Lyapunov exponents in continuum Bernoulli-Anderson models

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Abstract. We study one-dimensional, continuum Bernoulli-Anderson models with general single-site potentials and prove positivity of the Lyapunov exponent away from a discrete set of critical energies. The proof is based on Fürstenberg's Theorem. The set of critical energies is described explicitly in terms of the transmission and reflection coefficients for scattering at the single-site potential. In examples we discuss the asymptotic behavior of generalized eigenfunctions at critical energies.

1. The Main Result

We study Anderson-type random Schrödinger operators

$$H_\omega = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} q_n(\omega) f(x-n) \tag{1}$$

in $L^2(\mathbb{R})$. The *single site potential* f is assumed to be real, supported in $[-1/2, 1/2]$, locally in L^1, and not identical to 0 (in L^1-sense). The *coupling constants* $q_n(\omega)$, $n \in \mathbb{Z}$, are independent, identically distributed Bernoulli random variables, i.e. they have distribution μ with $\operatorname{supp}\mu = \{0, 1\}$. Thus the random potential $V_\omega(x) = \sum_n q_n(\omega) f(x-n)$ is in $L^1_{loc,unif}(\mathbb{R})$ for all ω, which allows to associate a unique selfadjoint operator with H_ω that may equivalently be defined by form methods or in the sense of Sturm-Liouville theory.

Our interest in Bernoulli-Anderson models of this type arises from the fact that their spectral properties are not as well understood, by mathematically rigorous standards, as those of Anderson models with continuous (or absolutely continuous) distribution μ. Furthermore, these discrete distributions are physically more relevant as they are modeling the charge numbers of nuclei.

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One of the central objects in the study of one dimensional random operators is the *Lyapunov exponent.* To define it for our model, let $g_E(n,\omega)$ be the transfer matrix of

$$-u'' + V_\omega u = Eu \tag{2}$$

from $n-1/2$ to $n+1/2$, i.e. for any solution of (2) one has

$$\begin{pmatrix} u(n+1/2) \\ u'(n+1/2) \end{pmatrix} = g_E(n,\omega) \begin{pmatrix} u(n-1/2) \\ u'(n-1/2) \end{pmatrix}.$$

Also, for $n \in \mathbb{N}$ let $U_E(n,\omega) = g_E(n,\omega)\dots g_E(1,\omega)$, and define the Lyapunov exponent at E by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log \|U_E(n,\omega)\|), \tag{3}$$

where $\mathbb{E}$ denotes expectation with respect to ω. Existence of $\gamma(E)$ follows from the subadditive ergodic theorem, e.g. [**5**]. Since $\|U_E\| \ge 1$, we have $\gamma(E) \ge 0$, and it can also be seen that one gets the same $\gamma(E)$ if transfer matrices on the negative half line are used analogously in (3).

For many applications it is crucial to know at which energies one has that $\gamma(E) > 0$, which corresponds to exponential growth (or decay) of the solutions to (2). Our main goal here is to show that under the above assumptions this holds for all but a discrete set M of energies E, and to explicitly describe M in terms of the *transmission* and *reflection coefficients* for scattering at the single site potential f. To define them, let $k \in \mathbb{C} \setminus \{0\}$ and $u_+(x,k)$ be the *Jost solution* of

$$-u'' + fu = k^2 u, \tag{4}$$

i.e. the solution satisfying

$$u_+(x,k) = \begin{cases} e^{ikx} & \text{for } x \le -1/2, \\ a(k)e^{ikx} + b(k)e^{-ikx} & \text{for } x \ge 1/2. \end{cases} \tag{5}$$

Since e^{ikx} and e^{-ikx} are linearly independent solutions of $-u'' = k^2 u$, this defines $a(k)$ and $b(k)$ uniquely. They are related to the transmission and reflection coefficients used in physics by $a = t^{-1}$ and $b = rt^{-1}$. In particular, vanishing of b is equivalent to vanishing of r. For real k we use constancy of the Wronskian to get

$$|a(k)|^2 - |b(k)|^2 = 1, \tag{6}$$

corresponding to the familiar $|r|^2 + |t|^2 = 1$.

It can also be seen from (5) that

$$\begin{pmatrix} a(k) \\ b(k) \end{pmatrix} = \frac{1}{2ik} e^{-ik/2} \begin{pmatrix} ik & -1 \\ ike^{ik} & -e^{ik} \end{pmatrix} \begin{pmatrix} u_+(1/2,k) \\ u_+'(1/2,k) \end{pmatrix}.$$

Since u_+ is a solution of (4), a linear differential equation which is analytic in k, and satisfies the analytic initial condition $(u_+(-1/2), u_+'(-1/2)) = (1, ik)$, we see that $u_+(1/2,k)$ and $u_+'(1/2,k)$ are entire in k. Thus $a(k)$ and $b(k)$ are analytic in $\mathbb{C} \setminus \{0\}$ with a possible pole at $k = 0$. Neither $a(k)$ nor $b(k)$ vanish identically. For $a(k)$ this is a trivial consequence of (6). For $b(k)$ it follows from the fact that a compactly supported $f \neq 0$ can not be *reflectionless* (i.e. a *soliton*), as follows from inverse scattering theory, e.g. [**7**], see also [**14**]. Thus the roots of $a(k)$ and $b(k)$ can not accumulate, neither at (the pole) 0 nor away from 0.

We are now ready to state our main result:

THEOREM 1. *Let $E \in \mathbb{R} \setminus M$, where the set of critical energies M is given by*

$$\begin{aligned} M \; := \; & \{\left(\frac{n\pi}{2}\right)^2 : n \in \mathbb{Z}\} \cup \{E = k^2 : k > 0, b(k) = 0\} & (7) \\ & \cup \{-\alpha^2 < 0 : a(i\alpha)a(-i\alpha)b(i\alpha)b(-i\alpha) = 0\}, & (8) \end{aligned}$$

then $\gamma(E) > 0$.

Note that the critical set M is the union of the discrete set $\{(n\pi/2)^2 : n \in \mathbb{Z}\}$ and all real numbers in $\{k^2 : k \in \mathbb{C} \setminus \{0\}, a(k) = 0 \text{ or } b(k) = 0\}$ (this uses that for real k, $a(k) \neq 0$ and $b(-k) = \overline{b(k)}$). Therefore M is discrete.

Our main tool for proving Theorem 1 in Section 2 will be Fürstenberg's Theorem, which has been used extensively in proofs of positivity of γ for discrete one dimensional random operators, see [**12**, Sec.14A] for a summary. For continuum models, the use of Fürstenberg's Theorem until recently seems to have been restricted to the special cases $f = \chi_{[-1/2,1/2]}$ and $f = \delta_0$, a δ-point-interaction [**1, 8**]. The first to have used Fürstenberg's Theorem systematically for continuum Anderson models with general classes of single sites f has been Kostrykin and Schrader [**9**]. While they state their general results for absolutely continuous distribution μ, they point out that most of their ideas can also be applied to discrete distributions. Our work here can be seen as an implementation of this fact with, as we feel, a minimal amount of technical effort.

A nice feature of using Fürstenberg's Theorem in the proof of Theorem 1 is that once we know that the Theorem holds under the assumption that $\operatorname{supp}\mu = \{0,1\}$, then it is immediately clear that the Theorem extends to the case where $\{0,1\} \subset \operatorname{supp}\mu$. In fact, for larger support the critical set M should be smaller. Kostrykin and Schrader note in [**10**] that M should be empty if $\operatorname{supp}\mu$ has at least one non-isolated point, but there doesn't seem to be a proof of this yet. While Theorem 1 does not state that $\gamma(E) = 0$ for all $E \in M$ (which in fact is not generally true), we will demonstrate in Section 3 that for Bernoulli-Anderson models one can indeed find many critical energies with $\gamma(E) = 0$. In the discrete case this has already been observed for the so-called *dimer model*, see [**2**]. In particular, Section 3 discusses the example $f = \lambda\chi_{[-1/2,1/2]}$, $\lambda \in \mathbb{R}$ a constant, which leads to two different types of critical energies. They can be classified by the asymptotics of solutions of (2). In some cases they are of plane wave type, in particular bounded, while other critical energies lead to solutions which grow like $\exp\sqrt{x}$, due to a connection with random walks.

In [**6**] we use a result much like Theorem 1 to prove exponential and dynamical localization for continuum Bernoulli-Anderson models, where in addition we build on ideas which were developed for discrete models in [**4**]. Discreteness of the critical set M is crucial to this approach. In particular, it shows that the result found in Theorem 1 is superior to the well known results of Kotani theory which prove that $\gamma(E) > 0$ for almost every E in a class of models containing ours. The methods of [**6**] also allow to extend Theorem 1 to the case where the support of μ is $\{a,b\}$, where $a \neq b$ are arbitrary real numbers. This requires considerably more technical effort, due to our use of scattering theory at a periodic background, and thereby somewhat obscures the rather simple ideas which we present here in Section 2.

2. Positivity of the Lyapunov Exponent

To prove positivity of γ, we need to understand properties of the transfer matrices. The transfer matrix from -1/2 to 1/2 of

$$-u'' + fu = Eu, \tag{9}$$

is the matrix, $g(E)$, for which

$$\begin{pmatrix} u(1/2) \\ u'(1/2) \end{pmatrix} = g(E) \begin{pmatrix} u(-1/2) \\ u'(-1/2) \end{pmatrix} \tag{10}$$

for any solution u of (9). By $g_0(E)$ we denote the corresponding transfer matrix of $-u'' = Eu$. Set $G(E)$ to be the closed subgroup of $SL(2,\mathbb{R})$ generated by $g_0(E)$ and $g(E)$. Let $P(\mathbb{R}^2)$ be the projective space, i.e. the set of the directions in $\mathbb{R}^2$ and $\overline{v}$ be the direction of $v \in \mathbb{R}^2 \setminus \{0\}$. Note that $SL(2,\mathbb{R})$ acts on $P(\mathbb{R}^2)$

by $g\overline{v} = \overline{gv}$. We say that $G \subset SL(2,\mathbb{R})$ is strongly irreducible if and only if there is no finite G-invariant set in $P(\mathbb{R}^2)$.

In order to prove Theorem 1 we will consider energies $E > 0$ and $E < 0$ separately. Note that $E = 0$ is contained in M, so we don't need to consider it.

The general approach is as follows: We will first prove that $G(E)$ is not compact by showing that a sequence of elements has unbounded norm. This argument will be valid for all $E \in \mathbb{R} \setminus M'$, where

$$\begin{aligned} M' \;&=\; \{(n\pi)^2 : n \in \mathbb{Z}\} \cup \{E = k^2 : k > 0, b(k) = 0\} \\ &\quad\; \cup \{-\alpha^2 < 0 : a(i\alpha)a(-i\alpha)b(i\alpha)b(-i\alpha) = 0\}. \end{aligned}$$

Once non-compact, then the group G is known to be strongly irreducible if and only if for each $\overline{v} \in P(\mathbb{R}^2)$,

$$\#\{g\overline{v} : g \in G\} \geq 3, \tag{11}$$

see [**3**]. In order to prove that this condition is also satisfied we will in addition have to exclude k's which are odd multiples of $\pi/2$. We then use Fürstenberg's Theorem which states, in our context, that if the group $G(E)$ is not compact and strongly irreducible then $\gamma(E) > 0$, see also [**3**].

We start with positive energies, i.e. $E = k^2$, $k > 0$, and express the transfer matrices over $[-1/2, 1/2]$ in terms of the Jost solutions (5), at $\pm k$. We have

$$g(E) = \begin{pmatrix} u_N(1/2,k) & u_D(1/2,k) \\ u_N'(1/2,k) & u_D'(1/2,k) \end{pmatrix}$$

where u_N and u_D are the solutions of (4) satisfying $u_N(-1/2,k) = u_D'(-1/2,k)$ $= 1$, $u_N'(-1/2,k) = u_D(-1/2,k) = 0$. Writing u_N and u_D as linear combinations of $u_+(x,k)$ and $u_-(x,k) = \overline{u_+(x,k)}$ and setting $z_\pm(k) = a(k)e^{ik} \pm b(k)e^{-ik}$, we see that

$$g(k^2) = \begin{pmatrix} \mathrm{Re}[z_+(k)] & \frac{1}{k}\mathrm{Im}[z_+(k)] \\ -k\mathrm{Im}[z_-(k)] & \mathrm{Re}[z_-(k)] \end{pmatrix}. \tag{12}$$

Clearly,

$$g_0(k^2) = \begin{pmatrix} \cos k & \frac{1}{k}\sin k \\ -k\sin k & \cos k \end{pmatrix}. \tag{13}$$

Note for $k > 0$,

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} a & \frac{1}{k}b \\ kc & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

Thus proving non-compactness for $G(k^2)$ is equivalent to proving non-compactness for the group $\tilde{G}(k^2)$ conjugate to $G(k^2)$ via (14).

If $b(k) = 0$ we have that $|a(k)| = 1$, and hence, we may write $a(k) = e^{i\phi}$. By (12), one has then that

$$\tilde{g}(k^2) = \begin{pmatrix} \cos(k+\phi) & \sin(k+\phi) \\ -\sin(k+\phi) & \cos(k+\phi) \end{pmatrix},$$

which is a rotation. The same holds for $g_0(k^2)$ by (13). Thus $\tilde{G}(k^2)$ is a group of rotations and thereby compact. This not only excludes an application of Fürstenberg, but shows $\gamma(k^2) = 0$ as a direct consequence of the definition (3).

Now assume $b(k) \neq 0$. Set $a(k) = Ae^{i\alpha}$, $b(k) = Be^{i\beta}$, and see that (6) implies $A^2 - B^2 = 1$. It follows then that $A > B > 0$, $A + B > 1$, and $A - B < 1$. With this notation, one may recalculate

$$\tilde{g}(k^2) = A \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} + B \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ -\sin(\psi) & -\cos(\psi) \end{pmatrix},$$

where we have set $\varphi := -k - \alpha$ and $\psi := k - \beta$.

We wish to show that a sequence of elements in $\tilde{G}(k^2)$ has unbounded norm. To do so, consider an arbitrary unit vector

$$v(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

The relation

$$\tilde{g}(k^2)v(\theta) = Av(\varphi + \theta) + Bv(-\psi - \theta)$$

shows that the image of the unit circle under $\tilde{g}$ is an ellipse, with the choices $\eta := -\frac{1}{2}(\varphi + \psi)$ and $\eta' := \frac{\pi}{2} - \frac{1}{2}(\varphi + \psi)$ identifying the semi-major and semi-minor axes as follows:

$$\tilde{g}(k^2)v(\eta) = (A + B)v(\tfrac{\varphi - \psi}{2}) \quad \text{and} \quad \tilde{g}(k^2)v(\eta') = (A - B)v(\tfrac{\pi}{2} + \tfrac{\varphi - \psi}{2}) \ .$$

Defining $R(\theta) := \|\tilde{g}(k^2)v(\theta)\|$, a short calculation, using $A^2 - B^2 = 1$, yields

$$R(\theta) - 1 = 2B\left[B + A\cos(2\theta + \varphi + \psi)\right].$$

As a consequence, we see that $R(\theta) - 1$ is π-periodic and has exactly two roots in $[0, \pi)$. In particular, $R(\theta) = 1$ if and only if $\cos(2\theta + \varphi + \psi) = -\frac{B}{A}$, which shows that the distance between the zeros of $R(\theta) - 1$ is not equal to $\frac{\pi}{2}$: recall $B \neq 0$ (and $B \neq A$). Similarly, $R(\theta) > 1$ if and only if $\cos(2\theta + \varphi + \psi) > -\frac{B}{A}$ and hence $|\{\theta \in [0, \pi) : R(\theta) > 1\}| > \frac{\pi}{2}$. As a result, there exists a compact interval K (not necessarily in $[0, \pi)$) with $|K| > \frac{\pi}{2}$ and $\|\tilde{g}(k^2)v(\theta)\| > c > 1$ for all $\theta \in K$.

Now, applying $\tilde{g}$ once to $v(\eta)$, η as above, produces a new vector with norm greater than one, but the direction, initially η, is possibly altered. A vector with this new direction may not increase in norm by directly applying $\tilde{g}$ again. However, as long as k is not an integer multiple of π, then finitely many applications of $\tilde{g}_0(k^2)$, which is rotation by k, produces a vector with direction in K. Once in K, a direct application of $\tilde{g}$ does increase the norm size uniformly by $c > 1$, as indicated above. In this manner, we can produce a sequence of elements in $\tilde{G}(k^2)$ with unbounded norm. Thus $\tilde{G}(k^2)$ is non-compact if $b(k) \neq 0$ and k is not an integer multiple of π.

To complete the proof of Theorem 1 for positive energies, we note that if in addition k is not an integer multiple of $\pi/2$, then the free transfer matrix produces three distinct elements in projective space, i.e. (11) is satisfied and Fürstenberg's Theorem implies $\gamma(k^2) > 0$.

We finally turn to energies $E < 0$. In this case $G(E)$ is non-compact for every $E < 0$, since the free transfer matrix

$$g_0(E) = \begin{pmatrix} \cosh(\alpha) & \frac{1}{\alpha}\sinh(\alpha) \\ \alpha\sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

has unbounded powers, where $\alpha = \sqrt{|E|}$. It remains to check (11), which we have to do for $E \in (-\infty, 0) \setminus \{-\alpha^2 : a(i\alpha)a(-i\alpha)b(i\alpha)b(-i\alpha) = 0\}$.

Let $E = -\alpha^2$ such that $a(\pm i\alpha) \neq 0$ and $b(\pm i\alpha) \neq 0$. By (5), this is equivalent to

$$g(E)\overline{v}_\pm \notin \{\overline{v}_+, \overline{v}_-\}, \tag{15}$$

where $\overline{v}_\pm$ are the directions of $v_\pm := (1, \pm\alpha)^t$. If $\overline{v} \notin \{\overline{v}_+, \overline{v}_-\}$, then $\#\{g_0(E)^n\overline{v} : n \in \mathbb{Z}\} = \infty$. If, on the other hand, $\overline{v} \in \{\overline{v}_+, \overline{v}_-\}$, then we use (15) to conclude that an initial application of $g(E)$ followed by iteration of $g_0(E)$ gives an infinite orbit. This shows (11) and completes the proof of Theorem 1.

3. Critical energies

In this section we discuss the appearance of critical energies in concrete examples. We thereby illustrate the following points: (i) critical energies with vanishing Lyapunov exponent do indeed exist, (ii) the structure and "size" of the set of energies with $b(k) = 0$ depends strongly on the concrete example, and (iii) at critical energies at least two different types of non-exponential asymptotics of solutions can be observed.

3.1. Zero reflection at degenerate gaps. Our first general observation is that if H^f_{per} is the Schrödinger operator with 1-periodic potential V satisfying $V(x) = f(x)$, $-1/2 \leq x \leq 1/2$ and H^f_{per} has a degenerate gap at energy $E = k^2$, i.e. either all solutions of (4) satisfy periodic boundary conditions on $[-1/2, 1/2]$ or all solutions satisfy anti-periodic boundary conditions on $[-1/2, 1/2]$, then $g(k^2) = I$ or $g(k^2) = -I$. By (12) this implies $(a(k), b(k)) = (1, 0)$ or $(-1, 0)$, respectively. Conversely, $(a(k), b(k)) = (\pm1, 0)$ necessarily requires a degenerate gap. Thus, the possibility of degenerate gaps in periodic potentials is one mechanism which leads to reflectionless energies. Whether all reflectionless energies arise in this way is equivalent to deciding if $b(k) = 0$ necessarily leads to $a(k) = \pm 1$. In Example 2 below we will see that this is not the case.

3.2. Example 1. Consider first $f := \lambda \chi_{[-1/2,1/2]}$ with $\lambda \in \mathbb{R}$. For $E = k^2 > \max\{0, \lambda\}$ we get with $\alpha := \sqrt{E - \lambda}$

$$\begin{aligned}
a(k)e^{ik} &= \cos\alpha + i\frac{2k^2 - \lambda}{2k\alpha}\sin\alpha \\
b(k) &= \frac{i\lambda}{2k\alpha}\sin\alpha \\
g_0(k^2) &= \begin{pmatrix} \cos k & \frac{1}{k}\sin k \\ -k\sin k & \cos k \end{pmatrix} \\
g(k^2) &= \begin{pmatrix} \cos\alpha & \frac{1}{\alpha}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{pmatrix}
\end{aligned}$$

We can now distinguish between the following types of critical energies:

Type 1a: k^2 such that $b(k) = 0$: This happens for $E = k^2 = (n\pi)^2 + \lambda$, $n \in \mathbb{N}$. In this case we have $g(k^2) = \pm I$, and it easily follows that the group $\langle g_0, g \rangle$ is bounded. This means that solutions to (2) are bounded and that $\gamma(k^2) = 0$.

Type 1b: $k = n\pi$, $n \in \mathbb{N}$: This means $g_0(k^2) = \pm I$, and again $\langle g_0, g \rangle$ bounded, $\gamma(k^2) = 0$. This type is "dual" to Type 1, since $k^2 = n^2\pi^2$ can be viewed as the energies where $-d^2/dx^2 + \lambda - \lambda\chi_{[-1/2,1/2]}$ is reflectionless with respect to $-d^2/dx^2 + \lambda$.

Type 2: $E = k^2$, where $k = (2n-1)\pi/2$, $n \in \mathbb{N}$. This leads to a critical energy with a rather different asymptotic behavior of solutions if also $\alpha = (2m-1)\pi/2$, $m \in \mathbb{N}$, that is, for specific values of $\lambda = k^2 - \alpha^2 = \pi^2(n-m)(n+m-1)$. In this case we have

$$g_0(k^2) = \pm \begin{pmatrix} 0 & -1/k \\ k & 0 \end{pmatrix}, \quad g(k^2) = \pm \begin{pmatrix} 0 & -1/\alpha \\ \alpha & 0 \end{pmatrix}.$$

Let $h_n(\omega) := g_{2n+1}(\omega)g_{2n}(\omega)$, where $g_n(\omega)$ is the transfer matrix of H_ω from $n-1/2$ to $n+1/2$. Then

$$h_n(\omega) = \begin{cases} \pm I & \text{with probability } p^2+q^2, \\ \pm\begin{pmatrix} k/\alpha & 0 \\ 0 & \alpha/k \end{pmatrix} & \text{with probability } pq, \\ \pm\begin{pmatrix} \alpha/k & 0 \\ 0 & k/\alpha \end{pmatrix} & \text{with probability } qp. \end{cases}$$

If $h_{n,1}$ and $h_{n,2}$ are the diagonal entries of h_n, then the sums of $\log|h_{n,i}|$, $i=1,2$, give a symmetric random walk (observing that $(p^2+q^2)\log 1 + pq\log(k/\alpha) + qp\log(\alpha/k) = 0$). Thus it follows that for every $\varepsilon > 0$ and almost surely

$$\limsup_{n\to\infty} n^{-1/2-\varepsilon} \log \|\prod_{k=1}^{n} h_k(\omega)\| = 0$$

and

$$\limsup_{n\to\infty} n^{-1/2+\varepsilon} \log \|\prod_{k=1}^{n} h_k(\omega)\| = \infty,$$

see e.g. [**13**]. The interpretation of this is that the hull of the (oscillatory) solutions of (2) asymptotically grows like $\exp c\sqrt{x}$ or decays like $\exp(-c\sqrt{x})$ for some $c > 0$, where a generalization of the Ruelle-Osceledec Theorem from [**11**, Section 8] is used.

3.3. Example 2. Consider $f := \lambda(\chi_{[-1/2,0]} - \chi_{(0,1/2]})$ with $\lambda > 0$. For $E = k^2 > \lambda$, we may again explicitly calculate the transfer matrix

$$g(E) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

and see that,

$$\begin{aligned} g_{11} &= \cos(\alpha_-/2)\cos(\alpha_+/2) - \frac{\alpha_-}{\alpha_+}\sin(\alpha_-/2)\sin(\alpha_+/2), \\ g_{12} &= \frac{1}{\alpha_-}\sin(\alpha_-/2)\cos(\alpha_+/2) + \frac{1}{\alpha_+}\cos(\alpha_-/2)\sin(\alpha_+/2), \\ g_{21} &= -\alpha_+\cos(\alpha_-/2)\sin(\alpha_+/2) - \alpha_-\sin(\alpha_-/2)\cos(\alpha_+/2), \\ g_{22} &= -\frac{\alpha_+}{\alpha_-}\sin(\alpha_-/2)\sin(\alpha_+/2) + \cos(\alpha_-/2)\cos(\alpha_+/2), \end{aligned}$$

for $\alpha_\pm := \sqrt{E \pm \lambda}$. From this and the boundary conditions of the Jost solution, i.e.

$$g\begin{pmatrix} e^{-ik/2} \\ ike^{-ik/2} \end{pmatrix} = \begin{pmatrix} a(k)e^{ik/2} + b(k)e^{-ik/2} \\ ika(k)e^{ik/2} - ikb(k)e^{-ik/2} \end{pmatrix},$$

one can show that

$$b(k) = 0 \text{ if and only if } \sin(\alpha_+/2) = 0 = \sin(\alpha_-/2).$$

Therefore, if we fix some $\lambda > 0$, then any $E = k^2 > \lambda$ for which $b(k) = 0$ must satisfy

$$\frac{\sqrt{E+\lambda}}{2} = n\pi, \text{ and } \frac{\sqrt{E-\lambda}}{2} = m\pi,$$

for some $n, m \in \mathbb{N}$. Thus λ must have the form

$$\lambda = 2\pi^2(n^2 - m^2), \text{ with } n, m \in \mathbb{N}, \tag{16}$$

and the corresponding reflectionless E's are then

$$E = 2\pi^2(n^2 + m^2). \tag{17}$$

From this we conclude that for most values of λ (e.g., Lebesgue almost every λ), no reflectionless energy exists above λ. On the other hand, every λ satisfying (16) has only finitely many reflectionless energies. In fact, if $N := \frac{\lambda}{2\pi^2} \in \mathbb{N}$, then $N = n^2 - m^2$ for only finitely many pairs $(n, m) \in \mathbb{N}^2$. But this finite number can be arbitrarily large, depending on λ: To find at least j exceptional energies, choose

$$\frac{\lambda_j}{2\pi^2} = N_j = 2^{j+1}(2^{j-1} + 1)$$

and see that for $0 \le \ell < j$ one may choose

$$n_\ell := \left(2^\ell(2^{j-1} + 1) + 2^{j-\ell-1}\right)$$

and

$$m_\ell := \left(2^\ell(2^{j-1} + 1) - 2^{j-\ell-1}\right)$$

satisfying $n_\ell^2 - m_\ell^2 = N_j$. Proof: Take $a := 2^\ell(2^{j-1} + 1)$ and $b := 2^{j-\ell-1}$ and see that $(a + b)^2 - (a - b)^2 = 4ab$. Therefore,

$$4ab = 2^2 \cdot 2^\ell(2^{j-1} + 1) \cdot 2^{j-\ell-1} = 2^{j+1}(2^{j-1} + 1) = N_j.$$

Recall from our remarks in 3.1 that in some cases, the condition $b(k) = 0$ implies that $a(k) = \pm 1$. To see that this is not always the case, we calculate $a(k)$ in this example for such values of k. For λ as in (16) and E as in (17), one can show that $a(k) = e^{-2\pi i\sqrt{n^2+m^2}}$ if $n + m$ is even and $a(k) = -e^{-2\pi i\sqrt{n^2+m^2}}$ if $n + m$ is odd.

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