}$ it is enough to check that $X \ni x \mapsto \hat{\alpha}_x(\hat{x})$ and $\hat{X} \ni \hat{x} \mapsto \hat{\alpha}_x(\hat{x})$ are continuous. The first one is obvious. Since $\alpha$ and $\gamma$ are continuous, the single critical fact is to check that if $\{x_\lambda\}_{\lambda\in\Lambda}$ is a net in $X$ such that $x_\lambda \to \infty$ and $(\sigma(x_\lambda), \tau(x_\lambda)) \to (s,t) \in S \times T$, then $(\sigma(x_\lambda + x), \tau(x_\lambda + x)) \to \beta_x(s,t) = (s, \gamma_x(t))$. By using (4-1) and the definition of a RTCT this follows immediately. $\square$

**[4.4] A decomposition result.**

We are placed in the framework presented at the beginning of [**4.3**]. By Corollary 3.5, we know that the C*-algebra $C(\hat{X})$ is canonically isomorphic with $C^{\text{asy}}_{C(S\times T)}(X)$, which is a translational invariant C*-subalgebra of $BC_u(X)$.

**Remark** By a straightforward application of the Stone-Weierstrass Theorem one may show that $C^{\text{asy}}_{C(S\times T)}(X)$ is isomorphic with the C*-subalgebra of $BC(X)$ generated by the union $C^{\text{asy}}_{C(S)}(X) \cup AP(X)$. We shall not make use of this fact.

**Theorem 4.5.** *We have a natural monomorphism:*

$$\left[C^{\text{asy}}_{C(S\times T)}(X) \rtimes X\right] \Big/ \left[C_0(X) \rtimes X\right] \hookrightarrow \prod_{s\in S} AP(X) \rtimes X.$$

*Proof.* For any $\varphi \in C^{\rm asy}_{C(S\times T)}(X)$ we denote by $\phi \in C(S\times T)$ its asymptotic function. For all $s \in S$ we set $\phi_s : T \to \mathbb{C}$, $\phi_s(t) := \phi(s,t)$ and interpret $\phi_s$ as an almost periodic function on $X$. Then $p : C^{\rm asy}_{C(S\times T)}(X) \to \prod_{s\in S} AP(X)$ given by $p[\varphi] := (\phi_s)_{s\in S}$ is a (non-surjective) *-morphism with kernel $C_0(X)$.

In fact, $p$ is even a $X$-morphism if we endow $\prod_{s\in S} AP(X)$ with the natural direct sum action. It follows that $p_{\rtimes} : C^{\rm asy}_{C(S\times T)}(X) \rtimes X \to \left[\prod_{s\in S} AP(X)\right] \rtimes X$ is also a *-morphism and (by Corollary 2.7.(b)) that its kernel is $C_0(X) \rtimes X$. Then we distribute the direct sum with respect to the crossed product by Proposition 2.8. $\square$

We have given the proof above because it gives the most transparent description of the monomorphism. Other approaches are also available. For instance, one may start with (3-6) and write $C(Z) \cong C(S) \otimes C(T)$. By taking advantage of the fact that the action of $X$ on $S$ is trivial, it is easy to see that

$$[C(S) \otimes C(T)] \rtimes_{id\times\gamma} X \cong C(S) \otimes [C(T) \rtimes_\gamma X] \cong C\left[S; AP(X) \rtimes_\alpha X\right]$$

and this gives immediately the result, just by replacing the usual topology on $S$ by the discrete one.

**[4.5] The essential spectrum.**

**Theorem 4.6.** *Let $h : X^\sharp \to \mathbb{R}$ a continuous function diverging at infinity. Let $V : X \to \mathbb{R}$ continuous, such that there exists a function $W$ in $C(S) \otimes AP(X)$ such that*

$$\lim_{x\to\infty} |V(x) - W(\sigma(x), x)| = 0. \tag{4-5}$$

*Set $H := h(P)+V$ and $H_s := h(P)+W_s$, where for each $s \in S$, the almost periodic function $W_s$ on $X$ is given by $W_s(x) := W(s,x)$. Then $\boldsymbol{\sigma}_{\rm ess}(H) = \overline{\bigcup_{s\in S}\boldsymbol{\sigma}[H_s]}$.*

*Proof.* We begin by giving a "representated version" of Theorem 4.5. By using the discusion in **[2.3]**, one can reformulate: There exists a *-morphism

$$q_{\rtimes} : \left\langle C^{\rm asy}_{C(S\times T)}(X) \cdot C_0(X^\sharp)\right\rangle \to \prod_{s\in S} \left\langle AP(X) \cdot C_0(X^\sharp)\right\rangle$$

with kernel $\left\langle C_0(X) \cdot C_0(X^\sharp)\right\rangle = K(\mathcal{H})$, sending $\varphi(Q)\psi(P)$ into $(\phi_s(Q)\psi(P))_{s\in S}$. The meaning of $\phi$ and $\phi_s$ is the same as in the proof of Theorem 4.5.

Let us set $H_0 := h(P)$. Obviously $(H_0 - z)^{-1} = \psi(P)$ for some explicit $\psi \in C_0(X^\sharp)$. To show that $H$ is affiliated to $\left\langle C^{\rm asy}_{C(S\times T)}(X) \cdot C_0(X^\sharp)\right\rangle$, it is enough to show that $(H-z)^{-1}$ belongs to this C*-algebra for some $z$. We choose $z$ such that the Neumann series

$$(H-z)^{-1} = \sum_{m\in\mathbb{N}} (-1)^m (H_0 - z)^{-1}\left[V(H_0-z)^{-1}\right]^m$$

converges. Now the affiliation is evident.

By applying the morphism $q_{\rtimes}$ term by term, one also sees that $q_{\rtimes}[H] = (H_s)_{s\in S}$ (see Definition 2.2). Thus, by the discussion at the end of **[2.1]**, we conclude that $\boldsymbol{\sigma}_{\rm ess}(H) = \overline{\bigcup_{s\in S}\boldsymbol{\sigma}[H_s]}$. $\square$

Now one gets easily Theorem 1.1 just by letting $X = \mathbb{R}^n$ and by remarking that (1-2) is equivalent to (4-5). The case $X = \mathbb{Z}^n$ is also interesting in physics.

## Appendix

We give here an inductive justification of the strategy we followed in Section 3. The general theory of extensions of C*-algebras (for which we send to [3], [4] and [11]) will be invoked to get a feeling of how one may construct a C*-algebra when "its quotient by an ideal is predetermined". The main tool is the Busby invariant. Only the case when the ideal is essential is relevant. The general, non-commutative situation is very instructive, but by taking crossed products we can reduce ourselves to the abelian case. Then, at the level of the spectrum of the algebras, the problem of extension becomes a matter of compactification. A natural tentative to define "approximatively split" extensions leads to the condition $[C]$ of Definition 3.1.

[**A.1**] Let us consider extensions of C*-algebras, i.e. exact sequences

$$0 \to \mathfrak{A} \xrightarrow{i} \mathfrak{B} \xrightarrow{r} \mathfrak{C} \to 0.$$

Here $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are C*-algebras, $i$ is a monomorphism, $r$ an epimorphism and we have $i(\mathfrak{A}) = \mathrm{Ker}(r)$. We assume that $\mathfrak{B}$ (hence also $\mathfrak{C}$) has a unit.

Let us recall *the Busby invariant.* There is also a standard extension

$$0 \to \mathfrak{A} \xrightarrow{I} \mathcal{M}(\mathfrak{A}) \xrightarrow{R} \mathcal{Q}(\mathfrak{A}) \to 0,$$

where $\mathcal{M}(\mathfrak{A})$ is the multiplier algebra and $\mathcal{Q}(\mathfrak{A})$ the corona (=outer multiplier) algebra of $\mathfrak{A}$. By the universal property of $\mathcal{M}(\mathfrak{A})$, there is a unique morphism $L : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ such that $L \circ i = I$. Our first extension is essential (i.e. $i(\mathfrak{A})$ is an essential ideal in $\mathfrak{B}$) iff $L$ is injective. Let us set $M : \mathfrak{C} \to \mathcal{Q}(\mathfrak{A})$, $M(C) := R[L(B)]$ for any $B \in \mathfrak{B}$ such that $r(B) = C$. One can verify that $M$ is a (well-defined) morphism, called the Busby invariant. It is indeed invariant under isomorphisms of extensions and one can recover the (class of the) extension by a pull-back procedure. This allows identifying classes of extensions of $\mathfrak{A}$ by $\mathfrak{C}$ with morphisms from $\mathfrak{C}$ to the corona algebra of $\mathfrak{A}$. It is known that an extension is essential iff its Busby invariant is injective and this is the case we shall consider. It is also known that the extension is split ($r$ has a right inverse which is a morphism) iff $M$ can be lifted to a morphism $N : \mathfrak{C} \to \mathcal{M}(\mathfrak{A})$.

[**A.2**] We come now to our abelian framework and take $\mathfrak{A} = C_0(X)$ and $\mathfrak{C} = C(Z)$, where $X$ is locally compact but non-compact and $Z$ is a compact space. To produce an essential extension of $C_0(X)$ by $C(Z)$ means exactly to construct a compactification $\hat{X}$ of $X$, with $Z = \hat{X} \setminus X$. Let us denote by $\Sigma X$ the Stone-Čech compactification and by $\Delta X$ the corona set $\Sigma X \setminus X$. This is relevant, since $\mathcal{M}[C_0(X)]$ can be identified with $C(\Sigma X) = BC(X)$ and $\mathcal{Q}[C_0(X)]$ with $C(\Delta X)$. The topological counterpart of the Busby invariant is a continuous function $\mu : \Delta X \to Z$; $M$ will just be "composition with $\mu$". Since we want $M$ to be injective (in order to get an essential extension), $\mu$ need to have a dense image; in fact it will be automatically surjective, since $\Delta X$ and $Z$ are compact. Of course, $X$ and $Z$ being given, the existence of such a $\mu$ is not granted (think of connexity for example).

The space $\Delta X$ is usually extremely complicated, so we would like to infer $\mu$ from a function defined on a simpler space.

The simplest situation (still giving some interesting examples) is *the trivial (split) extension*, when $\mu$ extends to a continuous function $\nu : \Sigma X = X \sqcup \Delta X \to Z$ (at the algebraic level it will give the lift $N$). Of course, what we have to do is just construct a continuous function $\zeta : X \to Z$. The Stone-Čech compactification is born to allow $\zeta$ to extend to a continuous function $\nu$ as above. We will just have to be sure that $\mu = \nu|_Z$ is surjective, and this at least implies that $\zeta$ has a dense image. But such a global $\zeta$ doesn't work in rather simple situations which interest us. Think of $X$ as a finite-dimensional, euclidian space and of $Z$ as the boundary of its radial compactification or as a product of this boundary with some other compact space. Obviously, there is no continuous function $\zeta$ defined globally on $X$ (which is simply connected) with dense image in $Z$. But it can be constructed if some neighbourhood of the origin in $X$ is left apart (think of $x \mapsto x/|x|$).

So let $X_\infty$ be a closed subset of $X$ with relatively compact complement and $\zeta : X_\infty \to Z$ a continuous function. $\zeta$ will admit a continuous extension to $\Sigma X_\infty$, the Stone-Čech compactification of $X_\infty$. Since any element of $BC(X_\infty)$ can be extended to a function in $B(X)$, it is easy to show that $\Sigma X_\infty = X_\infty \sqcup \Delta X$. Hence, by restriction, we get a continuous function $\mu : \Delta X \to Z$ and this is exactly what we needed to define the compactification and the extension. Now recall that $\mu$ has to be surjective (equivalently, to have a dense image) in order to produce an essential extension. It is easy to show that *this happens exactly when for any non-void, open set* $\mathcal{Z} \subset Z$, $\zeta^{-1}(\mathcal{Z})$ *is not relatively compact in* $X_\infty$ and this is exactly the condition $[C]$ we imposed on $\zeta$ in Definition 3.1.

## References


1. W. Amrein, A. Boutet de Monvel and V. Georgescu, *$C_0$-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser-Verlag, Basel-Boston-Berlin, 1996.
2. W. Amrein, M. Mantoiu and R. Purice, *Localization Properties Associated with Certain Ideals in C*-Algebras*, in preparation.
3. B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, New-York, 1986.
4. R. C. Busby, *Double Centralizers and Extensions of C*-Algebras*, Trans. Amer. Math. Soc. **132** (1968), 79-99.
5. J. Dixmier, *Les C*-algèbres et leurs representations*, Gauthier-Villars, Paris, 1964.
6. R. Ellis, *Locally Compact Transformation Groups*, Duke Math. J. **24** (1957), 119-125.
7. V. Georgescu and A. Iftimovici, *C*-Algebras of Energy Observables: I. General Theory and Bumps Algebras*, available as preprint 00-520 at $http://www.ma.utexas.edu/mp_arc/$.
8. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, Grundlehren der Mathematischen Wissenschaften, vol.115, Springer-Verlag Berlin, 1979.
9. G. K. Pedersen, *C*-Algebras and Their Automorphism Groups*, Academic Press, London, 1979.
10. B. Simon, *Some Pictorial Compactifications of the Real Line*, Amer.Math. Monthly **76** (1969), 536-538.
11. N. E. Wegge-Olsen, *K-Theory and C*-algebras*, Oxford University Press, Oxford, 1993.


PRESENT ADDRESS: DÉPARTEMENT DE PHYSIQUE THÈORIQUE, UNIVERSITÉ DE GENÈVE; 32, BD. D'YVOY; CH-1211, GENÈVE 4, SWITZERLAND. E-MAIL: MARIUS.MANTOIU@PHYSICS.UNIGE.CH