PROBABILISTIC DYNAMICS OF TWO-LAYER GEOPHYSICAL FLOWS

IGOR CHUESHOV, JINQIAO DUAN, AND BJÖRN SCHMALFUSS

ABSTRACT. The two-layer quasigeostrophic flow model is an intermidiate system between the single-layer 2D barotropic flow model and the continuously stratified, 3D baroclinic flow model. This model is widely used to investigate basic mechanisms in geophysical flows, such as baroclinic effects, the Gulf Stream and subtropical gyres. The wind forcing acts only on the top layer. We consider the two-layer quasigeostrophic model under stochastic wind forcing. We first transformed this system into a coupled system of random partial differential equations and then show that the asymptotic probabilistic dynamics of this system depends only on the top fluid layer. Namely, in the probability sense and asymptotically, the dynamics of the two-layer quasigeostrophic fluid system is determinied by the top fluid layer, or, the bottom fluid layer is slaved by the top fluid layer. This conclusion is true provided that the Wiener process and the fluid parameters satisfy a certain condition. In particular, this latter condition is satisfied when the trace of the covariance operator of the Wiener process is controled by a certain upper bound, and the Ekman constant r is sufficiently large. Note that the generalized time derivative of the Wiener process models the fluctuating part of the wind stress forcing on the top fluid layer, and the Ekman constant r measures the rate for vorticity decay due to the friction in the bottom Ekman layer.

Date: March 31, 2001.

1991 *Mathematics Subject Classification.* Primary 60H15, 76U05; Secondary 86A05, 34D35.

Key words and phrases. Stochastic geophysical flow models, convergence in probability, random dynamical systems, random wind forcing.

1. Introduction

The continuously stratified, three dimensional (3D) baroclinic quasigeostrophic flow model describes large scale geophysical fluid motions in the atmosphere and oceans. This model is much simpler than the primitive flow model or rotating Navier-Stokes flow model. When the fluid density is approximately constant, this model reduces to the barotropic, single-layer, two dimensional (2D) quasigeostrophic model. The two-layer quasigeostrophic flow model, in which the fluid consists of two homogeneous fluid layers of uniform but distinct densities ρ_1 and ρ_2, is an intermidiate system between the single-layer 2D barotropic flow model and the continuously stratified, 3D baroclinic flow model.

The two-layer quasigeostrophic flow model has been used as a theoretical and numerical model to understand basic mechanisms, such as baroclinic effects [29], wind-drven circulation [4, 5], the Gulf Stream [24], fluid stability [3] and subtropical gyres [30, 28], in large scale geophysical flows.

We consider the two-layer quasigeostrophic flow model ([29], p. 423):

$$\begin{aligned} q_{1t} + J(\psi_1, q_1 + \beta y) &= \nu\Delta^2\psi_1 + f + \dot{W} \ , \\ q_{2t} + J(\psi_2, q_2 + \beta y) &= \nu\Delta^2\psi_2 - r\Delta\psi_2 \ , \end{aligned} \tag{1}$$

where potential vorticities $q_1(x,y,t)$, $q_2(x,y,t)$ for the top layer and the bottom layer are defined via stream functions $\psi_1(x,y,t)$, $\psi_2(x,y,t)$, respectively,

$$\begin{aligned} q_1 &= \Delta\psi_1 - F_1 \cdot (\psi_1 - \psi_2), \\ q_2 &= \Delta\psi_2 - F_2 \cdot (\psi_2 - \psi_1). \end{aligned} \tag{2}$$

Here x, y are Cartesian coordinates in zonal (east), meridional (north) directions, respectively; $(x,y) \in O := (0,L) \times (0,L)$, where L is a positive number; F_1, F_2 are positive constants defined by (see also [31], p.87)

$$F_1 = \frac{f_0^2}{gh_1}\frac{\rho_0}{\rho_2 - \rho_1},$$

$$F_2 = \frac{f_0^2}{gh_2}\frac{\rho_0}{\rho_2 - \rho_1},$$

with g the gravitational acceleration; h_1, h_2 the depth of top and bottom layers, ρ_1, ρ_2 the densities ($\rho_2 > \rho_1$) of top and bottom layers, respectively; and L, ρ_0 the characteristic scales for horizontal length and density of the flows, respectively; $f_0 + \beta y$ (with f_0, β constants) is the Coriolis parameter and β is the meridional gradient of the Coriolis parameter, and $\nu > 0$ is viscosity.. Note that $r = f_0\frac{\delta_E}{2(h_1+h_2)}$ is the Ekman constant ([30], p.29) which measures the intensity of friction at the bottom boundary layer (the so-called Ekman layer) or the rate for vorticity decay due to the friction in the Ekman layer. Here $\delta_E = \sqrt{2\nu/f_0}$ is the Ekman layer thickness ([29], p.188). Moreover, $J(h,g) = h_x g_y - h_y g_x$ is the Jacobi operator and $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator. Finally, $f(x,y,t)$ is the mean (deterministic) wind forcing with average zero: $\int_O f dO = 0$

An important part of the above equation is the white noise term $\dot{W}$, describing the fluctuating part of the external forcing in the top fluid layer, see Hasselmann [22] and Arnold [2]. An example of these short time scale influences is the weather variability or wind forcing. A white noise is given as the generalized time derivative of a Wiener process $W(t)$ in a function space.

We assume periodic boundary conditions for ψ in x and y with period L. In addition, we impose that

$$\int_O \psi dO = 0.$$

We also assume an appropriate initial condition

$$q(x,y,0) = q_0(x,y).$$

Stochastically forced QGE has been used to investigate various phenomena in geophysical flows [23, 27, 33, 21, 15, 9]. Recently Salmon [32] introduced some generalized two-layer ocean flow models.

It is our aim to study the long time dynamics of the stochastic differential equation with random coefficients but without white noise which we obtain after a random coordinate transformation. The solution of this differential equation generates a random dynamical system. The structure of this differential equation enables us to prove the dissipativity of the system. In order to investigate the long time dynamics we will apply the method of *determining functionals* in a version for random dynamical systems which is based on the convergence in probability, see Chueshov et. al [13]. It follows that the asymptotic long time behavior is determined by the asymptotic behavior of finitely many functionals. For instance, for these functionals we can choose the Fourier modes given by the eigenfunctions of the Laplacian. According to this property we can further show that the functionals defined on the top fluid layer alone can *determine* the asymptotical behavior of the complete two-layer system, when the fluid parameters and the Wiener process satisfy a certain condition, such as the Ekman constant r is sufficiently large and the trace of the covariance operator of the Wiener process is controled by some upper bound. Note that the generalized time derivative of the Wiener process models the fluctuating part of the wind stress forcing on the top fluid layer, and the Ekman constant r measures the rate for vorticity decay due to the friction in the bottom Ekman layer.

We first recall some basic facts in random dynamical systems in Section 2. In Section 3, we establish the well-posedness of the stochastic two-layer quasigeostrophic model by transforming it into a coupled system of random partial differential equations. The main results on asymptotic probabilistic determining functionals are presented in Sections 4 and 5. Finally we summarize our conclusions in Section 6.

2. RANDOM DYNAMICAL SYSTEM AND DETERMINING FUNCTIONALS

It is our goal to study the long time dynamics of (1) which is influenced by random forces. Appropriate tools to treat this equation are given by the theory of *random dynamical systems.*

A random dynamical system consists of two components. The first component is a *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as a model for a noise, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable flow: we have

$$\theta_0 = \mathrm{id}, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t \theta_\tau$$

for t, $\tau \in \mathbb{R}$. The measure $\mathbb{P}$ is supposed to be ergodic with respect to θ. The second component of a random dynamical system is a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$-measurable mapping φ satisfying the *cocycle* property

$$\varphi(t+\tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x)), \qquad \varphi(0, \omega, x) = x,$$

where the phase space H is a separable metric space and x is chosen arbitrarily in H. We will denote this random dynamical system by symbol φ.

A standard model for such a noise θ is the two-sided *Brownian motion*: Let U be a separable Hilbert space. We consider the probability space

$$(C_0(\mathbb{R}, U), \mathcal{B}(C_0(\mathbb{R}, U)), \mathbb{P})$$

where $C_0(\mathbb{R}, U)$ is the Fréchet space of continuous functions on $\mathbb{R}$ of uniform convergence on compact intervals which are zero at zero and $\mathcal{B}(C_0(\mathbb{R}, U))$ is the corresponding Borel σ-algebra. Suppose that we have a covariance operator Q on U. Then $\mathbb{P}$ denotes the *Wiener measure* with respect to Q. Note that $\mathbb{P}$ is ergodic with respect to the flow

$$\theta_t \omega = \omega(\cdot + t) - \omega(t), \qquad \text{for } \omega \in C_0(\mathbb{R}, U) \tag{3}$$

which is called the *Wiener shift.*

A main source of a random dynamical system is a random differential equation. For example, let us consider the following evolution equation in some Hilbert space

$$\frac{du}{dt} = F(u, \theta_t \omega), \quad u(0) = x, \tag{4}$$

over some metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. If (4) is well-posed for every $\omega \in \Omega$ and solutions $u(t, \omega; x)$ depends measurably on (t, ω, x), then the operator

$$\varphi \; : \; (t, \omega, x) \to u(t, \omega; x)$$

defines a random dynamical system (cocycle) φ. For detailed presentation of random dynamical systems we refer to the monograph by L. Arnold [1].

Motivated by deterministic dynamical systems we introduce several useful notions from the theory of random dynamical systems.

An ω depending closed set B contained in the separable Hilbert space H is called random if the mapping $\omega \to \sup_{x \in B(\omega)} \|x - y\|_H$ is a random variable for any $y \in H$.

A random dynamical system is called a *dissipative* if there exists a *compact* random set B which is forward invariant:

$$\varphi(t,\omega,B(\omega)) \subset B(\theta_t\omega),\ t>0,$$

and which is absorbing: for any random variable $x(\omega) \in H$ there exists a $t_x(\omega) > 0$ such that if $t \geq t_x(\omega)$

$$\varphi(t,\omega,x(\omega)) \in B(\theta_t\omega).$$

For the following we need *tempered random variable*: A random variable x is called tempered if

$$t \to |x(\theta_t\omega)|$$

is only subexponentially growing:

$$\limsup_{t\to\pm\infty} \frac{\log^+|x(\theta_t\omega)|}{|t|} = 0 \quad \text{a.s.}$$

This condition is not a very strong restriction because the only alternative is that the above lim sup is ∞ which describes the degenerated case of stationarity, see Arnold [1], page 164 f.

Our main purpose will be to estimate the degree of freedom of the long time dynamics of a random dynamical systems which stem from the two-layer flow problem of the ocean introduced above. We will apply the theory of determining functionals to estimate the degree of freedom. Since dynamics of this model is influenced by random forces we will apply the theory of determining functionals related to the convergence in probability.
We now give our basic definition:

Definition 2.1. *Let V be a Banach space which is continuously embedded into H. Assume that there exists $\tau > 0$ such that $\varphi(t,\omega,x) \in L^2_{loc}(\tau,+\infty;V)$ for almost all $\omega \in \Omega$ and $x \in H$. A set $\mathcal{L} = \{l_j,\ j=1,\cdots,N\}$ of linear continuous and linearly independent functionals on V is called asymptotically determining in probability if*

$$(\mathbb{P})\lim_{t\to\infty}\int_t^{t+1} \max_j |l_j(\varphi(\tau,\omega,x_1(\omega)) - \varphi(\tau,\omega,x_2(\omega)))|^2 d\tau = 0$$

for two initial conditions $x_1(\omega)$, $x_2(\omega) \in H$ implies

$$(\mathbb{P})\lim_{t\to\infty}\|\varphi(t,\omega,x_1(\omega)) - \varphi(t,\omega,x_2(\omega))\|_H = 0.$$

The theory of determining functionals was started with the papers [19] and [26] devoted to $2D$ Navier-Stokes equations. Now this theory is well-developed for the deterministic systems (see, e.g., [7, 10, 11, 18, 20, 25] and the references therein). Some results are also available for stochastic systems (see, e.g., [8, 12, 13, 17]). One of the main advantages for this theory is the possibility to localize spatial domains or parameters which are responsible for long-time dynamics. The existence of a finite number of determining functionals means that the long-time behaviour of the system is finite-dimensional. Moreover the values of these functionals on solutions can be intertreted as a result of some measurament of the system. From applied point of view the finiteness of number of determining functionals means that we need only a finite number of devices to observe completely all dynamics of the system.

In this paper we rely on the following result.

Theorem 2.2. *Assume that a random dynamical system* φ *has an absorbing forward invariant random set* B *in* V *such that* $\sup_{x\in B(\omega)}\|x\|_V^2$ *is bounded by a tempered random variable and* $t\to\sup_{x\in B(\theta_t\omega)}\|x\|_V^2$ *is locally integrable. Let* $\mathcal{L}=\{l_j : j=1,...,N\}$ *be a set of linear continuous and linearly independent functionals on* V*. Suppose there exist constants* $c_{\mathcal{L}}>0$ *and* $C_{\mathcal{L}}>0$ *and a measurable function* $l_{\mathcal{L}}(x_1,x_2,\omega)$ *which maps* $V\times V\times\Omega$ *in* $\mathbb{R}$ *such that* $l_{\mathcal{L}}(x_1,x_2,\omega)\geq -c_{\mathcal{L}}$ *uniformly with respect to* $(x_1,x_2,\omega)\in V\times V\times\Omega$ *and for* $x_1(\omega)$, $x_2(\omega)\in B(\omega)$ *we have*

$$\begin{aligned} V(t,\omega)-V(s,\omega)\leq C_{\mathcal{L}}\cdot\int_s^t\mathcal{N}_{\mathcal{L}}(\tau,\omega)d\tau \\ +\int_s^t l_{\mathcal{L}}(\varphi(\tau,\omega,x_1),\varphi(\tau,\omega,x_2),\theta_\tau\omega)\cdot V(\tau,\omega)d\tau \end{aligned}\tag{5}$$

for all $t\geq s\geq 0$*, where*

$$V(t,\omega)=\|\varphi(t,\omega,x_1)-\varphi(t,\omega,x_2)\|_H^2,$$
$$\mathcal{N}_{\mathcal{L}}(t,\omega)=\max_{j=1,\dots,N}|l_j(\varphi(t,\omega,x_1)-\varphi(t,\omega,x_2))|^2.$$

Assume that

$$\frac{1}{t}\mathbb{E}\left\{\sup_{x_1,x_2\in B(\omega)}\int_0^t l_{\mathcal{L}}(\varphi(\tau,\omega,x_1),\varphi(\tau,\omega,x_2),\theta_\tau\omega)d\tau\right\}<0$$

for some $t>0$*. Then* $\mathcal{L}$ *is a set of asymptotically determining functionals in probability for random dynamical system* φ*.*

Proof. It is easy to find from relation (5) that

$$V(t,\omega)\leq V(0,\omega)e^{\int_0^t l_{\mathcal{L}}(s,\omega)ds}+C_{\mathcal{L}}\int_0^t\mathcal{N}_{\mathcal{L}}(\tau,\omega)e^{\int_\tau^t l_{\mathcal{L}}(s,\omega)ds}d\tau,$$

where $l(t,\omega)=l_{\mathcal{L}}(\varphi(t,\omega,x_1),\varphi(t,\omega,x_2),\theta_t\omega)$. Therefore we can apply the argument given in the proof of Theorem 2.2 [13]. ☐

In the next section, we return to the two-layer quasigeostrophic flow model.

3. MATHEMATICAL SETUP AND WELL-POSEDNESS OF THE RANDOM TWO-LAYER FLOWS

In the following, L^2_{per}, H^s_{per} for $s\in\mathbb{R}$ are the standard Sobolev spaces of L-periodic functions with the zero mean value, i.e. $\int_O\psi dO=0$. Let $(\cdot,\cdot)_0$

and $\|\cdot\|_0$ denote the standard scalar product and norm in L^2_{per}, respectively. Every element $u(x,y) \in L^2_{per}$ can be represented in the form

$$u(x,y) = \sum_{j\in\mathbb{Z}^2, j\neq 0} u_j L^{-1} \exp\left\{ i\frac{2\pi}{L}(j_1 x + j_2 y)\right\},$$

where the Fourier coefficients u_j possess the property $\bar{u}_j = u_{-j}$ (bar denotes the complex conjugation) and

$$\|u\|_0^2 := \int_O |u|^2 dO \equiv \sum_{j\in\mathbb{Z}^2, j\neq 0} |u_j|^2 < \infty.$$

Note that $L^{-1}\exp\left\{i\frac{2\pi}{L}(j_1x+j_2y)\right\}$ is the eigenelement of the $-\Delta$ with the eigenvalue $\lambda_1 \cdot (j_1^2 + j_2^2)$, where $\lambda_1 = \left(\frac{2\pi}{L}\right)^2$ is the smallest eigenvalue. The norm in H^s_{per} is defined by the formula

$$\|u\|_s^2 := \int_O |(-\Delta)^{\frac{s}{2}} u|^2 dO \equiv \lambda_1^s \sum_{j\in\mathbb{Z}^2, j\neq 0} (j_1^2 + j_2^2)^s |u_j|^2.$$

It is clear that

$$\|\nabla u\|_s^2 := \|\partial_x u\|_s^2 + \|\partial_y u\|_s^2 = \|u\|_{1+s}^2, \quad s\in\mathbb{R}. \tag{6}$$

We also denote $\mathbf{L}^2_{per} = L^2_{per}\times L^2_{per}$ and $\mathbf{H}^s_{per} = H^s_{per}\times H^s_{per}$.
We work on the phase space $\mathbf{H}^{-1}_{per}$ with the scalar product

$$(q,\bar{q})_* = h_1(\nabla\psi_1, \nabla\bar{\psi}_1)_0 + h_2(\nabla\psi_2,\nabla\bar{\psi}_2)_0 + p(\psi_1-\psi_2, \bar{\psi}_1 - \bar{\psi}_2)_0,$$

where $q=(q_1,q_2)$, $\bar{q} = (\bar{q}_1,\bar{q}_2)$ and $\psi = (\psi_1,\psi_2)\in\mathbf{H}^1_{per}$, $\bar{\psi} = (\bar{\psi}_1,\bar{\psi}_2)$. The relation between q (resp. $\bar{q}$) and ψ (resp. $\bar{\psi}$) is defined by (2). Here we also use the notation

$$p = \frac{f_0^2}{g}\frac{\rho_0}{\rho_2-\rho_1}.$$

Note that $F_1h_1 = F_2h_2 = p$. The norm induced by this scalar product

$$\|q\|_*^2 = (q,q)_* = h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2 + p\|\psi_1-\psi_2\|_0^2$$

is equivalent to the usual norm on $\mathbf{H}^{-1}_{per}$. Moreover, we have the estimate

$$h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2 \le \|q\|_*^2 \le a_0\left(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2\right), \tag{7}$$

where

$$a_0 = 1 + \frac{2p}{\lambda_1\min\{h_1,h_2\}} = 1 + \frac{2}{\lambda_1}\max\{F_1,F_2\}.$$

To treat the nonlinearity we need the following lemma:

Lemma 3.1. *The operator Jacobian verifies, for u, v, w in H^1_{per}*

$$J(u,v) = -J(v,u), \qquad (J(u,v),v)_0 = 0, \tag{8}$$

$$(J(u,v),w)_0 = (J(v,w),u)_0. \tag{9}$$

Moreover the following estimates hold:

$$|(J(u,v),\Delta u)_0| \le c_0\|\Delta v\|_0\cdot\|\nabla u\|_0\cdot\|\Delta u\|_0, \quad u,\, v\in H^2_{per}; \tag{10}$$

$$|(J(u,v),w)_0| \le c_1\|\Delta u\|_0 \cdot \|\Delta v\|_0 \cdot \|w\|_0, \quad u,\, v \in H^2_{per},\, w \in L^2_{per}; \tag{11}$$

$$|(J(u,v),w)_0| \le c_1\|\nabla u\|_0 \cdot \|\Delta v\|_0 \cdot \|\nabla w\|_0, \quad u,\, w \in H^1_{per}\,, v \in H^2_{per}. \tag{12}$$

Here above $c_0 = 2 + (\sqrt{2}\cdot\pi)^{-1}$ *and* $c_1 = c_0\lambda_1^{-\frac{1}{2}}$.

Proof. We start with (10). It is easy to see that

$$\begin{aligned} J(u,v)\cdot\Delta u =& \frac{1}{2}\cdot v_y\left\{\partial_x\left(u_x^2-u_y^2\right)+2\partial_y\left(u_xu_y\right)\right\} \\ &+\frac{1}{2}\cdot v_x\left\{\partial_y\left(u_x^2-u_y^2\right)-2\partial_x\left(u_xu_y\right)\right\}. \end{aligned}$$

Consequently relying on property (6) we have

$$\begin{aligned} |(J(u,v),\Delta u)_0| \le& \frac{1}{2}\cdot\|v_y\|_1\left(\|\partial_x\left(u_x^2-u_y^2\right)\|_{-1}+2\|\partial_y\left(u_xu_y\right)\|_{-1}\right) \\ &+\frac{1}{2}\cdot\|v_x\|_1\left(\|\partial_y\left(u_x^2-u_y^2\right)\|_{-1}+2\|\partial_x\left(u_xu_y\right)\|_{-1}\right) \\ \le& \frac{1}{2}\cdot\left(\|v_x\|_1+\|v_y\|_1\right)\cdot\left(\|u_x\|_{L^4}^2+\|u_y\|_{L^4}^2+2\|u_xu_y\|_0\right) \\ \le& \left(\|v_x\|_1+\|v_y\|_1\right)\cdot\left(\|u_x\|_{L^4}^2+\|u_y\|_{L^4}^2\right) \\ \le& \sqrt{2}\cdot\|\Delta v\|_0\cdot\left(\|u_x\|_{L^4}^2+\|u_y\|_{L^4}^2\right). \end{aligned}$$

Using the inequality (see, e.g., [18])

$$\|u\|_{L^4} \le a_1\cdot\|u\|^{1/2}\cdot\|\nabla u\|^{1/2} \quad \text{with} \quad a_1 = \left((2\pi)^{-1}+\sqrt{2}\right)^{1/2},$$

we obtain (10).

In a similar way we can establish the estimates

$$\begin{aligned} |(J(u,v),w)_0| &\le c_0\|\nabla u\|_0^{1/2}\cdot\|\Delta u\|_0^{1/2}\cdot\|\nabla v\|_0^{1/2}\cdot\|\Delta v\|_0^{1/2}\cdot\|w\|_0, \\ |(J(u,v),w)_0| &\le c_0\|u\|_0^{1/2}\cdot\|\nabla u\|_0^{1/2}\cdot\|\nabla v\|_0^{1/2}\cdot\|\Delta v\|_0^{1/2}\cdot\|\nabla w\|_0, \end{aligned}$$

which easily imply (11) and (12). □

We now transform the two-layer model (1) containing white noise into a system of coupled random partial differential equations. The reason for taking such a transformation is that we need some particular a priori estimates for the trajectories of the system. Often these a priori estimates can be calculated by the Gronwall lemma. For equations containing a white noise there exists no Gronwall lemma. But we can use this technique for the transformed random partial differential equations.

For this purpose we introduce an Ornstein-Uhlenbeck process $\eta(x,y,t,\omega)$ in L^2_{per}. This process is defined by the solution of the following linear stochastic partial differential equation

$$\eta_t = \nu(k+1)\Delta\eta + \dot{W}, \tag{13}$$

with periodic boundary condition and with some initial condition $x \in L^2_{per}$ where $k > 0$ is a free *control* parameter and W is a Wiener process in L^2_{per}.

We suppose that the covariance operator Q of this Wiener process has a finite trace. As mentioned in Section 2 such a Wiener process generates a metric dynamical system denoted by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ where θ is the Wiener shift introduced in (3). It is well known that this equation has a stationary solution which is generated by a *Gaussian* random variable η in H^1_{per}. In particular, the mapping

$$t \to \eta(\theta_t \omega) \in L^2_{loc}(-\infty, \infty; H^1_{per})$$

solves this equation. For moments of η we obtain:

$$\mathbb{E}\|\eta\|_1^2 \le \frac{\mathrm{tr}_0 Q}{2\nu(k+1)}, \quad \mathbb{E}\|\eta\|_1^{2n} \le C_n \left(\frac{\mathrm{tr}_0 Q}{\nu(k+1)}\right)^n, \ n \in \mathbb{N}, \ C_n > 0. \tag{14}$$

We introduce new variables

$$\tilde{q}_1 := q_1 - \eta, \ q_2, \ \tilde{\psi}_1 := \psi_1 + \xi_1, \ \tilde{\psi}_2 := \psi_2 + \xi_2, \tag{15}$$

where stationary process η solves the problem (13) and ξ_1 and ξ_2 are defined such that the elliptic equations (2) remain of the same form

$$\begin{aligned} \tilde{q}_1 &= \Delta\tilde{\psi}_1 - F_1 \cdot (\tilde{\psi}_1 - \tilde{\psi}_2), \\ q_2 &= \Delta\tilde{\psi}_2 - F_2 \cdot (\tilde{\psi}_2 - \tilde{\psi}_1), \end{aligned} \tag{16}$$

The processes ξ_1 and ξ_2 are solutions of the linear elliptic equations

$$\begin{aligned} \Delta\xi_1 - F_1 \cdot (\xi_1 - \xi_2) &= -\,\eta, \\ \Delta\xi_2 - F_2 \cdot (\xi_2 - \xi_1) &= 0 \end{aligned} \tag{17}$$

and can be presented in the form

$$\begin{aligned} \xi_1 &= \frac{1}{F_1 + F_2}\left\{F_2(-\Delta)^{-1} + F_1(-\Delta + F_1 + F_2)^{-1}\right\}\eta \\ \xi_2 &= \frac{F_2}{F_1 + F_2}\left\{(-\Delta)^{-1} - (-\Delta + F_1 + F_2)^{-1}\right\}\eta. \end{aligned}$$

Thus the processes ξ_1 and ξ_2 are smoother in spatial variables than η. In fact after simple calculations we have the estimates

$$\|\xi_i\|_{s+2} \le \|\eta\|_s, \quad i = 1, 2, \quad \text{and} \quad \|\xi_1 - \xi_2\|_{s+2} \le \|\eta\|_s \quad s \in \mathbb{R}. \tag{18}$$

Now we obtain the coupled random partial differential equations for new potential vorticities $\tilde{q}_1$, q_2:

$$\begin{aligned} \tilde{q}_{1t} + J(\tilde{\psi}_1 - \xi_1, \tilde{q}_1 + \eta + \beta y) &= \nu\Delta^2\tilde{\psi}_1 + f \\ &\quad - \nu\Delta^2\xi_1 - \nu(k+1)\Delta\eta\ , \\ q_{2t} + J(\tilde{\psi}_2 - \xi_2, q_2 + \beta y) &= \nu\Delta^2\tilde{\psi}_2 - r\Delta\tilde{\psi}_2 \\ &\quad - \nu\Delta^2\xi_2 + r\Delta\xi_2. \end{aligned}$$

We treat η, ξ_1, ξ_2 as known processes. Using (17) we have

$$\begin{aligned} -\nu\Delta^2\xi_1 - \nu(k+1)\Delta\eta &= -\,\nu F_1(\Delta\xi_1 - \Delta\xi_2) - \nu k\Delta\eta, \\ -\nu\Delta^2\xi_2 &= -\,\nu F_2(\Delta\xi_2 - \Delta\xi_1). \end{aligned}$$

For convenience, we drop the tilde and rewrite the above system. Thus we finally get the coupled system of random partial differential equations

$$\begin{aligned} q_{1t} + J(\psi_1 - \xi_1, q_1 + \eta + \beta y) =& \nu\Delta^2\psi_1 + f \\ & - \nu F_1(\Delta\xi_1 - \Delta\xi_2) - \nu k\Delta\eta \\ q_{2t} + J(\psi_2 - \xi_2, q_2 + \beta y) =& \nu\Delta^2\psi_2 - r\Delta\psi_2 \\ & - \nu F_2(\Delta\xi_2 - \Delta\xi_1) + r\Delta\xi_2 \end{aligned} \tag{19}$$

with

$$\begin{aligned} q_1 =& \Delta\psi_1 - F_1 \cdot (\psi_1 - \psi_2), \\ q_2 =& \Delta\psi_2 - F_2 \cdot (\psi_2 - \psi_1), \end{aligned} \tag{20}$$

in the class of L-periodic functions with initial data $q(x, y, 0) = q_0(x, y) \equiv (q_{01}(x, y), q_{02}(x, y)) \in \mathbf{H}^{-1}_{per}$, where η is the stationary solution to (13) and ξ_1 and ξ_2 are solves (17) in H^2_{per}.

For the rest of the paper, we work on this coupled system of random partial differential equations for stochastically forced two-layer quasigeostrophic fluid system.

The coefficients of coupled system (19), (20) have similar properties as the coefficients of the corresponding deterministic two-layer quasigeostrophic system (cf. [6], where N-layer model with other boundary conditions is considered). Therefore, similar to [6], using the Galerkin method and the compactness argument we can prove the following assertion on well-posedness of problem (19), (20) in the class of L-periodic functions.

Theorem 3.2. *Let $q_0 \in \mathbf{H}^{-1}_{per}$ and $f \in L^2_{per}$. Then for all $\omega \in \Omega$ and for all $T > 0$, the system (19), (20) has a unique solution $\{q(t), \psi(t)\}$ such that*

$$q \in C([0, T]; \mathbf{H}^{-1}_{per}) \cap L^2(0, T; \mathbf{L}^2_{per}) \cap L^2_{loc}(0, T; \mathbf{H}^1_{per}).$$

The function ψ associated to q by (20) satisfies

$$\psi \in C([0, T]; \mathbf{H}^1_{per}) \cap L^2(0, T; \mathbf{H}^2_{per}) \cap L^2_{loc}(0, T; \mathbf{H}^3_{per}).$$

The solution depends continuously on the initial condition $q_0 \in \mathbf{H}^{-1}_{per}$.

By the uniqueness assertion of the last Theorem the solution $t \to q(t)$ generates a random dynamical system φ with continuous $\mathbf{H}^{-1}_{per} \ni q \to \varphi(t, \omega, q)$ on the phase space $\mathbf{H}^{-1}_{per}$.

4. Dissipativity of the random dynamical system

Dynamical systems generated by many nonlinear parabolic differential equations have the dissipative property which means that there exists a compact set absorbing the states of the system in finite time. Usually one can choose such a set which is also forward invariant.

We now construct an absorbing forward invariant set for the random dynamical system generated by (19). This set will be a random set.

Theorem 4.1. *There exists a compact random set* $B(\omega) \subset \mathbf{H}^{-1}_{per}$ *such that*

$$\begin{aligned} &\varphi(t,\omega,B(\omega)) \subset B(\theta_t\omega) \quad \text{for } t \geq 0, \\ &\varphi(t,\omega,q(\omega)) \subset B(\theta_t\omega) \quad \text{for } t \geq t_0(\omega,q) \end{aligned} \tag{21}$$

where q *is a random variable with values in* $\mathbf{H}^{-1}_{per}$.

We now divide the proof of this theorem in some lemmata. We start with the following:

Lemma 4.2. *Let* $q(t)$ *be the solution of (19). Then* $q(t)$ *satisfies the following inequality*

$$\begin{aligned} \frac{d}{dt}\|q(t)\|_*^2 &+ \nu(h_1\|\Delta\psi_1(t)\|_0^2 + h_2\|\Delta\psi_2(t)\|_0^2) \\ &\leq d_0 \cdot \|\eta(\theta_t\omega)\|_0^2 \cdot (h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2) + m(\theta_t\omega), \end{aligned}$$

where

$$m(\omega) = d_1\|\eta(\omega)\|_0^4 + d_2\|\eta(\omega)\|_0^2 + d_3$$

and

$$\begin{aligned} d_0 =& \frac{6c_0^2}{\nu}\left(1 + \frac{p^2\nu}{\lambda_1^2\min\{h_1,h_2\}}\right), \\ d_1 =& \frac{6c_0^2h_1}{\nu\lambda_1}, \\ d_2 =& 9\left(\frac{\beta^2(h_1+h_2)}{\nu\lambda_1^3} + \frac{\nu p^2}{\lambda_1^2}\left(\frac{1}{h_1} + \frac{1}{5h_2}\right) + \frac{rh_2}{18\lambda_1} + k^2\nu h_1\right), \\ d_3 =& \frac{9h_1}{\nu\lambda_1}\|f\|_{-1}^2. \end{aligned}$$

Proof. Multiplying the equations (19) by $-2h_1\psi_1$, $-2h_2\psi_2$, respectively, and then adding together, we obtain

$$\begin{aligned} \frac{d}{dt}\|q(t)\|_*^2 &+ 2\nu(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) + 2rh_2\|\nabla\psi_2\|_0^2 \\ =& 2h_1(J(\psi_1 - \xi_1, q_1 + \eta + \beta y), \psi_1)_0 \\ &+ 2h_2(J(\psi_2 - \xi_2, q_2 + \beta y), \psi_2)_0 - 2h_1(f,\psi_1)_0 + 2h_1\nu k(\Delta\eta,\psi_1)_0 \\ &+ 2\nu p(\Delta\xi_1 - \Delta\xi_2, \psi_1)_0 + 2\nu p(\Delta\xi_2 - \Delta\xi_1, \psi_2)_0 \\ &- 2rh_2(\Delta\xi_2,\psi_2)_0. \end{aligned}$$

We have by Lemma 3.1

$$(J(\psi_1, q_1+\eta),\psi_1)_0 = 0, \ (J(\psi_2,q_2),\psi_2)_0 = 0.$$

A simple calculation shows that $(J(\psi_i,\beta y),\psi_i)_0 = 0$. We now estimate the right hand side terms. On account of (20), (10), (12), the bilinearity of J

and (18) we can estimate

$$
\begin{aligned}
-2\sum_{i=1}^{2} h_i(J(\xi_i, q_i), \psi_i)_0 = & -2\sum_{i=1}^{2} h_i(J(\xi_i, \Delta\psi_i), \psi_i)_0 + 2p(J(\xi_1 - \xi_2, \psi_2), \psi_1)_0 \\
\leq & 2c_0 \sum_{i=1}^{2} h_i \|\Delta\xi_i\|_0 \|\Delta\psi_i\|_0 \|\nabla\psi_i\|_0 + 2c_1 p \|\Delta(\xi_1 - \xi_2)\|_0 \|\nabla\psi_1\|_0 \|\nabla\psi_2\|_0 \\
\leq & \frac{\nu}{6}(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) + 6\frac{c_0^2}{\nu}\|\eta\|_0^2(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2) \\
& + \frac{c_1 p\|\eta\|_0}{\min\{h_1, h_2\}}(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2) \\
\leq & \frac{\nu}{3}(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) + d_0\|\eta\|_0^2(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2).
\end{aligned}
$$

Similarly, due to (11) and (18)

$$
-2h_1(J(\xi_1, \eta), \psi_1)_0 \leq 2c_1 h_1 \|\Delta\xi_1\|_0 \|\eta\|_0 \|\Delta\psi_1\|_0 \leq \frac{6c_1^2 h_1 \|\eta\|_0^4}{\nu} + \frac{\nu h_1}{6}\|\Delta\psi_1\|_0^2.
$$

By the Cauchy-Schwarz inequality

$$
\begin{aligned}
& -2\sum_{i=1}^{2} h_i(J(\xi_i, \beta y), \psi_i)_0 \leq 2\beta \sum_{i=1}^{2} h_i \|\nabla\xi_i\|_{-1} \|\nabla\psi_i\|_0 \\
& \leq \frac{9\beta^2(h_1 + h_2)}{\nu\lambda_1^3}\|\eta\|_0^2 + \frac{\nu}{9}(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2), \\
& -2h_1(f, \psi_1)_0 \leq 2h_1\|f\|_{-1}\|\nabla\psi_1\|_0 \leq \frac{9h_1}{\nu\lambda_1}\|f\|_{-1}^2 + \frac{\nu h_1}{9}\|\Delta\psi_1\|_0^2 \\
& + 2h_1\nu(k\Delta\eta, \psi_1)_0 \leq 9k^2\nu h_1\|\eta\|_0^2 + \frac{\nu h_1}{9}\|\Delta\psi_1\|_0^2.
\end{aligned}
$$

Using (18) once more

$$
\begin{aligned}
2\nu p(\Delta(\xi_1 - \xi_2), \psi_1)_0 & \leq 2\nu p\|\Delta\xi_1 - \Delta\xi_2\|_{-2}\|\Delta\psi_1\|_0 \\
& \leq \frac{9\nu p^2}{\lambda_1^2 h_1}\|\eta\|_0^2 + \frac{\nu h_1}{9}\|\Delta\psi_1\|_0^2, \\
-2\nu p(\Delta(\xi_1 - \xi_2), \psi_2)_0 & \leq \frac{9\nu p^2}{5\lambda_1^2 h_2}\|\eta\|_0^2 + \frac{5\nu h_2}{9}\|\Delta\psi_2\|_0^2, \\
-2rh_2(\Delta\xi_2, \psi_2)_0 & \leq \frac{rh_2}{2\lambda_1}\|\eta\|_0^2 + 2rh_2\|\nabla\psi_2\|_0^2.
\end{aligned}
$$

Adding all these inequalities together we obtain the conclusion. □

We now consider the random variable η defined in Section 2. Recall that η depends on the control parameter k. If k is chosen large enough then particular moments of η are small. Especially we can formulate:

Lemma 4.3. *Let* W *be a Wiener process in* L^2_{per} *with finite trace of the covariance. Then under assumptions*

$$\frac{2d_0 a_0 \mathrm{tr}_0 Q}{\lambda_1^2 \nu^2 (k+1)} < 1, \quad \frac{16 d_0 \mathrm{tr}_0 Q}{\lambda_1^2 \nu^2 (k+1)^2} < 1 \tag{22}$$

the random variable

$$R_0(\omega) := \int_{-\infty}^{0} e^{\frac{\nu\lambda\tau}{a_0}\tau + d_0 \int_\tau^0 \|\eta(\theta_{\tau'}\omega)\|_0^2 d\tau'} m(\theta_\tau \omega) d\tau$$

is finite and tempered. Moreover

$$\left(\mathbb{E}R_0^2\right)^{1/2} \le d_4^2 \left(\frac{3a_0}{2\nu\lambda_1}\right)^{3/2} \cdot \left(\frac{2\lambda_1\nu}{a_0} - \frac{4d_0 \mathrm{tr}_0 Q}{\lambda_1 \nu (k+1)}\right)^{-1/2},$$

where

$$d_4 = C_8^{1/4} \frac{d_1 (\mathrm{tr}_0 Q)^2}{\lambda_1^2 \nu^2 (k+1)^2} + C_4^{1/4} \frac{d_2 \mathrm{tr}_0 Q}{\lambda_1 \nu (k+1)} + d_3$$

is an estimate for $(\mathbb{E}m^4)^{1/4}$ *(the constants* C_8, C_4 *are defined in (14)).*

The proof of this lemma can be found in Chueshov et. al. [13] for an Ornstein-Uhlenbeck process in another Hilbert space. However the argument given there is of a general nature.

We now construct a set satisfying (21).

Lemma 4.4. *Let* $R(\omega) := aR_0(\omega)$ *for some* $a > 1$ *and* R_0 *as in Lemma 4.3. Then the closed* $\mathbf{H}^{-1}_{per}$*-ball* $B(0, R(\omega)^{\frac{1}{2}})$ *fullfills (21) provided conditions (22) hold.*

Proof. Using Lemma 4.2 and relation (7) we have

$$\frac{d}{dt}\|q(t)\|_*^2 \le \left(-\frac{\nu\lambda_1}{a_0} + d_0 \cdot \|\eta(\theta_t\omega)\|_0^2\right) \cdot \|q(t)\|_*^2 + m(\theta_t\omega).$$

Let $q_0 = q(0)$ and $\rho(t, \omega, \|q_0\|_*^2)$ be the solution of one dimensional random affine equation

$$\frac{d\rho(t)}{dt} + \frac{\nu\lambda_1}{a_0}\rho = d_0 \|\eta(\theta_t\omega)\|_0^2 \rho + m(\theta_t\omega), \quad \rho(0, \omega, \|q_0\|_*^2) = \|q_0\|_*^2. \tag{23}$$

A comparison argument gives that

$$\|\varphi(t, \omega, q_0)\|_*^2 \equiv \|q(t)\|_*^2 \le \rho(t, \omega, \|q_0\|_*^2).$$

Here φ is the dynamical system introduced in Section 3: $\varphi(t, \omega, q_0) = q(t)$, where $q(t)$ is the solution to (19) with the initial data q_0. Equation (23) has the stationary solution given by $t \to R_0(\theta_t\omega)$:

$$\rho(t, \omega, R_0(\omega)) = R_0(\theta_t\omega).$$

This can be checked by the variation of constants formula. This solution is exponentially attracting which follows from the variation of constants formula again:

$$
\begin{aligned}
|R_0(\theta_t\omega) - \rho(t,\omega,\|q_0\|_*^2)| &= |\rho(t,\omega,R_0(\omega)) - \rho(t,\omega,\|q_0\|_*^2)| \\
&\le e^{\int_0^t \left(d_0\|\eta(\theta_\tau\omega)\|_0^2 - \frac{\nu\lambda_1}{a_0}\right)d\tau} \left(R(\omega) + \|q_0\|_*^2\right)
\end{aligned}
$$

which tends to zero exponentially fast. Indeed, it follows from (14) and (22) that for a sufficient small $\varepsilon > 0$,

$$
\int_0^t d_0\|\eta(\theta_\tau\omega)\|_0^2 d\tau < \frac{\nu\lambda_1 - \varepsilon}{a_0}t, \qquad \frac{\nu\lambda_1 - \varepsilon}{a_0} > 0
$$

for large $t > 0$ under conditions (22). ☐

It remains to prove the existence of a compact set B satisfying (23).

Lemma 4.5. *Suppose that the random variable $R(\omega)$ is defined in Lemma 4.4. The set*

$$
B(\omega) := \overline{\varphi(1,\theta_{-1}\omega, B(0, R(\theta_{-1}\omega)^{\frac{1}{2}}))}
$$

is a compact absorbing forward invariant random set. Moreover

$$
\omega \mapsto \sup\left\{\|\Delta\psi_1\|_0^2 + \|\Delta\psi_1\|_0^2 \ : \ (q_1,q_2) \in B(\omega)\right\} \tag{24}
$$

is a tempered random variable (ψ_1 and ψ_2 are defined by (20)).

Proof. Since $\varphi(t,\omega,\cdot)$ is completely continuous for $t > 0$ (see the regularity assertion of Theorem 3.2) the sets $B(\omega)$ are compact. Since R is a random variable the ball $B(0, R^{\frac{1}{2}})$ is a random set. The continuity of $\varphi(t,\omega,\cdot)$ allows us to conclude that B is a random set. The construction of B ensures that that set in absorbing and forward invariant. The temperedness of (24) can be proved in the same way as in [13] for the $2D$ Navier-Stokes equations. ☐

Hence we have shown that the two-layer quasigeostrophic flow system is dissipative in the probabilistic sense. In the next section we consider asymptotic probabilistic determining functionals.

5. Determininging functionals

In this section, we compare two solutions as a way to understand asymptotic dynamics in the probabilistic sense. Consider the difference of two solutions $\hat{q}$ and $\bar{q}$, corresponding to the stream functions $\hat{\psi}$, $\bar{\psi}$. We set

$$
q = (q_1, q_2) = \hat{q} - \bar{q},
$$

corresponding to the stream function

$$
\psi = \hat{\psi} - \bar{\psi}.
$$

We get random partial differential equations for the solution difference $q = \hat{q} - \bar{q}$ from (19):

$$\begin{aligned} q_{1t} =& \nu\Delta^2\psi_1 \\ & - J(\psi_1, \hat{q}_1 + \beta y) - J(\bar{\psi}_1, q_1) - J(\psi_1, \eta) + J(\xi_1, q_1)\ , \end{aligned} \tag{25}$$

$$\begin{aligned} q_{2t} =& \nu\Delta^2\psi_2 - r\Delta\psi_2 \\ & - J(\psi_2, \hat{q}_2 + \beta y) - J(\bar{\psi}_2, q_2) + J(\xi_2, q_2)\ , \end{aligned} \tag{26}$$

Multiplying (25) by $-h_1\psi_1$, and (26) by $-h_2\psi_2$ and adding together, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|q\|_*^2 =& -\nu[h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2] - rh_2\|\nabla\psi_2\|_0^2 \\ & + h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 + h_2(J(\bar{\psi}_2 - \xi_2, q_2), \psi_2)_0. \end{aligned} \tag{27}$$

We start with

$$h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 \equiv I_1 + I_2,$$

where

$$I_1 = h_1(J(\bar{\psi}_1 - \xi_1, \Delta\psi_1), \psi_1)_0, \quad I_2 = p(J(\bar{\psi}_1 - \xi_1, \psi_2), \psi_1)_0.$$

By (10) we have

$$|I_1| \le c_0 h_1 \|\bar{\psi}_1 - \xi_1\|_2 \|\Delta\psi_1\|_0 \|\nabla\psi_1\|_0.$$

Using (12) we obtain

$$|I_2| \le c_1 p \|\bar{\psi}_1 - \xi_1\|_2 \|\nabla\psi_2\|_0 \|\nabla\psi_1\|_0.$$

Thus

$$|I_2| \le c_1 p \lambda_1^{-\frac{1}{2}} \|\bar{\psi}_1 - \xi_1\|_2 \|\Delta\psi_2\|_0 \|\nabla\psi_1\|_0.$$

Using the inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$, we get

$$\begin{aligned} h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 \le& \frac{\nu}{4}\left(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2\right) \\ & + \frac{h_1 c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2})\|\bar{\psi}_1 - \xi_1\|_2^2 \|\nabla\psi_1\|_0^2. \end{aligned} \tag{28}$$

In a similar way we have

$$\begin{aligned} h_2(J(\bar{\psi}_2 - \xi_2, q_2), \psi_2)_0 \le& \frac{\nu}{4}\left(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2\right) \\ & + \frac{h_2 c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2})\|\bar{\psi}_2 - \xi_2\|_2^2 \|\nabla\psi_2\|_0^2. \end{aligned} \tag{29}$$

Using (27), (28) and (29) we obtain

$$\begin{aligned} \frac{d}{dt}\|q\|_*^2 \le& -\nu[h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2] - 2rh_2\|\nabla\psi_2\|_0^2 \\ & + b_0\left(h_1\|\bar{\psi}_1 - \xi_1\|_2^2\|\nabla\psi_1\|_0^2 + h_2\|\bar{\psi}_2 - \xi_2\|_2^2\|\nabla\psi_2\|_0^2\right)\ , \end{aligned} \tag{30}$$

where $b_0 = \frac{2c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2})$. This estimate is the main point in the construction of determining functionals. It is our aim to obtain an integral equation like (5). To this end we first consider the case when parameter

$r > 0$ is arbitrary. Let $\mathcal{L} = \{l_j\}_1^N$ be a set of linearly independent bounded linear functionals on the space $\mathbf{H}^2_{per}$. Assume that the set $\mathcal{L}$ possesses the property

$$\|\psi\|_1 \le C_{\mathcal{L}} \max_j |l_j(\psi)| + \varepsilon_{\mathcal{L}} \|\psi\|_2, \tag{31}$$

for any $\psi \in \mathbf{H}^2_{per}$ with some positive constant $C_{\mathcal{L}}$ and $\varepsilon_{\mathcal{L}}$. We note that the best possible value of the parameter $\varepsilon_{\mathcal{L}}$ is called the completeness defect of the family $\mathcal{L}$ with respect of the pair of the spaces $\mathbf{H}^1_{per}$ and $\mathbf{H}^2_{per}$ (see [10, 11]) and the smallness of the parameter $\varepsilon_{\mathcal{L}}$ is crucial in the subsequent considerations. We refer to [10, 11] for general properties of completeness defect and for estimates of $\varepsilon_{\mathcal{L}}$ for several families of functionals on Sobolev spaces.

From (31) for $\psi = (\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)$ we have

$$\begin{aligned}\left(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2\right) \ge & \frac{1-\delta}{\varepsilon_{\mathcal{L}}^2}\left(h_1\|\nabla\psi_1\|_0^2 + \|h_2\nabla\psi_2\|_0^2\right) \\ & - C_{\delta,\mathcal{L}} \max_{j=1,\cdots,N} |l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)|^2.\end{aligned}$$

for any $0 < \delta < 1$ with appropriate positive constant $C_{\delta,\mathcal{L}}$. Therefore from (30) and (7) we obtain

$$\frac{d}{dt}\|q\|_*^2 \le l_{\mathcal{L}}^{(1)}(\bar{\psi},\omega)\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\cdots,N} |l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)|^2, \tag{32}$$

where

$$l_{\mathcal{L}}^{(1)}(\bar{\psi},\omega) = -\frac{1-\delta}{a_0\varepsilon_{\mathcal{L}}^2} + b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2\,, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}.$$

The application of Theorem 2.2 gives the following assertion.

Theorem 5.1. *Let $\mathcal{L} = \{l_j : j = 1, \cdots, N\}$ be a finite set of linearly independent continuous functionals on the space $\mathbf{H}^2_{per}$. We assume that this set satisfies (31). Let (22) hold. If*

$$\varepsilon_{\mathcal{L}} < \frac{\nu\sqrt{\min\{h_1,h_2\}}}{\sqrt{2a_0b_0\Sigma}},$$

where

$$\Sigma = d_0(\mathbb{E}\|\eta\|_0^4)^{\frac{1}{2}}(\mathbb{E}R^2)^{\frac{1}{2}} + \mathbb{E}m + \nu\min\{h_1,h_2\}\mathbb{E}\|\eta\|_0^2 \tag{33}$$

and m, involving the Wiener process through the Ornstein-Uhlenbeck process η, is defined in Lemma 4.2, then $\mathcal{L}^{h_1,h_2} = \{l_j^{h_1,h_2} : j = 1, \cdots, N\}$ where $l_j^{h_1,h_2}(\psi_1,\psi_2) := l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)$, $l_j \in \mathcal{L}$ is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system (19), (20).

Proof. We integrate (32). The conditions of Theorem (2.2) are satisfied if $\mathbb{E}l_{\mathcal{L}^{(1)}} < 0$. $l_{\mathcal{L}^{(1)}}$ can be estimated by

$$-\nu\frac{1-\delta}{a_0\varepsilon_{\mathcal{L}}^2}+\frac{2\nu b_0}{\nu\min\{h_1,h_2\}}(h_1\|\Delta\bar\psi_1\|_0^2+h_2\|\Delta\bar\psi_2\|_0^2)+2b_0\|\eta\|_0^2.$$

On account of Lemma 4.2

$$\sup_{x\in B(\omega)}\frac{1}{t}\nu\int_0^t(h_1\|\Delta\bar\psi_1(\tau,\omega,x)\|_0^2+h_1\|\Delta\bar\psi_2(\tau,\omega,x)\|_0^2)d\tau$$

has the bound

$$\frac{1}{t}\left(R(\omega)+\int_0^t(d_0\|\eta(\theta_\tau\omega)\|_0^2R(\theta_\tau\omega)+m(\theta_\tau\omega))d\tau\right)$$

since B is forward invariant. Note that the expectation of R/t can be made arbitrarily small if t is large. Calculation the expectation of the last expression and choosing t sufficiently large yields the conclusion. □

Remark 5.2. i) It is easily seen that there exists a set $\hat{\mathcal{L}}^{h_1,h_2}$ of linearly independent linear bounded functionals $\{\hat{l}_j\}_1^N$ which are determining with respect to q. Indeed, q and ψ are connected by a linear homeomorphism Λ from $\mathbf{L}_{per}^2$ to $\mathbf{H}_{per}^2$ defined by (20) such that we can set $\hat{l}_j^{h_1,h_2}=l_j^{h_1,h_2}\circ\Lambda$ where $\hat{\mathcal{L}}=\{l_j\}_1^N$ defines the set of determining functionals introduced in Theorem 5.1.

ii) If $h_1=h_2$ then the set $\mathcal{L}$ itself is determining in the sense of Definition 2.1.

iii) Assumption (22) holds, if

$$\frac{4d_0a_0\mathrm{tr}_0Q}{\lambda_1^2\nu^2(k+1)}<1,\quad k+1>\frac{4}{a_0}, \tag{34}$$

for example. In this case it is easy to see that $\left(\mathbb{E}R_0^2\right)^{1/2}\le 4d_4^2\left(\frac{3a_0}{2\nu\lambda_1}\right)^2$ (we choose $a=4/3$ in Lemma 4.4). Therefore using (14) we obtain

$$\Sigma\le C_4^{1/2}\frac{\mathrm{tr}_0Q}{\nu(k+1)}\cdot\left(4d_0d_4^2\left(\frac{3a_0}{2\nu\lambda_1}\right)^2+\nu\min\{h_1,h_2\}\right)+d_4.$$

Using (34) again we have the estimate

$$d_4\le\frac{\mathrm{tr}_0Q}{k+1}\cdot\left(C_8^{1/4}\frac{d_1}{4d_0a_0}+C_4^{1/4}\frac{d_2}{\lambda_1\nu}\right)+d_3.$$

Therefore under conditions (34) in the deterministic limit $\mathrm{tr}_0Q\to 0$ we obtain estimate

$$\varepsilon_{\mathcal{L}}<\frac{\nu^2\lambda_1\sqrt{\min\{h_1,h_2\}}}{6\|f\|_{-1}c_0\sqrt{h_1}}\cdot\left\{(\lambda_1+2\max\{F_1,F_2\})\left(1+\frac{F_1F_2}{\lambda_1^2}\right)\right\}^{-1/2}. \tag{35}$$

If this estimate holds, then functionals $\{l_j^{h_1,h_2}\}$ are determining for the *deterministic* two-layer model. Moreover under condition (35) these functionals are also determining for our stochastic two-layer model (1) and (2) provided the noise parameter $\mathrm{tr}_0 Q$ is small enough.

At the end of this section, we will discuss appropriate values for fluid parameters. We will estimate the sufficient condition in Theorem 5.1, namely the inequality (35), in terms of fluid parameters.
We further show that the long-time dynamics of the two-layer geophsysical fluid system is determined by the long time dynamics of the top layer alone , when the fluid parameters and the Wiener process satisfy certain conditons. In this case the long time dynamics will also be determined by only finitely many functionals.
Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a finite set of linearly independent continuous functionals on the space H^2_{per} of the top stream functions. We assume that

$$\|\nabla\psi_1\|_0 \le \varepsilon_{\mathcal{L}} \cdot \|\Delta\psi_1\|_0 + C_{\mathcal{L}} \cdot \max_{j=1,...,N} |l_j(\psi_1)|, \quad \psi_1 \in H^2_{per}, \tag{36}$$

where $C_{\mathcal{L}} > 0$ is a constant depending on $\mathcal{L}$. As above we have

$$h_1\|\Delta\psi_1\|_0^2 \ge \frac{1-\delta}{\varepsilon_{\mathcal{L}}^2} h_1\|\nabla\psi_1\|_0^2 - C_{\delta,\mathcal{L}} h_1 \max_{j=1,\cdots,N} |l_j(\psi_1)|^2.$$

for any $0 < \delta < 1$ with appropriate positive constant $C_{\delta,\mathcal{L}}$.
Thus under the condition (36) and using the inequality $\|\nabla\psi_2\|_0^2 \le \lambda_1\|\Delta\psi_2\|_0^2$, the estimate (30) for the solution difference $\hat{q} - \bar{q}$ becomes

$$\begin{aligned} &\frac{d}{dt}\|q\|_*^2 + \nu h_1 \frac{1-\delta}{\varepsilon_{\mathcal{L}}^2}\|\nabla\psi_1\|_0^2 + (\nu\lambda_1 + 2r)h_2\|\nabla\psi_2\|_0^2 \\ &\qquad \le b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\cdots,N} |l_j(h_1\psi_1)|^2. \end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\|q\|_*^2 \le l_{\mathcal{L}}^{(2)}(\bar{\psi}, \omega)\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\cdots,N} |l_j(h_1\psi_1)|^2,$$

where

$$\begin{aligned} l_{\mathcal{L}}^{(2)}(\bar{\psi}, \omega) = &- \min\left\{\nu\frac{1-\delta}{a_0\varepsilon_{\mathcal{L}}^2}, \frac{\nu\lambda_1 + 2r}{a_0}\right\} \\ &+ b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}. \end{aligned}$$

Again applying Theorem 2.2 we obtain the main result.

Theorem 5.3. *Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a finite set of linearly independent continuous functionals on the space H^2_{per}. We assume that this set satisfies the following condition involving only with the top fluid layer dynamical variable, i.e., the stream function $\psi_1(x, y, t)$*

$$\|\nabla\psi_1\|_0 \le \varepsilon_{\mathcal{L}} \cdot \|\Delta\psi_1\|_0 + C_{\mathcal{L}} \cdot \max_{j=1,...,N} |l_j(\psi_1)|, \quad \psi_1 \in H^2_{per}. \tag{37}$$

If

$$\Sigma < \min\left\{\frac{\nu}{\varepsilon_{\mathcal{L}}^2}, (\nu\lambda_1 + 2r)\right\}\frac{\nu\min\{h_1, h_2\}}{2a_0b_0}, \tag{38}$$

where Σ is given by (33), then $\mathcal{L}$ is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system (19), (20).

The proof of this theorem is the same as the proof of Theorem 5.1.

Relation (38) holds if

$$\frac{\nu}{\varepsilon_{\mathcal{L}}^2} \geq \nu\lambda_1 + 2r \tag{39}$$

and

$$\Sigma < (\nu\lambda_1 + 2r)\frac{\nu\min\{h_1, h_2\}}{2b_0a_0}. \tag{40}$$

The parameter Σ depends on r via d_2. Therefore it is not clear whether (40) holds for some r. However as in Remark 5.2 in the deterministic limit $\mathrm{tr}_0 Q \to 0$ the estimate (40) turns into the relation

$$\nu\lambda_1 + 2r > \frac{36c_0^2\|f\|_{-1}^2 h_1}{\nu^3\lambda_1\min\{h_1, h_2\}} \cdot \left(1 + \frac{2}{\lambda_1}\max\{F_1, F_2\}\right)\left(1 + \frac{F_1F_2}{\lambda_1^2}\right). \tag{41}$$

This observation leads to the following assertion.

Corollary 5.4. *Assume that (41) holds. Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a finite set of linearly independent continuous functionals on the space H_{per}^2. We assume that this set satisfies (37) with the parameter $\varepsilon_{\mathcal{L}}$ satisfying (39). Then there exists $\gamma > 0$ such that $\mathcal{L}$ is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system (19), (20) provided* $\mathrm{tr}_0 Q \leq \gamma$.

Consequencely, the asymptotic probabilistic dynamics of the stochastically forced two-layer quasigeostrophic fluid system is determined only by the top fluid layer.

Remark 5.5. Note that the main task is to prove the existence of determining functionals of the main equation (1). It is easily seen by the structure the transformations (15) a set $\mathcal{L}$ is determining in probability for (19) if and only if the same set $\mathcal{L}$ is determining in probability for (1); see [13].

As Theorems 5.1 and 5.3 show the problem of describing of finite families of determining functionals is reduced to the study of sets of functionals for which the estimate (31) (or (36)) holds with $\varepsilon_{\mathcal{L}}$ small enough. It is also important to calculate the best possible value for $\varepsilon_{\mathcal{L}}$ for the given family of functional. For this parameter there is the estimate from below depending only on a number of functionals [10] and this estimate coinside with $\varepsilon_{\mathcal{L}}$ when functionals are modes (see Example 1 below). We refer to [10, 11] for further discussion concerning an optimal choice of families of functionals with properties like (31) or (36). We also note that the deterministic counterparts of Theorems 5.1 and 5.3 were proved in [7] for other boundary conditions.

Example 1 (Determining modes) Let $\{e_i(x,y)\}_{i=1}^{\infty}$ be the basis of eigenfunctions of $-\Delta$ with the periodic boundary conditions in O such that the corresponding eigenvalues possesses the property

$$0 < \lambda_1 \leq \lambda_2 \leq \dots . \tag{42}$$

We note that every eigenfunction has the form $L^{-1}\exp\{\frac{2\pi}{L}(j_1 x + j_2 y)\}$. However we numerate them such that (42) holds. Suppose $\mathcal{L}_N = \{l_j : j = 1, ..., N\}$ is the set of the functionals on H^2_{per} of the form

$$l_j(u) = \int_O u(x,y) e_j(x,y) dO, \quad j = 1, \dots, N.$$

Then one can prove (see, e.g. [10]) that estimate (36) holds for $\mathcal{L}_N$ with $\varepsilon_{\mathcal{L}_N} = \lambda_{N+1}^{-1/2}$ and this value is the best possible among all families of functionals consisting of N elements. Since $\lambda_N \sim c_0 N L^{-2}$ for N large enough, we have that $\varepsilon_{\mathcal{L}_N} \sim \tilde{c}_0 L/\sqrt{N}$ with some absolute constant $\tilde{c}_0$. Therefore we can use Theorem 5.3 to estimate number of determining modes. We also note that using the family $\mathcal{L}_N$ of the functionals on H^2_{per} we can easily construct a family $\tilde{\mathcal{L}}_N$ of the functionals on $\mathbf{H}^2_{per} = H^2_{per} \times H^2_{per}$ such that (31) holds with $\varepsilon_{\mathcal{L}_N} = \lambda_{N+1}^{-1/2}$.

Example 2 (Determining nodes) Let us consider the nodes

$$(x_i, y_j) = \frac{L}{\sqrt{N}} \cdot (i,j) \in O, \quad 1 \leq i,j \leq \sqrt{N},$$

and define functionals l_{ij} on H^2_{per} as δ-functions at (x_i, y_j), i.e by the formulas $l_{ij}(u) = u(x_i, y_j)$. Let $\mathcal{L} = \{l_{ij}\}$. One can prove (see, e.g. [10] or [25]) that estimate (36) holds for $\mathcal{L}$ with $\varepsilon_{\mathcal{L}} = \bar{c}_0 L/\sqrt{N}$ where $\bar{c}_0 = \frac{1}{2}$ is an absolute constant [25]. Thus $\varepsilon_{\mathcal{L}}$ has the same order for large N as is the best possible value $\varepsilon_{\mathcal{L}_N}$ for families of functionals consisting of N elements.

We also note that the estimates for the completeness defect $\varepsilon_{\mathcal{L}}$ and the Ekman constant r given in Theorems 5.1 and 5.3 are rather crude. In the two-layer quasigeostrophic model considered above, the parameter ν is the molecular viscosity. For ocean water, $\nu = 10^{-6} m^2 s^{-1}$. However, fluid turbulence at small scales can act as an extra dissipative mechanism, thus calling for the substitution of the molecular viscosity by a much larger eddy viscosity. For example, in the two-layer quasigeostrophic model flow simulation in [28], the eddy viscosity (we still use the same notation as the molecular viscosity) is taken as $\nu = 50 m^2 s^{-1}$. The gravitational acceleration $g = 9.81 m s^{-2}$.

At mid-latitude (45 degree$N$), $f_0 = 8 \times 10^{-5} s^{-1}$ and $\beta = 2.3 \times 10^{-11} m^{-1} s^{-1}$.

For large-scale flows at mid-latitude, such as the Gulf Stream in the Atlantic ocean, the horizontal spatial scale L is at the order of $1000km$. Moreover, layer depth h_1 and h_2 are at the order of $500m$ each for large scale flows such as the Gulf Stream at the mid-latitude in the Atlantic ocean. For the eddy viscosity $50m^2s^{-1}$ in [28], the Ekman constant r is then at the order of $10^{-5}s^{-1}$. The number r is large when the eddy viscosity is taken to be large.

The ocean water mean density ρ_0 is about $1025kgm^{-3}$ or $1.025gcm^{-3}$. ρ_1 and ρ_2 are in the vicinity of ρ_0. The density difference $\rho_2 - \rho_1 \approx 25kgm^{-3}$. But it is this small density difference that in turn determines pressure differences and thereby drive the ocean circulation [34].

The mean wind forcing f, i.e., the deterministic part of the curl of the wind stree on the top fluid layer is usually taken as a stationary, or being time-averaged and even also zonally averaged, sinusoidal function. For example [16],

$$f = \frac{2\pi\tau_0}{\rho_0 h_1 L} \sin \frac{2\pi y}{L},$$

where the wind tension τ_0 is of order 1 $dyne/cm^2$ or of order 0.1 N/m^2. With this mean wind forcing and physical parameters specified above, the inequality (35) turns into the estimate $\varepsilon_{\mathcal{L}} \leq 1.35 \cdot 10^{-3}m$ and therefore in the case of Examples 1 or 2 for the number N of functionals we obtain the estimate $N \approx 10^{18}$. Thus Theorem 5.1 should be only considered as an qualitative assertion about finite-dimensionality of the long-time behaviour of the stochastically forced two-layer quasigeostrophic fluid system. As for Theorem 5.3, a similar calculations show that the condition (41) can be valid under some special choice of parameters. Therefore Theorem 5.3 only predicts the possibility of a situation when the bottom fluid layer is slaved by the top layer.

6. SUMMARY

We have considered asymptotic probabilistic dynamics of the stochastically forced two-layer quasigeostrophic fluid system. We first transformed this system into a coupled system of random partial differential equations and then show that the asymptotic probabilistic dynamics of this system depends only on the top fluid layer, provided that the Wiener process and the fluid parameters satisfy a certain condition, i.e., the inequality (38); see Theorem 5.3. In particular, this latter condition is satisfied when the trace of the covariance operator of the Wiener process is controled by a certain upper bound (see Corollary 5.4) and the Ekman constant r is sufficiently large (see the inequality (41)). Note that the generalized time derivative of the Wiener process models the fluctuating part of the wind stress forcing on the top fluid layer, and the Ekman constant r measures the rate for vorticity decay due to the friction in the bottom Ekman layer.

Acknowledgement. A part of this work was done at the Oberwolfach Mathematical Research Institute, Germany, while J. Duan and B. Schmalfuß were Research in Pairs Fellows, supported by *Volkswagen Stiftung.* J. Duan would like to thank Tamay Ozgokmen, University of Miami, for helpful discussions. This work was partly supported by the NSF Grant DMS-9973204.

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(Igor Chueshov) INSTITUTE FÜR DYNAMISCHE SYSTEME, FB3, UNIVERSITÄT BREMEN, D-28334 BREMEN, GERMANY. ON LEAVE FROM DEPARTMENT OF MECHANICS ANDMATHEMATICS, KHARKOV UNIVERSITY, 310077 KHARKOV, UKRAINE
E-mail address, Igor D. Chueshov: `chueshov@math.uni-bremen.de`

(Jinqiao Duan) DEPARTMENT OF APPLIED MATHEMATICS, ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO, IL 60616, USA.
E-mail address, Jinqiao Duan: `duan@iit.edu`

(Björn Schmalfuß) DEPARTMENT OF APPLIED SCIENCES, UNIVERSITY OF TECHNOLOGY AND APPLIED SCIENCES, GEUSAER STRASSE, D–06217 MERSEBURG, GERMANY,
E-mail address, Björn Schmalfuß: `schmalfuss@in.fh-merseburg.de`