HETEROCLINIC AND PERIODIC CYCLES IN A PERTURBED CONVECTION MODEL

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Abstract. Vivancos and Minzoni [30] proposed a singularly perturbed rotating convection system to model the Earth's dynamo process. Numerical simulation shows that the perturbed system is rich in chaotic and periodic solutions. In this paper, we show that if the perturbation is sufficiently small, the system can only have simple heteroclinic solutions and two types of periodic solutions near the simple heteroclinic solutions. One looks like a figure "Delta" and the other looks like a figure "Eight".

Due to the fast–slow characteristic of the system, the reduced slow system has a relay nonlinearity [21]–Solutions to the slow system are continuous but their derivative changes abruptly at certain junction surfaces. We develop new types of Melnikov integral and Lyapunov–Schmidt reduction methods which are suitable to study heteroclinic and periodic solutions for systems with relay nonlinearity.

1. Introduction

1.1. The system. We start with the well-known Guckenheimer and Holmes system (1.1). This system was first obtained by Busse and Clever and is successful model for the Rayleigh-Bénard convection [12] and rotation convection problems [5, 6]. In their system the variables (x, y, z) represent the amplitude of a convective velocity field.

$$\begin{aligned} \dot{x} &= x(1 - ax^2 - by^2 - cz^2), \\ \dot{y} &= y(1 - ay^2 - bz^2 - cx^2), \\ \dot{z} &= z(1 - az^2 - bx^2 - cy^2). \end{aligned} \tag{1.1}$$

System (1.1) possesses interesting symmetry properties, which ensure some rich dynamical behavior: There are 12 heteroclinic solutions connecting 6 equilibria (not including the origin). The equilibria are on the coordinate axes and the heteroclinic solutions are on the coordinate planes $x = 0$, $y = 0$ or $z = 0$. The system has eight heteroclinic cycles (homoclinic cycles in some literature), formed by three consecutive heteroclinic solutions. The cycles are stable provided the parameters a, b and c satisfy: $0 < b < a < c,\ 2a < b + c,\ a + b + c = 1$ [14, 15]. Several papers are devoted to the bifurcation of such heteroclinic cycles under various perturbations [20, 26, 7].

In this paper we study what is called a transverse perturbation of the heteroclinic cycles. The term transverse refers to adding a fourth equation for a new variable that evolves in the direction transverse to the original three dimensional phase space.

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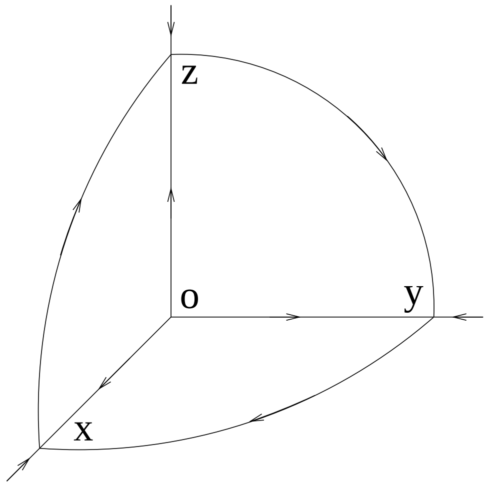

FIGURE 1.1. A stable homoclinic cycle in the first octant.

Although several authors [1, 20, 26] have studied such type of perturbations within the context of equivariant bifurcation and symmetry breaking theory, our new approach is to consider a specific transverse perturbation which gives rise to a singularly perturbed system:

$$\begin{aligned} \dot{x} &= x(1 - ax^2 - by^2 - cz^2) + \bar{d}yzu, \\ \dot{y} &= y(1 - ay^2 - bz^2 - cx^2) + \bar{d}zxu, \\ \dot{z} &= z(1 - az^2 - bx^2 - cy^2) + \bar{d}xyu, \\ \delta\dot{u} &= f(x, y, z, u). \end{aligned} \tag{1.2}$$

Here, $\bar{d} \in R$ denotes a parameter, $0 < \delta << 1$, and $f(x, y, z, u)$ is $\mathcal{C}^r$ where $r > 3$ will be specified later. We call (1.1) the unperturbed system and refer to (1.2) as the perturbed system.

Numerical simulations for system (1.2) have revealed some interesting dynamic behavior such as chaotic trajectories and strange attractors [30]. In this paper, we shall only consider the existence, uniqueness and stability of heteroclinic and periodic orbits. Besides the interest from the dynamical point of view, system (1.2) is closely related to a problem in magnetohydrodynamics (MHD). This was the physical motivation of [30] and also this work. For the sake of completeness, we include a short paragraph explaining the physics underlying our system.

1.2. **The physical motivation.** The physical motivation of this work comes from nonlinear magnetohydrodynamics which is the study, on a macroscopic scale, of electrically conducting flows. The magnetohydrodynamic equations have been known for a long time but still constitute a difficult problem far from being solved. The full hydromagnetic problem has been partially decoupled into the kinematic dynamo problem and the problem of generation by convection of appropriate dynamo amplifying flow fields. We refer the reader to Ghil and Childress [13] for a complete survey in this topic.

In its simplest form, the dynamo effect is modeled by the so-called induction equation: $\frac{\partial b}{\partial t} = \Delta b + curl(v \times b)$ where v and b denote the velocity and the magnetic field respectively. The amplification of small magnetic fields can be studied in models which neglect all coupling between v and b. This idea, studying the effect of a pre-existing velocity field in the context of dynamo theory, defines the kinematic dynamo problem. The purpose, of course, is to learn what choices of v are likely to cause magnetic activity. By numerical simulations for the magnetic field of fluids confined in spherical domains, Friedrich and Haken [11], Buses [4] have shown the existence of periodic attractors. In a later work, Armbruster and Chossat [1] conjectured that the dynamical behavior observed for the magnetic field in some experiments could result as a perturbation of convective flows possessing heteroclinic cycles. These kinds of cycles have been proven to exist in the particular case of mode interaction of the first spherical modes. Our approach is quite different from theirs and is similar to the so-called smoothing method. Roughly speaking, this method is based on a decomposition of the magnetic field into two different scales. One scale, ℓ, is smaller than the core value, L, the scale of the smooth part of the magnetic field. The study of the organization of spatial structure of the resulting magnetic field requires large-scale flows in the convection zone in addition to the small-scale buoyant turbulence, which also determines the strictly regular field direction.

In the conventional nonlinear theory for the MHD, one starts from the unstable equilibrium and calculates the ensuing nonlinear evolution and saturation of small initial perturbations. However, this approach is rather unrealistic. Since typical times for nonlinear saturation are much shorter than the time scales of equilibrium evolution, the unstable mode will grow as soon as the marginal point is passed. Two different situations may then occur, either the equilibrium bifurcates in the simplest case by a transcritical or pitchfork bifurcation, or a catastrophe occurs corresponding to a local loss of equilibrium. In our approach, we follow the nonlinear development of the magnetic field from a specific dynamical state, namely a flow possessing heteroclinic cycles, which is an initial unstable state. This approach has some justification because it may occur that, owing to additional processes, the instability is temporarily suppressed such that the system continues to evolve on the unstable branch up to some point where the instability finally sets in.

The main physical assumption we have taken to justify our model equations from a convection model is that there exist states where velocity and magnetic field fluctuations are strongly correlated. This relaxed state is likely to occur dynamically, for instance, in satellite observations of the solar wind. The small-scale chaotic behavior may be considered as an ensemble of Alfven waves propagating along the average magnetic field, see Burlage & Turner [3].

Hence, for our purposes, we will assume that the smooth part of the magnetic field represented by the variables (x, y, z) follows these heteroclinic cycles, while the non-smooth part that evolves in the large scale is denoted by u. As it will become apparent, small singular perturbations of the original Guckenheimer and Holmes cycle can lead to modulated waves with very long periods close to the equilibrium points. For such flows, states with reversed polarities are successively explored which is reminiscent, for instance, of the Earth's magnetic field physics. For these reasons we believe that the mathematical model given in this work is relevant for the study of magnetohydrodynamics.

1.3. **The singularly perturbed system.** We present a sketch of the magnetic field in Figure 1.2. Because of the small constant δ, (1.1) is a singularly perturbed system.

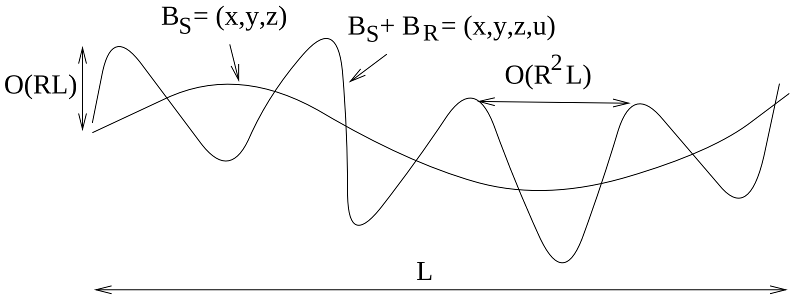

FIGURE 1.2. A sketch of the magnetic field with two length scales.

Vivancos and Minzoni proposed that:

$$f(x, y, z, u) = u(\mu - u^2 + p_2(x^2 + y^2 + z^2)) + p_3xyz. \tag{1.3}$$

Observe that we have the symmetry

$$(x, y, z, u) \to (y, z, x, u),$$
$$(x, y, z, u) \to (-x, y, z, -u).$$

Numerical simulation shows that the singularly perturbed system exhibits various periodic and chaotic solutions. A further simplified system derived from (1.2) and (1.3) was considered and chaotic solutions were shown analytically in their paper.

Before going into the analysis of (1.2), some remarks are useful. When $u = 0$, the unperturbed system has a heteroclinic cycle [14]. Adding a new variable u to that cycle destroys this structure and gives rise to very rich dynamics. Sandstede and Scheel [26] studied this kind of system within the context of the equivariant bifurcation theory. They found slow drifting periodic orbits in the non singular case while in this work we deal with fast relaxation oscillations along a slow periodic solution. Also Melbourne et al [20] proved the existence and stability of periodic orbits in a similar context, but regarding the case of a quadratic equation in the transverse direction to the heteroclinic cycle.

To simplify the system further, we assume that $x^2 + y^2 + z^2$ is a constant since it is uniformly bounded away from 0 and ∞. After some rescaling, we are led to the following u equation:

$$\delta\dot{u} = u - u^3 - xyz/D, \quad D > 0. \tag{1.4}$$

The small constant $\bar{d}$ in (1.2) controls the perturbation terms to the first three equations, and the small constant D in (1.4) controls the amplitude of oscillation. Again numerical simulation shows that (1.2) and (1.4) possess rich periodic and chaotic solutions.

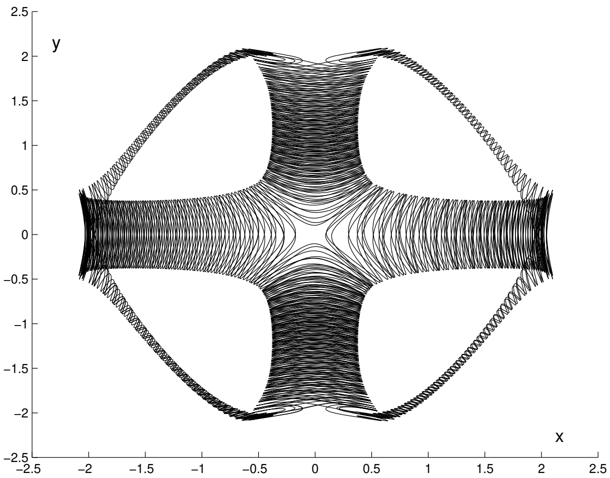

FIGURE 1.3. Chaotic or long periodic solutions for (1.2)

It is still difficult to prove rigorously the existence of many exotic types of periodic and chaotic orbits observed in this system. Notice that the chaotic solution in Figure 1.3 stays near the unperturbed heteroclinic cycles. This motivates us to further study solutions that can directly bifurcate from the original heteroclinic cycles. We will demonstrate that only simple heteroclinic orbits and two kinds of periodic orbits can bifurcate from the unperturbed heteroclinic cycles. Other complicated periodic and chaotic orbits observed numerically must be created by different mechanisms, e. g., the tangential intersection of the orbit with junction surfaces [9, 30]. We hope that this will be a first step towards the better understanding of the dynamics of the system.

Equation $u - u^3 = 2w/(3\sqrt{3})$ has three branches of inverse: $u = u^*(w, s), s = -1, 0, 1$, where $s = 0$ is the branch near zero and $s = \pm 1$ are the positive and negative branches. The domains of $u^*(w, -1), u^*(w, 0)$ and $u^*(w, 1)$ are $[-1, \infty), [-1, 1]$ and $(-\infty, 1]$ respectively. See Fig. 1.4.

Let $\epsilon := 2D/(3\sqrt{3})$. In the singular limit $\delta = 0$, the equation for u becomes

$$u - u^3 = \frac{xyz}{\epsilon} \frac{2}{3\sqrt{3}}.$$

The solution of the above is

$$u = u^*(\frac{xyz}{\epsilon}, s), \quad |xyz| \leq \epsilon.$$

Notice that if $s = \pm 1$ then u is uniformly away from zero, that is, $|u^*| \geq \sqrt{1/3}$.

In order to smoothly follow a heteroclinic solution to its birth place–an unperturbed heteroclinic solution, we simultaneously scale down the constant $\bar{d}$ by letting $\bar{d} = \epsilon d$. This leads to the following "slow system" on a three–dimensional submanifold in the

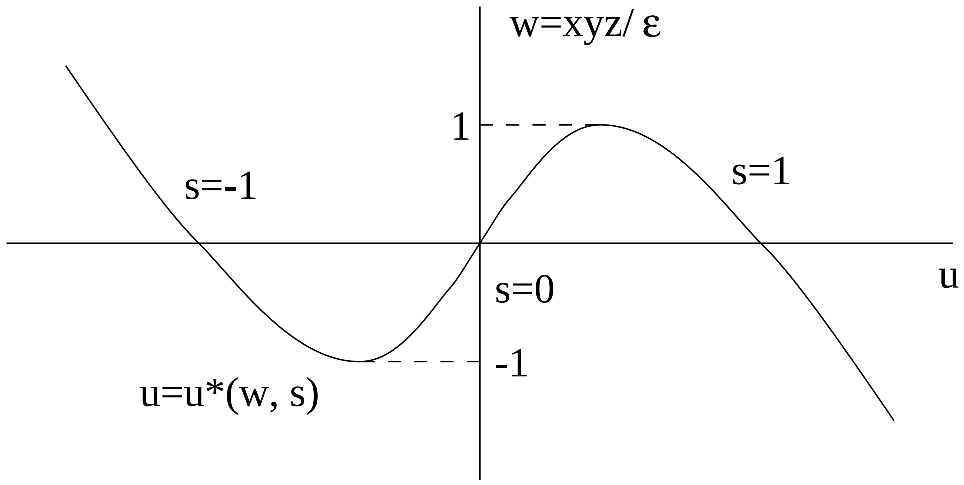

FIGURE 1.4. The curves $u = u^*(w, s)$

four dimensional space:

$$\begin{aligned}
\dot{x} &= x(1 - ax^2 - by^2 - cz^2) + \epsilon dyzu^*(xyz/\epsilon, s), \\
\dot{y} &= y(1 - ay^2 - bz^2 - cx^2) + \epsilon dzxu^*(xyz/\epsilon, s), \quad s = -1, 0, +1, \\
\dot{z} &= z(1 - az^2 - bx^2 - cy^2) + \epsilon dxyu^*(xyz/\epsilon, s).
\end{aligned} \tag{1.5}$$

The slow manifolds corresponding to $s = \pm 1$ are attracting, while the one corresponding to $s = 0$ is repelling. This is evident if we write the u equation in the stretched time $\tau = t/\delta$:

$$\frac{du}{d\tau} = u - u^3 - xyz/D.$$

In the stretched time, (x, y, z) are constants. The stability of the u equation depends on the sign of $1 - 3u^2$, which is negative if $u^2 > 1/3$ (or $s = \pm 1$), and positive if $u^2 < 1/3$ (or $s = 0$).

A well-known result indicates that if δ is small but nonzero, system (1.2) exhibits a relaxation oscillation similar to a singularly perturbed Van der Pol equation [21]. For any given initial data, the solution will quickly jump close to one of the stable branches of the slow manifold where $u = u^*(xyz/\epsilon, s)$, $s = \pm 1$ and the (x, y, z) variable will then follow the flow on the slow manifold defined by (1.5). If the flow hits the boundaries of this branch defined by junction surfaces $xyz = 2D/(3\sqrt{3}) = \pm\epsilon$, then the solution will quickly jump close to another stable branch of the slow manifold. Thus any heteroclinic or periodic orbit of (1.2) is near an orbit of the same kind of (1.5). On the other hand, near any periodic or heteroclinic orbit of the reduced system (1.5), there exists a same kind of orbit of (1.2). For this reason, we will study the reduced system (1.5) on the two stable branches of the slow manifold in the rest of this paper.

1.4. **Assumptions and the main results.** Throughout this paper, we assume that $d \geq 0$. Numerical work shows that if an initial data is in the region $|xyz| > \epsilon$ with

$s(0) = -1$ or $+1$, the orbit will move along the slow manifold until it is trapped in the region $|xyz| \le \epsilon$. We will expand on this point later. The solution will then zig-zag between the junction surfaces and $s(t)$ will alternate between -1 and $+1$ whenever $|xyz| = \epsilon$. In particular, if xyz hits ϵ from below at $t = \tilde{t}$, then $s(\tilde{t}-) = +1, s(\tilde{t}+) = -1$, and if it hits $xyz = -\epsilon$ from above then $s(\tilde{t}-) = -1, s(\tilde{t}+) = +1$. The change of s brings the orbit back to the region $|xyz| \le \epsilon$. A projection of the reduced flow to the xyz-plane is plotted in Fig. 1.5. We will restrict our attention to the orbits that hit the junction surfaces transversely, or "the switching is normal" according to [21]. Otherwise the reduced flow on the xyz-plane is not well defined.

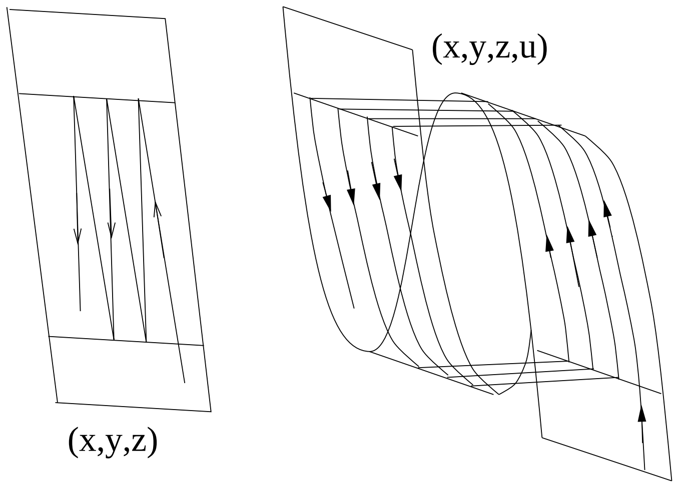

FIGURE 1.5. Projection of the 4-d system to the (xyz)-plane.

The unperturbed system ($\epsilon = 0$) has 12 heteroclinic solutions connecting six equilibria $(\pm\sqrt{1/a}, 0, 0)$, $(0, \pm\sqrt{1/a}, 0)$ and $(0, 0, \pm\sqrt{1/a})$. The following assumption ensures that every equilibrium is hyperbolic with one unstable eigenvalue that is weaker than the two stable eigenvalues, which we assume throughout this paper:

H 1. $a + b + c = 1, \ 0 < b < a < c, \ 2a < b + c.$

After adding small $\epsilon > 0$, numerical simulation shows that around the unperturbed heteroclinic orbits, there seems to exist a small thin invariant band that looks like a two dimensional manifold. This is explained in the following two lemmas. The first lemma shows that if the orbit starts on one of the stable slow manifolds, it will move towards an invariant region $0 < m \le (x^2 + y^2 + z^2) \le M$ and will remain there. The result is similar to that of [14, 30]. The second lemma shows that under certain conditions, the solution will be trapped in the region $|xyz| \le \epsilon$.

Lemma 1.1. *There exist $0 < m(\epsilon) < M(\epsilon)$ such that for any initial data satisfying $s(0) = 1, xyz \le \epsilon$ or $s(0) = -1, xyz \ge -\epsilon$, the region $m \le x^2 + y^2 + c_3^2 \le M$ is (forward) invariant and attracting. Moreover, as $\epsilon \to 0$,*

$$M(\epsilon) \to \frac{1}{a}, \quad m(\epsilon) \to \frac{2}{b+c}.$$

Lemma 1.2. *If $\eta > 0$ is a constant satisfying*

$$0.5|d\sqrt{1/3}|\eta m > \max\{|3 - 1/a|, |3 - 2/(b+c)|\}, \tag{1.6}$$

then for any solution satisfying $m \le (x^2 + y^2 + z^2) \le M$ and $\min\{x^2 + y^2, y^2 + z^2, z^2 + x^2\} \ge \eta$, the region $|xyz| \le \epsilon$ is invariant. Moreover, the solution satisfies $\frac{d}{dt}(xyz) > 0$ if $s = 1$ and $\frac{d}{dt}(xyz) < 0$ if $s = -1$.

The proofs of the two lemmas are given in Appendix A. We remark that $\frac{d}{dt}(xyz)$ may change signs if the orbit is near an equilibrium.

Under the conditions of Lemmas 1.1 and 1.2, if a solution is not near any of the equilibria, it will not leave the region bounded by $|xyz| \le \epsilon$ and $m \le x^2+y^2+z^2 \le M$, and will travel from one junction surface to another.

If the solution is near an equilibrium so that $\min\{x^2 + y^2, y^2 + z^2, z^2 + x^2\} \le \eta$, Lemma 1.2 cannot be used. However, numerical works indicate that solutions are still trapped in the region $|xyz| \le \epsilon$. This can be explained as follows. Any orbit starting in the neighborhood of an equilibrium is attracted to its unstable manifold and then leaves the neighborhood following closely to that unstable manifold. The unstable manifold is $O(\epsilon)$ close to the unperturbed heteroclinic orbits issuing from the equilibrium $(\sqrt{1/a}, 0, 0)$ while all the other orbits are $O(e^{-\lambda\tau/\epsilon})$ close to the unstable manifold, where τ is the time the orbits are near the equilibrium. This mechanism keeps the orbit trapped in the region $|xyz| \le \epsilon$ and hence, near the unperturbed heteroclinic cycle.

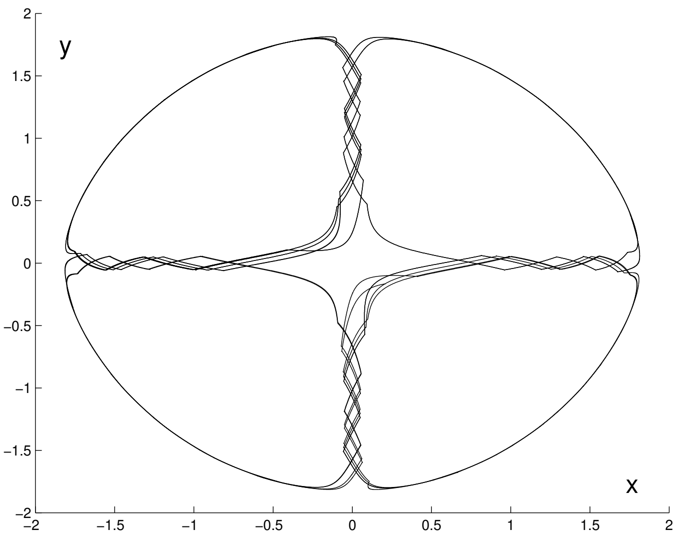

FIGURE 1.6. Chaotic or long periodic solutions, note the tangential intersection of the orbits with the junction surface at $(x = \pm 0.3, y = \pm 0.2)$

Figure 1.6 shows the two dimension plot of a long periodic or chaotic solution. At $(x, y) = (-0.3, 0.2)$, $(0.3, -0.2)$, we find that the solution is nearly tangent to the junction surfaces $xyz = \pm\epsilon$. This causes a drastic splitting of orbits near these points. Bo Deng [9] and Vivancos & Minzoni [30] showed that this mechanism can cause chaos in some model systems. To study pure heteroclinic bifurcation, we will avoid tangential intersection of orbits with junction surfaces. We also assume that in the regions we are interested in, orbits move monotonically from one junction surface to another, i.e., $d(xyz)/dt \neq 0$. These conditions will be imposed on the "singular heteroclinic orbits" of the system (3.2)–(3.4) in §3 and will be labeled Condition F1. The singular heteroclinic orbits are the limits of heteroclinic orbits as $\epsilon \to 0$ and are characterized by how many times they hit the junction surfaces, i.e., how many times $s(t)$ switches between ± 1.

Under the condition F1, which will be stated in §3 when all the notations are defined, we can show that for every integer $m \geq 0$, there exists a unique d_m such that (3.2)–(3.4) has exactly two singular heteroclinic orbits, related by the symmetry $(x, y_1, z, s) \to (x, -y_1, z, -s)$. Each of the singular orbits moves monotonically between junction surfaces, and the switching is normal at the junction surfaces.

For certain ranges of (a, b, c, d), we have numerically verified condition F1, see Appendix B. However, it seems to be very hard to verify F1 analytically. In fact, there may be some parameter regions where F1 is not satisfied.

The construction of heteroclinic and periodic solutions is based on the perturbation of singular heteroclinic solutions and is rigorous. We can show that immediately after bifurcating from singular heteroclinic orbits, there can only be two types of periodic solutions. One looks like a figure "Δ", the other looks like a figure "8". In Figure 1.7, the figure "Δ" is computed with $d = -1.07$ while the figure "8" with $d = -1.88$. Both with $\epsilon = 0.077$.

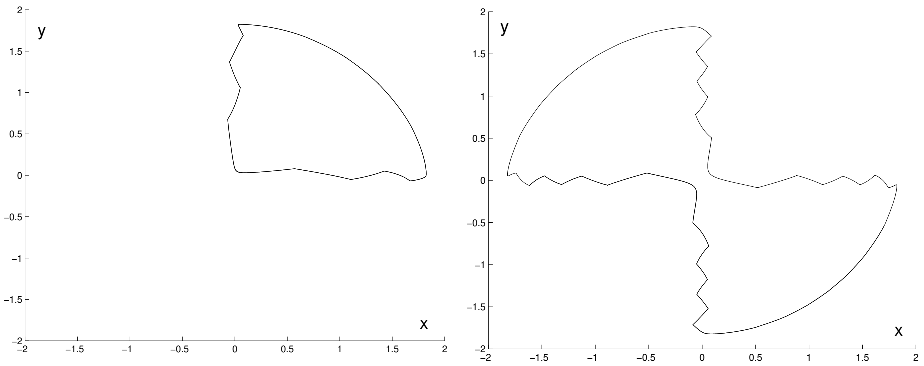

FIGURE 1.7. Periodic orbits bifurcate from the heteroclinic cycles look like a figure "Δ" or a figure "8".

Theorem 1.3. *(Main Results) Assume H1 and F1 are satisfied, then for any integer $m \geq 0$, there exists $d_m \geq 0$ such that precisely two singular heteroclinic solutions of (3.2)–(3.4) exist. They hit the junction surfaces with normal switchings exactly m times. Moreover, for any $0 < \epsilon \leq \epsilon_0$, we have:*

(1) There exists a unique $d_m(\epsilon)$, with $\lim_{\epsilon \to 0} d_m(\epsilon) = d_m$, such that the system has exactly 2 heteroclinic solutions connecting $X^+ \to Z^+$, hitting $\Sigma = \{xyz = \pm\epsilon\}$ m

times if and only if $d = d_m(\epsilon)$. The total number of the heteroclinic solutions for each m is 24.

(2) There exists a neighborhood $N_m(\epsilon)$ of $d_m(\epsilon)$ s.t. if $d \in N_m(\epsilon)\backslash d_m(\epsilon)$ then

(i) if m is even, then there exists figure "Δ" periodic solutions.

(ii) if m is odd, then there exists figure "8" periodic solutions.

The number of figure "Δ" periodic solutions is eight, and the number of figure "8" periodic solutions is four.

The periodic solutions constructed in this paper are stable, due to the information of the eigenvalues at equilibria from H1. The proof of the stability is not included in this paper.

This paper is organized as follows. In §2, we present some basic lemmas and introduce a Melnikov theory for the systems with relay nonlinearity. In §3, we discuss the existence of heteroclinic solutions. In §4, we construct figure "Δ" and figure "8" periodic solutions near heteroclinic cycles. Appendix A is devoted to the proofs of Lemmas 1.1 and 1.2. Appendix B shows how we compute the regions where F1 is satisfied.

The collaboration of Lin and Vivancos was suggested by Jack Hale when they were visiting Georgia Tech. We are grateful for the advise from Hale and the support from the Center for Dynamical Systems and Nonlinear Studies in GT. We thank Marty Golubitsky for the discussion on some systems relevant to this paper.

2. Preliminaries and some lemmas

To find a heteroclinic solution, we want to choose d so that $W^u(E_1) \cap W^s(E_2)$ is nonempty. However, since both manifolds are reflected by the junction surfaces $xyz = \pm\epsilon$ many times, the method of computing the gap between $W^u(E_1)$ $W^s(E_2)$ or its linearization (Melnikov integral) is not available in literature. To fill in the gap, we present some fundamental lemmas which will be useful in this paper, and hopefully in other systems with relay nonlinearity.

Consider a general system with relay nonlinearity:

$$\dot{y} = f(y, t, \mu, s), \quad y \in \mathbb{R}^n,\ \mu \in \mathbb{R},\ s \in \{0, 1, \ldots, m\}. \tag{2.1}$$

Assume that f is smooth in an open set adjacent to the junction surfaces, but has continuous one-sided limit to these surfaces. Let the solution map be $\phi(t, \tau, \mu, s; \bar{y})$, where τ and t are the beginning and ending times, and $\bar{y}$ is the initial data at τ. The rule determining $s(t)$ is as follows: Let the junction surfaces $\Gamma_i = \{(t, y) : F_i(t, y) = 0\}, i = 1, \ldots, m$, be cross sections of the product flow of (2.1) and $\dot{t} = 1$. This means that the one-sided limit of the vector field is non zero and is not tangent to Γ_i. This condition is also called normal switching in [21]. If y_0 is a solution that starts with $t = \tau < t_1$ and $s(t) = 0$ for $t < t_1$, and successively hits Γ_i at $t = t_i, 1 \le i \le m$, then each time the solution hits a cross section, the value of $s(t)$ is increased by one. Thus, $(t_i, y_0(t_i)) \in \Gamma_i$, and $s(t) = i$ for $t_i < t < t_{i+1}$. For convenience, let $t_0 = -\infty, t_{m+1} = \infty$.

In singular perturbation problems, the vector field is the reduced flow on the slow manifold and may only be defined on one side of $F_i = 0$, say $F_i(t, y) > 0$, but has a one–sided limit to $F_i(t,y)=0$. In this case, we can extend the flow to the side $F_i(t,y)<0$ so that (2.1) is defined on both sides of the surface Γ_i. In the problem of this paper, the surface is $xyz=\pm\epsilon$ and the flow is undefined for $xyz>\epsilon$, $s(t)=-1$ or $xyz<-\epsilon$, $s(t)=1$. To extend the flow we simply let $u^*(w,s)=u^*(w,\pm 1)$ if $|s|>1$.

Denote the evolution operator for the linearized flow by $\Phi=\partial\phi/\partial\bar{y}$.

Let $y_0(t)+\Delta y(t)$ be a solution near $y_0(t)$ with the parameter $\mu_0+\Delta\mu$. The smallness of $\Delta y(t)$ implies that both solutions intersect Γ_i transversely and change the values of s at the intersection. Let $y_0(t)+\Delta y(t)$ intersects Γ_i at $t_i+\Delta t_i, 1\le i\le m$ at the point $P(t_i+\Delta t_i):=y_0(t_i+\Delta t_i)+\Delta y(t_i+\Delta t_i)$.

$$F_i(t_i+\Delta t_i, y_0(t_i+\Delta t_i)+\Delta y(t_i+\Delta t_i))=0.$$

Define a time cross section $\{t=t_i\}$. The solution $y_0(t)$ of course hits $t=t_i$ at $y_0(t_i)$. Let $\pi_i^-:\{t=t_i\}\to\Gamma_i$ be a Poincaré mapping related to the flow of (2.1) with $s=i-1$. Since the flow at t_i^- is transverse to the cross section Γ_i, π_i^- is well defined. Let $\pi_i^+:\Gamma_i\to\{t=t_i\}$ be the Poincaré mapping related to $s=i$. Define the virtual hitting point vP and virtual reflection point rP as follows:

$$\begin{aligned} vP(t_i)&=(\pi_i^-)^{-1}P(t_i+\Delta t_i),\\ rP(t_i)&=\pi_i^+P(t_i+\Delta t_i). \end{aligned}$$

Note that vP or rP may not be on the solution orbit. If $\Delta t_i<0$, the solution $y_0(t)+\Delta y(t)$ follows the new vector field with $s(t)=i$ before hitting $t=t_i$. If $\Delta t_i>0$, it starts to follow the new vector field after time t_i. For this reason they are called virtual points.

For every orbit that is near $y_0(t)$, there corresponds a unique virtual orbit which hits $\{t=t_i\}$ at $vP(t_i)$ then jumps to $rP(t_i)$ by the reflection law:

$$rP(t_i)=r_i^*(vP(t_i))=\pi_i^+\circ\pi_i^-\circ vP(t_i).$$

The virtual and the original orbits differ only between t_i and $t_i+\Delta t_i$ and will be denoted by the same symbol.

For every heteroclinic solution near y_0 there corresponds a unique virtual heteroclinic solution that starts from E_1 at $t=-\infty$, jumps from $vP(t_i)$ to $rP(t_i)=r_i^*(vP(t_i))$ at each $\{t=t_i\}, i=1,\dots,m$ then approaches E_2 as $t\to\infty$.

It is convenient to write

$$vP(t_i)=y_0(t_i)+\Delta y(t_i^-),\quad rP(t_i)=y_0(t_i)+\Delta y(t_i^+).$$

Using linearization,

$$\Delta t_i=-\frac{\partial_y F_i\cdot\Delta y(t_i^-)}{\partial_t F_i+\partial_y F_i\cdot y_0(t_i^-)}+O(|\Delta y(t_i^-)|^2), \tag{2.2}$$

$$P(t_i+\Delta t_i)=vP(t_i)+\dot{y}_0(t_i)\Delta t_i+O(|\Delta y(t_i^-)|^2+|\Delta\mu|^2), \tag{2.3}$$

$$rP(t_i)=vP(t_i)+\dot{y}_0(t_i^-)\Delta t_i-\dot{y}_0(t_i^+)\Delta t_i+O(|\Delta y(t_i^-)|^2+|\Delta\mu|^2). \tag{2.4}$$

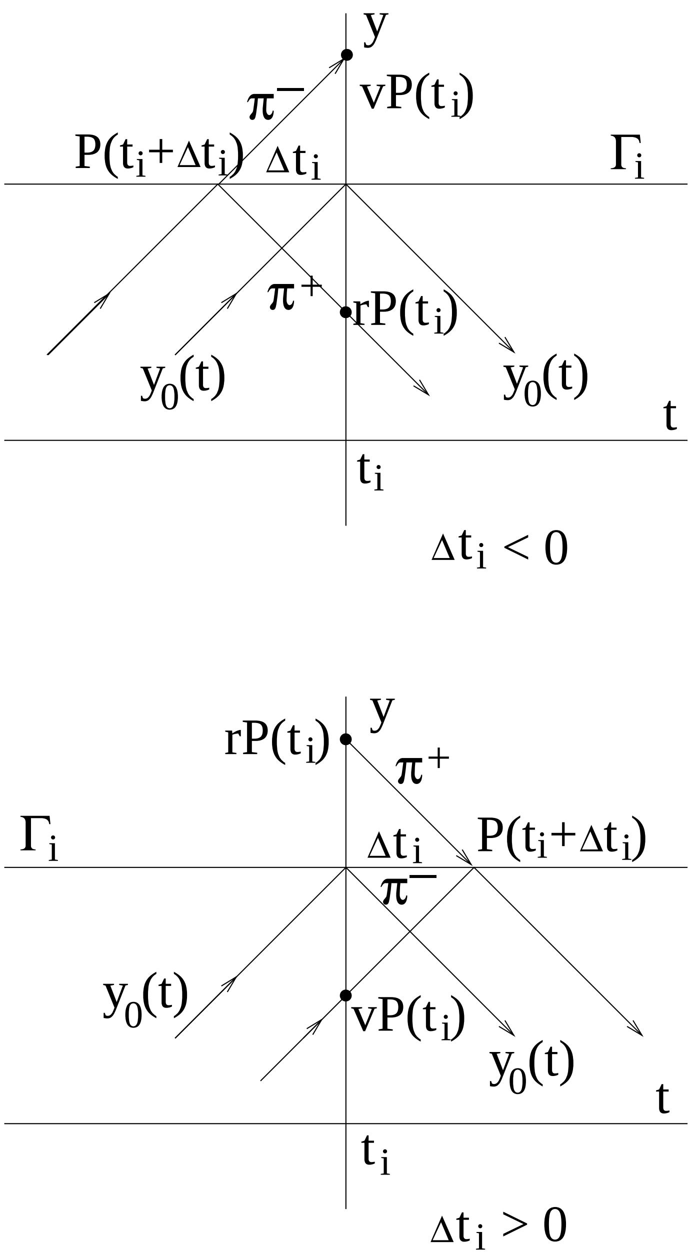

FIGURE 2.1. Define the virtual hitting and reflection points

Thus,

$$\begin{aligned} rP(t_i) - vP(t_i) &= \Delta y(t_i^+) - \Delta y(t_i^-) \\ &= (\dot{y}_0(t_i^-) - \dot{y}_0(t_i^+))\Delta t_i + O(|\Delta y(t_i^-)|^2 + |\Delta\mu|^2). \end{aligned}$$

We have derived the reflection law for virtual orbits at t_i:

Lemma 2.1. *The virtual orbits near y_0 jump from $vP(t_i)$ to $rP(t_i)$ according to the following reflection law:*

$$rP(t_i) = r_i^*(vP(t_i)) = \pi_i^+ \circ \pi_i^- \circ vP(t_i). \tag{2.5}$$

If we denote the derivative of r_i^* by R_i, we have

$$\Delta y(t_i^+) = R_i \Delta y(t_i^-) + O(|\Delta y(t_i^-)|^2 + |\Delta\mu|^2).$$

The $n \times n$ matrix R_i has the form

$$R_i = I + \frac{\dot y_0(t_i^+) - \dot y(t_i^-)}{\partial_t F_i + \partial_y F_i \dot y_0(t_i^-)} \partial F_i.$$

We have the expression:

$$\Delta y(t_i^+) = \Delta y(t_i^-) + (\dot y_0(t_i^+) - \dot y(t_i^-)) \frac{\partial_y F_i \cdot \Delta y(t_i^-)}{\partial_t F_i + \partial_y F_i \cdot y_0(t_i^-)} + O(|\Delta y(t_i^-)|^2 + |\Delta\mu|^2). \tag{2.6}$$

Lemma 2.2. *If the flow is also transverse to Γ_i, then the matrix R_i is non-singular.*

Proof. In this case the Poincaré mappings π_i^- and π_i^+ are diffeomorphisms. □

Consider a linear system in $\mathbb{R}^n$

$$\dot Y = A(t)Y + h(t), \quad t \in \mathbb{R}, \tag{2.7}$$

where $A(t) := \partial_y f(y, t, \mu, s(t))$ and $h(t)$ are piecewise continuous and uniformly bounded. Assume that there exists $T > 0$ such that (2.7) has exponential dichotomies on $(-\infty, -T]$ and $[T, \infty)$ respectively. For an introduction to exponential dichotomies and their applications, please see [8, 22]. Denote the projections to stable and unstable subspaces by $P_s(t), P_u(t), t \in (-\infty - T]$ or $[T, \infty)$. Let $\dim \mathcal{R} P_u(-T) = k^-$ and $\dim \mathcal{R} P_u(T) = k^+$. Suppose that the sequence $\{t_i\}_1^m$ and T satisfy

$$-\infty < -T < t_1 < t_2 < \cdots < t_m < T < \infty.$$

Let $T_1 > T$ and $T_2 > T$. Let $t_0 = -T_1$ and $t_{m+1} = T_2$.

We look for piecewise continuous, bounded solutions of (2.7) that satisfy the equation for $t \in (t_i, t_{i+1})$, and the nonhomogeneous reflection law at t_i:

$$\begin{aligned} Y(t_i^+) &= R_i Y(t_i^-) i + g_i \\ &= Y(t_i^-) + (\dot y_0(t_i^+) - \dot y_0(t_i^-)) \frac{\partial_y F_i \cdot Y(t_i^-)}{\partial_t F_i + \partial_y F_i \cdot \dot y_0(t_i^-)} + g_i. \end{aligned} \tag{2.8}$$

Let $C^0(\{t_i\})$ be the space of functions which are continuous in $(t_i, t_{i+1}), i = 0, \ldots, m$, and admit left and right sided limits at each t_i. Let $C^1(\{t_i\})$ be the space of functions in $C^0(\{t_i\})$ that has a first order derivative in $C^0(\{t_i\})$. Let

$$\begin{aligned} |h|_{C^0(\{t_i\})} &= \sup\{|h(t)| : t \in \mathbb{R}\}, \\ |h|_{C^1(\{t_i\})} &= |h|_{C^0(\{t_i\})} + |h'|_{C^0(\{t_i\})}. \end{aligned}$$

Define the operator $\mathcal{F} : Y \to (h, \{g_i\}, \phi_s, \phi_u)$ with

$$\begin{aligned} h(t) &= \dot{Y}(t) - A(t)Y(t), \\ g_i &= Y(t_i^+) - R_i Y(t_i^-), \\ \phi_s &= P_s(-T_1)Y(-T_1), \quad \phi_u = P_u(T_2)Y(T_2). \end{aligned}$$

The following is the main tool to prove the existence of periodic solutions bifurcating from a singular heteroclinic cycle.

Lemma 2.3. *The operator $\mathcal{F} : C^1(\{t_i\}) \to C^0(\{t_i\}) \times \mathbb{R}^m \times \mathcal{R}P_s(-T_1) \times \mathcal{R}P_u(T_2)$ is Fredholm with Index$(\mathcal{F}) = k^- - k^+$.*

Moreover, if we let the adjoint equation of (2.7) be

$$\dot{\psi} + A^*(t)\psi = 0, \tag{2.9}$$

then $(h, \{g_i\}, \phi_s, \phi_u) \in C^0(\{t_i\}) \times \mathbb{R}^m \times \mathcal{R}P_s(-T_1) \times \mathcal{R}P_u(T_2)$ is in the range of $\mathcal{F}$ if and only if for any bounded solution ψ of (2.9) that satisfies a dual reflection law at each t_i:

$$\begin{aligned} \psi(t_k^-) &= R_i^* \psi(t_k^+) \\ &= \psi(t_i^+) + (\psi(t_i^+) \cdot (y_0(t_i^+) - y_0(t_i^-))) \frac{\partial_y F_i}{\partial_t F_i + \partial_y F_i \cdot \dot{y}_0(t_i^-)}, \end{aligned}$$

the following condition is satisfied:

$$\int_{-T_1}^{T_2} < \psi(t), h(t) > dt + \sum_1^m \psi(t_i^+) g_i + < \psi(-T_1), \phi_s > - < \psi(T_2), \phi_u >= 0. \tag{2.10}$$

If (2.10) is satisfied and if phase conditions are posed so that the solution is also unique, then

$$|Y|_{C^1(\{t_i\})} \leq C(|h|_{C^0(\{t_i\})} + \sum_i |g_i| + |\phi_s| + |\phi_u|).$$

The constant C does not depend on T_1 or T_2.

Remark. The left hand side of (2.10) generalizes the Melnikov's integral and shall still be called a "Melnikov's integral".

Proof. For $\tau, t \notin \{t_1, \ldots, t_m\}$ define an evolution operator $\Theta(t, \tau)$ by

$$\Theta(t,\tau) = \begin{cases} \Phi(t,\tau), & \text{if } t_i < \tau \leq t < t_{i+1}, \\ \Phi(t,t_k)R_k\Phi(t_k,t_{k-1})\ldots\Phi(t_{i+1},t_i)R_i\Phi(t_i,\tau), & \text{if } \tau < t_i \leq t_k < t. \end{cases}$$

$\Theta(t, \tau)$ is continuous with respect to $t, \tau \notin \{t_1, \ldots, t_m\}$, and has one sided limits as $t, \tau \to t_i, i = 1, \ldots, m$.

Without loss of generality, assume that $0 \notin \{t_1, \ldots, t_m\}$. Using $\Theta(t, \tau)$ we can extend the exponential dichotomies from $(-\infty, -T]$ to $\mathbb{R}^-$ and from $[T, \infty)$ to $\mathbb{R}^+$ by extending the stable and unstable subspaces ($w = u$ or s):

$$\begin{aligned} \mathcal{R}P_w(t) &= \Theta(t, -T)\mathcal{R}P_w(-T), \quad t \leq 0, \\ \mathcal{R}P_w(t) &= \Theta(t, T)\mathcal{R}P_w(T), \quad t \geq 0. \end{aligned}$$

The rest of the proof is similar to that of [22, 18]. Let $\eta > 0$ be a small positive constant. Let $\tilde{h}(t) = h(t) + \sum_1^m \delta(t - t_i - \eta) g_i$ where δ is the delta function in the theory of distributions. The solution Y we are looking for satisfies $\dot{Y} = A(t)Y + \tilde{h}(t)$ and the homogeneous reflection law $Y(t_i^+) = R_i Y(t_i^-)$. We then have

$$Y(t) = \begin{cases} \int_{-T_1}^t \Theta(t,\tau) P_s(\tau)\tilde{h}(\tau)d\tau + \Theta(-T_1, t)\phi_s \\ + \int_0^t \Theta(t,\tau)P_u(\tau)\tilde{h}(\tau)d\tau + \Theta(t,0)\phi_3, & \text{if } -T_1 < t < 0, \\ \int_{T_2}^t \Theta(t,\tau)P_u(\tau)\tilde{h}(\tau)d\tau + \Theta(t,T_2)\phi_u \\ + \int_0^t \Theta(t,\tau)P_s(\tau)\tilde{h}(\tau)d\tau + \Theta(t,0)\phi_4, & \text{if } 0 < t < T_2, \end{cases}$$

where $\phi_4 \in \mathcal{R}P_s(0+),\ \phi_3 \in \mathcal{R}P_u(0-)$.

Thus

$$Y(0^-) = \int_{-T_1}^0 \Theta(0,\tau)P_s(\tau)\tilde{h}(\tau)d\tau + \Theta(0,-T_2)\phi_s + \phi_3,$$
$$Y(0^+) = \int_{T_2}^0 \Theta(0,\tau)P_u(\tau)\tilde{h}(\tau)d\tau + \Theta(0,T_2)\phi_u + \phi_4.$$

Note in the above $\Theta(0,-T_1)\phi_s + \int_{-T_1}^0 \cdots \in \mathcal{R}P_s(0^-)$ and $\Theta(0,T_2)\phi_u + \int_{T_2}^0 \cdots \in \mathcal{R}P_u(0^+)$. The jump at $t = 0$ is $Y(0^-) - Y(0^+)$, which depending on the choice of (ϕ_3, ϕ_4), will be denoted $G(\phi_3,\phi_4)$. Let $H = \mathcal{R}P_u(0^-) + \mathcal{R}P_s(0^+)$. Define projections $P_H + P_{H^\perp} = id$ according to the splitting $H \oplus H^\perp = \mathbb{R}^n$. Then by choosing (ϕ_3,ϕ_4) we have $P_H G(\phi_3,\phi_4) = 0$. In order that $P_{H^\perp}G(\phi_3,\phi_4) = 0$, we must have $< \psi(0), G(\phi_3,\phi_4) >= 0$ for every $\psi(0) \in H^\perp$. Using the fact $\psi(0) \perp (\mathcal{R}P_u(0^-) + \mathcal{R}P_s(0^+))$, we have $\phi(0) \in \mathcal{R}P_s^*(0^-) \cap \mathcal{R}P_u^*(0^+)$. If $\psi(t) = (\Theta(0,\tau))^*\psi(0), t \in \mathbb{R}$, then $\psi(t)$ is a bounded solution to the adjoint system (2.9) and satisfies the dual reflection law at each t_i.

The necessary and sufficient condition for (2.7) to have a bounded solution is

$$\begin{aligned}
&< \psi(0), \Theta(0,-T_1)\phi_s + \int_{-T_1}^0 \Theta(0,\tau)P_s(\tau)\tilde{h}(\tau)d\tau - \Theta(0,T_2)\phi_u - \int_{T_2}^0 \Theta(0,\tau)P_u(\tau)\tilde{h}(\tau)d\tau > \\
&= \int_{-T_1}^0 < (\Theta(0,\tau))^* P_s^*(0^-)\psi(0), \tilde{h}(\tau) > d\tau + \int_0^{T_2} < (\Theta(0,\tau))^*P_u^*(0^+)\psi(0), \tilde{h}(\tau) > d\tau \\
&\quad + < (\Theta(0,-T_1))^*\psi(0), \phi_s > - < (\Theta(0,T_2))^*\psi(0), \phi_u > \\
&= \int_{-T_1}^{T_2} < \psi(t), \tilde{h}(t) > dt + < \psi(-T_1), \phi_s > - < \psi(T_2), \phi_u > \\
&= \int_{-T_1}^{T_2} < \psi(t), h(t) > dt + \sum_1^m \psi(t_i + \eta) g_i + < \psi(-T_1), \phi_s > - < \psi(T_2), \phi_u > \\
&= 0.
\end{aligned}$$

Letting $\eta \to 0$, we have obtained (2.10) which is necessary and sufficient for $(h, \{g_i\}, \phi_s, \phi_u) \in \mathcal{RF}$.

Let the dimension of $\mathcal{R}P_u(0^-) \cap \mathcal{R}P_s(0^+)$ be ν. Then $\dim[\mathcal{R}P_u(0^-) + \mathcal{R}P_s(0^+)] = k^- + (n - k^+) - \nu$, and $\dim H^\perp = n - \dim H = \nu - (k^- - k^+)$. This shows that the codimension of $\mathcal{RF} = \nu - (k^- - k^+)$. Since dim$\mathcal{KF} = \nu$, the index of the Fredholm operator $\mathcal{F}$ is clearly $k^- - k^+$. □

Consider the case T_1 and $T_2 \to \infty$. Let $t_0 = -\infty$ and $t_{m+1} = \infty$. Let $C^0(\{t_i\})$ and $C^1(\{t_i\})$ be defined as before with an additional condition that the functions are also uniformly bounded in $(-\infty, t_1)$ and (t_m, ∞). From Lemma 2.3, we have the following corollary:

Corollary 2.4. *If we define the operator* $\mathcal{F} : Y \to (h, \{g_i\})$, $h(t) = \dot{Y}(t) - A(t)Y(t)$, $g_i = Y(t_i^+) - R_iY(t_i^-)$, *then* $\mathcal{F} : C^1(\{t_i\}) \to C^0(\{t_i\}) \times \mathbb{R}^m$ *is Fredholm with Index*$(\mathcal{F}) = k^- - k^+$.

Moreover, let Ψ *be the linear space of all the bounded solutions of the adjoint equation (2.7) which satisfy a dual reflection law as above. Then* $(h, \{g_i\})$ *is in the range of* $\mathcal{F}$ *if and only if for any* $\psi \in \Psi$, *we have*

$$\int_{-\infty}^{\infty} < \psi(t), h(t) > dt + \sum_{1}^{m} \psi(t_i^+)g_i = 0. \tag{2.11}$$

Corollary 2.5. *Assume that* $0 \notin \{t_1, \dots, t_m\}$. *Under the same conditions of Lemma 2.3, if* $\mathcal{F}$ *is also of codimension one and* ψ *is, up to a constant multiple, a unique nonzero bounded solution to the adjoint equation (2.9), then for every* $(h, \{g_i\}, \phi_s, \phi_u)$, *there exists a unique generalized solution* Y *such that* $Y \perp Ker\mathcal{F}$, *and* Y *has a jump at* $t = 0$ *along the given direction* $\psi(0)/\|\psi(0)\|^2$:

$$Y(0^-) - Y(0^+) = G(h, \{g_i\}, \phi_s, \phi_u)\psi(0)/\|\psi(0)\|^2.$$

Here G *is equal to the left hand side of (2.10)–the Melnikov's integral.*

Proof. Assume that Y has one more jump of the following form at $t = 0$:

$$Y(0^-) - Y(0^+) = \tilde{g}\psi(0)/\|\psi(0)\|^2, \quad \tilde{g} \in \mathbb{R}.$$

The Melnikov integral in Lemma 2.3 gains one more term $-\tilde{g}$. One can uniquely solve for $\tilde{g} := G(h, \{g_i\}, \phi_s, \phi_u)$ so that the condition in Lemma 2.3 is satisfied. □

3. Heteroclinic solutions

We now study the existence of perturbed heteroclinic solutions connecting the equilibrium $X^+ := (\sqrt{1/a}, 0, 0)$ to $Z^+ := (0, 0, \sqrt{1/a})$. It can be constructed as the perturbation of a "singular heteroclinic solution" $(x_0, \epsilon y_1, z_0)$. The construction of the singular heteroclinic solution relies on some numerical aid, but the perturbation analysis of the singular heteroclinic solution to the exact heteroclinic solution is rigorous.

3.1. **Construction of singular heteroclinic solutions.** Numerical computation shows that the heteroclinic solution connecting X^+ to Z^+ is narrow in the y direction. We blow up the y direction by writing it as $(x, \epsilon y_1, z)$.

$$\begin{aligned} \dot{x} &= x(1 - ax^2 - \epsilon^2 by_1^2 - cz^2) + \epsilon^2 dy_1 zu^*(xy_1z, s), \\ \dot{y}_1 &= y_1(1 - \epsilon^2 ay_1^2 - bz^2 - cx^2) + dzxu^*(xy_1z, s), \\ \dot{z} &= z(1 - az^2 - bx^2 - \epsilon^2 cy_1^2) + \epsilon^2 dxy_1u^*(xy_1z, s). \end{aligned} \tag{3.1}$$

By setting $\epsilon = 0$, we have a system satisfied by the "singular heteroclinic solution":

$$\dot{x} = x(1 - ax^2 - cz^2), \tag{3.2}$$

$$\dot{y}_1 = y_1(1 - bz^2 - cx^2) + dzxu^*(xy_1z, s), \tag{3.3}$$

$$\dot{z} = z(1 - az^2 - bx^2). \tag{3.4}$$

At the equilibrium X^+, the eigenvalues and eigenvectors for (3.2)–(3.4) are:

eigenvalue: $\lambda^{--} = -2$ eigenvector: $(1, 0, 0)$,
eigenvalue: $\lambda^{-} = 1 - c/a < 0$ eigenvector: $(0, 1, 0)$,
eigenvalue: $\lambda^{+} = 1 - b/a > 0$ eigenvector: $(0, d\sqrt{1/a}u^*(0, s), (c - b)/a)$.

Here the first vector is in the normal direction, the next two are tangent to the weakest stable and unstable manifold respectively. The angle between the unstable eigenvector and the z axis is of $O(d)$.

Similarly, at Z^+, the eigenvalues and eigenvectors are:

eigenvalue: $\lambda^{-} = 1 - c/a < 0$ eigenvector: $((b - c)/a, d\sqrt{1/a}u^*(0, s), 0)$,
eigenvalue: $\lambda^{+} = 1 - b/a > 0$ eigenvector: $(0, 1, 0)$,
eigenvalue: $\lambda^{--} = -2$ eigenvector: $(0, 0, 1)$.

The third eigenvector is in the normal direction. The first two eigenvectors are tangent to the weakest stable and unstable manifolds respectively. The angle between the stable eigenvector and the x axis is of $O(d)$.

Based on the eigenvalues at X^+ and Z^+, we know that for each $s = \pm 1$, there exists a one-dimensional local unstable manifold of the equilibrium X^+ and a two-dimensional local stable manifold of Z^+,. We extend the local unstable manifold forward and the local stable manifold backward by the flow with relay nonlinearity and call the result the (global) unstable (or stable) manifold of X^+ (or Z^+). If $u(t)$ with $s_1(t)$ is a solution on the unstable manifold and $v(t)$ with $s_2(t)$ is a solution on the unstable manifold, then we say that these two manifolds have the same polarity if $s_1(t) = s_2(t)$ as $t \to \infty$; otherwise we say that the two manifolds have different polarity. We seek the intersection of $W^u(X^+)$ and $W^s(Z^+)$, with the same polarity. For some values of d, the limiting system (3.2)-(3.4) has exactly two singular heteroclinic solutions that connect X^+ to Z^+. The two solutions are related by the symmetry: $(x, y_1, z, s) \to (x, -y_1, z, -s)$. It suffices to consider the singular heteroclinic solution $y_1(t, d)$ with $s(t) = 1$ for t near $-\infty$.

Equations (3.2) and (3.4) form a system that does not depend on y_1. It is known to possess a heteroclinic solution (x_0, z_0) connecting $(\sqrt{1/a}, 0)$ to $(0, \sqrt{1/a})$. We impose the phase condition $x_0(0) = z_0(0)$ so that the solution is unique. We assume that the solution (x_0, z_0) approaches $(0, \sqrt{1/a})$ along the eigenspace corresponding to the weakest stable eigenvalue, based on numerical simulations.

Let $\theta(t) = \tan^{-1}(x_0(t)/z_0(t))$, $0 < \theta < \pi/2$. Then

$$d\theta/dt = \frac{x_0(t)z_0(t)((b - a)z_0^2(t) + (a - c)z_0^2(t))}{x_0^2(t) + z_0^2(t)}.$$

Using $b < c < a$, we have $d\theta/dt < 0$. The function $\theta(t)$ is invertible with the inverse denoted by $t^*(\theta)$. The mapping $(x_0(t^*(\theta)), z_0(t^*(\theta)))$ is one-to-one between the interval $0 < \theta < \pi/2$ and the heteroclinic orbit in xz-space.

After determining $(x, z) = (x_0(t), z_0(t))$, we need to find a "heteroclinic solution" $y_1(t, d)$ of the time-dependent equation (3.3) that approaches 0 as $t \to \pm\infty$. Equation (3.3) is a relay system where $s(t)$ changes sign when it hits the junction surfaces

$$\Gamma_1 := \{y_1 = 1/(x_0(t)z_0(t))\}, \quad \Gamma_2 := \{y_1 = -1/(x_0(t)z_0(t))\}.$$

If $y_1(t)$ is such a solution with the corresponding $s(t)$, then $(-y_1(t), -s(t))$ is also a solution. For simplicity, we assume that $s(t) = 1$ for $t \approx -\infty$ in the future.

Observe that for any $d \geq 0$, there exists a unique $y_1(t)$ that approaches 0 as $t \to -\infty$ and with $s(-\infty) = 1$. In fact, if (x_0, y_1, z_0) is the unique solution on the one-dimensional unstable manifold of (3.2)-(3.4), then its restriction to $y_1(t)$ is such a solution. This solution shall be denoted $y_1^u(t, d)$. The trajectory of $y_1^u(t, d)$ is called the unstable manifold of the time-dependent equation (3.3). Similarly, for each $d \geq 0$ and any $\nu = \pm 1$, there exists a unique $y_1(t) = y_1^s(t, d, \nu)$ such that $y_1(t) \to 0$ as $t \to \infty$ with $s(\infty) = \nu$. The trajectories of $y_1^s(t, d, \nu)$ are called the stable manifold of (3.3) with the polarity $\nu = \pm 1$. In fact, the local two-dimensional stable manifold for (3.2)–(3.4) can be written as $y_1 = \mathcal{S}(x, z, d, \nu)$ where (x, z) is in a neighborhood of zero. Then $y_1^s(t, d, \nu) = \mathcal{S}(x_0(t), z_0(t), d, \nu)$. We need to find d such that $y_1^u(t, d) \to 0$ as $t \to \infty$, that is, the unstable manifold of (3.2) meets the stable manifold with the correct polarity. The latter means that if $s(t)$ is related to $y_1^u(t, d)$, then $s(t) = \nu$ for $t \approx \infty$.

Let us introduce $w = x_0(t)y_1(t)z_0(t)$. The junction surfaces can be written as $w = \pm 1$. We consider the region $|w| < 1$. We say $y_1(t)$ moves monotonically in this region if $dw/dt \neq 0$.

Observe that for fixed (t, y), if d increases, then the slope dy_1/dt increases if $s = 1$ and decreases if $s = -1$. This property will be used in the comparison argument below without further mention.

Using a ODE solver, we gradually increase d from 0 and find that for every non-negative integer m, there exists $d = d_m \geq 0$ such that $y_1^u(t, d_m) \to 0$ as $t \to \infty$ and the orbit moves monotonically between the junction surfaces and will hit the junction surfaces exactly m times. The switching is normal at the points where the orbit hits $\Gamma_\nu, \nu = \pm 1$, as defined in §2. We also find that there exists $\tilde{d}_m$ between d_m and d_{m+1} such that $y_1^u(t, \tilde{d}_m)$ is tangent to the junction surfaces Γ_1 or Γ_2. The increasing of number of switches must come through a tangential intersection of $y_1^u(t, d)$ to Γ_ν. In order to ensure that the tangential intersection of the vector field with the junction surfaces will not occur in the region where we search for heteroclinic orbits, we will find the region where the vector field is monotone and the relation between this region and $y_1^u(t, d)$ and $y_1^s(t, d, -1)$. The following facts are discovered numerically and are assumed to be satisfied by the system we study.

F1) If $s = 1$ and $d > \tilde{d}_1$, then:
(1) In the region $y_1 \leq 0$, $dw/dt > 0$. The region $y_1 > 0$ is divided by curves X^+T_1 and Z^+T_2 into three parts as in Figure 3.1. In the mid part between the two curves, we have $dw/dt > 0$, while in the two outer parts we have $dw/dt < 0$.

(2) The unstable manifold enters the mid region of $y_1 > 0$ and remains there until it hits Γ_1. The stable manifold related to $s = -1$ enters the mid region of $y_1 > 0$ backward in time and remains there until it hits Γ_1 at Q, which is to the right of T_1. The mapping $t \to \theta$ is a diffeomorphism. If we plot the slope field of (3.3) on the (θ, y_1) coordinates, then to the right means larger θ.

The observations in **F1** can be extended to the case $s = -1$ by using the symmetry $(s, y_1) \to (-s, -y_1)$ satisfied by (3.3). In Figure 3.1, we also plot the two solutions $y_1^u(t, d)$ denoted by W^u and $y_1^s(t, d, -1)$ denoted by W^s to confirm that W^s hits Γ_1 at Q which is to the right of T_1.

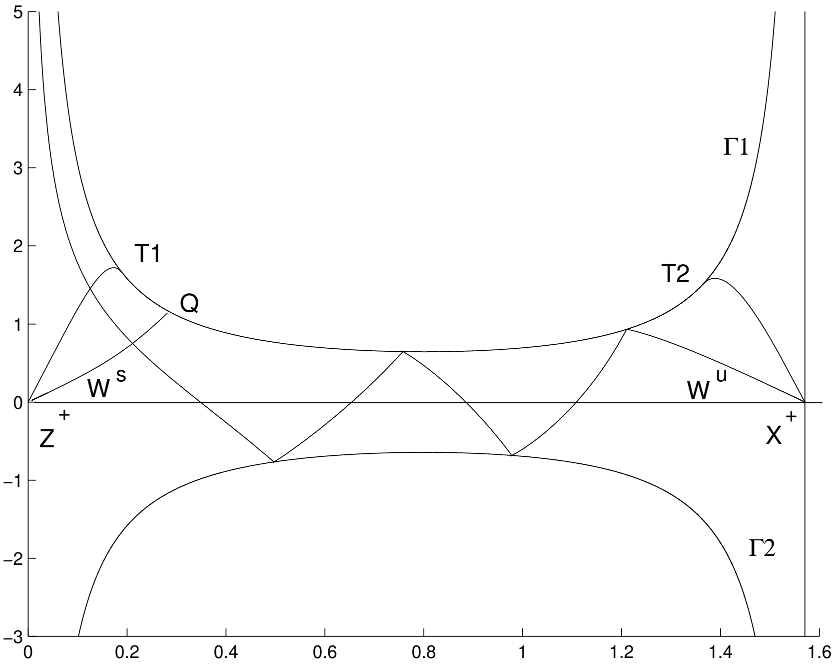

FIGURE 3.1. The unstable and stable manifolds enter the mid region where $dw/dt > 0$. We choose d to be slightly above d_4 and the heteroclinic $y_1(t, d_4)$ just breaks. We plot $y_1^u(t, d)$ which approaches ∞ as $t \to \infty$. As we further increase d, $y^u(t, d)$ first hits Γ_1 tangentially at T_1, and then hooks with $y_1^s(t, d, -1)$ at Q, creating a new heteroclinic solution for $d = d_5$.

Assuming that the system we study satisfies **F1**, we can prove the existence of d_m and the corresponding heteroclinic solution $y_1(t, d_m)$ by a shooting and continuation method.

First, when $m = 0$, $y_1^u(t, 0) = 0$ with $d_0 = 0$ is clearly a desired solution. Gradually increasing d, the zero heteroclinic solution breaks and $y_1^u(t, d) \to \infty$ as $t \to \infty$. If we further increase d, $y_1^u(t, d)$ will hit Γ_1 tangentially at T_1 for some minimum value $d = \tilde{d}_0 > 0$. From **F1**, $y_1^s(t, \tilde{d}, -1)$ will cross $y_1^u(t, d)$ and hit Γ_1 at Q, which is to the right of T_1.

If we increase d further from $\tilde{d}_0$, then, from **F1** again, $y_1^u(t,d)$ will hit Γ_1 transversely at P. Furthermore the point P will move to the right and Q will move to the left. At certain value $d = d_1$, we have $P = Q$ where $y_1^u(t, d_1) = y_1^s(t, d_1, -1)$.

We proceed by induction. Suppose we have found d_m and the corresponding $y_1^u(t, d_m)$. Assuming m is even first. If we continue to increase d from d_m, the heteroclinic $y_1(t, d_m)$ breaks and $y_1^u(t,d) \to \infty$ as $t \to \infty$; then $y_1^u(t,d)$ hits Γ_1 tangentially at the point T_1 first, then hits Γ_1 transversely at P next (not shown on the figure). The point P moves towards $Q = y_1^s(t, d, -1) \cap \Gamma_1$, and finally P meets Q at a unique value $d = d_{m+1}$. At this value we have $y_1^u(t, d_{m+1}) = y_1^s(t, d_{m+1}, -1)$ with the correct polarity. See Figure 3.1. We have observed numerically that W^u and T_2 both move to the right as d increases so that $y_1^u(t,d)$ never hits Γ_1 tangentially. The case m being odd can be considered similarly. One only need to replace $y_1^u(t,d) \to \infty$ by $y_1^u(t,d) \to -\infty$, Γ_1 by Γ_2, $y_1^s(t,d,-1)$ by $y_1^s(t,d,+1)$.

3.2. **Properties of the singular heteroclinic solution.** Suppose that $y_1(t, d_m)$ is a heteroclinic solution of (3.3). Assume also that the solution hits the junction surfaces at $t = t_1, t_2, \ldots, t_m$, and $0 \neq t_1, \ldots, t_m$. As before, assume that the corresponding $s(t)$ satisfies $s(-\infty) = 1$. As in §2, define Poincaré sections for this equation:

$$\Gamma_i := \{(t,y) | y = \pm 1/(x_0(t) z_0(t))\}, \quad i = 1, \ldots, m,$$

where $+1$ is used if i is odd and -1 is used if i is even. That is, $\Gamma_i = \Gamma_1$ (or Γ_2) if i is odd (or even).

If we define time cross sections

$$\mathcal{T}_i := \{(t,y) | t = t_i\}, \quad i = 1, \ldots, m,$$

then $(t_i, y_1(t_i, d_m)) \in \Gamma_i \cap \mathcal{T}_i$.

We now consider a perturbation of $y_1(t, d_m)$. For clarity, assume that m is even. Let $d = d_m + \Delta d$ and $y_1^u(t,d)$ and $y_1^s(t,d,-1)$ be the unstable and stable manifold of (3.3). Both y_1^u and y_1^s can be considered as virtual orbits that obey reflection laws at $t = t_i,\ i = 1, \ldots, m$. Without loss of generality, assume $0 \notin \{t_1, \ldots, t_m\}$. To have a heteroclinic orbit we must have

$$G(d) := y_1^u(0,d) - y_1^s(0,d,-1) = 0.$$

Consider a linear variational equation of (3.3):

(3.5)
$$\dot{Y} = Y\{(1 - bz_0^2 - cx_0^2) + d_m x_0 z_0 \frac{\partial}{\partial w} u^*(x_0 y_1 z_0, s)\} + \Delta d x_0 z_0 u^*(x_0 y_1 z_0, s) + f(t).$$

The solution Y satisfies the reflection law at each t_i

$$Y(t_i^+) = R_i^{(y)}(y) Y(t_i^-) + g_i^{(y)}, \quad g_i^{(y)} \in \mathbb{R}, \tag{3.6}$$

where $R_i^{(y)} = 1 + \frac{\dot{y}_1(t_i^-) - y_1(t_i^+)}{\dot{y}^*(t_i) - \dot{y}_1(t_i^-)}$. Let $\psi^{(y)}(t)$ be the solution for the adjoint equation

$$\dot{\psi} + \psi\{(1 - bz_0^2 - cx_0^2) + d_m x_0 z_0 \frac{\partial}{\partial w} u^*(x_0 y_1 z_0, s)\} = 0, \tag{3.7}$$

that satisfies $\psi(0)=1$ and the dual reflection law

$$\psi(t_i^-)=R_i^{(y)}\psi(t_i^+).$$

Observe that the coefficients of (3.7) have the following asymptotic limits:

$$(3.8)\qquad \begin{aligned}&(1-bz_0^2-cx_0^2)+d_m x_0 z_0\frac{\partial}{\partial w}u^*(x_0y_1z_0,s)\to(1-c/a)<0,\quad t\to-\infty,\\ &(1-bz_0^2-cx_0^2)+d_m x_0 z_0\frac{\partial}{\partial w}u^*(x_0y_1z_0,s)\to(1-b/a)>0,\quad t\to\infty.\end{aligned}$$

Based on (3.8), $\psi(t)\to 0$ as $t\to\pm\infty$.

Lemma 3.1. *(1) For any $i\in\{1,\dots,m\}$, $\psi^{(y)}(t)$ changes signs between $t_{i-1}<t<t_i$ and $t_i<t<t_{i+1}$, assuming $t_0=-\infty, t_{m+1}=\infty$.*
(2) $\int_{-\infty}^{\infty}<\psi^{(y)}(t),x_0(t)z_0(t)u^*(x_0y_1z_0,s)>dt\neq 0$. The sign of the integral is equal to the sign of $s(0)$.
(3) for any $f\in C^0(\{t_i\})$ and $\{g_i^{(y)}\}\in\mathbb{R}^m$, there exists a unique Δd so that (3.5) has a bounded solution $Y\in C^1(\{t_i\})$. Moreover,

$$|\Delta d|+|Y|_{C^1(\{t_i\})}\le C(|f|_{C^0(\{t_i\})}+\sum_i|g_i^{(y)}|).$$

Proof. We use Lemma 2.3 with $n=1$. From the asymptotic limits of the coefficient of (3.5), the homogeneous part of the equation has exponential dichotomies for $t\ge T$ and $t\le -T$ where $T>\max\{|t_i|\}$. Also $\dim\mathcal{R}P_u(-T)=0$ and $\dim\mathcal{R}P_u(T)=1$. The index of the Fredholm operator defined in Lemma 2.3 is -1. It is obvious that the kernel of the Fredholm operator is zero dimensional. Thus the codimension of its range is one. There exists a unique bounded solution to the adjoint equation (3.7), denoted $\psi^{(y)}$, that satisfies $\psi^{(y)}(0)=1$.

We will now demonstrate $R_i^{(y)}<0$. Since the flow at t_i^- and t_i^+ are transverse to the boundary $\Gamma_i=\{y=y^*(t)\}$, we have either

$$\dot y_1(t_i^-)<\dot y^*(t_i)<\dot y_1(t_i^+)\quad\text{ or }\quad \dot y_1(t_i^+)<\dot y^*(t_i)<\dot y_1(t_i^-).$$

In both cases, one can show that $\frac{\dot y_1(t_i^-)-y_1(t_i^+)}{\dot y^*(t_i)-\dot y_1(t_i^-)}<-1$. This implies that $R_i^{(y)}<0$, hence part (1) of this lemma.

Since $\psi^{(y)}(t)$ changes sign when crossing each t_i and the function $u^*(x_0y_1z_0,s)$ does so as well, we find that $<\psi^{(y)}(t),x_0(t)z_0(t)u^*(x_0y_1z_0,s)>$ does not change sign. Moreover, $\psi^{(y)}(0)=1$ and $u^*>0$ at $t=0$ if $s(0)=1$. This proves part (2) of this lemma.

From Lemma 2.3, the necessary and sufficient condition for (3.5) with reflection law (3.6) to have a bounded solution $Y(t)$ is

$$\int_{-\infty}^{\infty}<\psi^{(y)}(t),f(t)+\Delta d x_0(t)z_0(t)u^*(x_0y_1z_0,s)>dt+\sum_i\psi^{(y)}(t_i^+)g_i^{(y)}=0.$$

From part (2) of this lemma, we can solve for Δd from the above.

The estimate for the solution follows from the standard theory of Fredholm operators. $\square$

3.3. **Existence of perturbed heteroclinic solutions.** Let $(x_0(t), \epsilon y_1(t, d_m), z_0(t))$ be the singular heteroclinic solution for a given m. We now consider the existence of an exact heteroclinic solution near the singular heteroclinic solution, written as

$$x = x_0 + \epsilon^2 X,\ y = \epsilon y_1 + \epsilon^3 Y,\ z = z_0 + \epsilon^2 Z, \quad \text{with } d = d_m + \epsilon^2 D.$$

We also need a phase condition for the heteroclinic solution. Let

$$\Sigma := \{(x, y, z)|x = z\}$$

be a cross section of the flow. We assume that the heteroclinic solution satisfies $(x(0), y(0), z(0)) \in \Sigma$. In particular,

$$X(0) = Z(0). \tag{3.9}$$

The system for (X, Y, Z, D) can be written as:

$$\dot{X} = X(1 - 3ax_0^2 - cz_0^2) - 2cx_0z_0Z + h_1(t) + \epsilon^2 N_1, \tag{3.10}$$

$$\dot{Y} = Y[(1 - bz_0^2 - cx_0^2) + d_m x_0^2 z_0^2 \frac{\partial u^*}{\partial w}] + Dx_0z_0u^*(x_0y_1z_0, s) + h_2(t) + \epsilon^2 N_2, \tag{3.11}$$

$$\dot{Z} = Z(1 - 3az_0^2 - bx_0^2) - 2bx_0z_0X + h_3(t) + \epsilon^2 N_3, \tag{3.12}$$

where $h_j, j = 1, 3$ is a given function of t that does not depend on (X, Y, Z, D, ϵ) while h_2 is a function of t and X, Z but not Y:

$$\begin{aligned} h_1(t) &= -bx_0y_1^2 + d_m y_0 z_0 u^*(x_0y_1z_0, s), \\ h_2(t, X, Z) &= -ay_1^3 + d_m z_0 X u^*(x_0y_1z_0, s) + d_m x_0 Z u^*(x_0y_1z_0, s), \\ h_3(t) &= -cz_0y_1^2 + d_m x_0 y_1 u^*(x_0y_1z_0, s). \end{aligned}$$

The nonlinear terms (N_1, N_2, N_3) are functions of (X, Y, Z, D, ϵ) and satisfy

$$|N_1| + |N_2| + |N_3| = O(|X| + |Y| + |Z| + |D|),$$

$$\sum_{j=1}^{3} |N_j(X_1, Y_1, Z_1, D_1) - N_j(X_2, Y_2, Z_2, D_2)| \le C(|\Delta X| + |\Delta Y| + |\Delta Z| + |\Delta D|).$$

Here ΔU denotes $U_1 - U_2$ with $U = X, Y, Z, D$.

The solutions must also satisfy $(X, Y, Z) \to 0$ as $t \to \pm\infty$. Due to the hyperbolicity of the equilibria, it suffices to show that (X, Y, Z) is uniformly small and bounded.

The heteroclinic solution hits Γ_i successively at the time $t_i + \Delta t_i$, and switches between $s = -1$ and $s = 1$. This is the relay system studied in §2. For each heteroclinic solution, there corresponds a unique virtual heteroclinic solution that satisfies the reflection law:

$$(x(t_i^+), y(t_i^+), z(t_i^+)) = r_i^*((x(t_i^-), y(t_i^-), z(t_i^-))).$$

The above can be written as

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} (t_i^+) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_i^{(y)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} (t_i^-) + \epsilon^2 \begin{pmatrix} M_i^{(x)} \\ M_i^{(y)} \\ M_i^{(z)} \end{pmatrix}.$$

The nonlinear terms satisfy

$$(M_i^{(x)}, M_i^{(y)}, M_i^{(z)}) = O(X^2 + Y^2 + Z^2 + D^2),$$
$$|\Delta M_i^{(x)}| + |\Delta M_i^{(y)}| + |\Delta M_i^{(z)}| \le C(|\Delta X| + |\Delta Y| + |\Delta Z| + |\Delta D|).$$

We use $\Delta M_i^{(x)}$ to denote the variation of $M_i^{(x)}$ when (X_1, Y_1, Z_1, D_1) and (X_2, Y_2, Z_2, D_2) are the arguments.

We first study a linear nonhomogeneous system:

$$\dot X = X(1 - 3ax_0^2 - cz_0^2) - 2cx_0z_0Z + f_1, \tag{3.13}$$

$$\dot Y = Y[(1 - bz_0^2 - cx_0^2) + d_m x_0^2 z_0^2 \frac{\partial u^*}{\partial w}] + Dx_0z_0u^*(x_0y_1z_0, s) + f_2, \tag{3.14}$$

$$\dot Z = Z(1 - 3az_0^2 - bx_0^2) - 2bx_0z_0X + f_3. \tag{3.15}$$

The forcing term $(f_1, f_2, f_3) \in C^0(\{t_i\})$. The solution is in $C^1(\{t_i\})$ and satisfies the nonhomogeneous reflection law:

$$X(t_i^+) = X(t_i^-) + g_i^{(x)},$$
$$Y(t_i^+) = R_i^{(y)} Y(t_i^-) + g_i^{(y)},$$
$$Z(t_i^+) = Z(t_i^-) + g_i^{(z)},$$

at each $t_i, i = 1, \dots, m$; and the phase condition (3.9).

Observe that system (3.13) and (3.15) have exponential dichotomies in $(-\infty, 0]$ and $[0, \infty)$ respectively. The stable subspace of the dichotomy in $(-\infty, 0]$ is one-dimensional. There exists a unique bounded solution $(X(t, D), Z(t, D)), t \le 0$ that satisfies the phase condition $(X(0), Z(0)) \perp (\dot x_0(0), \dot z_0(0))$. The solution can be continued to $t = T$ by using (3.13), (3.15) and the reflection law at each $t_i, 1 \le i \le m$. Finally, since $f_1(t), f_3(t)$ is uniformly bounded for $t \ge T$ and the stable subspace of the dichotomy in $[T, \infty)$ is two-dimensional, one can show that the solution $(X(t), Z(t))$ is uniformly bounded for $t \ge T$. Also

$$|X| + |Z| \le C(|f_1| + |f_3| + \sum_i (|g_i^{(x)}| + |g_i^{(z)}|)).$$

According to Lemma 3.1, the function Y and the parameter D are uniquely determined by (3.14) and the reflection law at each t_i: $Y(t_i^+) = R_i^{(y)} Y(t_i^-) + g_i^{(y)}$. Moreover,

$$|Y|_\infty + |D| \le C(|f_2|_\infty + \sum_i |g_i^{(y)}|).$$

Denote the solution of system (3.13)–(3.15)

$$X = G_1(f_1, f_3, \{g_i^{(x)}\}, \{g_i^{(z)}\}), \quad (D, Y) = G_2(f_2, \{g_i^{(y)}\}), \quad Z = G_3(f_1, f_3, \{g_i^{(x)}\}, \{g_i^{(z)}\}).$$

The nonlinear system (3.10)-(3.12) can be written as a fixed point problem of the mapping $\mathfrak{P} : (X, Y, Z, D) \to (X_1, Y_1, Z_1, D_1)$:

$$
\begin{aligned}
X_1 &= G_1(h_1(t) + \epsilon^2 N_1, h_3(t) + \epsilon^2 N_3, \epsilon^2 M_i^{(x)}, \epsilon^2 M_i^{(z)}),\\
Z_1 &= G_3(h_1(t) + \epsilon^2 N_1, h_3(t) + \epsilon^2 N_3, \epsilon^2 M_i^{(x)}, \epsilon^2 M_i^{(z)}),\\
(D_1, Y_1) &= G_2(h_2(t, X_1, Z_1) + \epsilon^2 N_2, \epsilon^2 M_i^{(y)}).
\end{aligned}
$$

Here $N_j, 1 \le j \le 3$ and $M_i^{(x)}, M_i^{(y)}, M_i^{(z)}$ are functions of (X, Y, Z, D, ϵ).

Let

$$
\begin{aligned}
\xi_1(t) &= G_1(h_1(t), h_3(t), 0, 0),\\
\xi_3(t) &= G_3(h_1(t), h_3(t), 0, 0),\\
(\bar{D}, \xi_2(t)) &= G_2(h_2(t, \xi_1(t), \xi_3(t)), 0).
\end{aligned}
$$

Then $h_2(t, X_1, Z_1) - h_2(t, \xi_1, \xi_3)$ is a linear homogeneous function of $(X_1 - \xi_1, Z_1 - \xi_3)$, denoted $\tilde{h}(X_1 - \xi_1, Z_1 - \xi_3)$. The mapping $\mathfrak{P}$ cam be written as

$$
\begin{aligned}
X_1 &= \xi_1 + \epsilon^2 G_1(N_1, N_3, M_i^{(x)}, M_i^{(z)}),\\
X_3 &= \xi_3 + \epsilon^2 G_3(N_1, N_3, M_i^{(x)}, M_i^{(z)}),\\
(D, Y_1) &= (\bar{D}, \xi_2) + \epsilon^2 G_2(\tilde{h}(G_1, G_3) + N_2, M_i^{(y)}),
\end{aligned}
$$

where $\tilde{h}(G_1, G_3) = \tilde{h}(G_1(N_1, N_3, M_i^{(x)}, M_i^{(z)}), G_3(N_1, N_3, M_i^{(x)}, M_i^{(z)}))$.

Let B_δ be a δ ball centered at $(\xi_1, \xi_2, \xi_3.\bar{D})$:

$$
B_\delta := \{(X, Y, Z, D) : |X - \xi_1| + |Y - \xi_2| + |Z - \xi_3| + |D - \bar{D}| \le \delta\}.
$$

If ϵ is small, it is easy to see that $\mathfrak{P}$ maps B_δ into itself and is a contraction mapping. Thus, there exists a small $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\mathfrak{P}$ has a unique fixed point (X^*, Y^*, Z^*, D^*) in B_δ. This fixed point corresponds to a unique heteroclinic solution near the first approximation of the heteroclinic solution $(x_0, \epsilon y_1, z_0)$. Denote this heteroclinic solution by $(x^*, \epsilon y^*, z^*)$ corresponding to $d_m(\epsilon) = d_m + \epsilon^2 D^*$.

4. Bifurcation of heteroclinic cycle to periodic solutions

Let

$$
\begin{aligned}
\kappa_1 &:= (x, y, z, s) \to (-x, y, z, -s),\\
\kappa_2 &:= (x, y, z, s) \to (x, -y, z, -s),\\
\kappa_3 &:= (x, y, z, s) \to (x, y, -z, -s),\\
r &:= (x, y, z, s) \to (y, z, x, s).
\end{aligned}
$$

System (1.5) respects the symmetry group $(\mathbf{Z}_2)^3 \times \mathbf{Z}_3$ where $(\mathbf{Z}_2)^3$ and $\mathbf{Z}_3$ are reflection and rotation subgroups:

$$
\begin{aligned}
(\mathbf{Z}_2)^3 &:= \{\kappa_1^i \kappa_2^j \kappa_3^k,\ i, j, k = 0, 1\},\\
\mathbf{Z}_3 &:= \{r^\ell,\ \ell = 0, 1, 2\}.
\end{aligned}
$$

There are 24 elements in the group $(\mathbf{Z}_2)^3 \times \mathbf{Z}_3$.

System (1.5) has 6 nonzero equilibria on the three coordinate axes, denoted by

$$X^{\pm} = (\pm\sqrt{1/a}, 0, 0), \quad Y^{\pm} = (0, \pm\sqrt{1/a}, 0), \quad Z^{\pm} = (0, 0, \pm\sqrt{1/a}).$$

Since $s(t)$ is part of a solution, we denote a heteroclinic solution by $H = (q(t), s(t))$ where $q : \mathbb{R} \to \mathbb{R}^3$. In §3, we have shown the existence of a heteroclinic solution, with $s(-\infty) = 1$, connecting X^+ to Z^+. Denote this solution by

$$H_1 := (q_1(t), s(t)) = (x(t), y(t), z(t), s(t)).$$

There is another heteroclinic solution connecting X^+ to Z^+, but with $s(-\infty) = -1$. It is related to H_1 by

$$\kappa_2 H_1 := (x(t), -y(t), z(t), -s(t)).$$

If we apply the 24 elements of the group $(\mathbf{Z}_2)^3 \times \mathbf{Z}_3$ on H_1, we have 24 heteroclinic solutions connecting the 6 equilibria. Any of the two equilibria not symmetric about the origin are connected by exactly 2 heteroclinic solutions which are associated by one of the reflections $\kappa_j, j = 1, 2, 3$. For clarity, we use $H(E_1 \to E_2, s(-\infty) = \pm 1)$ to denote the heteroclinic solution connecting E_1 to E_2 starting with $s(-\infty) = \pm 1$.

System (1.5) can have many complicated periodic solutions. However, periodic solutions that bifurcate from heteroclinic cycles by changing parameter d slightly from $d_m(\epsilon)$ can be described as follows. Without loss of generality, we consider a heteroclinic cycle that starts with $H_1 = H(X^+ \to Z^+, s(-\infty) = 1)$.

Theorem 4.1. *Suppose that $\{H_i\}_1^\ell$, with $H_1 = H(X^+ \to Z^+, s(-\infty) = 1)$, is a heteroclinic cycle that can bifurcate into a periodic solution, and suppose that each H_i hits the junction surfaces m times, $m \geq 0$. Then:*

(1) If m is even, we have $\ell = 3$ and either $H_{i+1} = rH_i$ or $H_{i+1} = \kappa_1\kappa_2 rH_i$ for all $1 \leq i \leq \ell$. Here r, κ_1, κ_2 are the group elements defined above. The corresponding cycle is either

$$\begin{aligned} H_1 &= H(X^+ \to Z^+, s(-\infty) = 1), \\ H_2 &= H(Z^+ \to Y^+, s(-\infty) = 1), \\ H_3 &= H(Y^+ \to X^+, s(-\infty) = 1), \end{aligned}$$

or

$$\begin{aligned} H_1 &= H(X^+ \to Z^+, s(-\infty) = 1), \\ H_2 &= H(Z^+ \to Y^-, s(-\infty) = 1), \\ H_3 &= H(Y^- \to X^+, s(-\infty) = 1). \end{aligned}$$

This kind of heteroclinic cycle looks like a figure "Δ".

(2) If m is odd, then $\ell = 6$. Either $H_{i+1} = \kappa_1 r H_i$ or $H_{i+1} = \kappa_2 r H_i$ for all $1 \le i \le \ell$. The cycles is

$$\begin{aligned}
H_1 &= H(X^+ \to Z^+, s(-\infty) = +1),\\
H_2 &= H(Z^+ \to Y^+, s(-\infty) = -1),\\
H_3 &= H(Y^+ \to X^-, s(-\infty) = +1),\\
H_1 &= H(X^- \to Z^-, s(-\infty) = -1),\\
H_2 &= H(Z^- \to Y^-, s(-\infty) = +1),\\
H_3 &= H(Y^- \to X^+, s(-\infty) = -1),
\end{aligned}$$

or

$$\begin{aligned}
H_1 &= H(X^+ \to Z^+, s(-\infty) = +1),\\
H_2 &= H(Z^+ \to Y^-, s(-\infty) = -1),\\
H_3 &= H(Y^- \to X^-, s(-\infty) = +1),\\
H_1 &= H(X^- \to Z^-, s(-\infty) = -1),\\
H_2 &= H(Z^- \to Y^+, s(-\infty) = +1),\\
H_3 &= H(Y^+ \to X^+, s(-\infty) = -1).
\end{aligned}$$

This kind heteroclinic cycle looks like a figure "8".

Applying elements of the symmetry group to the above, we can generate all the other figure "Δ" and figure "8" heteroclinic cycles. There are eight figure "Δ" and four figure "8" heteroclinic cycles which can bifurcate into a periodic solution. Observe that figure "8" heteroclinic cycles are symmetric about the origin while figure "Δ" heteroclinic cycles are near the boundaries of an octant.

The proof of Theorem 4.1 is presented in the rest of this section.

Suppose that $H_1, \dots, H_\ell$ is a heteroclinic cycle that can bifurcate into a periodic solution $P(t) = (p(t), s(t))$, $p : t \to \mathbb{R}^3$. In this cycle, we have

$$q_i(+\infty) = q_{i+1}(-\infty), \quad \text{and} \quad s_i(+\infty) = s_{i+1}(-\infty).$$

We say H_{i+1} follows H_i, or q_{i+1} follows q_i.

Assume that $H_{i+1} = \gamma_i H_i$ where γ_i is a mapping in the symmetry group. For convenience, let $H_{\ell+1} = H_1, H_0 = H_\ell$ and $\gamma_{\ell+1} = \gamma_1, \gamma_0 = \gamma_\ell$. Recall that $H_i = (q_i, s_i)$ is four-dimensional. We often want to drop the s_i and consider only the (x, y, z) component of a solution. If $(q_{i+1}, s_{i+1}) = \gamma_i(q_i, s_i)$, then q_{i+1} is uniquely determined by γ_i and q_i. By restricting ourselves to q_i, we will write $q_{i+1} = \gamma_i q_i$, with some abuse of notation.

Assume that q_i connects the equilibrium E_i to E_{i+1}, with $E_{\ell+1} = E_1$. Let Σ_i be a 2-dimensional plane passing through $q_i(0)$ and perpendicular to $\dot{q}_i(0)$. The periodic solution must hit Σ_i successively. Let the time $P(t)$ spent traveling from Σ_{i-1} to Σ_i be $2\omega_i$. Assume that the orbit of $P(t)$ is the union of $p_i(t), i = 1, \dots, \ell$ with $p_i(t)$ near $q_i(t)$ for $-\omega_i \le t \le \omega_{i+1}$. See Figure 4.1.

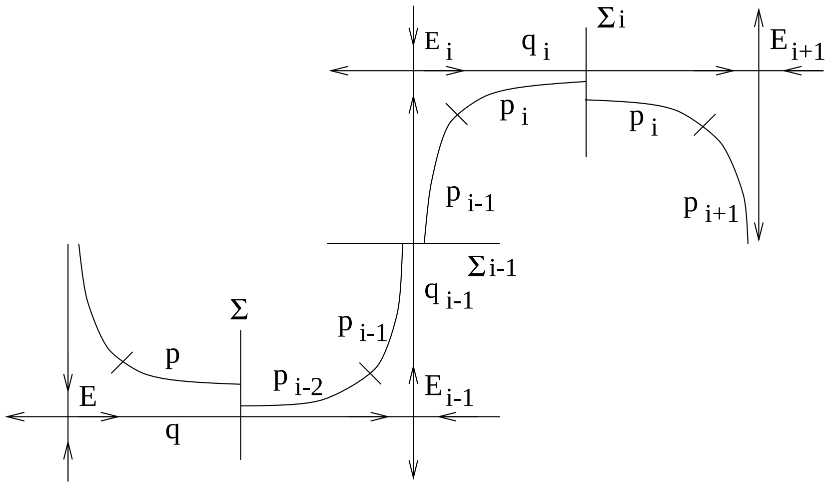

FIGURE 4.1. A generalized periodic solution near $\{q_i\}$. At $t = 0$, $p_i(t)$ may have a jump.

4.1. **Existence of figure "Δ" and figure "8" periodic solutions.** Assuming that ϵ is small but fixed, we want to find all the sufficiently large $\{\omega_i\}_1^\ell$ and sufficiently small perturbations of d such that there is a corresponding periodic solution $P(t)$. We will first assume that $\omega_i = \omega$ for all i. This assumption will be removed in the next sub-section.

Let $p_i = q_i + Q_i$ where $Q_i = (X_i, Y_i, Z_i), i = 1, \dots, \ell$. We derive the equation for Q_1 and omit the subscript to simplify the notation. First, the heteroclinic solution $H_1 = (q_1, s_1)$ is a perturbation of the singular heteroclinic solution corresponding to $d_m(\epsilon) = d_m + \epsilon^2 D^*$. Recall that the heteroclinic solution has the following form:

$$\begin{aligned} x^*(t) &= x_0(t) + \epsilon^2 X^*(t), \\ \epsilon y^*(t) &= \epsilon y_1(t) + \epsilon^3(t) Y^*(t), \\ z^*(t) &= z_0(t) + \epsilon^2 Z^*(t), \end{aligned}$$

where (X^*, Y^*, Z^*, D^*) is the fixed point of the contraction mapping $\mathfrak{P}$ as in §3. Next, write $p_1(t)$ as

$$x^*(t) + X, \quad \epsilon y^*(t) + Y, \quad z^*(t) + Z, \quad \text{with } d = d_m(\epsilon) + D.$$

We study the virtual orbit related to $p_1(t)$. Suppose that the heteroclinic solution $q_1(t)$ hits the junction surfaces at $\{t_i\}_{i=1}^m$. Using (x, y, z, d) for $(x^*, y^*, z^*, d_m(\epsilon))$ to simplify the notation, we find that for $t \neq t_i, i = 1, \ldots, m$,

$$\begin{aligned}
\dot{X} &= [X(1 - 3ax^2 - cz^2) - 2cxzZ] \\
&\quad - b\epsilon^2 y^2 X - 2b\epsilon xyY + \epsilon^2 Dyzu^* + \epsilon dYzu^* + \epsilon^2 dyZu^* \\
&\quad + \epsilon dyz\frac{\partial u^*}{\partial w}\dot{}(X\epsilon yz + xYz + x\epsilon yZ) + N^{(x)}, \\
\dot{Y} &= [Y(1 - bz^2 - cx^2) + dx^2z^2\frac{\partial u^*}{\partial w}Y + \epsilon Dxzu^*] \\
&\quad - 3a\epsilon^2 y^2 Y - 2b\epsilon yZ - 2c\epsilon yX + \epsilon dXzu^*\epsilon xZu^* \\
&\quad + \epsilon dxz\frac{\partial u^*}{\partial w}\dot{}(Xyz + xyZ) + N^{(y)}, \\
\dot{Z} &= [Z(1 - 3az^2 - bx^2) - 2bxzX] \\
&\quad - c\epsilon^2 y^2 Z - 2c\epsilon yzY + \epsilon^2 Dxyu^* + \epsilon^2 dXyu^* + \epsilon dxYu^* \\
&\quad + \epsilon dxy\frac{\partial u^*}{\partial w}(X\epsilon yz + xYz + x\epsilon yZ) + N^{(z)}, \\
N &:= (N^{(x)}, N^{(y)}, N^{(z)}) = O(|X|^2 + |Y|^2 + |Z|^2 + |D|^2).
\end{aligned}$$

The nonlinear terms $N^{(x)}, N^{(y)}, N^{(z)}$ are functions of (X, Y, Z, D). Let $\epsilon D = \tilde{D}$. Then the linear terms outside $[\ldots]$ are of $O(\epsilon)$. The linear terms inside $[\ldots]$ are similar in form to system (3.10)–(3.12) but the coefficients are ϵ perturbations of those of the system (3.10)–(3.12).

With $Q = (X, Y, Z)$ and $B(t) = (\epsilon yzu^*, xzu^*, \epsilon xyu^*)^\tau$, we can write the above as

$$\dot{Q} = A(t)Q + B(t)(\epsilon D) + N. \tag{4.1}$$

At $t = \{t_j\}_{j=1}^m$, we have the reflection law:

$$Q(t_j^+) = R_i Q(t_j^-) + M_j.$$

The nonlinear terms satisfy

$$\begin{aligned}
M_j &= O(X^2 + Y^2 + Z^2 + D^2), \\
\Delta M_j &= O(|\Delta X| + |\Delta Y| + |\Delta Z| + |\Delta D|).
\end{aligned}$$

Here ΔM_j denotes the change of M_j due to the change of (X, Y, Z, D).

The linear homogeneous part of (4.1) has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ with $\mathcal{R}P_u(0^-)$ and $\mathcal{R}P_u(0^+)$ being one dimensional. The adjoint system of (4.1)

$$\dot{\psi} + A^*(t)\psi = 0 \tag{4.2}$$

has exponential dichotomies on $\mathbb{R}^\pm$ with $\mathcal{R}P_u(0^\pm)$ being two dimensional. Let $\psi(t)$ be the unique bounded solution to (4.2) that satisfies $\psi(0) = (O(\epsilon), 1, O(\epsilon))$. We have the following lemma:

Lemma 4.2. *$\int_{-\infty}^{\infty} \psi(t)B(t)dt$ is nonzero with a sign equal to that of $s(0)$.*

Proof. The lemma is true if $\epsilon = 0$ for, in this case, the heteroclinic solution is the unperturbed one and the result follows from Lemma 3.1. The general case $\epsilon \neq 0$

follows since both $\psi(t)$ and $B(t)$ are $O(\epsilon)$ perturbations of unperturbed ones. In particular $B(t) = (0, x_0 z_0 u^*, 0)^\tau + O(\epsilon)$. $\square$

Similarly, we find that $\{Q_i\}_1^\ell$ satisfies the following system

$$\dot{Q}_i = A_i(t)Q_i + B_i(t)\epsilon D + N_i(Q_i, D), \tag{4.3}$$

$$Q_i(t_{ij}^+) = R_{ij}Q_i(t_{ij}^-) + M_{ij}(Q_i, D), \quad j = 1, \ldots, m, \tag{4.4}$$

and the matching conditions

$$Q_i(\omega) - Q_{i+1}(-\omega) = d_i := q_{i+1}(-\omega) - q_i(\omega). \tag{4.5}$$

In the above, N_i and M_{ij} are nonlinear functions of Q_i.

Since the diffeomorphism γ_i maps the flow in a neighborhood of q_i to that of q_{i+1}, it is easy to verify the following invariance properties:

$$\begin{aligned} &A_{i+1}(t) = \gamma_i A_i(t)\gamma_i^{-1}, \quad B_{i+1} = \gamma_i B_i(t), \\ &N_{i+1} = \gamma_i N_i \gamma_i^{-1}, \\ &R_{i+1,j} = \gamma_i R_{ij}\gamma_i^{-1}, \quad M_{i+1,j} = \gamma_i M_{ij}\gamma_i^{-1}. \end{aligned} \tag{4.6}$$

Note also that the linear homogeneous part of (4.3) has exponential dichotomies on $\mathbb{R}^\pm$ with projections $P_{i,s}(t)$ and $P_{i,u}(t)$, which satisfy

$$P_{i+1,u}(t) = \gamma_i P_{i,u}(t)\gamma_i^{-1}, \quad P_{i+1,s}(t) = \gamma_i P_{i,s}(t)\gamma_i^{-1}$$

We say that $\{Q_i\}_1^\ell$ is a generalized solution if each Q_i is allowed to have a jump at $t = 0$ along the direction of $\psi_i(0)$, which is orthogonal to $\mathcal{R}P_{i,u}(0-) + \mathcal{R}P_{i,s}(0+)$:

$$Q_i(0^-) - Q_i(0^+) = \delta_i \psi_i(0)/\|\psi_i(0)\|^2, \ \delta_i \in \mathbb{R}.$$

Following [18], we can show that for each given sequence $\{\omega_i\}$ and D, there exists a unique generalized solution $\{Q_i\}_1^\ell$. The proof is outlined as follows.

First, consider a linear system associated to (4.3), (4.4),

$$\dot{Q}_i = A_i(t)Q_i + B_i(t)\epsilon D + f_i(t), \tag{4.7}$$

$$Q_i(t_{ij}^+) = R_{ij}Q_i(t_{ij}^-) + g_{ij}, \quad j = 1, \ldots, m, \tag{4.8}$$

and the matching conditions (4.5). In the above, f_i is a given function of $t \in [-\omega, \omega]$, g_{ij} and d_i are given vectors in $\mathbb{R}^3$. We look for a generalized solution $\{Q_i\}_1^\ell$ where each Q_i has a jump along the direction of $\psi_i(0)$. Assume also $Q_i(0) \perp \dot{q}_i(0)$. For each $1 \leq i \leq \ell$, let us introduce

$$\phi_{i,s} = P_{i,s}(-\omega)Q_i(-\omega), \quad \phi_{i,u} = P_{i,u}(\omega)Q_i(\omega).$$

Using Lemma 2.3 and Corollary 2.5, with $(\phi_{i,s}, \phi_{i,u})$ undetermined, for each $1 \leq i \leq \ell$, the generalized solution of (4.7) and (4.8) uniquely exists and is denoted by $Q_i(t, \phi_{i,s}, \phi_{i,u})$. To satisfy the matching condition (4.5), we look for $\{(\phi_{i,s}, \phi_{i,u})\}_1^\ell$ so that

$$\phi_{i,u} + P_{i,s}(\omega)Q_i(\omega, \phi_{i,s}, \phi_{i,u}) - [\phi_{i+1,s} + P_{i+1,u}(-\omega)Q_{i+1}(-\omega, \phi_{i+1,s}, \phi_{i+1,u})] = d_i. \tag{4.9}$$

By linearity, we first solve (4.7), (4.8) with the nonhomogeneous terms $\{f_i\}, \{g_{ij}\}$ and D without considering (4.9). We then solve (4.7), (4.8) and (4.9) with $\{f_i\} = 0, \{g_{ij}\} = 0$ and $D = 0$.

Let $\lambda^+ > 0$ and $\lambda^- < 0$ be the unstable and weakest stable eigenvalues of each equilibrium. From H1, $\lambda := \min\{\lambda^+, |\lambda^-|\} = \lambda^+$.

From the existence of exponential dichotomies of (4.7) and the integral representation of solutions in the proof of Lemma 2.3, we have

$$|P_{i,s}(\omega)Q_i(\omega, \phi_{i,s}, \phi_{i,u})| + |P_{i+1,u}(-\omega)Q_{i+1}(-\omega, \phi_{i+1,s}, \phi_{i+1,u})| \leq Ce^{-\lambda\omega} \max_i(|\phi_{i,s}| + |\phi_{i,u}|).$$

If ω is sufficiently large, (4.9) is clearly a small Lipschitz perturbation of the equation

$$\phi_{i,u} - \phi_{i+1,s} = d_i := q_{i+1}(-\omega) - q_i(\omega). \tag{4.10}$$

Since the subspaces $\mathcal{R}P_{i,u}(\omega)$ and $\mathcal{R}P_{i+1,s}(-\omega)$ where $\phi_{i,u}$ and $\phi_{i+1,s}$ belong are exponentially close to the unstable and stable eigenspaces at the equilibrium E_{i+1}, (4.10) has a unique solution

$$|\phi_{i,u}| + |\phi_{i+1,s}| \leq C|d_i|,$$

with the constant C independent of ω. In particular, $\phi_{i,u} \sim q_{i+1}(-\omega)$ and $\phi_{i,s} \sim q_i(\omega)$. Therefore, system (4.9) has a unique solution $(\{\phi_{i,s}\}_1^\ell, \{\phi_{i,u}\}_1^\ell)$ which is bounded by $\max_i |d_i|$.

Substituting $(\{\phi_{i,s}\}_1^\ell, \{\phi_{i,u}\}_1^\ell)$ into $Q_i(t, \phi_{i,s}, \phi_{i,u})$, we have proved the existence of the generalized solution, denoted by

$$Q_i = Q_i^*(t, \{f_i\}, \{g_{ij}\}, \{d_i\}, \omega, D).$$

The existence of the generalized solution of the original nonlinear system can be obtained by a contraction mapping principle. Using the generalized solution, the gap functions $\delta_i := G_i(\omega, D)$ are functions of ω and D. From Lemma 2.3,

$$\begin{aligned} G_i(\omega, D) = \int_{-\omega}^{\omega} &< \psi_i(t), B_i(t)\epsilon D + N_i(Q_i^*, D) > dt \\ &+ \sum_{j=1}^{m} \psi_i(t_{ij}^+) M_{ij}(Q_i^*, D) + \psi_i(-\omega)\phi_{i,s} - \psi_i(\omega)\phi_{i,u}. \end{aligned} \tag{4.11}$$

To have a true solution, we must solve the "bifurcation equations" $G_i(\omega, D) = 0,\ i = 1, \ldots, m$.

We now show that if the bifurcation equations have a solution, we must have $\gamma_i \equiv \gamma_1$ for all i. To this end, we use an asymptotic estimate from [18] which is still valid in our case with an almost identical proof.

$$\begin{aligned} \int_{-\omega}^{\omega} &< \psi_i(t), N_i(Q_i^*, D) + \epsilon D B_i(t) > dt + \sum_{j=1}^{m} \psi_i(t_j^+) M_{ij} \\ &+ \psi_i(-\omega) q_{i-1}(\omega) - \psi_i(\omega) q_{i+1}(-\omega) + o(e^{-2\lambda\omega}) = 0. \end{aligned} \tag{4.12}$$

As $\omega \to \infty$,

$$\psi_i(-\omega) \sim C_1 \mathbf{a} e^{\lambda^- \omega}, \quad q_{i-1}(\omega) \sim C_2 \mathbf{b} e^{\lambda^- \omega},$$

where **a** or **b** is a left or right stable eigenvector of the matrix $A_i(-\infty)$. We have $|\psi_i(-\omega)q_{i-1}(\omega)| \sim C_-e^{2\lambda^-\omega}$. Similarly, $|\psi_i(\omega)q_{i+1}(-\omega)| \sim C_+e^{-2\lambda^+\omega}$. Since $|\lambda^-| > \lambda^+$, we have $|\psi_i(-\omega)q_{i-1}(\omega)| << |\psi_i(\omega)q_{i+1}(-\omega)|$ as $\omega \to \infty$. In [18], it is shown that the solution $Q_i(t)$ is larger at $= \pm\omega$ and decays exponentially as $t \to 0$. We can drop the nonlinear terms N_i, $\{M_{ij}\}_1^m$ and the boundary term $\psi_i(-\omega)q_{i-1}(\omega)$ and only introduce an error of $o(e^{-2\lambda\omega})$. We end with

$$\int_{-\omega}^{\omega} < \psi_i(t), \epsilon DB_i(t) > dt - \psi_i(\omega)q_{i+1}(-\omega) + o(e^{-2\lambda\omega}) = 0.$$

Another observation is that $\int_{-\omega}^{\omega} \psi_i(t)B_i(t)dt$ does not dependent on i. This is due to $\psi_i(t) = \gamma_i^*\psi_{i+1}(t)$ and $B_{i+1}(t) = \gamma_i B_i(t)$. Denote

$$k(\omega) := \int_{-\omega}^{\omega} < \psi_1(t), B_1(t) > dt.$$

We have derived the following important condition

$$\text{sign}\{\psi_i(\omega)q_{i+1}(-\omega)\} = \text{sign}\{k(\omega)\}. \tag{4.13}$$

Lemma 4.3. *If the bifurcation equations have a solution* (ω, D)*, then*

$$\gamma_i = \gamma_1, \text{ for all } i = 1, \ldots, \ell.$$

Proof. Observe that for certain ℓ', the finite sequence $\{\gamma_1^j q_1\}_1^{\ell'}$ forms a cycle. The proof is as follows. Since $q_2 = \gamma_1 q_1$ follows q_1, applying γ_1^j, we find that $\gamma_1^{j+1}q_1$ follows $\gamma_1^j q_1$. Since there are only finitely many heteroclinic solutions, the sequence must be periodic for certain ℓ'. However, we have not proven that $\ell = \ell'$.

Now observe that as $\omega \to \infty$, $\psi(\omega)$ approaches one of the two directions orthogonal to $W^s(E_{i+1})$, while $q_{i+1}(-\omega)$ follows one of the two directions of the unstable manifold $W^u(E_{i+1})$. With (4.13), the direction q_{i+1} must follow is uniquely determined. Moreover $s_{i+1}(-\omega) = s_i(\omega)$. Thus H_{i+1}, which follows H_i, is uniquely determined by (4.13). Since $\epsilon D > 0$, we have

$$\begin{aligned} k(\omega) &\simeq < \psi_i(\omega), q_{i+1}(-\omega) > \\ &= < \gamma_i^*\psi_{i+1}(\omega), q_{i+1}(-\omega) > \\ &= < \psi_{i+1}(\omega), \gamma_i q_{i+1}(-\omega) > . \end{aligned}$$

Here $\simeq$ means that the signs on both sides are the same. Notice that

$$\begin{aligned} k(\omega) &\simeq < \psi_{i+1}(\omega), q_{i+2}(-\omega) > \\ &= < \psi_{i+1}(\omega), \gamma_{i+1}q_{i+1}(-\omega) > . \end{aligned}$$

Based on this, we can show that $\gamma_i = \gamma_1$. Let $i = 1$ first. At the beginning of this proof, we showed that $\gamma_1 q_2 = \gamma_1^2 q_1$ must follow q_2. By the definition of γ_2, $\gamma_2 q_2$ must also follow q_2 . They are placed on the same side of the stable manifold, since their inner products with $\psi_{i+1}(\omega)$ are of the same sign. This shows $\gamma_2 = \gamma_1$. The proof follows by induction on the index i. □

Using $\gamma_i \equiv \gamma_1$ for all i, we can show that if $\{Q_i\}$ is a generalized solution, then $Q_{i+1} = \gamma_1 Q_i$. To this end, applying γ_1 to (4.7), (4.8) and (4.5), we find that $\{Q_i'\}$, with $Q_i' = \gamma_1 Q_{i-1}$, is also a generalized solution to the same system. The desired result follows from the uniqueness of the generalized solution.

Lemma 4.4. *For all $1 \le i \le \ell$, $G_{i-1}(\omega, D) = G_i(\omega, D)$.*

Proof. We examine the first term (integral term) in (4.11)

$$G_i = \int_{-\omega}^{\omega} < \psi_i(t), B_i(t)\epsilon D + N_i(Q_i^*, D) > dt + \dots .$$

Using the invariance properties (4.6), $Q_i = \gamma_1 Q_{i-1}$ and $\gamma_1^* \psi_i(t) = \psi_{i-1}(t)$, we have

$$\int_{-\omega}^{\omega} < \psi_i(t), B_i(t)\epsilon D + N_i(Q_i^*, D) > dt = \int_{-\omega}^{\omega} < \psi_{i-1}(t), B_{i-1}(t)\epsilon D + N_{i-1}(Q_{i-1}^*, D) > dt.$$

Similarly, we can show that every term of G_i is equal to a corresponding term of G_{i-1}. ∎

Suppose now that γ_1 is a mapping in the symmetry group, and $H_2 = \gamma_1 H_1$ follows H_1. By going through all the possible H_2 that can follow H_1, we find that either (1) $\gamma_1 = r$ or $\gamma_1 = \kappa_1\kappa_2 r$ or (2) $\gamma_1 = \kappa_1 r$ or $\gamma_1 = \kappa_2 r$. In Case (1), $s_1(-\infty) = s_2(-\infty) = 1$. In Case (2), $s_1(-\infty) = 1$ but $s_2(-\infty) = -1$. Recall that $s_1(\infty) = s_2(-\infty)$ since the orbit does not hit the junction surfaces when close to an equilibrium. Therefore, in Case (2), $s_1(-\infty)$ and $s_1(+\infty)$ change signs but not in Case (1), which can happen if and only if q_i hits the junction surfaces an odd number of times. This proves the types of heteroclinic cycles that can bifurcate into a periodic solution, as stated in Theorem 4.1.

Theorem 4.5. *Assume that $\{H_i\}_1^{\ell}$, where $\ell = 3$ or 6, is a heteroclinic cycle as in Theorem 4.1 and that $\epsilon > 0$ is sufficiently small. Then there exists $\Omega(\epsilon)$ which approaches ∞ as $\epsilon \to 0$, such that for all $\omega \ge \Omega(\epsilon)$, there exists a unique $D > 0$ such that with $d = d_m(\epsilon) + D$, the system has a periodic solution $P(t)$ that is close to the period cycle $\{H_i\}_1^{\ell}$, and the time $P(t)$ travels from Σ_{i-1} to Σ_i is 2ω.*

Proof. We use the estimate in Lemma 3.2 of [18], which in our case has almost an identical proof:

$$\frac{\partial G_i}{\partial D} = \int_{-\infty}^{\infty} \psi(t)_i \epsilon B_i(t) dt + O(e^{-\lambda\omega} + |D|).$$

Thus, there exists $C > 0$ such that if $e^{-\lambda\omega} + |D| < C\epsilon$, the latter being small but fixed, then $\frac{\partial G_i}{\partial D} \neq 0$. Let $i = 1$. The equation $G_1 = 0$ clearly has a solution $\omega = \infty$ and $D = 0$. By the Implicit Function Theorem, $G_1(\omega, D) = 0$ can be solved in a neighborhood of $(\omega, D) = (\infty, 0)$. The proof of the theorem follows since $G_i = G_1$ for all $1 \le i \le \ell$.

∎

4.2. **Nonexistence of some solutions.** In this sub-section, we show that if D is sufficiently small, the time a periodic solution spends from Σ_i to Σ_{i+1} is the same. This can be proved by an estimate on the derivative of bifurcation functions. We also want to show that multiple heteroclinic solutions do not exist. A multiple heteroclinic solution starts from an equilibrium, passes near one or more equilibria before approaching a final equilibrium. To summarize, the figure "Δ" and figure "8" periodic solutions are the only interesting solutions that can directly bifurcate from heteroclinic cycles.

Consider two cases: (1) $P(t)$ is a periodic solution near the heteroclinic cycle $\{H_i\}_1^\ell$; (2) $P(t)$ is a multiple heteroclinic solution near a heteroclinic sequence $\{H_i\}_1^\ell$, $\ell \geq 2$. Let $P(t)$ be the union of $\{p_i(t)\}_1^\ell$ with $p_i(0) \in \Sigma_i$ as in §4. We assume that the time $P(t)$ travels form Σ_{i-1} to Σ_i is $2\omega_i$. Thus $p_i(t)$ is near $q_i(t)$ for $\omega_i \leq t \leq \omega_{i+1}$. In Case (1), we assume that $\omega_1 = \omega_{\ell+1}$; while in Case (2), $\omega_1 = -\infty$ and $\omega_{\ell+1} = \infty$.

Let $\omega = \min\{\omega_1, \dots, \omega_\ell\}$ in Case (1) and $\omega = \min\{\omega_2, \dots, \omega_\ell\}$ in Case (2). Let $\omega = \omega_k$ for certain (nonunique) $1 \leq k \leq \ell$. We use $o(1)$ to denote a number that approaches zero as $\omega \to \infty$.

Lemma 4.6. *Suppose that $\epsilon D > 0$ but small. Then in Case (1), $\omega_i = \omega + o(1)$ for all i; while in Case (2), $\omega_i = \omega + o(1)$ for $2 \leq i \leq \ell + 1$.*

Proof. As in §4, $Q_i = p_i - q_i$ satisfies (4.3) for $t \in (-\omega_i, \omega_{i+1}) \backslash \{t_{i1}, \dots, t_{im}\}$, and reflection law (4.4) at $\{t_{ij}\}_{j=1}^m$. The matching condition (4.5) becomes

$$Q_i(\omega_{i+1}) - Q_{i+1}(-\omega_{i+1}) = d_i := q_{i+1}(-\omega_{i+1}) - q_i(\omega_{i+1}).$$

The above is true for $i = 1, \dots, \ell$ in the Case (1), and is true for $i = 1, \dots, \ell - 1$ in Case (2). Moreover, we have

$$Q_1(-\infty) = Q_\ell(\infty) = 0, \quad \text{in Case (2).}$$

If $\omega := \min\{\omega_i\}$ is sufficiently large and D is sufficiently small, with an almost identical proof as in [18], we can show that there exists a unique generalized solution $\{Q_i\}_1^\ell$ that satisfies phase conditions $Q_i(0) \perp \dot{q}_i(0)$ for each i, but is allowed to have a jump along the direction $\psi_i(0)$ at $t = 0$. The gap $Q_i(0+) - Q_i(0-)$ depends on $(\{\omega\}, D)$:

$$G_i(\{\omega\}, D) = \int_{-\omega_i}^{\omega_{i+1}} < \psi_i(t), B_i(t)\epsilon D + N_i(Q_i^*, D) > dt$$
$$+ \sum_{j=1}^m \psi_i(t_{ij}^+) M_{ij}(Q_i^*, D) + \psi_i(-\omega_i)\phi_{i,s} - \psi_i(\omega_{i+1})\phi_{i,u}.$$

Let $L(\infty) = \int_{-\infty}^\infty \psi_i(t) B_i(t) dt$ which does not depend on i. Then $\int_{-\omega_i}^{\omega_{i+1}} \psi_i(t) B_i(t) dt = (1 + o(e^{-\lambda\omega}))L(\infty)$. The bifurcation equations have the following asymptotic form

$$(1 + o(e^{-\lambda\omega}))L(\infty)\epsilon D + \psi_i(-\omega_i) q_{i-1}(\omega_i) - \psi_i(\omega_{i+1}) q_{i+1}(-\omega_{i+1}) + o(e^{-2\lambda\omega}) = 0.$$

Since $\omega_i \geq \omega$ for all i and the weakest stable eigenvalue at an equilibrium satisfies $|\lambda^-| > \lambda$, we have $\psi_i(-\omega_i) q_{i-1}(\omega_i) = o(e^{2\lambda\omega})$. Thus

$$L(\infty)\epsilon D = \psi_i(\omega_{i+1}) q_{i+1}(-\omega_{i+1}) + o(e^{2\lambda\omega}).$$

Observe that $\psi_i(\omega_{i+1}) q_{i+1}(-\omega_{i+1}) \sim C e^{-2\lambda\omega_{i+1}}$. We have

$$L(\infty)\epsilon D = C e^{-2\lambda\omega_{i+1}} + o(e^{2\lambda\omega}), \quad 1 \leq i \leq \ell. \tag{4.14}$$

Let $i+1=k$ where $\omega_k=\min\{\omega_i\}$. Then

$$L(\infty)\epsilon D = Ce^{-2\lambda\omega}+o(e^{2\lambda\omega}). \tag{4.15}$$

The estimate $\omega_i=\omega+o(1)$ follows from (4.14) and (4.15). $\square$

As a consequence of Lemma 4.6, multiple heteroclinic solutions ($\omega_{\ell+1}=\infty$) do not exist.

To show $\omega_i\equiv\omega$ for all i, we need a nonzero lower bound for the partial derivative $\frac{\partial G_i}{\partial\{\omega\}}$. See [31]. A similar result has been obtained in [25].

Lemma 4.7. *If the sequence of time $\{\omega_i\}$ satisfies $\frac{1}{k}<\frac{\omega_i}{\omega_{i+1}}<k$ for some constant $k>1$, then*

$$\frac{\partial G_i}{\partial\omega_j}=\frac{\partial}{\partial\omega_j}(<\psi_i(-\omega_i),q_{i-1}(\omega_i)>-<\psi_i(\omega_{i+1}),q_{i+1}(\omega_{i+1})>)+o(e^{2\lambda^+\omega_{i+1}})+o(e^{-2\lambda^-\omega_i}).$$

The condition of Lemma 4.7 is satisfied since ℓ is finite.

From our assumption, the heteroclinic solution is tangent to the eigenvector corresponding to the weakest negative eigenvalue λ^-. Therefore, $q_i(t)\sim a_ie^{-\lambda t}$ for $t\approx-\infty$ and $q_i(t)\sim b_ie^{\lambda^- t}$ for $t\approx\infty$. Recall that $\omega_i\approx\omega$. Thus

$$\frac{\partial G_i}{\partial\omega_j}=-\frac{\partial}{\partial\omega_j}<\psi_i(\omega_{i+1}),q_{i+1}(-\omega_{i+1})>)+o(e^{2\lambda\omega}).$$

It is now clear that $\frac{\partial G_i}{\partial\omega_j}$ is of order $e^{-2\lambda\omega}$ if $j=i+1$ and is of order $o(e^{-2\lambda\omega})$ if $j\neq i+1$. Using this, we can prove that there do not exist two distinct sequences $\{\omega_i^k\}, k=1,2$ that solve the bifurcation equations for the same D. Otherwise, let $\Delta\omega_i=\omega_i^2-\omega_i^1$ and let $\omega_i(\mu)=\omega_i^1+\mu\Delta\omega_i$. We have

$$G_i(\{\omega_i(\mu)\},D)=0,\quad \mu=0,1, \text{ and for all } i.$$

There exists $0<\bar\mu<1$ such that $\frac{dG_i}{d\mu}(\{\omega_j(\bar\mu)\},d)=0$. Let $\Delta\omega_\nu=\max\{\Delta\omega_j, j=1,\dots,\ell\}$. Then

$$\sum_j\frac{\partial G_{\nu-1}}{\partial\omega_j}\Delta\omega_j=0,$$

$$\frac{\partial G_{\nu-1}}{\partial\omega_\nu}\Delta\omega_\nu=-\sum_{j\neq\nu}\frac{\partial G_{\nu-1}}{\partial\omega_j}\Delta\omega_j.$$

But from Lemma 4.7, the left side of the last equation is of $e^{-2\lambda\omega}$ while the right side is of $o(e^{-2\lambda\omega})$. We have reached a contradiction, which shows that $\Delta\omega_j=0$ for all j.

Appendix A. Proof of Lemma 1.1 and Lemma 1.2.

Proof of Lemma 1.1. We start from

$$\begin{aligned}0.5(x^2+y^2+z^2)'=(x^2+y^2+z^2)-a(x^4+y^4+z^4)\\ -(b+c)(x^2y^2+y^2z^2+z^2x^2)+3\epsilon duxyz.\end{aligned} \tag{A.1}$$

Using $2a < b+c$, we have

$$0.5(x^2+y^2+z^2)' \le (x^2+y^2+z^2) - a(x^2+y^2+z^2)^2 + 3\epsilon duxyz. \tag{A.2}$$

Assuming that $x^2+y^2+z^2 \ge M$, we want to show $(x^2+y^2+z^2)' < 0$. Assume first $s(t) = 1$ and $xyz \le 0$. Then since $d \ge 0$ and $u > 0$, we have $3\epsilon duxyz \le 0$. Since

$$(x^2+y^2+z^2) \le \frac{(x^2+y^2+z^2)^2}{M}, \tag{A.3}$$

if $\frac{1}{M} - a < 0$, then the right hand side of (A.2) is negative. Thus $\frac{d}{dt}(0.5(x^2+y^2+z^2)) < 0$.

We then assume that $s(t) = 1$ and $0 < xyz < \epsilon$. We need an elementary inequality

$$|xyz| \le \left(\frac{x^2+y^2+z^2}{3}\right)^{1.5}. \tag{A.4}$$

In this case, the value $|u|$ is bounded uniformly with respect to ϵ. Based on this,

$$|3\epsilon duxyz| \le 3\epsilon|du|\frac{(x^2+y^2+z^2)^2}{M^{0.5}3^{1.5}}.$$

Using (A.3) again, we see that the right hand side of (A.2) is negative if $\frac{1}{M} + \frac{3\epsilon|du|}{M^{0.5}3^{1.5}} \le a$. In this case, we also have $0.5(x^2+y^2+z^2)' < 0$. The condition is true if M is sufficiently large.

Finally, it is easy to see that M can be any constant larger than $1/a$ if ϵ is sufficiently small.

Similar arguments can be applied to the case $s(t) = -1$ and $xyz \ge -\epsilon$. This completes the proof of $\frac{d}{dt}(x^2+y^2+z^2) < 0$ if $x^2+y^2+z^2 \ge M$.

Assuming that $x^2+y^2+z^2 \le m$, we want to show that $(x^2+y^2+z^2)' > 0$. From (A.1), using $b+c > 2a$ again, we have

$$0.5(x^2+y^2+z^2)' \ge (x^2+y^2+z^2) - \frac{b+c}{2}(x^2+y^2+z^2)^2 + 3\epsilon dxyzu. \tag{A.5}$$

First assume that $s(t) = 1$ and $xyz/\epsilon \le -1$. Since $u^3 - u = -\frac{xyz}{\epsilon}\frac{2}{3\sqrt{3}}$, for this range of xyz, there exists $k > 0$ such that $ku^3 < -\frac{xyz}{\epsilon}\frac{2}{3\sqrt{3}}$. Thus there exists $K > 0$, independent of ϵ, such that $0 < u < K(-xyz)^{1/3}/\epsilon^{1/3}$. From (A.4),

$$\begin{aligned} 3\epsilon|duxyz| &\le 3K|d|\epsilon^{2/3}(xyz)^{4/3} \\ &\le 3|d|K\epsilon^{2/3}\frac{(x^2+y^2+z^2)^2}{3^2} \\ &\le 3|d|K\epsilon^{2/3}\frac{(x^2+y^2+z^2)m}{9}. \end{aligned}$$

Thus $0.5(x^2+y^2+z^2)' > 0$ if

$$\frac{b+c}{2}m + \frac{1}{3}\epsilon^{2/3}|d|Km < 1.$$

The latter is true if $m > 0$ is sufficiently small. It is also clear that m can be any positive constant smaller than $2/(b+c)$ if ϵ is sufficiently small.

Next, assume that $s(t) = 1$ and $|xyz| \leq \epsilon$. The proof is easy in this case since the value $|u|$ is bounded uniformly with respect to ϵ . We will leave this to the readers.

A similar argument can be applied to the case $s = -1$. □

Proof of Lemma 1.2. Let $w = xyz$, then

$$\dot{w} = (3 - (x^2 + y^2 + z^2))w + \epsilon du(x^2y^2 + y^2z^2 + z^2x^2). \tag{A.6}$$

First, suppose that a solution starts with $s = -1$ and $|xyz| \leq \epsilon$. Then since $\min\{x^2 + y^2, y^2 + z^2, z^2 + x^2\} > \eta > 0$, we have

$$\begin{aligned}
&|3 - (x^2 + y^2 + z^2)| \leq \max\{|3 - 1/a|, |3 - 2/(b + c)|\} + O(\epsilon),\\
&\qquad |0.5du((x^2 + y^2)z^2 + (y^2 + z^2)x^2 + (z^2 + x^2)y^2)| \geq 0.5|du|\eta(x^2 + y^2 + z^2)\\
&\qquad\qquad \geq 0.5|du|\eta(m + O(\epsilon)).
\end{aligned}$$

Since $|w| \leq \epsilon$ which is sufficiently small, using condition (1.6), the sign of $\dot{w}$ is determined by $du(x^2y^2 + y^2z^2 + z^2x^2)$. Therefore, $\dot{w} < 0$ until the orbit hits $w = -\epsilon$.

Similarly, if $s(t) = 1$ and $|xyz| \leq \epsilon$, then $\dot{w} > 0$, until the orbit hits $w = \epsilon$. □

Appendix B. Determine the regions where $dw/dt > 0$

Since we do not have formulas for $(x_0(t), z_0(t))$, we will accept some observations from numerical computations as basic facts. For all the parameter values of (a, b, c) we tested that satisfy H1, we find

F2) $(x_0(t), z_0(t))$ are positive and $dx_0(t)/dt < 0$ and $dz_0(t)/dt > 0$ for all $t \in \mathbb{R}$. The solution (x_0, z_0) is tangent to the weakest stable eigenvector $(1, 0)$ as $t \to \infty$. Moreover, $x_0^2(t) + z_0^2(t) > 3$.

We shall assume that the systems we study satisfy **F2**.

Assume $s = 1$ and we try to determine the sign of dw/dt for $-1 \leq w \leq 1$. Elementary computation shows:

$$dw/dt = w(3 - x_0^2(t) - z_0^2(t)) - dx_0^2(t)z_0^2(t)u^*(w, s).$$

Thus

$$\text{sign}\frac{dw}{dt} = \text{sign}[d\frac{x_0^2(t)z_0^2(t)}{x_0^2(t) + z_0^2(t) - 3} - \frac{w}{u^*(w, 1)}].$$

If $-1 < w \leq 0$, it is clear that $dw/dt > 0$. We now discuss the case $0 < w < 1$. Using the change of variable $(t, w) \to (\theta, v)$, where $v = \frac{w}{u^*(w,1)}$, and letting $f(\theta) = d\frac{x_0^2(t)z_0^2(t)}{x_0^2(t)+z_0^2(t)-3}$, we plot $f(\theta)$ and $v = \frac{1}{u^*(1,1)}$ in Figure B.1. By comparing the two graphs, we find the region $\{(\theta, v)|0 \leq \theta \leq \pi/2, 0 \leq v \leq w/u^*(w, 1)\}$ is divided into three parts. See Figure B.1 for $(a, b, c, d) = (0.3, 0.15, 0.55, 1.2)$. In the subset bounded by $Z^+ \to T_1 \to T_2 \to X^+ \to Z^+$, we have $dw/dt > 0$ while in the two outer regions $dw/dt < 0$. Map the region back to (θ, y_1) coordinates, in Figure 3.1, we see that between the y_1 axis and Γ_1, the sign of $dw/dt > 0$ in the region bounded by $Z^+ \to T_1 \to T_2 \to X^+ \to Z^+$ and is negative to the left of the curve Z^+T_1 or to the right of T_2X^+. The two points T_1 and T_2 are important, for the flow is tangent to Γ_1 at T_1 and T_2. The points T_1 and T_2 divide Γ_1 into three parts. In the middle part, the flow is pointing outwards of the junction surface and in the two outer parts, the flow points inwards Γ_1. In the region below the y_1 axis and above Γ_2, we always have $dw/dt > 0$ (if $s = 1$).

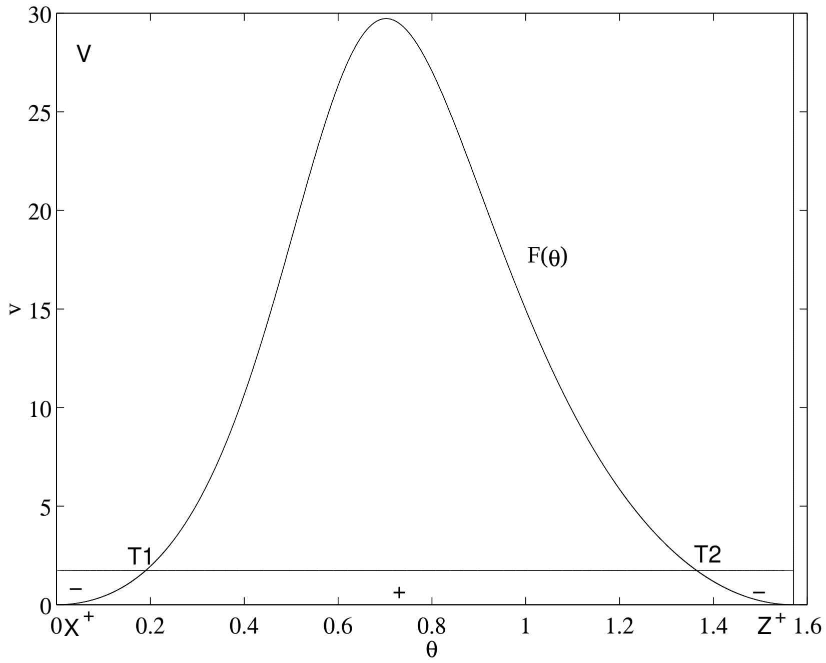

FIGURE B.1. The regions in the (θ, v) coordinates.

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