Sharp upper bounds on the number of resonances near the real axis for trapped systems

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1 Introduction

This paper is devoted to a detailed study of the behavior of the resonances and resonant states near the real axis. We work mainly in the semi-classical setting but most results can be easily translated into the classical one. By resonances near the real axis we mean resonances in a "box" $\Omega(h) = [a_0, b_0] + i[-S(h), 0]$, where $0 < S(h) = O(h^K)$, $K \gg 1$. Such resonances may exist only for trapping geometries. We accept the convention here that resonances lie in the lower half-plane.

For simplicity, we consider compact perturbation $P(h)$ of the Laplacian only. The basic properties are established in the abstract "black box scattering" setting introduced by Sjöstrand and Zworski [SjZ] (see next section). It is well known that if $z(h)$ is a resonance, then there exists a $z(h)$-outgoing resonant state $u(h)$ satisfying the equation $(P(h) - z(h))u(h) = 0$. By [B1], [St3], if $-\operatorname{Im} z = O(h^\infty)$, then

$$u(h) = O(h^\infty) \quad \text{for } R_1 \le |x| \le R_2, \tag{1.1}$$

where $R_0 < R_1 < R_2$ are such that the scatterer is included in the ball $B(0, R_0)$. For simplicity, in this introduction we will assume $S(h) = O(h^\infty)$. This does not immediately imply that the same is true for the generalized "eigenfunctions", i.e., for the solutions of $(P(h) - z(h))^k u(h) = 0$ with some $k > 1$. We call those generalized "eigenfunctions" (with infinite energy) resonant states as well. Since $-\operatorname{Im} z = O(h^\infty)$, one can expect that

$$(P(h) - z(h))u(h) = O(h^\infty) \quad \text{for any resonant state.} \tag{1.2}$$

The estimate above has to be considered in the following sense — u is normalized in $B(0, R)$, $R > R_0$ and then the r.h.s. has to be $O(h^\infty)$ in the same ball. If one tries to carry out some recursive procedure for proving those two estimates, it is quite likely to get exponentially big (with respect to $1/h$) terms , because for the number of steps k we have $k = O(h^{-n^\#})$, $n^\# \ge n$, where $n \ge 2$ is the dimension. However obtaining such estimates even under the assumption that $m(h)$ is uniformly bounded does not follow easily from known results. The third question we are trying to answer is about the degree of linear independence of resonant states corresponding to resonances too close to each other, for example two resonances $z_1(h)$ and $z_2(h)$ in $\Omega(h)$ with $z_z(h) - z_2(h) = O(h^\infty)$. In other words, can we control how small is the angle between two (or more) resonant states $u_1(h)$, $u_2(h)$ corresponding to such resonances? If we can, then we would know that a linear combination of $u_1(h)$, $u_2(h)$ satisfying (1.1) would also satisfy (1.1) for any $z(h)$ that is $O(h^\infty)$ close to $z_1(h)$ and $z_2(h)$. This is also related to the stability of the property of linear independence of such resonant states and is crucial in providing link between resonances and eigenvalues of a reference operator below.

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Instead of working with single resonances, we work with clusters of resonances. We exploit the following argument: since the number of resonances in $\Omega(h)$ is $O(h^{-n^{\#}})$, we can always group the resonances in clusters of diameter $d(h) = O(h^{\infty})$ with distance between two clusters at least $ch^{n^{\#}}d(h)$ with some $c > 0$. Then we prove (see Proposition 3.3 and Theorem 3.3) estimates (1.1) and (1.2) for any linear combination of resonant states corresponding to resonances in such cluster. For technical reasons we work with the complex scaled operator $P_\theta(h)$. In Proposition 3.4 we give an affirmative answer to the third question above for resonant states associated with different clusters. Within a single cluster, we still cannot control the angle between resonant states corresponding to two different resonances in this cluster, but we can simply choose an orthonormal basis and our results show that we still get states satisfying (1.1) and (1.2). Note that in particular, even though we work with clusters, we get those estimates for a resonant state corresponding to any single resonance $z(h)$.

In Theorem 3.1 and the proof of Theorem 3.2 we show that existence of resonances in $\Omega(h)$ near the real axis implies existence of at least the same number of real quasimodes on an interval slightly wider than the projection of $\Omega(h)$ on the real axis. The corresponding quasimode states are linearly independent in a stable way under small perturbation and this allows us to prove in Theorem 3.2 that they generate at least the same number of eigenvalues of a suitably chosen reference operator $P^{\#}(h)$. This generalizes the result in [St4], where the number of quasimodes is not estimated. In some sense, this is result, converse to that in [St1], that says that existence of asymptotically orthogonal real quasimodes implies existence of at least the same number of resonances nearby. It is implicit in [St1] that the asymptotic orthogonality can be replaced by the condition of linear independence stable under certain small perturbations. By quasimodes states we mean approximate solutions with error $O(h^N)$, $N \gg 1$ or $N = \infty$ supported in a fixed compact.

In sections 4-6 we study a differential elliptic second order operator $P(h)$ that is a compact perturbation of $-h^2\Delta$ as before and satisfies the black box assumptions. Using (1.1) and propagation of singularities arguments, we show that the wave front set of any linear combination of resonant states as above is contained in the set $\mathcal{T}$ of trapped bicharacteristics of $P(h)$. In particular, this is true for any resonant state. The upper bound for $N(\Omega(h))$ established in Theorem 3.2 in terms of upper spectral bound for a self-adjoint reference operator $P^{\#}(h)$ says that the number of eigenvalues of $P^{\#}(h)$ in a small neighborhood of $[a_0, b_0]$ is at least the same as the number of resonances $N(\Omega(h))$ in $\Omega(h)$. This provides us with effective ways to get sharp upper bounds of the counting function $N(\Omega(h))$ by using suitably chosen reference operators. One obvious choice for $P^{\#}(h)$ is the Dirichlet realization of $P(h)$ in the ball $B(0, R)$, $R > R_0$ but it does not provide sharp bounds. We choose $P^{\#}(h)$ to be $P(h)$ plus additional h-ΨDO with principal symbol that vanishes near the trapped rays and increases quickly outside some small neighborhood of them. This allows us to use known spectral asymptotics for $P^{\#}(h)$ to get upper bound of the type

$$N(\Omega(h)) \leq (2\pi h)^{-n}\left(\operatorname{meas}\left(\mathcal{T} \cap p_0^{-1}[a_0 - 0, b_0 + 0]\right) + o(1)\right), \tag{1.3}$$

where p_0 is the principal symbol of $P(h)$, and $S(h) = h^K$, $K \gg 1$ (see Theorem 5.1). In one special case in section 6 we can actually prove an asymptotic formula for $N(\Omega(h))$. Let $P(h) = -h^2\Delta + V(x)$ and assume that $p^{-1}[a_0, b_0]$ has a bounded component, assume also that the unbounded one is non-trapping and a_0, b_0 are non-critical values of $p(h)$. Then the inequality above turns into asymptotics, in this case actually, $\mathcal{T} \cap p_0^{-1}[a_0, b_0]$ is the volume of the union of the compact components of $p_0^{-1}[a_0, b_0]$. To prove the asymptotic formula, we use the bound (1.3) and the fact that the eigenfunctions of a suitable reference operator serve as quasimodes for $P(h)$, therefore we get a sharp lower bound for $N(\Omega(h))$ as well using the results in [St1]. We also prove that there is a resonance free strip $O(h^{\infty}) = S(h) \leq -\operatorname{Im} z \leq Mh$, $\forall M > 0$, $0 < h \leq h(M)$, $a_0 \leq \operatorname{Re} z \leq b_0$. This is done in Theorem 6.1.

We notice that the fact that the counting function of the resonances (not only near the real line) is essentially bounded by the spectral counting function of certain reference operator, which is $P(h)$, modified for $|x| > R_0$, has been proven by Sjöstrand and Zworski [SjZ]. Sjöstrand [Sj1], under certain assumptions involving hyperbolicity of the bicharacteristic flow, showed that the resonances in a box of height δ, $C_0 h \leq \delta \leq 1/C_0$ is $O(\delta^d h^{-n})$, where d is the Minkowski codimension of the trapped rays. Numerical study of this and other phenomena can be found in [LZ]. M. Zerzeri [Ze] obtained in the classical case an upper bound in a sector in $\mathbf{C}$ related to the measure of the trapped rays for the case of classical scattering but the notion of trapped rays that he uses is weaker than the common one, in particular, $-\Delta + V(x)$ with $0 \neq V \in C_0^\infty$ can be trapping, also a non-trapping kidney-shaped domain is also trapping according to that definition. Lower bounds of the type ch^{-n} near the real axis can be obtained any time we have asymptotically orthogonal quasimodes with the same density [St1]. Such bound in terms of the measure of the periodic trajectories is proven in [PZ] with different methods. In obstacle classical scattering under the assumption of existence of elliptic degenerate periodic ray, one has lower bound cr^n with c equal to the measure of invariant tori up to a constant factor. One can interpret this measure as the measure of a subset of the trapped rays near the elliptic one and this well corresponds to the classical version of (1.3).

Resonances connected to potential well, considered in section 6 have been studied extensively, mainly for energy levels close to a non-degenerate minimum of $V(x)$ (see e.g. ... [**?**]...). If the potential is real analytic, the convergence to the real axis is exponentially fast [CP]. In section 6 we consider however energy levels not necessarily close to the bottom of the well.

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2 Assumptions, Black-Box Scattering

We work in the general framework of *black-box scattering* proposed by Sjöstrand and Zworski [SjZ] (see also [Sj2], [TZ1]). In fact, we consider only compactly supported perturbations. Let $\mathcal{H}$ be a complex Hilbert space of the form

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0,R_0)),$$

where $R_0 > 0$ is fixed and $B(0,R_0)$ is the ball centered at the origin with radius R_0. We think of $\mathcal{H}_{R_0}$ as the space where the scatterer is supported. We consider a family of self-adjoint unbounded operators $P(h)$ in $\mathcal{H}$ with common domain $\mathcal{D}$, whose projection onto $L^2(\mathbf{R}^n \setminus B(0,R_0))$ is $H^2(\mathbf{R}^n \setminus B(0,R_0))$. Denote by $\mathbf{1}_{B(0,R_0)}$ the orthogonal projector onto $\mathcal{H}_{R_0}$ and similarly, we define $\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)}$. Then we assume that

$$\mathbf{1}_{B(0,R_0)} \left(P(h)+i\right)^{-1} : \mathcal{H} \to \mathcal{H}$$

is compact. Outside $\mathcal{H}_{R_0}$, $P(h)$ is assumed to coincide with the semiclassical Laplacian, i.e.,

$$\mathbf{1}_{B(0,R_0)} P(h) u = -h^2 \Delta \left(u|_{\mathbf{R}^n\setminus B(0,R_0)}\right)$$

Finally, we assume that $P(h) > -C_0$, $C_0 > 0$. Under those assumptions, one can define (the semi-classical) resonances $\mathrm{Res}P(h)$ of $P(h)$ in conic neighborhoods of the real axis by the method of complex scaling (see [SjZ] and the above cited papers). An outline of the complex scaling technique is given below. Resonances are also poles of the meromorphic continuation of the resolvent $(P(h)-z)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\, z > 0$ into a conic neighborhood of the real line. We will denote the so continued resolvent by $R(z,h)$. In this paper we adopt the convention that resonances lie in the lower half-plane $\mathrm{Im}\, z < 0$. In classical scattering, we consider P as above independent of h by formally assuming that $h = 1$. Then P has classical resonances $\mathrm{Res}P$ defined as the poles of the meromorphic continuation of the resolvent $(P-\lambda^2)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$ from $\mathrm{Im}\,\lambda > 0$ to a neighborhood of the real line. For such P we then set $P(h) = h^2 P$ and define resonances $z(h)$ as above. Then the semi-classical resonances and the classical ones are related by $\lambda^2 = h^{-2} z$.

As in [SjZ], we construct a reference selfadjoint operator $P^\#(h)$ from $P(h)$ on $\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0,R_0)$, where $M = (\mathbf{R}/R\mathbf{Z})^n$ for some $R > R_0$. Then for the number of eigenvalues of $P^\#$ in a given interval $[-\lambda,\lambda]$, we assume

$$\#\{z \in \mathrm{Spec}P^\#(h);\ -\lambda \le z \le \lambda;\ \} \le C(\lambda/h^2)^{n^\#/2}, \quad \lambda \ge 1,$$

with some $n^\# \ge n$. This implies (see [SjZ] and [Sj2]) that

$$\#\{z \in \mathrm{Res}P(h);\ 0 < a_0 \le \mathrm{Re}\, z \le b_0;\ 0 \le -\mathrm{Im}\, z \le c_0\} \quad \le \quad C(a_0,b_0,c_0)h^{-n^\#}, \tag{2.1}$$

$$\#\{\lambda \in \mathrm{Res}P;\ 1 \le |\lambda| \le r;\ 0 \le -\mathrm{Im}\,\lambda \le 1\} \quad \le \quad Cr^{n^\#}, \quad r > 1. \tag{2.2}$$

Polynomial estimates of this type have been proved also in [M], [Z1], [SjZ], [V], [Sj2].

In this paper we will often omit the dependence on h, i.e., we will write P instead of $P(h)$, z_0 instead of $z_0(h)$ where it is clear from the context that we work with h-dependent objects. We denote by C various positive constants, that may change from line to line.

For any resonance $z(h)$ there is an outgoing solution $u(h)$ to $(P(h)-z(h))u(h)=0$ and possibly "generalized eigenvectors" $v(h)$ satisfying $(P(h)-z(h))^{k(h)}v(h)=0$. We will call u and v *resonant states.* Given $\Omega(h)\subset \mathbf{C}$, $N(\Omega(h))$ will denote the number of resonances in $\Omega(h)$ counted with their multiplicities defined as the rank of the residue of the cut-off resolvent at any resonance. Given a self-adjoint *reference operator* $P^{\#}(h)$ (this notion is defined later) with point spectrum, $N^{\#}([a,b])$ denotes the number of eigenvalues, counting multiplicities, of $P^{\#}(h)$ in the interval $[a,b]$. With some abuse of notation and terminology, given $\chi\in C_0^\infty(\mathbf{R}^n)$, we will denote the operator $\mathbf{1}_{B(0,R_0)}\oplus \chi|_{\mathbf{R}^n\setminus B(0,R)}$ by χ. If $\operatorname{supp}\big(\mathbf{1}_{\mathbf{R}^n\setminus B(0,R_0)}f\big)\subset K$, where $K\supset B(0,R_0)$, we will say that $\operatorname{supp} f\subset K$. By $H^s(\mathbf{R}^n)$, $s=0,1,\ldots$, we denote the Sobolev space with semiclassical norm $\|f\|^2_{H^s}=\sum_{|\alpha|\le s}\|(hD)^\alpha f\|^2$

3 Estimates on the resonant states

In this section we obtain estimates on the resonant states in the semiclassical case corresponding to resonances $z(h)$ in a box $\Omega(h)=[a(h),b(h)]+i[-c(h),0]$, where $0<b(h)-a(h)=O(h^N)$, $0<c(h)=O(h^N)$, $N\gg 1$. Our goal is to prove that the resonant states are essentially supported near the scatterer and solve $(P(h)-z_0)f=O(h^{N_1})$, where $z_0\in[a(h),b(h)]$, where $N_1\le N$ depends on N. As explained in the Introduction, instead of studying single resonances, we study clusters of them close to each other and the resonances in domains of the type $\Omega(h)$ will be considered later as such clusters. We prove those estimates for each linear combination of resonant states corresponding to resonances in $\Omega(h)$. In particular, this allows us to choose asymptotically orthogonal system of such linear combinations that form quasimode states for $P(h)$ with quasimode $z_0(h)$ of multiplicity equal (at least) to the total multiplicity of $\mathrm{Res}P(h)\cap\Omega(h)$. At the end of this section we study wider domains, where $a(h)=a_0$, $b(h)=b_0$ are independent of h by grouping them in clusters contained in "small" domains $\Omega_k(h)$ as above.

3.1 Brief review of Complex Scaling

We follow here [SjZ] and [Sj2]. Fix $A<B$, such that

$$R_0+1<A<B-1.$$

We will perform complex scaling for $r:=|x|>B$. Choose a real-valued increasing C^∞- function $\kappa(r)$, $r\ge 0$, with the properties:

(i) $\kappa(r)=0$ for $0\le r\le B$,
(ii) $\kappa(r)=1$ for $r\ge T_0$, with some $T_0>B$,
(iii) $0\le\kappa(r)\le 1$,
(iv) $\kappa(r)=e^{-1/(r-B)^2}$ for $B\le r\le B+\epsilon_0$ with some $\epsilon_0\ll 1$.

We would like to note that another choice in (iv) of $\kappa(r)$ near $r=B$ might influence the error term in Proposition 3.1 below. Set $\theta=\theta(r):=\theta_0\kappa(r)$, where $0<\theta_0\ll 1$ will be chosen later. Define $f_\theta(r):=re^{i\theta(r)}$. Note that for $\theta_0>0$ small enough, $df_\theta/dr\neq 0$. As in [SjZ], [Sj2], we perform the analytic dilation by considering the map

$$\mathbf{R}^n\ni x=r\omega\mapsto f_\theta(r)\omega\in\mathbf{C}^n,\quad \omega\in S^{n-1}. \tag{3.1}$$

Under the action of the map (3.1), the operator P is transformed into an operator P_θ on $\mathcal{H}_{R_0}\oplus L^2(\Gamma_\theta\setminus B(0,R_0))$, where Γ_θ is the image of (3.1). We refer to the above mentioned papers for details. We will identify Γ_θ with $\mathbf{R}^n$ (in other words, we parameterize Γ_θ by r and ω). Then inside the ball $B(0,B):=\{x;\ |x|<B\}$ the operator P_θ coincides with P, while outside that ball in polar coordinates we have

$$P_\theta=P(f_\theta(r),(f'_\theta(r))^{-1}D_r,D_\omega), \tag{3.2}$$

where $P(r, D_r, D_\omega)$ is the semiclassical symbol of P in polar coordinates. Since $P = -h^2\Delta$ for $|x| > R_0$, equation (3.2) implies that

$$P_\theta|_{r>R_0} = \left(\frac{1}{f_\theta'}D_r\right)^2 - \frac{n-1}{f_\theta f_\theta'}iD_r + \frac{1}{f_\theta^2}D_\omega^2. \tag{3.3}$$

The operator P_θ is elliptic, closed in $\mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$ which we identify with $\mathcal{H}$ with domain $\mathcal{D}_\theta$ that is actually the same as the domain of P after that identification. It is known that for a fixed $h > 0$ and $z \neq 0$ with $\arg(z) \neq -2\theta_0$, the operator P_θ is Fredholm with index 0. Moreover, for $z \neq 0$ with $-\arg(z) < 2\theta_0$, we have that z is a resonance of P if and only if z is an eigenvalue of P_θ and the multiplicities coincide. Since we are going to work with resonances with $\operatorname{Im} z = o(1)$, for h small enough those resonances will be always eigenvalues of P_θ.

3.2 An absorption estimate, after N. Burq [B2]

Next proposition is a refinement of [B2, Prop. 6.1]. Let $\rho(r)$ be a smooth function equal to 1 for $r < A - 1/2$ and equal to $r^{(n-1)/2}$ for $r > A$. Set $\tilde{P} := \rho P \rho^{-1}$. Then $\tilde{P}$ is self-adjoint for the measure $d\mu := \rho^{-2} r^{n-1} dr d\omega$ and we denote by $\tilde{\mathcal{H}} = \rho\mathcal{H}$ the corresponding Hilbert space. Furthermore,

$$\tilde{P}|_{r>A} = h^2\Big(-\partial_r^2 - \frac{\Delta_w}{r^2} + \frac{(n-1)(n-3)}{4r^2}\Big), \quad d\mu|_{r>A} = dr d\omega.$$

Let $\tilde{P}_\theta$ be the operator obtained from $\tilde{P}$ by analytic dilation for $r \geq B$ and denote by $\tilde{\mathcal{D}}$ its domain. In fact, $\tilde{P}_\theta = \rho(f_\theta) P_\theta \rho^{-1}(f_\theta)$. Here, for n even, the branch of $f_\theta^{1/2}$ is chosen in an obvious way. Fix $a_0 > 0$.

Proposition 3.1 *For* $h > 0$ *and* $\theta_0 > 0$ *small enough,* $\operatorname{Re} z \geq a_0$, $\operatorname{Im} z \leq 0$, *and for any* $u \in \tilde{D}_\theta$ *we have*

$$\begin{aligned} C \int \big((\theta + r\theta')|h\partial_r u|^2 + \theta(|hr^{-1}\nabla_\omega u|^2 + |u|^2)\big) dr d\omega & \\ \leq \; -\operatorname{Im} \left(e^{i\theta}(\tilde{P}_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} + \left(-\operatorname{Im} z + e^{-h^{-1/3}}\right) \|u\|_{\tilde{\mathcal{H}}}^2, & \end{aligned} \tag{3.4}$$

where $C = \min(a_0, 1)/2$ *and the inner product and the norm are taken in* $\tilde{\mathcal{H}}$.

Remark. It follows from the proof that $e^{-h^{-1/3}}\|u\|^2_{\tilde{\mathcal{H}}}$ can be replaced by $e^{-h^{-1/3}} \int_{B \leq r \leq B + ch^{1/6}} |u|^2 dr d\omega$ and that $e^{-h^{-1/3}}$ can be replaced by $e^{-h^{-2/3+\epsilon}}$, $\epsilon > 0$, by choosing $\kappa(r)$ near $r = B$ in a different way.

Proof. As mentioned earlier, we follow the proof of [B2, Prop. 6.1]. Write

$$\begin{aligned} -\operatorname{Im} \left(e^{i\theta}(\tilde{P}_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} &= -\operatorname{Im} \left(e^{i\theta}(\tilde{P}_\theta - \operatorname{Re} z)u, u\right)_{\tilde{\mathcal{H}}} - (-\operatorname{Im} z)(\cos\theta u, u)_{\tilde{\mathcal{H}}} \\ &\geq -\operatorname{Im} \left(e^{i\theta}(\tilde{P}_\theta - \operatorname{Re} z)u, u\right)_{\tilde{\mathcal{H}}} - (-\operatorname{Im} z)\|u\|^2_{\tilde{\mathcal{H}}}. \end{aligned} \tag{3.5}$$

Therefore, it is enough to prove the proposition for z real. So, let $z \geq a_0$. Note next that one can assume that u is supported in $r > A$, where in particular, $d\mu = dr d\omega$. Indeed, choose a smooth cut-off function $0 \leq \chi \leq 1$ such that $\chi(x) = 0$ for $|x| < A$ and $\chi(x) = 1$ for $|x| > A + 1/2$. Since $e^{i\theta}(\tilde{P}_\theta - z)$ is symmetric on $\operatorname{supp}(1-\chi)$, writing $u = \chi u + (1-\chi)u$ we can see as in [B2] that $(1-\chi)u$ contributes nothing to $\operatorname{Im}(e^{i\theta}(\tilde{P}_\theta - z)u, u)_{\tilde{\mathcal{H}}}$, so we may replace u by χu there.

For $e^{i\theta}\tilde{P}_\theta$ we have

$$e^{i\theta}\tilde{P}_\theta = -\frac{1}{1 + ir\theta'} h\partial_r \frac{e^{-i\theta}}{1 + ir\theta'} h\partial_r - e^{-i\theta}\frac{h^2\Delta_\omega}{r^2} + e^{-i\theta}h^2\frac{(n-1)(n-3)}{4r^2}.$$

Integrating by parts we get

$$\begin{aligned}-\mathrm{Im}\,\Big(e^{i\theta}(\tilde{P}_\theta - z)u,u\Big)_{\tilde{\mathcal{H}}} &= \int\Big(\mathrm{Im}\Big(-\frac{e^{-i\theta}}{(1+ir\theta')^2}\Big)|h\partial_r u|^2+\sin\theta|hr^{-1}\nabla_\omega u|^2\Big)drd\omega \\ &\quad+\int\mathrm{Im}\,\Big(e^{i\theta}z-e^{-i\theta}h^2\frac{(n-1)(n-3)}{4r^2}\Big)|u|^2drd\omega - h\mathrm{Im}\,(g(r)h\partial_r u,u) \\ &= I_1+I_2+I_3\end{aligned} \tag{3.6}$$

with

$$g(r)=\frac{d}{dr}\Big(\frac{1}{1+ir\theta'}\Big)\frac{e^{-i\theta}}{1+ir\theta'}=\frac{-i(r\theta''+\theta')e^{-i\theta}}{(1+ir\theta')^3}. \tag{3.7}$$

It is easy to see that if $\theta_0>0$ is small enough we have

$$I_1\geq\frac{3}{4}\int\left((\theta+2r\theta')|h\partial_r u|^2+\theta|hr^{-1}\nabla_\omega u|^2\right)drd\omega. \tag{3.8}$$

For I_2 we get for h small enough,

$$I_2\geq z\int\sin\theta|u|^2drd\omega-Ch^2\int\theta|u|^2drd\omega\geq\frac{3}{4}a_0\int\theta|u|^2drd\omega \tag{3.9}$$

For I_3 we obtain

$$I_3=-\frac{h}{i}((\mathrm{Re}\,g)h\partial_r u,u)_{\tilde{\mathcal{H}}}-\frac{h^2}{2i}(\mathrm{Re}\,g'u,u)_{\tilde{\mathcal{H}}}+\frac{h^2}{2}(\mathrm{Im}\,g'u,u)_{\tilde{\mathcal{H}}}. \tag{3.10}$$

Since I_3 is real, we have

$$I_3=-\mathrm{Im}\,h((\mathrm{Re}\,g)h\partial_r u,u)_{\tilde{\mathcal{H}}}+\frac{h^2}{2}(\mathrm{Im}\,g'u,u)_{\tilde{\mathcal{H}}}=I_3^{(1)}+I_3^{(2)}. \tag{3.11}$$

The function g admits the following estimates

$$\begin{aligned}|\mathrm{Re}\,g| &\leq C(|\theta'|+|\theta''|)(|\theta|+|\theta'|)\leq C|\theta|, &(3.12)\\ |g'| &\leq C(|\theta'|+|\theta''|+|\theta'''|), &(3.13)\end{aligned}$$

The second estimate (3.13) and the first part of (3.12) follow directly from (3.7). The second part of (3.12) holds trivially for $A<r<B$, where $\theta=0$ and for $B+\epsilon_0<r$, where $|\kappa|>1/C$, and is also true for $B<r<B+\varepsilon_0$, where $\theta=\theta_0e^{-1/(r-B)}$. Now (3.12) implies that $I_3^{(1)}$ in (3.10) can be estimated by

$$|I_3^{(1)}|\leq Ch\int\theta(|h\partial_r u|^2+|u|^2)drd\omega \tag{3.14}$$

and for $h\ll1$ this can be absorbed by the r.h.s. of (3.8) and (3.9). Next, to estimate $I_3^{(2)}$, we will show that for $h\ll1$

$$|g'|\leq e^{-h^{-1/3}}+h^{-3/2}\theta,\quad \text{for } B\leq r. \tag{3.15}$$

Set $t=r-B\geq0$. Then, since $\theta=\theta_0e^{-1/t^2}$ for $0\leq t\ll1$, (3.13) implies that for $0\leq t^2\leq h^{1/3}/2$ we have $|g'|\leq Ct^{-9}e^{-1/t^2}\leq e^{-1/(2t^2)}\leq e^{-h^{-1/3}}$ for $h\ll1$. On the other hand, for $h^{1/3}/2\leq t^2\leq\epsilon_0^2$ we have $|g'|\leq Ct^{-9}e^{-1/t^2}\leq Ch^{-3/2}\theta$. Since (3.15) is trivially true for $r>B+\varepsilon_0$, this proves it for all $r\geq B$. Therefore,

$$|I_3^{(2)}|\leq e^{-h^{-1/3}}\|u\|^2_{\tilde{\mathcal{H}}}+h^{1/2}\int\theta|u|^2drd\omega \tag{3.16}$$

and the integral above can be absorbed by the r.h.s. of (3.9). Combining (3.6) with (3.8), (3.9), (3.10), (3.14) and (3.16), we complete the proof. □

Remark. Estimate (3.4) remains true for $\mathrm{Im}\, z > 0$ if we replace $-\mathrm{Im}\, z$ there by $-\cos\theta_0 \mathrm{Im}\, z$ (see (3.5)). In particular, this implies that for $\theta_0 \ll 1$ and any $u \in \tilde{\mathcal{D}}$

$$\frac{1}{2}\left(\mathrm{Im}\, z - e^{-h^{-1/3}}\right) \|u\|^2_{\tilde{\mathcal{H}}} \leq -\mathrm{Im}\, \left(e^{i\theta}(\tilde{P}_\theta - z)u, u\right)_{\tilde{\mathcal{H}}} \leq \|(\tilde{P}_\theta - z)u\|_{\tilde{\mathcal{H}}} \|u\|_{\tilde{\mathcal{H}}}$$

and after replacing u by ρu, we get

$$\|(P_\theta - z)^{-1}\| \leq \frac{2}{\mathrm{Im}\, z - e^{-h^{-1/3}}}, \quad \mathrm{Im}\, z > e^{-h^{-1/3}}. \tag{3.17}$$

3.3 Estimates on the spectral projector

Fix $0 < a_0 < b_0$. Choose some $a(h)$, $b(h)$ and $c(h)$ such that

$$0 < a_0 \leq a(h) \leq b(h) \leq b_0, \quad b(h) - a(h) = o(1), \quad 2e^{-h^{-1/3}} \leq c(h) \leq Ch^{2n^\# + 1}. \tag{3.18}$$

and let

$$\Omega(h) := [a(h), b(h)] + i[-c(h), 0]. \tag{3.19}$$

Let $z_1(h), \dots, z_p(h)$ be all distinct resonances in $\Omega(h)$ with multiplicities $m_1(h), \dots, m_p(h)$. Set $m(h) := m_1 + \ldots + m_p$. Assume that there are no resonances on $\partial\Omega$. Consider the spectral projector associated with the eigenvalues of P_θ in Ω

$$\Pi_\Omega := \frac{1}{2\pi i} \oint_{\partial\Omega} (z - P_\theta)^{-1} dz,$$

where $\partial\Omega$ is assumed to be positively oriented. Denote $\mathcal{H}_\Omega := \mathrm{Ran}\Pi_\Omega$. Then it is well known (see e.g. [K]) that P_θ acts invariantly on $\mathcal{H}_\Omega$ that is the span of all eigenvectors and generalized eigenvectors of P_θ with eigenvalues in Ω. Generalized eigenvectors corresponding to distinct eigenvalues in Ω are linearly independent. The dimension $m(h)$ of $\mathcal{H}_\Omega$ is finite, bounded by $Ch^{-n^\#}$, and equal to the sum of the multiplicities of $z_j \in \Omega$. Set $P_\Omega := P_\theta|_{\mathcal{H}_\Omega}$. Then P_Ω is a finite rank operator (matrix) and we denote by $\|\cdot\|_\Omega$ the operator norm in $\mathcal{H}_\Omega$. The spectrum of P_Ω consists of $\{z_1, \dots, z_p\}$ with the same multiplicities. The following estimate is due to Zworski [Z2] (in this generality, see the proof of Lemma 1 in [TZ1])

$$\|(z - P_\theta)^{-1}\| \leq Ce^{Ch^{-n^\#}\ln(1/g)} \quad \text{for } z \in \Omega_0,\ \mathrm{dist}(z, \mathrm{Res}P(h) \cap \Omega_0) \geq g(h),\ g(h) \ll 1,$$

where Ω_0 is any simply connected precompact subset of $-\pi < -\arg z < 2\theta_0$ (independent of h). In our analysis we always work in the box $\Omega_0 := [a_0, b_0] + i[-c_0, c_0]$ with fixed $0 < a_0 < b_0$ and fixed $0 < c_0 \ll 1$, therefore the constant C above will be uniform. As a consequence, the resolvent of P_Ω satisfies the following estimate

$$\|(z - P_\Omega)^{-1}\|_\Omega \leq Ce^{Ch^{-n^\#}\ln(1/g)} \quad \text{for } z \in \Omega_0,\ \mathrm{dist}(z, \mathrm{Res}P(h) \cap \Omega_0) \geq g(h),\ g(h) \ll 1. \tag{3.20}$$

This allows us to apply the "semiclassical maximum principle" ([TZ1], [TZ2]) as in [St3, Lemma 2] to get the following.

Proposition 3.2 *Assume that $c(h) \leq S(h) \leq h^{(3n^\# + 1)/2} w(h)$, $w(h) = o(1)$, as $h \to 0$, where $a(h)$, $b(h)$ and $c(h)$ are as in (3.18). Then*

$$\|(z - P_\Omega)^{-1}\|_\Omega \leq \frac{C}{S(h)} \quad \text{on } \partial\tilde{\Omega},$$

where $\tilde{\Omega} := [a(h) - w(h), b(h) + w(h)] + [-h^{-n^\#} S(h), S(h)]$

Sketch of the Proof: We follow closely the proof of [St3, Lemma 2]. Let

$$\tilde{z}_j(h) := \bar{z}_j(h) + 2iS(h), \quad j = 1 \ldots p,$$

where the bar denotes complex conjugate. Then z_j and $\tilde{z}_j$ are symmetric about the line $\operatorname{Im} z = S(h)$ and on that line we have $\|(z - P_\Omega)^{-1}\|_\Omega \le 4/S(h)$ by (3.18) and (3.17) . Set

$$G(z, h) := \frac{(z - z_1)^{m_1} \ldots (z - z_p)^{m_p}}{(z - \tilde{z}_1)^{m_1} \ldots (z - \tilde{z}_p)^{m_p}}.$$

We observe first that

$$|G(z, h)| \le 1 \quad \text{for } \operatorname{Im} z \le S(h). \tag{3.21}$$

The function $F := G(z - P_\Omega)^{-1}$ is holomorphic below the line $\operatorname{Im} z = S(h)$, and in particular in $\Omega(h)$. Our goal is to apply the "semiclassical maximum principle" [TZ1] in the form presented in [St3, Lemma 1] to the function F in the domain $\Omega_1 := [a(h) - 5w(h), b(h) + 5w(h)] + i[-S(h)h^{-2n^\# - 1}, S(h)]$. To this end, we need to modify $w(h)$ and $S(h)$ to be sure that F satisfies an exponential estimate in this region with a constant independent of the region. The only obstacle to that would be existence of resonances too close to the boundary. To this end we extend Ω_1 by shifting the sides, staying in the fixed h-independent neighborhood Ω_0 of $\Omega(h)$, such that the closest resonance stays at distance at least $g(h) = h^{n^\# + 1}$. This is possible in view of (2.1). Then we apply (3.20) with $\ln(1/g) = (n^\# + 1)\ln(1/h)$ and using (3.21), we see that $\|F\| = O(\exp(Ch^{-n}\ln h^{-1}))$ on the boundary of the extended domain. By the maximum principle, this is true inside it, and in particular in Ω_1. Now we are in position to apply [St3, Lemma 1]. Since by (3.17), $\|F\| \le 4/S(h)$ on the upper part of Ω_1,we deduce that for h small enough

$$\|G(z)(z - P_\Omega)^{-1}\| \le 2e^3/S(h), \quad \forall z \in \tilde{\Omega}(h). \tag{3.22}$$

Next step is to show that

$$1/C \le |G(z, h)| \quad \text{on } \partial\tilde{\Omega}(h). \tag{3.23}$$

It is enough to estimate $(z - \tilde{z}_j)/(z - z_j)$ on $\partial\tilde{\Omega}(h)$. Observe first that $|z_j - \tilde{z}_j| \le 4S(h)$, $\forall j$. Next, the distance from each z_j from the three sides $\operatorname{Im} z = -S(h)h^{-n^\#}$, $\operatorname{Re} z = a - w$, $\operatorname{Re} z = b + w$ of $\tilde{\Omega}$ is bounded below by $S(h)h^{-n^\#}/2$ for $h \ll 1$. Therefore,

$$\left|\frac{z - \tilde{z}_j}{z - z_j} - 1\right| = \left|\frac{z_j - \tilde{z}_j}{z - z_j}\right| \le \frac{4S(h)}{S(h)h^{-n^\#}/2} = 8h^{n^\#}, \quad \forall z \in \partial\tilde{\Omega}(h) \setminus \{\operatorname{Im} z = S(h)\}$$

This yields

$$\left|\frac{z - \tilde{z}_j}{z - z_j}\right|^{m_j} \le (1 + 8h^{n^\#})^{m_j}, \quad \forall z \in \partial\tilde{\Omega}(h) \setminus \{\operatorname{Im} z = S(h)\} \tag{3.24}$$

On the fourth side $\operatorname{Im} z = S(h)$ of $\partial\tilde{\Omega}$ we have $|(z - \tilde{z}_j)/(z - z_j)| = 1$, thus (3.24) is trivially true there. Since $(1 + x)^{1/x} < e$, $0 < x < \infty$, we get

$$|1/G(z, h)| \le (1 + 8h^{n^\#})^{m_1 + \ldots + m_p} = (1 + 8h^{n^\#})^m \le (1 + 8h^{n^\#})^{Ch^{-n^\#}} \le e^{8C}.$$

This proves (3.23). Estimates (3.22) and (3.23) together imply the proof of the proposition. □

This proposition allows us to estimate $\|P_\Omega - z_0\|_\Omega$ for $z_0 \in [a(h), b(h)]$. We have

$$z_0 - P_\Omega = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}} (z_0 - P_\Omega)(z - P_\Omega)^{-1} dz = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}} (z_0 - z)(z - P_\Omega)^{-1} dz,$$

therefore,

$$\|z_0 - P_\Omega\|_\Omega \le \frac{|\partial\tilde{\Omega}|}{2\pi}|z_0 - z|\|(z - P_\Omega)^{-1}\|_\Omega \le C\frac{b - a + w}{2\pi}(b - a + w)\frac{1}{S} = C\frac{(b - a + w)^2}{S}. \tag{3.25}$$

Choosing $w(h) = h^{-(3n^{\#}+1)/2}S(h)$, estimate (3.25) implies the following

$$\|(P_\theta - z_0)f\| \le C\left(\frac{(b(h)-a(h))^2}{S(h)} + h^{-3n^{\#}-1}S(h)\right)\|f\|, \quad \forall f \in \mathrm{Ran}\Pi_\Omega. \tag{3.26}$$

If $c(h)h^{-(3n^{\#}+1)/2} \le b(h) - a(h) = o(1)$, then we choose $S(h) = h^{(3n^{\#}+1)/2}(b(h) - a(h))$ (then $S(h) \ge c(h)$ as required). If $b(h) - a(h) \le c(h)h^{-(3n^{\#}+1)/2}$, then we set $S(h) = c(h)$. This choice of $S(h)$ implies the following.

Proposition 3.3 *Let* Ω *and* Π_Ω *be as above. Then for* $z_0 \in [a(h), b(h)]$ *we have*

$$\|(P_\theta - z_0)f\| \le Ch^{-(3n^{\#}+1)/2}\max\{b(h) - a(h), c(h)\}\|f\|, \quad \forall f \in \mathrm{Ran}\Pi_\Omega. \tag{3.27}$$

In particular, we get the following.

Corollary 3.1 *Let* $z_0(h)$ *be a resonance with* $0 < a_0 \le \mathrm{Re}\, z_0(h) \le b_0 < \infty$, $-\mathrm{Im}\, z_0(h) = o(1)h^{(3n^{\#}+1)/2}$, *and let* $f = f(h)$ *be any generalized eigenfunction of* $P_\theta(h)$ *corresponding to the eigenvalue* $z_0(h)$ *(a function such that* $(P_\theta - z_0)^k f = 0$ *for some* $k \ge 0$*). Then*

$$\|(P_\theta(h) - z_0(h))f\| \le Ch^{-(3n^{\#}+1)/2}\max\{-\mathrm{Im}\, z_0(h), e^{-h^{-1/3}}\}\|f\|.$$

Note that the r.h.s. in (3.27), measuring the "error", is "small" only if the width of $\Omega(h)$ does not exceed h^N, $N \gg (3n^{\#}+1)/2$. This does not allow us to control the linear independence of the generalized eigenfunctions under small perturbation by integration by parts as in Proposition 3.5 below in wider domains, for example, if $a(h)$ and $b(h)$ are independent of h. Next proposition plays a crucial role in proving that resonances in "wide" boxes generate at least as many eigenvalues of the reference operator nearby. It states that the spectral projectors Π_Ω related to suitably chosen clusters of resonances contained in those boxes are polynomially bounded when restricted to the generalized eigenfunctions corresponding to eigenvalues in the "wide" box, see (3.30) below. Under the additional assumption of "well separated" resonances [TZ2] or, more generally, if we assume the existence of a resonance-free strip $c(h) \le -\mathrm{Im}\, z \le h^{-2n^{\#}-1}c(h)$ below $\Omega(h)$ [St3], then the spectral projectors Π_Ω are polynomially bounded on the whole space $\mathcal{H}$. In general, however, we do not know whether this is true, but fortunately, the proposition below is all we need for our purposes later.

Let

$$a_0 \le a(h) < b(h) \le b_0, \quad 2e^{-h^{-1/3}} \le c(h) \le o(1)h^{(5n^{\#}+1)/2} \tag{3.28}$$

(without the requirement that $b(h) - a(h) = O(h^N)$, $N \gg 1$). Let $\Omega(h)$ be as in (3.19). Assume that there are no resonances on $\partial\Omega(h)$. A direct consequence from (2.1) is that one can group the resonances in $\Omega(h)$ into clusters contained in the interiors of the boxes

$$\Omega_k(h) = [a_k(h), b_k(h)] + i[-c(h), 0], \quad k = 1, \ldots, K(h),\ K = O(h^{-n^{\#}}), \tag{3.29}$$

where $\Omega_k(h)$ do not intersect, moreover, $\mathrm{dist}\{\Omega_k, \Omega_m\} \ge 4w(h)$ for $k \ne m$ and $\mathrm{width}(\Omega_k) = b_k - a_k \le Ch^{-n^{\#}}w(h)$, where $0 < w(h) = o(1)h^{n^{\#}}$ is fixed in advance. There are no resonances in Ω outside Ω_k's. Denote as before by Π_{Ω_k} the spectral projectors related to the eigenvalues of P_θ in Ω_k and let P_Ω, $\mathcal{H}_\Omega$ be as before.

We know that the subspaces $\mathrm{Ran}\Omega_k(h)$ are linearly independent. The following proposition basically gives us control over the lower bound of the angles between them.

Proposition 3.4 *Under the assumptions above, if* $w(h) = h^{-(3n^{\#}+1)/2}c(h)$, *then there exists a constant* $A = A(a_0, b_0)$, *such that*

$$\|\Pi_{\Omega_k}\|_{\mathcal{H}_\Omega} \le Ah^{-(5n^{\#}+1)/2}, \quad k = 1, \ldots, K. \tag{3.30}$$

For any $f_k \in \mathrm{Ran}\Omega_k(h)$, $k = 1, \ldots, K$, *and for any* k_0 *we have*

$$\|f_{k_0}\| \le Ah^{-(5n^{\#}+1)/2}\|f_1 + \ldots + f_K\|,$$

Proof. Following the proof of Proposition 3.2, we get that

$$\|(z-P_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \leq \frac{C}{c(h)} \quad \text{on } \partial\tilde{\Omega}_k(h),\ \forall k, \tag{3.31}$$

where $\tilde{\Omega}_k(h) := [a_k(h)-w(h), b_k(h)+w(h)] + i[-h^{-n^\#}c(h), c(h)]$. Note that $\tilde{\Omega}_k(h)$ have the same properties as $\Omega_k(h)$ concerning the distance between two such domains and their widths, with $w(h)$ replaced by $w(h)/2$. To justify (3.31), it is enough to note that in the proof of Proposition 3.2 we used the fact that there are no poles of $(z-P_\Omega)^{-1}$ above $\operatorname{Im} z = c(h)$ only, and the fact that there might be poles to the left or right of Ω_k does not play any role as far as those poles are separated by distance $2w(h)$ (see also [St3]). Notice also that the constant C in (3.31) is independent of k. Since there are no eigenvalues of P_Ω in $\tilde{\Omega}_k \setminus \Omega_k$, one can define $\Pi_{\Omega_k}|_{\mathcal{H}_\Omega}$ as integrals of $(z-P_\Omega)^{-1}$ over $\partial\tilde{\Omega}_k$. A direct estimation of that integral, using (3.31), yields the proof of the first part.

To prove the second part, write

$$f_{k_0} = \Pi_{\Omega_{k_0}}(f_1 + \ldots + f_K)$$

and use the estimate on $\Pi_{\Omega_{k_0}}$. □

3.4 From resonances to quasimodes

We are ready now to formulate and prove some consequences of the estimates proven so far. The first one, roughly speaking, says that if P has m resonances in Ω, then P has m real quasimodes q_j in $[a(h), b(h)]$ with compactly supported asymptotically orthogonal quasimode states. Note that this theorem is in some sense converse to [St1, Theorem 1] that states that locally existence of quasimodes implies existence of resonances nearby.

Theorem 3.1 *Let*

$$0 < a_0 \leq a(h), \quad 0 < b(h)-a(h) = o(1)h^{n^\#/2+1}, \quad 2e^{-h^{-1/3}} \leq c(h) \leq Ch^{2n^\#+1}.$$

and set

$$\Omega(h) = [a(h), b(h)] + i[-c(h), 0].$$

Suppose that $P(h)$ has $m(h) = N(\Omega(h))$ resonances (counting multiplicities) in $\Omega(h)$. Fix $z_0(h) \in [a(h), b(h)]$. Then z_0 is a quasimode of multiplicity $m(h)$ for $P(h)$, i.e., there exist $m(h)$ quasimode states ψ_j, $j = 1, \ldots, m(h)$, such that for all $i, j = 1, \ldots, m(h)$ we have

(a) $\operatorname{supp}\psi_j \subset B(0, B+1)$, $\|\psi_j\| = 1$,

(b) $\|(P(h)-z_0(h))\psi_j\| \leq \epsilon(h)$,

(c) $|(\psi_i, \psi_j) - \delta_{ij}| \leq \epsilon(h)$,

where $\epsilon(h) = h^{-n^\#/2-1}\{\max\{b(h)-a(h), c(h)\}\}^{1/3}$.

Proof. Without loss of generality we can assume that there are no resonances on $\partial\Omega$. Since we have m resonances in Ω, the spectral projector Π_Ω has rank m. Choose m mutually orthogonal normalized functions f_j, $j = 1, \ldots, m$ in $\operatorname{Ran}\Pi_\Omega = \mathcal{H}_\Omega$. Our goal next is to cut off f_j by preserving the property $(P_\theta - z_0)f_j = O(h^N)$, $N \gg 1$, such that the new functions will form an orthonormal set up to a "small" error. To this end, we will use Proposition 3.1 that says that u is "small" where $-\operatorname{Im} z + e^{-h^{-1/3}} \ll \theta(r)$, so we will cut off for r such that this is true. This would give us compactly supported quasimodes for P_θ. On the other hand, we want them to be quasimodes for P as well, so we have to perform the cut-off in a set, where $\theta = O(h^N)$, $N \gg 1$, that would guarantee us that replacing P_θ by P leads to "small" error. This explains the choice of the cut off function below.

Choose $\phi \in C^\infty$, $0 \leq \phi \leq 1$ such that $\phi(s) = 1$ for $s \leq 1$ and $\phi(s) = 0$ for $s \geq 2$. Then $\chi(t) = \phi((t-t_1)/(t_2-t_1)+1)$ vanishes for $t \leq t_1$ and is equal to 1 for $t > t_2$, where $0 < t_1 < t_2$. Fix $\mu(h) \to 0$, as $h \to 0$ and let $t_1(h)$ and $t_2(h)$ be determined by the equations $\theta(B+t_1) = \mu(h)$, $\theta(B+t_2) = 2\mu(h)$.

Since $\theta(B+t) = e^{-1/t^2}$ for $0 \le t < \epsilon_0$ (with ϵ_0 independent of h), we have $t_1^{-2} = -\ln \mu$, $t_2^{-2} = -\ln 2\mu$ for $h \ll 1$. Then we define the cut-off function $\chi = \chi(t,h)$ as above. Elementary calculations imply that $(\ln 2/2)t_1^3 \le t_2 - t_1$. We have therefore arranged the properties

$$\begin{aligned} \chi(t,h) &= \begin{cases} 1 & \text{for } t \le t_1(h), \\ 0 & \text{for } t \ge t_2(h), \end{cases} \\ \frac{d^k}{dt^k}\chi(t,h) &\le C_k \left(\ln \frac{1}{\mu(h)}\right)^{3k/2}, \quad k = 0, 1, \dots, \\ \theta(t_1(h)) &= \mu(h), \quad \theta(t_2(h)) = 2\mu(h). \end{aligned} \tag{3.32}$$

Define

$$\psi_j = \chi(|x| - B, h) f_j. \tag{3.33}$$

We claim that ψ_j have the required properties. Set $\epsilon_1(h) := h^{-(3n^\#/2+1)} \max\{b(h) - a(h), c(h)\}$ (see (3.27)). First, observe that by Proposition 3.1 and Proposition 3.3,

$$\mu(h) \int_{t_1 \le |x| - B} \left(|h\nabla f_j|^2 + |f_j|^2\right) dx \le C\left(\epsilon_1(h) + e^{-h^{-1/3}}\right) \|f_j\|^2 \le C\epsilon_1(h), \quad \forall j = 1, \dots, m(h) \tag{3.34}$$

with $C > 0$ independent of j (and h). Next, $(P_\theta - z_0)\psi_j = [P_\theta, \chi] f_j + \chi(P_\theta - z_0) f_j$ and using (3.32) we get

$$\|[P_\theta, \chi] f_j\| \le C h \ln^3 \frac{1}{\mu} \left(\int_{t_1 \le |x| - B \le t_2} \left(|h\nabla f_j|^2 + |f_j|^2\right) dx \right)^{1/2} \le C h \left(\frac{\epsilon_1(h)}{\mu(h)/\ln^6(1/\mu)} \right)^{1/2}.$$

Estimate (3.34) also implies that $\|f_j - \psi_j\|_{H^1}$ can be similarly estimated by the r.h.s. above. Choose $\mu(h) > 0$, such that $\mu(h) \to 0$ and the r.h.s. above tends to 0, as $h \to 0$. Then we get

$$|(\psi_i, \psi_j) - \delta_{ij}| + \|(P_\theta - z_0)\psi_j\| \le C h \left(\frac{\epsilon_1(h)}{\mu(h)/\ln^6(1/\mu)} \right)^{1/2} + C\epsilon_1(h). \tag{3.35}$$

To pass from P_θ to P, observe that for $|x| \le B + t_2(h)$ the coefficients of $P_\theta - P$ are bounded by $C(\theta + \theta' + \theta'') \le C t^{-6}\theta$, $0 \le t = r - B \le t_2$. Arguing as in (3.15), we get that $C(\theta + \theta' + \theta'') \le e^{-h^{-1/3}} + C h^{-1}\theta$ for $h \ll 1$ and by (3.32), the latter can be estimated by $e^{-h^{-1/3}} + C h^{-1}\mu$ for $t \le B + t_2(h)$. Combining this with the ellipticity of P_θ and the fact that $P_\theta \psi_j = O(1)$, we see that $\|(P_\theta - P)\psi_j\| \le C(e^{-h^{-1/3}} + h^{-1}\mu)$. We will choose below $\mu > e^{-h^{-1/3}}$ so we may omit the exponential here. Set

$$\tilde\varepsilon(h) := h \left(\frac{\epsilon_1(h)}{\mu(h)/\ln^4(1/\mu)} \right)^{1/2} + \epsilon_1(h) + h^{-1}\mu(h). \tag{3.36}$$

We have therefore $\|(P - z_0)\psi_j\| \le C\tilde\varepsilon(h)$. By minimizing $\tilde\varepsilon(h)$ w.r.t. μ, we see that a convenient choice for μ is $\mu = h^{2/3}\epsilon_1^{1/3}$. This and (3.35) prove the theorem. □

Remark. It is worth noting that the proof also implies that

$$\|\psi_j - f_j\|_{H^1} \le \epsilon(h). \tag{3.37}$$

Theorem 3.1 implies immediately the following fact. Let $P^\#(h)$ be equal to $P(h)$ in $\mathcal{H}_{R_0} \oplus L^2(B(0,R) \setminus B(0,R_0))$ with Dirichlet boundary conditions on $\partial B(O,R)$, $R > R_0$. Then $P^\#(h)$ has at least $m(h) = N(\Omega(h))$ eigenvalues (counting multiplicities) in the interval $[a(h) - \delta(h), a(h) + \delta(h)]$, with $\delta(h) \ge C h^{-n^\#}\epsilon(h)$, $C \gg 1$, if $\epsilon(h) h^{-n^\#} \ll 1$ (see [L, Proposition 32.4]). This is useful, only if the width $b(h) - a(h)$ of $\Omega(h)$ is $O(h^N)$, $N \gg 1$. We will generalize this in two directions. First, we will consider more general *reference* *operators* than the Dirichlet realization of $P(h)$ in a large ball, and secondly, we will prove this property for larger domains of width that can be independent of h, for example.

Let $\Omega(h)$ and $\Omega_k(h)$ be as in Proposition 3.4. Apply Theorem 3.1 to each Ω_k. In view of the upper bound of $b_k(h)-a_k(h)$ that we have, we see that we can replace $\epsilon(h)$ with the function $\epsilon(h) = Ch^{-(8n^{\#}+7)/6}c^{1/3}(h)$, where C depends only on the constant $C(a_0, b_0, c_0)$ in (2.1). This gives us a system of ψ_{jk}, $k = 1, \dots, K(h)$, $j = 1, \dots, m_k(h)$ supported in $B(0, B+1)$, such that for each k it satisfies (a), (b), (c) in Theorem 3.1 with $z_0(h)$ there replaced by $z_k(h) \in [a_k(h), b_k(h)]$ and $\epsilon(h)$ as above.

Definition 3.1 *Let $\mathcal{H}^{\#}$ be a Hilbert space that can be expressed as $\mathcal{H}^{\#} = \mathcal{H}_{R_0} \oplus L^2(B(0,R)\backslash B(0,R_0)) \oplus \mathcal{H}_{\mathrm{ext}}$, $R > B+1$, with $\mathcal{H}_{\mathrm{ext}}$ another Hilbert space, and assume that $P^{\#}(h)$ is a selfadjoint operator in $\mathcal{H}^{\#}$ with discrete spectrum in a h-independent neighborhood of the interval $[a(h), b(h)]$, where $a_0 \le a(h) < b(h) \le b_0$. We call $P^{\#}(h)$ a reference operator for $P(h)$ in $\Omega(h)$ with discrepancy $\delta(h)$, if for some decomposition of $\Omega(h)$ as above one has $\|(P^{\#}(h) - z_k(h))\psi_{jk}(h)\| \le \delta(h) \to 0$, as $h \to 0$, where $z_k(h) \in \Omega_k(h) \cap \mathbf{R}$.*

We would like to note here that the requirements of the definition are fulfilled, if we assume that $\|(P^{\#}(h) - z_k(h))\psi_k(h)\| \le \delta(h) \to 0$ for any $f_k \in \mathrm{Ran}\Omega_k$, where ψ_k is related to f_k by (3.33).

The notion reference operator depends on the choice of the discrepancy function $\delta(h)$. The Dirichlet realization of $P(h)$ in a large ball, considered above, is an example of a reference operator with discrepancy function $\delta(h) = \epsilon(h)$. Clearly, we have a lot of freedom to choose the reference operator $P^{\#}$, for example one can impose other type of selfadjoint boundary conditions on $\partial B(0,R)$, or to extend $P(h)$ on a perturbed torus as in [SjZ]. The more complicated definition of reference operator that we give is justified by our desire later to obtain $P^{\#}$ from P not only by modifying it for large x but also by modifying it outside the wave front set of the resonant states.

Next theorem is a "global" version of Theorem 3.1, i.e., it applies to resonances in wider domains $\Omega(h)$.

Theorem 3.2 *Let*

$$0 < a_0 \le a(h) < b(h) \le b_0, \quad 2e^{-h^{-1/3}} \le c(h) \le Ch^{2n^{\#}+1}.$$

and set

$$\Omega(h) = [a(h), b(h)] + i[-c(h), 0].$$

Let $P^{\#}(h)$ be a reference operator in $\Omega(h)$ with discrepancy $\delta(h) \le h^{7n^{\#}/2+1}$. Then

$$N(\Omega(h)) \le N^{\#}[a(h) - \delta_1(h), b(h) + \delta_1(h)] \quad \textit{for } h \ll 1,$$

where $\delta_1(h) = h^{-7n^{\#}/2-1}\delta(h)$.

Proof. The basic argument in the proof is that the property that the resonant states corresponding to resonances in $\Omega(h)$ are linearly dependent is stable under small perturbations. For the resonant states corresponding to each $\Omega_k(h)$ this is true in a trivial way if $c(h) = O(h^N)$, $N \gg 1$, because they can be chosen asymptotically orthogonal as in Theorem 3.1. We do not know however whether resonant states corresponding to $\Omega_{k_1}(h)$ and $\Omega_{k_1}(h)$, where $k_1 \ne k_2$, have the same property, unless they are far enough from each other (see Proposition 3.5). Instead, we use Proposition 3.4.

Let ψ_{jk} be as in Definition 3.1. Then ψ_{jk} are cut-off functions $f_{jk} \in \mathrm{Ran}\Pi_{\Omega_k}$ as in (3.33). By (3.37), $\|f_{jk} - \psi_{jk}\| \le \epsilon(h)$. Let $\Pi^{\#}(h)\{[a,b]\}$ be the spectral projector of $P^{\#}(h)$ corresponding to the interval $[a,b]$. Set

$$v_{jk}(h) = \Pi^{\#}(h)\big\{[a_j(h) - \delta_1(h), b_j(h) + \delta_1(h)]\big\}\psi_{jk}(h), \quad j = 1, \dots, m_k(h),\ k = 1, \dots, K(h). \tag{3.38}$$

We claim that $v_{jk}(h)$ are linearly independent. Assume the opposite. Then

$$\alpha_1 v_1 + \ldots + \alpha_K v_K = 0 \tag{3.39}$$

with at least one coefficient non-zero, where for each k, $v_k = \sum_j \beta_{jk} v_{jk}$. Then v_k can be obtained as in (3.38) as projections of $\psi_k := \sum_j \beta_{jk}\psi_{jk}$, and in turn, ψ_k are defined as in (3.33) using $f_k = \sum_j \beta_{jk} f_{jk} \in \operatorname{Ran}\Pi_{\Omega_k}$. Without loss of generality we may assume that v_k are normalized. Then $\|(P^{\#}(h)-a_k(h))\psi_k\| \le \delta(h)$. Next, we may also assume that the largest coefficient in (3.39) by absolute value is α_1. Dividing by it, we get

$$v_1 = \beta_2 v_2 + \ldots + \beta_K v_K, \quad |\beta_j| \le 1. \tag{3.40}$$

We have as above $\|f_k - \psi_k\| \le \epsilon(h)$. Since $\|(P^{\#}(h)-a_k(h))\psi_k\| \le \delta(h)$, we get that $\|\psi_k - v_k\| \le \delta(h)/\delta_1(h) = h^{7n^{\#}/2+1}$ as in [St1, sec. 3] . Therefore, $\|f_k - v_k\| \le 2h^{7n^{\#}/2+1}$.

Denote by v_k' the orthogonal projections of v_k onto the space $\operatorname{Ran}\Pi_\Omega$. Projecting (3.40) onto this space, we get

$$v_1' = \beta_2 v_2' + \ldots + \beta_K v_K', \quad |\beta_j| \le 1, \tag{3.41}$$

and as before, $\|f_k - v_k'\| \le 2h^{7n^{\#}/2+1}$ since f_k already belongs to this space. Let us apply the projector Π_{Ω_1} to both sides of (3.41). Applying Proposition 3.4, we get

$$\begin{aligned} \|f_1\| - 2h^{7n^{\#}/2+1}h^{-(5n^{\#}+1)/2} \le \|\Pi_{\Omega_1} v_1'\| &\le Ch^{-n^{\#}} \max_{k\ge 2} \|\Pi_{\Omega_1} v_k'\| \\ &\le Ch^{-n^{\#}} \max_{k\ge 2} \|\Pi_{\Omega_1}(v_k' - f_k)\| \\ &\le Ch^{-n^{\#}} h^{-(5n^{\#}+1)/2} 2h^{7n^{\#}/2+1} = Ch^{1/2}. \end{aligned}$$

This contradicts the fact that $\|f_k\| = 1 + o(1)$ for small h. ☐

3.5 Decay of the resonant states away from the scatterer

Next we study the behavior of the resonant states f_j before the complex scaling. Let $\Omega(h)$ be as in Theorem 3.1, and let $f \in \operatorname{Ran}\Pi_\Omega$, $\|f\| = 1$ and set as in (3.33), $\psi = \chi(|x| - B, h)f$. Then $\psi|_{|x|<B} = f$, $\|\psi\| = 1 + o(1)$ and $(P - z_0)\psi = g$ with $\|g\| \le \epsilon(h)$. Both g and ψ are supported in $B(0, B+1)$ for small h. Introduce a smooth cut-off function χ_1 with the properties $\chi_1(x) = 1$ for $A < |x|$ and $\chi_1(x) = 0$ for $|x| \le A - 1/2$. Set

$$\psi_1 = \psi - \chi_1 R_0(z_0)\chi_1 g,$$

where $R_0(z) = (-h^2\Delta - z_0 - i0)^{-1}$ is the free outgoing resolvent. Then $\psi_1 \in \mathcal{D}_{\mathrm{loc}}$ and

$$(P - z_0)\psi_1 = g_1, \quad g_1 := (1 - \chi_1^2)g + [h^2\Delta, \chi_1]R_0(z_0)\chi_1 g.$$

The standard estimates on $R_0(z)$ imply that $\|g_1\| = O(\epsilon(h))$. Moreover, $g_1 = 0$ for $|x| > A$ thus in particular, ψ_1 is an outgoing solution of $(-h^2\Delta - z_0(h))\psi_1 = 0$ there. We will need here the Green's formula for "black boxes":

$$(P(h)u, v)_K - (u, P(h)v)_K = -h^2 \int_{\partial K} \left(\frac{\partial u}{\partial \nu}\bar{v} - u\frac{\partial \bar{v}}{\partial \nu} \right) dS_x \tag{3.42}$$

for any bounded domain $K \supset\supset B(0, R_0)$ with smooth boundary. Here ν is the outward unit normal on ∂K and $(u, v)_K = (u_K, v_K)$, where $u_K = \mathbf{1}_{B(0,R_0)}u + \mathbf{1}_{K\setminus B(0,R_0)}u$, and we define similarly v_K. To prove (3.42), let $\chi_1 + \chi_2$ be a smooth partition of unity such that $\chi_1(x) = 1$ for $|x| \le R_0 + \varepsilon$, and $\chi_2(x) = 1$ for $|x| \ge R_0 + 2\varepsilon$. Set $u_{1,2} = \chi_{1,2}u$ and $v_{1,2} = \chi_{1,2}v$. Plug $u = u_1 + u_2$ and $v = v_1 + v_2$ into the l.h.s. of (3.42). As in the proof of Proposition 3.1, since $P(h)$ is self-adjoint and u_1, v_1 vanish near ∂K, all terms involving either u_1 or v_1 cancel out. Thus

$$\begin{aligned} (P(h)u, v)_K - (u, P(h)v)_K &= (P(h)u_2, v_2)_K - (u_2, P(h)v_2)_K \\ &= -h^2 \int_{K\setminus B(0,R_0+\varepsilon)} ((\Delta u_2)\bar{v}_2 - u_2\Delta\bar{v}_2)\, dx \\ &= -h^2 \int_{\partial K} \left(\frac{\partial u}{\partial \nu}\bar{v} - u\frac{\partial \bar{v}}{\partial \nu} \right) dS_x, \end{aligned}$$

where in the last step we applied the classical Green's formula and used the fact that $u_2 = v_2 = 0$ on $\partial B(0, R_0 + \varepsilon)$. This proves (3.42).

Let $A \leq a \leq B'$ with some $B' > A$. By Green's formula (3.42) applied for $u = v = \psi_1$,

$$-\int_{|x|=a} \operatorname{Im}(\bar{\psi}_1 \partial_r \psi_1) dS_x \leq h^{-2} \int_{|x|\leq a} |\psi_1(x)||g_1(x)|dx \leq Ch^{-2}\epsilon(h). \tag{3.43}$$

We are going to use here Proposition 2.2 in [B1][1]: since ψ_1 is the outgoing solution to $(-h^2\Delta - z_0)\psi_1 = 0$ for $|x| > A$,

$$\frac{C}{h}\int_{|x|=a} \left(|\psi_1|^2 + |h\nabla\psi_1|^2\right) dS_x \leq -\int_{|x|=a} \operatorname{Im}(\bar{\psi}_1 \partial_r \psi_1) dS_x + e^{-c/h}\int_{|x|=a-1/2} \left(|\psi_1|^2 + |h\nabla\psi_1|^2\right) dS_x. \tag{3.44}$$

Note that the constant above can be chosen independent of $a \in [A, B']$. Using the ellipticity of P, it is easy to see that the last term above can be estimated by $Ce^{-c/h}$ which can be absorbed by $\epsilon(h)$. Combining (3.43) and (3.44), we get

$$\int_{|x|=a} \left(|\psi_1|^2 + |h\nabla\psi_1|^2\right) dS_x \leq Ch^{-1}\epsilon(h), \quad A \leq a \leq B'. \tag{3.45}$$

Integrating with respect to $a^{n-1}da$ from $a = A$ to $a = B'$, we get a similar estimate in the annulus $A \leq |x| \leq B'$. This immediately leads to a similar estimate on ψ, and by the remark after Theorem 3.1, we get such an estimate on f there:

$$\int_{A<|x|<B'} \left(|f|^2 + |h\nabla f|^2\right) dS_x \leq C_{B'}h^{-1}\epsilon(h), \quad A < B'. \tag{3.46}$$

We are ready to formulate the following analogue of Theorem 3.1 for the (non-scaled) resolvent.

Theorem 3.3 *Let $\Omega(h)$, $a(h)$, $b(h)$, $c(h)$, $z_0(h)$ and $\epsilon(h)$ be as in Theorem 3.1. Let A and B be such that $R_0 < A < B$. Set*

$$\pi_\Omega := \frac{1}{2\pi i}\oint_{\partial\Omega} R(z,h)dz : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{loc}}$$

where $R(z,h)$ is the outgoing resolvent. Then for any g with $\operatorname{supp} g \subset B(0,B)$ and for $0 < h < h_0$ small enough we have

$$\int_{A\leq|x|\leq B} \left(|u|^2 + |h\nabla u|^2\right) dx \leq Ch^{-1}\epsilon(h)\|u\|^2_{\mathcal{H}_B}, \quad u := \pi_\Omega g \tag{3.47}$$

where $\mathcal{H}_B = \mathcal{H}_{R_0} \oplus L^2(B(0,B) \setminus B(0,R_0))$, and $C > 0$, $h_0 > 0$ do not depend on $\Omega(h)$ and g.

Moreover, for any $z(h) \in \Omega(h)$ we have

$$\|(P(h) - z(h))u(h)\|_{\mathcal{H}_B} \leq C\epsilon(h)\|u\|_{\mathcal{H}_B}. \tag{3.48}$$

Proof. For $|x| < B$ we have $u = \Pi_\Omega g =: f$. Estimate (3.47) then follows from (3.46), if we can show that $\|f\| \leq C\|u\|_{\mathcal{H}_B}$. To prove the latter, write $\|f\|^2 = \|u\|^2_{\mathcal{H}_B} + \|f\|^2_{L^2(\mathbf{R}^n\setminus B(0,B))}$. It is enough to show that the last term is smaller than $\|f\|^2/2$ for small h. For $B \leq |x| \leq B'$ with some $B' > B$ this follows from (3.46) (where $\|f\| = 1$), while for $|x| > B + 1$ this follows from the remark after Theorem 3.1 since $\psi = 0$ there.

Estimate (3.48) follows in a similar way from Theorem 3.1(b). □

Theorem 3.3 remains true for a resonant state associated with a single resonance. For any such resonance $z_0(h)$, denote by π_{z_0} the residue of $R(z,h)$ at $z = z_0$ (defined as π_Ω with $\partial\Omega$ replaced by a small contour around z_0).

---

[1] The author thanks M. Zworski for pointing out that Burq's estimate provides simpler proof than the use of a parametrix for the Neumann operator.

Corollary 3.2 *Let* $z_0(h)$ *be a resonance with* $0 < a_0 \leq \operatorname{Re} z_0(h) \leq b_0 < \infty$, $-\operatorname{Im} z_0(h) = o(1) h^{(3n^{\#}+1)/2}$, *and let* $u = \pi_{z_0} g$, $\operatorname{supp} g \subset B(0, B)$ *(then* $(P - z_0)^k u = 0$ *for some* $k = O(h^{-n^{\#}})$*). Then*

$$\int_{A \leq |x| \leq B} \left(|u|^2 + |h\nabla u|^2\right) dx \leq C h^{-n^{\#}/2-2} \max\{-\operatorname{Im} z_0, e^{-h^{-1/3}}\}^{1/3} \|u\|^2_{\mathcal{H}_B}, \tag{3.49}$$

and

$$\|(P(h) - z_0(h))u(h)\|_{\mathcal{H}_B} \leq C h^{-n^{\#}/2-1} \max\{-\operatorname{Im} z_0, e^{-h^{-1/3}}\}^{1/3} \|u\|_{\mathcal{H}_B}. \tag{3.50}$$

Proof. Set $f := \Pi_{z_0} g$, where Π_{z_0} is the scaled projector. Then $\Pi_\Omega f = f$ and (3.49) holds for f. Using the same arguments as above, we derive it for u as well. The proof of (3.50) is similar. □

3.6 Asymptotic orthogonality of resonant/quasimode states

Theorem 3.1 shows that one can choose $m(h)$ "almost orthogonal", or even orthogonal quasimodes, $m(h)$ being the total multiplicity of resonances in Ω provided that the size of Ω is "small". However, those quasimode states are not necessarily cut-off single resonant states, in fact they are cut-off linear combinations of such resonant states. This result is non-trivial, as explained in the Introduction, since we do not know how to control the angles between resonant states corresponding to resonances too close to each other ($|z_1 - z_2| \ll \min\{-\operatorname{Im} z_1, -\operatorname{Im} z_2\}$).

As mentioned in the Introduction, we do not know whether one can construct "almost orthogonal" quasimodes corresponding to $\Omega(h)$ with larger size (for example $b(h) - a(h) = O(1)$) by keeping their number the same as the total multiplicity of resonances in Ω. A simple argument based on integration by parts however, shows that resonant states or quasimode states, respectively, corresponding to $\Omega_1(h)$ and $\Omega_2(h)$ with $\operatorname{dist}\{\Omega_1, \Omega_2\} \gg \operatorname{diam}\Omega_{1,2}$ are "almost orthogonal". We are not going to use the proposition below in our analysis, its purpose is more to stress on the fact that for domains $\Omega_1(h)$, $\Omega_2(h)$ too close to each other, integration by parts argument does not provide asymptotic orthogonality. Note that the control that we have on the lower bound of $\operatorname{dist}\{\Omega_k, \Omega_m\}$ for the domains (3.29) is not enough to guarantee asymptotic orthogonality.

Proposition 3.5 *Let* $\Omega_1(h)$ *and* $\Omega_2(h)$ *be two domains as in Theorem 3.1 and* $\epsilon_1(h)$, $\epsilon_2(h)$ *be related to* Ω_1, Ω_2 *as in Theorem 3.1.*

(a) Let ψ_{jk}, $j = 1, \dots, m_k(h)$; $k = 1, 2$ *be the quasimode states whose existence is guaranteed by Theorem 3.1 corresponding to* $\Omega_k(h)$, $k = 1, 2$. *Then for* $h \ll 1$,

$$|(\psi_{i1}, \psi_{j2}| \leq 2 \frac{\epsilon_1(h) + \epsilon_2(h)}{\operatorname{dist}\{\Omega_1(h), \Omega_2(h)\}}$$

(b) Let $f_1 = \pi_{\Omega_1} g_1$, $f_2 = \pi_{\Omega_2} g_2$, *supp*$g_k \subset B(0, R)$, $k = 1, 2$, $R > R_0$. *Then*

$$|(f_1, f_2)_R| \leq C_R h^{-3/2} \frac{\epsilon_1^{1/2}(h) + \epsilon_2^{1/2}(h)}{\operatorname{dist}\{\Omega_1(h), \Omega_2(h)\}},$$

where $(\cdot, \cdot)_R$ *is the scalar product in* $\mathcal{H}_{R_0} \oplus L^2(B(0, R) \setminus B(0, R_0))$.

Proof. (a) We have $(P - z_{0k})\psi_{jk} = g_{jk}$ with $\|g_{jk}\| \leq \epsilon_k(h)$, $k = 1, 2$. Therefore,

$$z_{01}(\psi_{i1}, \psi_{j2}) = (P\psi_{i1} - g_{j1}, \psi_{j2}) = (\psi_{i1}, P\psi_{j2}) - (g_{i1}, \psi_{j2}) = z_{02}(\psi_{i1}, \psi_{j2}) + (\psi_{i1}, g_{j2}) - (g_{i1}, \psi_{j2}).$$

The proof of (b) follows in a similar way from Theorem 3.3 by applying Green's formula. □

4 Trapped geodesics and wave front set of the resonant states

In this section we will show that in some classical situations the wave front set of the resonant states is contained in the union of the trapped rays (see also [CdV] for quasimodes related to eigenfunctions on compact Riemannian manifolds).

We consider self-adjoint differential operators of the form

$$P(h)=\sum_{i,j=1}^{n}a_{ij}(x)hD_{x_i}hD_{x_j}+\sum_{i=1}^{n}b_i(x)hD_{x_i}+V(x)+P_1(h) \tag{4.1}$$

with smooth coefficients such that $\sum a_{ij}(x)\xi_i\xi_j>0$ for any $x\in\mathbf{R}^n$ and $a_{ij}-\delta_{ij}=b_i=V(x)=0$ for $|x|>R_0$ with some $R_0>0$. Here $P_1(h)=\sum\tilde{b}_j(x,h)hD_{x_j}+\tilde{V}(x,h)$ is assumed to be a differential operator of first order the same type as $P(h)$ but with coefficients supported in $B(0,R_0)$, depending polynomially on h with the lowest power h, that would guarantee us that $P_1(h)$ is of no more than first order, considered as a h-ΨDO. The operator $P(h)$ is self-adjoint in $L^2(\mathbf{R}^n)$ and satisfies the black-box assumptions. Resonances of $P(h)$ are the poles of the meromorphic extension of $(P(h)-z)^{-1}:L^2_{\mathrm{comp}}\to L^2_{\mathrm{loc}}$ form $\operatorname{Im}z>0$ to $\mathbf{C}$, if n is odd, and to the logarithmic plane, if n is even. We are interested in the resonances near the real axis only.

We will use propagation of singularities results to get microlocal estimates of $\pi_\Omega f/\|\pi_\Omega f\|$ away from the trapping trajectories.

One can define $\mathrm{WF}^s(u)$ and $\mathrm{WF}(u)$ of a tempered u as in [I]. We note that here we will work with finite points of the semiclassical wave front set only, because we study operators with characteristic variety bounded in the ξ variable. Consider the bicharacteristics of $P(h)$ related to its the semi-classical principal symbol $p_0(x,\xi)=\sum a_{ij}\xi_i\xi_j+\sum b_i\xi_i+V$. They are the integral curves of the Hamiltonian vector field $H_{p_0}=(\partial_\xi p_0)\partial_x-(\partial_x p_0)\partial_\xi$. We call a bicharacteristic $t\mapsto\gamma(t)$ *non-trapped*, if there exists τ (positive or negative), such that $\gamma(\tau)$ lies outside $T^*B(0,R_0)$. We call all other bicharacteristics *trapped*. Denote by $\mathcal{T}$ the trapped subset of $T^*\mathbf{R}$, i.e, $(x,\xi)\in\mathcal{T}$ if and only if the bicharacteristic passing through (x,ξ) is trapped.

Theorem 4.1 *Let*

$$0<a_0\le a(h)<b(h)\le b_0,\quad b(h)-a(h)\le h^M,\quad M\ge 3n/2+3,$$

and set

$$\Omega(h)=[a(h),b(h)]+i[-h^N,0],\quad N\ge 2n+1,\quad N\ge M.$$

Let $P(h)$ *be the operator defined above and let* $f=f(h)\in\operatorname{Ran}\Pi_\Omega$, *and* $\|f\|=1$. *Then for* $s=M/3-n/2-1$, $\mathrm{WF}^s(f)$ *is supported in the set of trapped bicharacteristics of* $P(h)$ *on energy levels in* $p_0^{-1}[a_0,b_0]$ *uniformly with respect to the choice of* u. *More precisely, for any zeroth order symbol* $q(x,\xi)$ *with support disjoint from* $\mathcal{T}\cap p_0^{-1}[a_0,b_0]$ *there exists* $C>0$ *such that* $\|q(x,hD)f\|\le Ch^s$ *for any* $f(h)$ *as above.*

If $\Omega(h)$ *is as above with* $0<a_0\le a(h)<b(h)\le b_0$, $N>(5n+1)/2$ *but without smalness assumptions on* $b(h)-a(h)$, *then the statement of the theorem is true with* $s=N/3-(13n^\#+4)/2$.

The proof of Theorem 4.1 is based on a propagation of singularities argument. The following lemma follows directly from [I], if the energy level $z_0(h)$ is independent of h.

Lemma 4.1 (propagation of singularities) *Let* $(P(h)-z_0(h))u(h)=g(h)$ *in* $B(0,R)$ *with* $0<a_0\le z_0(h)\le b_0<\infty$, $R_0<R$, *and* $\|u(h)\|_{\mathcal{H}_\mathbf{R}}\le C$. *Let* $(x_0,\xi_0)\in T^*B(0,R)$. *Assume that* $(x_1,\xi_1)\in T^*B(0,R)$ *can be connected with* (x_0,ξ_0) *by a bicharacteristic (of finite length) lying in* $T^*B(0,R)\setminus\mathrm{WF}^{s+1}(g)$. *Then, if* $(x_0,\xi_0)\notin\mathrm{WF}^s(u)$, *we also have* $(x_1,\xi_1)\notin\mathrm{WF}^s(u)$ *(and therefore, if* $(x_0,\xi_0)\in\mathrm{WF}^s(u)$, *then* $(x_1,\xi_1)\in\mathrm{WF}^s(u)$*). The estimates that those inclusions imply are uniform with respect to the choice of* $u(h)$.

Proof. Let $[0,T]\ni t\mapsto\gamma(t)\subset T^*B(0,R)\setminus \mathrm{WF}^{s+1}(g)$ be the bicharacteristic such that $\gamma(0)=(x_0,\xi_0)$ and $\gamma(T)=(x_1,\xi_1)$. Denote by Φ^t the bicharacteristic flow. Let $q_0(x,hD)$ be such that $q_0(x,hD)u(h)=O(h^s)$, $q_)(x,\xi)=1$ near (x_0,ξ_0). We will construct a symbol $q_t(x,\xi)$ such that $q_t=q_0$ for $t=0$, for each $t\in[0,T]$, the principal symbol of $q_t(x,\xi)$ is equal to 1 near $\Phi^t(x,\xi)$ and q_t has support contained in a small neighborhood of $\Phi^t(x,\xi)$, such that $\operatorname{supp} q_t\subset T^*B(0,R)\setminus \mathrm{WF}^{s+1}(g)$.

Set

$$\alpha(t)=\|q_t(x,hD)u\|.$$

We know that $\alpha(0)=O(h^s)$. Our goal is to construct q_t such that $\alpha(t)=O(h^s)$, $\forall t\in[0,T]$. We have

$$\frac{d}{dt}\frac{\alpha^2(t)}{2}=\mathrm{Re}\left(\frac{d}{dt}q_t(x,hD)u,q_t(x,hD)u\right).$$

Set $Q_t=q_t(x,hD)$. We will choose q_t so that

$$ih\frac{d}{dt}Q_t=[P,Q_t]+R_t,\quad Q_t|_{t=0}=Q_0, \tag{4.2}$$

with R_t is of order $-s-1$. Suppose that we have Q_t with those properties. Then

$$\begin{aligned}\frac{d}{dt}\frac{\alpha^2(t)}{2}&=h^{-1}\mathrm{Im}\,(([P-z_0,Q_t]+R_t)u,Q_tu)\\&=-h^{-1}\mathrm{Im}\,(Q_t(P-z_0)u,Q_tu)+h^{-1}\mathrm{Im}\,(R_tu,Q_tu)\\&=-h^{-1}\mathrm{Im}\,(Q_tg,Q_tu)+h^{-1}\mathrm{Im}\,(R_tu,Q_tu).\end{aligned}$$

Since $\operatorname{supp} q_t$ is disjoint from $\mathrm{WF}^{s+1}(g)$, we get

$$\alpha(t)\frac{d}{dt}\alpha(t)\le Ch^s\alpha(t)+Ch^{-1}\|R_tu\|\alpha(t)\quad\Longrightarrow\quad\frac{d}{dt}\alpha(t)\le Ch^s+Ch^{-1}\|R_tu\|.$$

Since $\|R_t\|=O(h^{s+1})$,

$$\frac{d}{dt}\alpha(t)\le Ch^s\quad\Longrightarrow\quad\alpha(t)\le Ch^s\quad\text{for }0\le t\le T.$$

It remains to solve (4.2). We look for $q_t(x,\xi)$ of the form $q_t=q_t^{(0)}+hq_t^{(1)}+\ldots+h^{s-1}q_t^{(s-1)}$. The principal symbol $q_t^{(0)}$ of Q_t must solve the equation

$$(\partial_t+H_{p_0})\,q_t^{(0)}=0,\quad q_t^{(0)}|_{t=0}=q_0.$$

Therefore, we define $q_t^{(0)}(x,\xi))=q_0(\Phi^{-t}(x,\xi))$. This implies that

$$ih\frac{d}{dt}Q_t^{(0)}=[P,Q_t^{(0)}]+R_t^{(0)},\quad Q_t^{(0)}|_{t=0}=Q_0,$$

where $R_t^{(0)}$ is of order -2. Note that the symbol of $R_t^{(0)}$ is included in the set $\Gamma=\cup_{0\le t\le T}\Phi^t(\operatorname{supp} q_0)$. We next solve

$$ih\frac{d}{dt}Q_t^{(1)}=[P,Q_t^{(1)}]-h^{-1}R_t^{(0)},\quad Q_t^{(0)}|_{t=0}=0 \tag{4.3}$$

on principal symbol level, which gives us the equation

$$(\partial_t+H_{p_0})\,q_t^{(1)}=ir_t^{(0)},\quad q_t^{(1)}|_{t=0}=0,$$

where $h^2r_t^{(0)}$ is the principal symbol of $R_t^{(0)}$. This is an ODE along the bicharacteristics of p_0 and the solution is again supported in Γ. Then $Q_t^{(1)}=hq_t^{(1)}(x,hD)$ solves (4.3) up to a remainder $R_t^{(1)}$ of order -3. We complete the construction of the solution to (4.2) by induction. □

Proof of Theorem 4.1. Let ψ be related to f as in (3.33). By Theorem 3.1, f solves $(P(h)-a(h))f(h) = O(h^{M/3-n/2-1})$, and $f = 0$ for $|x| > B+1$. We can therefore apply Lemma 4.1 with $s = M/3 - n/2 - 1$, and we have $s > 0$ because $M > 3n/2 + 3$. On the other hand, $\mathrm{WF}^s(f)$ is contained in $p_0^{-1}[a_0, b_0]$ because $P(h) - a(h)$ is elliptic outside any neighborhood of this set. This proves the first part of the theorem.

To prove the second part, let us group all resonances in Ω in subdomains $\Omega_k(h)$ of width $O(h^M)$, where $M = N - (5n+1)/2$ as in (3.29). Then we have the conclussion of the theorem for each normalized $f_k \in \mathrm{Ran}\Pi_{\Omega_k}$. Let $f \in \mathrm{Ran}\Pi_\Omega$. Then $f = f_1 + \ldots + f_K$, $K = O(h^{-n^\#})$. Therefore, for $q(x, hD)$ as in the theorem, we have by Proposition 3.4(b),

$$\|q(x,hD)f\| \le C \sum \|q(x,hD)f_j\| \le Ch^s h^{-n^\#} \max \|f_j\| \le Ch^{s-(7n^\#+1)/2}\|f\|$$

with $s = M/3 - n/2 - 1 = N/3 - 3n - 3/2$. □

5 Upper bounds on the number of resonances close to the real axis

Let $P(h)$ be the operator (4.1). In this section we are going to establish an upper bound of the resonances of $P(h)$ in a box of width independent of h and height h^N, $N \gg 1$ in terms of the measure of the trapped set $\mathcal{T}$, where the measure is considered in $T^*\mathbf{R}^n$. To this end we choose a suitable reference operator $P^\#(h)$ that imposes a barrier outside a small neighborhood of the trapped set $\mathcal{T}$ by modifying $P(h)$ there. Since the resonant states are "small" there, the resonant states will be quasimodes for the new operator. An application of Theorem 3.2 then will imply an upper bound and the well-known asymptotics for the eigenvalues of selfadjoint h-ΨDOs will relate this bound with $\mathrm{meas}(\mathcal{T})$.

Fix $0 < a_0 < b_0$, $N \ge (5n+1)/2$ (!) and let

$$\Omega(h) = [a_0, b_0] + i[-h^N, 0].$$

The resonances in $\Omega(h)$ are contained in $\cup_{k=1}^{K(h)} \Omega_k(h)$, where $\Omega_k(h)$ are as in (3.29) and $b_k - a_k \le h^{N-(5n+1)/2}$, $\mathrm{dist}\{\Omega_{k_1}, \Omega_{k_2}\} \ge 4h^{N-(3n+1)/2}$. Each Ω_k satisfies the assumptions of Theorem 3.1. The corresponding discrepancy function (see the paragraph before Definition 3.1) is given by

$$\epsilon(h) = h^{N/3-(8n+7)/6}.$$

Denote

$$\mathcal{T}^\nu = \mathcal{T} \cap p_0^{-1}[a_0 - \nu, b_0 + \nu], \quad \mathcal{T}^\nu_\mu = \{\zeta \in T^*\mathbf{R}^n;\ \mathrm{dist}\{\zeta, \mathcal{T}^\nu\} < \mu\},$$

where $\nu > 0$, $\mu > 0$ are small parameters. Fix $\nu > 0$. Clearly, $\mathcal{T}^\nu$ is a closed set. Assume that it is non-empty (but it may have zero measure). On the other hand, $\mathcal{T}^\nu_\mu$ is an open set of positive measure. We claim that $\mathrm{Vol}(\mathcal{T}^\nu_\mu) \to \mathrm{meas}(\mathcal{T}^\nu)$, as $\mu \to 0$. Indeed, $\forall \varepsilon > 0$, there exists an open set $U_\varepsilon \supset \mathcal{T}^\nu$, such that $\mathrm{Vol}(U_\varepsilon) \le \mathrm{meas}(\mathcal{T}^\nu) + \varepsilon$. Then $T^*\mathbf{R}^n \setminus U_\varepsilon$ is at positive distance from $\mathcal{T}^\nu$ and therefore for $\mu = \mu(\varepsilon) \ll 1$ we have $\mathcal{T}^\nu \subset \mathcal{T}^\nu_\mu \subset U_\varepsilon$ which implies that $\mathrm{meas}(\mathcal{T}^\nu) \le \mathrm{Vol}(\mathcal{T}^\nu_\mu) \le \mathrm{meas}(\mathcal{T}^\nu) + \varepsilon$. This proves our claim.

Denote $-C_0 = \min p_0(x,\xi)$ and let $q_\mu(x,\xi)$ be a smooth function such that $q_\mu = 0$ on $\mathcal{T}^\nu_\mu$, $q_\mu = 2b_0 + C_0$ outside $\mathcal{T}^0_{2\mu}$, and $0 \le q_\mu \le 2b_0 + C_0$. Set

$$p_\mu(x,\xi) = p_0(x,\xi) + q_\mu(x,\xi).$$

Then $P^\#_\mu(h) := p_\mu(x, hD)$ has discrete spectrum in $(-\infty, 2b_0)$, because $\{p_\mu(x,\xi) \le M\}$ is compact for any $M < 2b_0$. Moreover, we have the following estimate for the number $N^\#_\mu[a_0, b_0]$ of eigenvalues of $P^\#_\mu(h)$ (see [DSj])

$$\frac{1}{(2\pi h)^n}\left(V^\#_-([a_0,b_0]) + o(1)\right) \le N^\#_\mu[a_0,b_0] \le \frac{1}{(2\pi h)^n}\left(V^\#_+([a_0,b_0]) + o(1)\right), \tag{5.1}$$

where

$$V^{\#}_{\pm}([a_0,b_0]) = \lim_{\pm\epsilon\searrow 0}\int_{p_\mu(x,\xi)\in[a_0-\epsilon,b_0+\epsilon]} dx d\xi. \tag{5.2}$$

If a_0 and b_0 are not critical values for $p_\mu(x,\xi)$, then $V^{\#}_{-}([a_0,b_0]) = V^{\#}_{+}([a_0,b_0])$ and the remainder is actually $O(h)$. In particular,

$$N^{\#}_{\mu}([a_0-\nu/2, b_0+\nu/2]) \le \frac{1}{(2\pi h)^n}\left(\mathrm{Vol}\left(p_\mu^{-1}[a_0-\nu, b_0+\nu]\right) + o(1)\right), \quad \text{as } h\to 0. \tag{5.3}$$

Since $\mathcal{T}^\nu \subset p_\mu^{-1}[a_0-\nu, b_0+\nu] \subset \mathcal{T}^\nu_{2\mu}$, we obtain that $\mathrm{Vol}(p_\mu^{-1}[a_0-\nu, b_0+\nu]) \to \mathrm{meas}(\mathcal{T}^\nu)$, as $\mu\to 0$.

We claim that $P^{\#}_{\mu}(h)$ is a reference operator in $\Omega(h)$ with discrepancy $\epsilon(h)$. Indeed, fix k and a normalized $f_k \in \mathrm{Ran}\Omega_k(h)$. Let $\psi_k(h)$ be the corresponding cut-off resonant state given by (3.33). Then by Theorem 3.1, $(P(h)-a_k(h))\psi_k = O(\epsilon(h))$ and by Theorem 4.1, $\mathrm{WF}^s(\psi_k)\subset \mathcal{T}^0$, where $s = N/3-n/2-1$. The latter implies that replacing $P(h)$ by $P^{\#}_{\mu}(h)$ would keep the estimate $\|(P^{\#}_{\mu}(h)-a_k(h))\psi_k\| \le C\epsilon(h)$ with different constant C. This constant depends on μ and ν but is independent of k and on the choice of ψ_k. Therefore, $P^{\#}_{\mu}(h)$ is a reference operator with discrepancy $C_{\mu,\nu}\epsilon(h)$. We can therefore apply Theorem 3.2, if $N > 29n/2+13/2$ to conclude that the number of eigenvalues of $P^{\#}_{\mu}(h)$ in $[a_0-\delta_1(h), b_0+\delta_1(h)]$, where $\delta_1(h) = h^{N/3-29n/6-13/6}$, is at least equal to the number of resonances in Ω. In particular,

$$N(\Omega(h)) \le N^{\#}_{\mu}([a_0-\nu/2, b_0+\nu/2]) \quad \text{for } 0<h\le h_0(\mu,\nu). \tag{5.4}$$

Relations (5.3) and (5.4) imply

$$N(\Omega(h)) \le \frac{1}{(2\pi h)^n}\left(\mathrm{Vol}\left(p_\mu^{-1}[a_0-\nu, b_0+\nu]\right) + o(1)\right) \quad \text{for } 0<h\le h_0(\mu,\nu). \tag{5.5}$$

Therefore, $\limsup_{h\to 0}(2\pi h)^n N(\Omega(h)) \le \mathrm{Vol}\left(p_\mu^{-1}[a_0-\nu, b_0+\nu]\right)$ and taking the limit $\mu\to 0$, $\nu>0$ fixed, we get by the remark after (5.3) that $\limsup_{h\to 0}(2\pi h)^n N(\Omega(h)) \le \mathrm{meas}(\mathcal{T}^\nu)$. Taking the limit $\nu\to 0$, we get that the latter converges to $\mathrm{meas}\left\{\mathcal{T}\cap p_0^{-1}[a_0-0, b_0+0]\right\}$, where $p_0^{-1}[a_0-0, b_0+0] := \cap_{\nu>0} p_0^{-1}[a_0-\nu, b_0+\nu]$. We have therefore proved the following.

Theorem 5.1 *Let $0<E_1<E_2$ be fixed and $N>(29n+13)/2$. Let $P(h)$ be as in (4.1) and set*

$$\Omega(h) = [E_1, E_2] + i[-h^N, 0].$$

Then

$$N(\Omega(h)) \le \frac{1}{(2\pi h)^n}\left(\mathrm{meas}\left\{\mathcal{T}\cap p_0^{-1}[E_1-0, E_2+0]\right\} + o(1)\right), \quad \text{as } h\to 0,$$

where $\mathcal{T}$ is the trapped set related to $P(h)$. If E_1 and E_2 are non-critical values of p_0, then $p_0^{-1}[E_1-0, E_2+0]$ can be replaced by $p_0^{-1}[E_1, E_2]$.

In the formulation of this theorem we passed to the commonly used notation E for the energy levels. We would like to make the obvious remark here that if $\mathrm{meas}\{p_0^{-1}(E_1)\} = \mathrm{meas}\{p_0^{-1}(E_2)\} = 0$, which is true in particular if E_1 and E_2 are non-critical values of p_0, then one can replace $\mathrm{meas}\left\{\mathcal{T}\cap p_0^{-1}[E_1-0, E_2+0]\right\}$ above by $\mathrm{meas}\left\{\mathcal{T}\cap p_0^{-1}[E_1, E_2]\right\}$.

6 Asymptotic formula in the case of potential well

In this section we study the resonances of the operator

$$P(h) = -h^2\Delta + V(x),$$

where $V \in C_0^\infty(\mathbf{R}^n)$, with additional non-trapping assumption for the unbounded component of $p_0^{-1}[E_1, E_2]$. Notice that $P(h)$ belongs to the class (4.1). In this case we obtain an asymptotic formula for the resonances near the real line and a resonance free zone. This situation can be considered as a generalized potential well. Similar situation was studied by Shu Nakamura [N], where he obtains asymptotic for the spectral shift function. Perhaps his results, combined with the techniques developed by V. Petkov and M. Zworski [PZ] could provide different approach to proving (b) in the theorem below.

Fix two energy levels $0 < E_1 < E_2$, and denote by $p(x, \xi) = |\xi|^2 + V(x)$ the semiclassical symbol of $P(h)$. Assume that E_1 and E_2 are non-critical values of p. Then

$$p^{-1}[E_1, E_2] = W_{\mathrm{int}} \cup W_{\mathrm{ext}},$$

is a unbounded closed set in $T^*\mathbf{R}^n$ with smooth boundary, where we denote by W_{ext} the unbounded connected component, and the union of the bounded ones, if exist, is denoted by W_{int}. Then W_{int} is a compact with smooth boundary and is associated with the potential well. The following assumption loosely speaking says that the system is non-trapping outside the potential well:

(**H**) All points in W_{ext} are non-trapping.

Notice that all points in W_{int} are trapping, therefore Theorem 5.1 provides an upper bound for the number of resonances in a box $[E_1, E_2] + i[-h^N, 0]$, $N \gg 1$ of the kind $(2\pi h)^{-n}(\mathrm{Vol}(W_{\mathrm{int}}) + o(1))$. In the following theorem we can actually say more.

Theorem 6.1 *Under the assumptions above, for any $M > 0$, there exists a function $0 < S(h) = O(h^\infty)$, and a constants $h_0 = h_0(M) > 0$, such that:*

(a) There are no resonances in

$$[E_1, E_2] + i[-Mh, -S(h)] \quad \textit{for } 0 < h < h_0. \tag{6.1}$$

(b) Let $\Omega(h) = [E_1, E_2] + i[-S(h), 0]$. Then

$$N(\Omega(h)) = \frac{1}{(2\pi h)^n} \left(\mathrm{Vol}(W_{\mathrm{int}}) + O(h)\right), \quad \textit{as } h \to 0.$$

Remark. The proof actually shows that the asymptotics in (b) holds if we choose $S(h) = e^{-d/h}$, where $d > 0$ is any constant less than the Agmon distance between W_{int} and W_{ext}on energy level E_2. Then for any other $S(h) = O(h^\infty)$, the number of resonances in Ω with $e^{-d/h} \le -\mathrm{Im}\, z \le S(h)$ is $O(h^{1-n})$.

Proof. We can always assume that W_{int} is non-empty because if it is empty, then all energy levels in $[E_1, E_2]$ are non-trapping and then for any $M > 0$, there are no resonances in $[E_1, E_2] + i[-Mh, 0]$ for $0 \le h \le h_0(M)$ (see [B3]), thus the theorem holds in a trivial way.

It is easy to see that our assumptions imply that $E_2 < \max V$ and moreover, the set $\{V(x) \le E_2\}$ is not connected.

Let X_{ext} be the unbounded connected component of $\{x;\ V(x) \le E_2\}$, and let X_{int} be the union of the bounded connected components of the same set. The distance between X_{int} and X_{ext} is positive. Let $\chi_{\mathrm{int}}^2 + \chi_{\mathrm{ext}}^2 = 1$ be a partition of unity associated with those two closed sets, i.e., $\chi_{\mathrm{int}} = 1$ in a neighborhood of X_{int}, and $\chi_{\mathrm{ext}} = 1$ in a neighborhood of X_{ext}. Define

$$\begin{aligned} P_{\mathrm{int}}(h) &= -h^2\Delta + V_{\mathrm{int}}(x), \qquad & V_{\mathrm{int}}(x) &:= \chi_{\mathrm{int}}(x)V(x) + 2E_2\chi_{\mathrm{ext}}(x), \\ P_{\mathrm{ext}}(h) &= -h^2\Delta + V_{\mathrm{ext}}(x), \qquad & V_{\mathrm{ext}}(x) &:= 2E_2\chi_{\mathrm{int}} + \chi_{\mathrm{ext}}V(x). \end{aligned}$$

Then E_1 and E_2 are not critical values for neither symbol $p_i = |\xi|^2 + V_i(x)$, $i = int, ext$, and $p_{\mathrm{int}}^{-1}[E_1, E_2] = W_{\mathrm{int}}$, $p_{\mathrm{ext}}^{-1}[E_1, E_2] = W_{\mathrm{ext}}$. Moreover, $P_{\mathrm{int}}(h)$ and $P_{\mathrm{ext}}(h)$ are selfadjoint, $P_{\mathrm{int}}(h)$ has discrete spectrum in $[E_1, E_2]$, while $P_{\mathrm{ext}}(h)$ is non-trapping for energy levels in $[E_1, E_2]$.

Fix $M > 0$ and assume that $z(h)$ is a resonance in the domain (6.1). Then there exists an outgoing $u(h)$ belonging locally to the domain of $P(h)$ such that $(P(h) - z(h))u(h) = 0$. Let us normalize $u(h)$ by requiring that $\|u\|_{L^2(B(0,R))} = 1$ with a fixed $R > R_0$, where $R_0 > 0$ is such that $\operatorname{supp} V \subset B(0, R_0)$. Notice that $P(h) - z(h)$ is elliptic for $x \notin X_{\text{int}} \cup X_{\text{ext}}$. This yields $\text{WF}(u) \subset (X_{\text{int}} \cup X_{\text{ext}}) \times \mathbf{R}^n$. To prove the latter, choose a cut-off function $0 \leq \chi(x) \leq 1$ equal to 1 in a neighborhood of $B(0, R_0)$ and having support in $B(0, R)$. Then $(P(h) - z(h))\chi u(h) = w(h)$, where $w(h) = [P(h), \chi]u(h)$ is supported in $B(0, R) \setminus B(0, R_0)$ and $\|\chi u(h)\| \leq 1$. Now, $(P(h) - z(h))\chi u(h) = 0$ in a neighborhood of $\mathbf{R}^n \setminus (X_{\text{int}} \cap X_{\text{ext}}) \subset B(0, R_0)$, moreover $P(h) - z(h)$ is elliptic there, and therefore for the wave fron set of the compactly supported $\chi u(h)$ we get $\text{WF}(\chi u) \subset (X_{\text{int}} \cup X_{\text{ext}}) \times \mathbf{R}^n$. This implies the same for $\text{WF}(u)$.

Choose the smooth cut-off function χ'_{ext} so that $\chi_{\text{ext}} = 1$ on $\operatorname{supp} \chi'_{\text{ext}}$ and $\chi'_{\text{ext}} = 1$ in a neighborhood of X_{ext}. Then $(P(h) - z(h))\chi'_{\text{ext}} u(h) = v(h)$, where $v = [P(h), \chi'_{\text{ext}}]u(h) = O(h^\infty)$ is supported in $B(0, R)$ and $\text{WF}(\chi'_{\text{ext}} u) \subset X_{\text{ext}} \times \mathbf{R}^n$. Then $(P(h) - z(h))\chi'_{\text{ext}} u(h) = (P_{\text{ext}}(h) - z(h))\chi'_{\text{ext}} u(h) = v(h)$. Since $\chi'_{\text{ext}} u(h) = u(h)$ for large $|x|$, we get that $\chi'_{\text{ext}} u(h)$ is $z(h)$-outgoing. Therefore, $\chi'_{\text{ext}} u(h) = R_{\text{ext}}(z(h), h)v(h)$, where $R_{\text{ext}}(z, h)$ is the outgoing resolvent of $P_{\text{ext}}(h)$. By [B3, Theorem 2], $\|\chi'_{\text{ext}} u(h)\|_{L^2(B(0,R))} \leq (C/h)\|v(h)\| = O(h^\infty)$. By the ellipticity of $P(h)$ we have similar estimate for the H^2 norm of $u(h)$ near $\partial B(0, R)$ and by the trace theorem, the H^1 norm of $u(h)$ on $\partial B(0, R)$ is $O(h^\infty)$ as well. An application of the Green's formula (3.42) in the ball $B(0, R)$ then yields $-\operatorname{Im} z = O(h^\infty)$. This proves part (a) of the theorem.

To prove part (b), fix $\Omega(h)$ with some $0 < S(h) = O(h^\infty)$. We will first prove an upper bound for $N(\Omega)$. Observe that $P_{\text{int}}(h)$ is a reference operator for $P(h)$ with discrepancy $O(h^\infty)$. Indeed, let $\Omega_k(h)$ be as in (3.29). Since $c(h) = O(h^\infty)$ in our case, we get that the error in Theorem 3.1, applied to each Ω_k, is $\epsilon(h) = O(h^\infty)$. Choose the smooth cut-off function χ'_{int} so that $\chi_{\text{int}} = 1$ on $\operatorname{supp} \chi'_{\text{int}}$ and $\chi'_{\text{int}} = 1$ in a neighborhood of X_{int}. Given $f_k(h) \in \operatorname{Ran} \Pi_{\Omega_k}$ with $\|f_k\| = 1$, we have $\text{WF}(f_k) \subset W_{\text{int}}$ (uniformly in k) by Theorem 4.1, thus $(P_{\text{int}}(h) - z_k(h))f_k = O(h^\infty)$, where $z_k(h) \in \Omega_k(h) \cap \mathbf{R}$. According to Definition 3.1, this means that $P_{\text{int}}(h)$ is a reference operator in $\Omega(h)$ with $\delta(h) = O(h^\infty)$. By Theorem 3.2, for $h \ll 1$, $N(\Omega(h)) \leq N^\#([E_1 - \delta_1(h), E_2 + \delta_1(h)]) \leq N^\#([E_1, E_2]) + O(h^{1-n})$, with $\delta_1(h) = O(h^\infty)$, where $N^\#$ is the counting function of the eigenvalues of $P_{\text{int}}(h)$. Here we used the fact that the eigenvalues of $P_{\text{int}}(h)$ in an interval of length $O(h)$ are $O(h^{1-n})$ [DSj]. Using (5.1) and the remark after it, we get

$$N(\Omega(h)) \leq \frac{1}{(2\pi h)^n} \left(\text{Vol}(W_{\text{int}}) + O(h)\right) \tag{6.2}$$

In order to complete the proof of part (b) of the theorem, we have to estimate $N(\Omega(h))$ from below. To this end, we will use the eigenfunctions of $P_{\text{int}}(h)$ as quasimodes of $P(h)$. Let $v_j(h)$, $j = 1, \dots, m(h)$ be a full system of orthonormal eigenfunctions of $P_{\text{int}}(h)$ corresponding to all eigenvalues $e_j(h)$ of $P_{\text{int}}(h)$ in $[E_1 + \delta(h), E_2 - \delta(h)]$, where $0 < \delta = O(h^K)$, $K \gg 1$, will be chosen later. For small h, we have an asymptotics for $m(h)$ as in (b) because in intervals of length $\delta(h)$ near a non-critical energy, there are only $O(h^{1-n})$ resonances. Moreover, v_j decay as $O(e^{-c/h})$ for $\operatorname{dist}(x, X_{\text{int}}) > \epsilon > 0$ with $c = c(\epsilon)$ in the H^1 norm (see e.g. [DSj], where the requirement that $e(h) \to E_2$, as $h \to 0$ for any eigenvalue for each the estimates apply, can be replaced by $\max\{e(h), E_2\} \to E_2$, as $h \to 0$). Therefore, if χ_{int} is as above, then $w_j := \chi_{\text{int}} v_j$ form quasimodes for $P(h)$, i.e., $(P(h) - e_j(h))w_j = O(h^\infty)$, $(w_i, w_j) = \delta_{ij} + O(h^\infty)$, and $\operatorname{supp} w_j \subset B(0, R)$ for any j. By [St1, Theorem 1], $P(h)$ has at least $m(h)$ resonances in $\Omega_1(h) := [E_1 - \delta(h) + h^K, E_2 + \delta(h) + h^K] + i[-S_1(h), 0]$ with some positive $S_1(h) = O(h^\infty)$ and $K \geq 1$. If $\delta(h)$ was chosen in advance so that $\delta(h) = h^K$, then we would have

$$N(\Omega_1(h)) \geq \frac{1}{(2\pi h)^n} \left(\text{Vol}(W_{\text{int}}) + O(h)\right), \quad \Omega_1(h) := [E_1, E_2] + i[-S_1(h), 0]. \tag{6.3}$$

Let $S(h)$ be the function which existence is guaranteed by (a). If $S(h) \leq S_1(h)$ for h small enough, then (6.3) is true with $\Omega_1(h)$ replaced by $\Omega(h)$, because there are no resonances in Ω_1 in the strip $S(h) \leq -\operatorname{Im} z \leq S_1(h)$. If $S_1(h) \leq S(h)$, then we can do the same because replacing $\Omega_1(h)$ in (6.3) by $\Omega(h)$ may only increase the l.h.s. Therefore, (6.3) is true for $\Omega(h)$ as well and (6.2), (6.3) together complete the proof of Theorem 6.1. □

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