UPPER CRITICAL FIELD AND LOCATION OF SURFACE NUCLEATION OF SUPERCONDUCTIVITY

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Abstract

In this paper we improve the estimate obtained by Lu-Pan on the value of the upper critical field $H_{C_3}(\kappa)$ for a cylindrical superconductor with cross section Ω being an arbitrary 2-dimensional smooth bounded domain. We also show that, when a homogeneous magnetic field is applied along the axis of the cylinder with magnitude of the field close to H_{C_3}, superconductivity nucleates first at the surface of the sample where the curvature of $\partial\Omega$ is maximal.

§1. Introduction

Let us consider a cylindrical superconducting sample of type 2 with cross section Ω being an *arbitrary bounded* smooth domain in $\mathbb{R}^2$, and place the sample in an applied magnetic field. It is well-known that, if the applied field is very strong then the sample loses superconducting property. As the field is gradually reduced to a certain value H_{C_3} called the *upper critical field*, the nucleation of superconductivity occurs, see [dG, SST, T].

We are interested in the estimate of the value of $H_{C_3}(\kappa)$ when the Ginzburg-Landau parameter κ is large, and in the localization of the nucleation of superconductivity. These problems have been studied by many physicists, see Saint-James and De Gennes [SdG], Saint-James and Sarma [SST], and Tinkham [T]. More recently, a lot of papers devoted to the mathematical analysis on these problems have appeared. Among them we mention the works of Chapman [C] and Bernoff-Sternberg [BS] based on some formal analysis, Bauman-Philips-Tang [BPT] for the rigorous analysis on disks, Giorgi-Phillips [GP], Lu-Pan [LP1-6] and del Pino-Felmer-Sternberg [DFS] for rigorous analysis on general domains. Our main concern is the effect of the domain geometry on the value of H_{C_3} and on the location of superconductivity nucleation.

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Before stating our main results we recall that, for a cylindrical superconductor of infinite height with cross section Ω and subject to an applied magnetic field along the cylindrical axis, the behavior of superconductivity can be described by a (global) minimizer $(\psi, \mathcal{A})$ of the Ginzburg-Landau functional

$$\mathcal{G}(\psi, \mathcal{A}) = \int_\Omega \{|\nabla_{\kappa\mathcal{A}}\psi|^2 + \kappa^2|\text{curl }\mathcal{A} - \mathcal{H}|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2\}dx.$$

The Euler equation is the 2-dimensional Ginzburg-Landau system (see [GL, dG, SST, CHO, DGP])

$$\begin{cases} -(\nabla - i\kappa\mathcal{A})^2\psi = \kappa^2(1 - |\psi|^2)\psi, & \\ \text{curl}^2\mathcal{A} = -\dfrac{i}{2\kappa}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\mathcal{A} + \text{curl }\mathcal{H}, & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} - i\kappa\mathcal{A}\psi \cdot \nu = 0, & \\ \text{curl }\mathcal{A} - \mathcal{H} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here ψ is a complex-valued function called order parameter, $\mathcal{A}$ is a real vector field called magnetic potential, $\mathcal{H}$ is the applied magnetic field, $i = \sqrt{-1}$, κ is the Ginzburg-Landau parameter given by the ratio of the London penetration depth and the coherence length of the superconductor, ν is the unit out-normal vector at the boundary of Ω. Here we use the notations

$$\begin{aligned} &\partial_j = \frac{\partial}{\partial x_j}, \qquad \nabla_{\mathcal{A}}\psi = \nabla\psi - i\psi\mathcal{A}, \\ &\nabla^2_{\mathcal{A}}\psi = (\nabla - i\mathcal{A})^2\psi = \Delta\psi - i[2\mathcal{A}\cdot\nabla\psi + \psi\text{ div }\mathcal{A}] - |\mathcal{A}|^2\psi, \\ &\text{curl }\mathcal{A} = \partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1, \\ &\text{curl}^2\mathcal{A} = (\partial_2(\text{curl }\mathcal{A}), -\partial_1(\text{curl }\mathcal{A})). \end{aligned}$$

Note that in (1.1) the unit of length is the penetration depth. Also note that in [LP1-6] a more general boundary condition introduced by de Gennes was considered. For simplicity we consider the homogeneous Neumann boundary condition. However, the ideas and methods used in this paper apply as well for this boundary condition.

Notations. *In the following, a (global) minimizer $(\psi, \mathcal{A})$ of the Ginzburg-Landau functional is called a minimal solution of the Ginzburg-Landau system (1.1). We call a complex-valued function ψ an order parameter if there exists a real vector field $\mathcal{A}$ such that $(\psi, \mathcal{A})$ is a minimal solution of the Ginzburg-Landau system (1.1).*

Throughout this paper we assume

$$\mathcal{H}(x) \equiv \sigma\mathbf{e}_3, \tag{1.2}$$

where σ is a constant. We shall treat σ as a parameter and consider the behavior of minimizers as $\sigma \to \infty$. It is well-known that there exists a unique smooth vector field $\mathbf{F}$ on $\overline{\Omega}$ such that

$$\text{curl }\mathbf{F} = 1 \quad \text{and} \quad \text{div }\mathbf{F} = 0 \quad \text{in } \Omega, \qquad \mathbf{F}\cdot\nu = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Note that, under the assumption, $(0, \sigma\mathbf{F})$ is a trivial critical point of the functional $\mathcal{G}$. Moreover, $(0, \sigma\mathbf{F})$ is the only minimizer if σ is large enough, which means that a sufficiently strong applied magnetic field penetrates the entire superconductor and completely destroys the superconductivity (see [LP4], or Theorem A below).

A mathematical definition for H_{C_3} was given in [LP4] :

$$H_{C_3}(\kappa) = \inf\{\sigma > 0 : (0, \sigma\mathbf{F}) \text{ is the only minimizer of } \mathcal{G}\}. \tag{1.4}$$

Theorem A ([LP4]). *For any bounded 2-dimensional simply-connected domain Ω of class C^3 we have :*

1. $\displaystyle\lim_{\kappa\to\infty} \frac{H_{C_3}(\kappa)}{\kappa} = \frac{1}{\beta_0}$, *where β_0 is the lowest eigenvalue of (2.6).*
2. *If the homogeneous applied field is sufficiently close to H_{C_3}, superconductivity nucleates at the surface of the sample.*

Conclusion (2) in Theorem A is called surface nucleation. The precise meaning is the following. Assume that the applied field $\mathcal{H}$ is given in (1.2), with the magnitude σ satisfying

$$(1 + o(1))\frac{\kappa}{\beta_0} \le \sigma < H_{C_3}(\kappa).$$

Then, for any $\delta > 0$, there exists $\kappa(\delta) > 0$ such that, for all $\kappa > \kappa(\delta)$, the order parameters ψ must satisfy

$$\|\psi\|_{L^\infty(\Omega)} < \delta.$$

For any $x \in \Omega$ and any $\eta > 0$, there exists $\kappa(x, \eta) > 0$ such that, for all $\kappa > \kappa(x, \eta)$,

$$\frac{|\psi(x)|}{\|\psi\|_{L^\infty(\Omega)}} < \eta.$$

Moreover, the order parameters exhibit a boundary layer in the neighborhood of $\partial\Omega$ within distance $O(\frac{1}{\kappa})$ to $\partial\Omega$. For more details see [LP4].

However, the information given in (2) does not give the optimal localization of the surface nucleation. It turns out that the localization depends on the gap between the applied field and H_{C_3}. In fact, it was shown in [LP4] that if the gap between the applied field and H_{C_3} is not very small, one can observe superconductivity uniformly along the entire surface of the sample. For a more precise statement, see [LP4] (Section 5). On the other hand, if the gap is small, the lower bound estimates on H_{C_3} (which are related to the upper bound estimates of the lowest eigenvalue of Schrödinger operator with magnetic field) given in [BS] and [LP4] suggest that superconductivity can only be observed near the maximum points of the curvature of $\partial\Omega$.

In this paper, based on the very recent work by Helffer-Morame [HMor] on the asymptotical behavior of the lowest eigenvalue, we shall improve the estimate of H_{C_3} given in [LP4], and show that, under a homogeneous magnetic field $\mathcal{H} = H\mathbf{e}_3$ parallel to the axis direction $\mathbf{e}_3$ of the cylinder with magnitude close to H_{C_3}, superconductivity nucleates first at the points in the surface of the sample where the curvature of $\partial\Omega$ is maximal.

Throughout this paper Ω is a smooth simply-connected bounded domain in $\mathbb{R}^2$. Let $\kappa_r(x)$ be the curvature function of $\partial\Omega$, and let us introduce

$$\kappa_{max} = \max_{x\in\partial\Omega} \kappa_r(x), \qquad \kappa_{min} = \min_{x\in\partial\Omega} \kappa_r(x),$$
$$\mathcal{N}(\partial\Omega) = \{x \in \partial\Omega : \kappa_r(x) = \kappa_{max}\}.$$

Denote by $s = s(x)$ the arclength of $\partial\Omega$, denote by $x(s)$ a point at $\partial\Omega$, and write the curvature function $\kappa_r(x(s))$ by $\kappa_r(s)$. We say that a maximum point $x(s_0)$ of the curvature is *non-degenerate*, if $\kappa_r''(s_0) \neq 0$.

For a C^1 domain Ω, there is a geometric constant $\varepsilon(\Omega) > 0$ such that the distance function to the boundary is regular in the domain $\{x \in \overline{\Omega} : \text{dist }(x, \partial\Omega) < \varepsilon(\Omega)\}$, and for any $0 < \delta < \varepsilon(\Omega)$, we can introduce a new coordinates (s, t) in a neighborhood of $\partial\Omega$

$$\Omega_\delta = \{x \in \overline{\Omega} : \text{ dist } (x, \partial\Omega) < \delta\}$$

to straighten the boundary $\partial\Omega$, where $s = s(x)$ measures the arclength and $t = t(x) = \text{dist }(x, \partial\Omega)$ measures the normal distance to the boundary, see [HMor] (Appendix B). Let us fix $\delta_0 < \varepsilon(\Omega)$. We shall identify $x(s, 0)$ with $x(s)$. To every point $x = x(s, t) \in \Omega_{\delta_0}$, we can assign a unique point $x(s) = x(s, 0) \in \partial\Omega$. Hence, for $0 < \delta \le \delta_0$, we denote the set $\{x(s, t) \in \Omega_\delta : \kappa_{max} - \kappa_r(x(s)) > \eta\}$ by $\{x \in \Omega_\delta : \kappa_{max} - \kappa_r(s) > \eta\}$. We can measure the tangential distance from the point $x(s)$ to $\mathcal{N}(\partial\Omega)$ along $\partial\Omega$, and we shall denote this distance by $d_t(x(s), \mathcal{N}(\partial\Omega))$. Then we define the *tangential distance* from $x = x(s, t)$ to $\mathcal{N}(\partial\Omega)$ by

$$d_t(x(s, t), \mathcal{N}(\partial\Omega)) \equiv d_t(x(s), \mathcal{N}(\partial\Omega)).$$

Our main results in this paper are Theorems 1.1, 1.2 and 1.3. Theorem 1.1 gives an estimate of $H_{C_3}(\kappa)$ up to the second term. Theorems 1.2 and 1.3 give a quantitative version of the property that superconductivity nucleates at the maximum points of the curvature as the applied magnetic field decreases from H_{C_3}, and the condensation behavior of the order parameters depends on the gap $H_{C_3} - \sigma$ between the applied field σ and the upper critical field H_{C_3}.

Theorem 1.1 (Asymptotics of H_{C_3}). *Assume that Ω is a bounded simply-connected 2-dimensional domain of class C^4. There exists a universal constant $C_1 > 0$ such that, for κ large, we have*

$$H_{C_3}(\kappa) = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + O(\kappa^{-1/3}). \tag{1.5}$$

Theorem 1.2 (Location of nucleation: general domains). *Let Ω be as in Theorem 1.1 and let*

$$\rho = \frac{\beta_0^{3/2}}{C_1}[H_{C_3}(\kappa) - \sigma].$$

The order parameters ψ have the following concentration behaviors.

1. *Assume that $0 < \rho = o(1)$ as $\kappa \to \infty$. Then there exist positive constants a_0, ℓ_0, κ_0 and M_0 such that, for all $\kappa > \kappa_0$, we have :*

$$\int_\Omega |\psi|^2 \exp(a_0\sqrt{\kappa}[\kappa_{max} - \kappa_r(s)])dx \le M_0 \exp(\ell_0\rho\sqrt{\kappa}) \int_\Omega |\psi|^2 dx. \tag{1.6}$$

(2) *Assume that* $\varepsilon_0 < \rho < \kappa_{max} - \kappa_{min}$, *where* $\varepsilon_0 > 0$. *Then there exist positive constants* a, M *and* κ_0' *such that, for all* $\kappa > \kappa_0'$ *we have*[1] *:*

$$\int_\Omega |\psi|^2 \exp(a\sqrt{\kappa}[\kappa_{max} - \kappa_r(s) - \rho]_+^{3/2})dx \le M\kappa^{1/3} \int_\Omega |\psi|^2 dx. \tag{1.7}$$

Theorem 1.3 (Location of nucleation: non-degenerate domains). *Assume that* Ω *is a bounded simply-connected* 2*-dimensional domain of class* C^4 *and* $\mathcal{N}(\partial\Omega)$ *consists of non-degenerate maximum points of curvature. Let*

$$\rho = \frac{\beta_0^{3/2}}{C_1}[H_{C_3}(\kappa) - \sigma].$$

The order parameters ψ *have the following concentration behaviors.*

(1) *Assume that* $0 < H_{C_3}(\kappa) - \sigma \le L_1\kappa^{-1/3}$ *for some constant* $L_1 > 0$. *Then there exist positive constants* a_1, r_1, ℓ_1, B_1 *and* κ_1 *such that, for all* $\kappa > \kappa_1$, *we have :*

$$\int_{\overline{\Omega}\setminus\{x\in\overline{\Omega}:\mathrm{dist}(x,\partial\Omega)<r_1\kappa^{-5/6},d_t(x,\mathcal{N}(\partial\Omega))<\ell_1\kappa^{-1/6}\}} |\psi|^2 dx \le B_1 \exp(-a_1\kappa^{1/6})\|\psi\|^2_{L^2(\Omega)}. \tag{1.8}$$

(2) *Assume that*[2] *:* $L_2\kappa^{-1/3} \ll H_{C_3} - \sigma = o(1)$ *for some constant* $L_2 > 0$. *Then there exist positive constants* a_2, r_2, ℓ_2, κ_2 *and* B_2 *such that, for all* $\kappa > \kappa_2$, *we have :*

$$\int_{\overline{\Omega}\setminus\{x\in\overline{\Omega}:\mathrm{dist}(x,\partial\Omega)<r_2\rho\kappa^{-1/2},d_t(x,\mathcal{N}(\partial\Omega))<\ell_2\sqrt{\rho}\}} |\psi|^2 dx \le B_2 \exp(-a_2\rho\sqrt{\kappa})\|\psi\|^2_{L^2(\Omega)}. \tag{1.9}$$

Remark 1.4. We will see in Proposition 4.2 that, if the applied field is close to but below H_{C_3}, the order parameters ψ decay exponentially in the normal direction and have a boundary layer of the order $O(\kappa^{-1})$. Thus superconductivity nucleates in a boundary layer with thickness of order $O(\kappa^{-1})$. Theorems 1.2 and 1.3 describe in a weak sense the exponential decay of the order parameters in the tangential direction within the boundary layer, and hence describe the way in which superconductivity nucleates and expands as the gap $H_{C_3}(\kappa) - \sigma$ between the applied field and H_{C_3} increases. In the case where all the maximum points of the curvature $\kappa_r(s)$ are non-degenerate, we have the following conclusions :

(1.) If the gap is of order $O(\kappa^{-1/3})$, then superconductivity nucleates near the set $\mathcal{N}(\partial\Omega)$ of the maximum points of the boundary curvature, with a normal distance of order $O(\kappa^{-5/6})$ and with a tangential distance of order $O(\kappa^{-1/6})$ to $\mathcal{N}(\partial\Omega)$.

(2.) If the gap tends to zero but is still much larger than the order $O(\kappa^{-1/3})$, then superconductivity still nucleates near the maximum points of the boundary curvature. But now it is located within a neighborhood of $\mathcal{N}(\partial\Omega)$ with a normal distance $O(\kappa^{-1/2}(H_{C_3} - \sigma))$ and with a tangential distance of order $(H_{C_3} - \sigma)^{1/2}$.

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[1] We use the notation $a_+ = \max\{a, 0\}$.

[2] Our convention is that, for two positive quantities depending on a parameter κ, we will write $a \ll b$ if $\frac{a}{b} = o(1)$ as $\kappa \to \infty$.

(3.) If the gap does not tend to zero but remains bounded below by $\frac{C_1}{\beta_0^{3/2}}a$, where $0 < a < \kappa_{max} - \kappa_{min}$, then superconductivity is localized inside a neighborhood, with thickness $O(\kappa^{-1/2})$, of the subset of the surface where $\kappa_r(s) \geq \kappa_{max} - a$.

(4.) We believe that, if the gap is greater than $\frac{C_1}{\beta_0^{3/2}}(\kappa_{max} - \kappa_{min})$, namely, if

$$\sigma < \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{min},$$

then order parameters concentrate uniformly along the entire boundary. We should mention that uniform concentration along boundary was proved in [LP4] under a stronger condition : $\frac{1}{\beta_0}\kappa^{2/3} - \sigma\kappa^{-1/3} \to +\infty$ as $\kappa \to +\infty$. This condition means that : $\sigma \ll \frac{\kappa}{\beta_0} - L\kappa^{1/3}$ for some $L > 0$.

Remark 1.5. We believe that the error term in (1.5) is $O(\kappa^{-1/2})$. In fact, using Proposition 2.3, we have, for some positive constants m_1 and m_2,

$$\frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - m_1\kappa^{-1/2} < H_{C_3}(\kappa) < \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + m_2\kappa^{-1/3}.$$

Remark 1.6. The estimates given in Theorems 1.2 and 1.3 are not optimal.

(1) Inequalities (1.6) and (1.8) indicate in a weak version that order parameters concentrate near the maximum points of curvature within a tangential distance of order $O(\kappa^{-1/6})$. We believe that the concentration is within a tangential distance of order $O(\kappa^{-1/2})$ to the maximum points of curvature.

(2) We believe that, under the condition $0 < H_{C_3}(\kappa) - \sigma < L_1\kappa^{-1/3}$, it holds that

$$\|\psi\|_{L^\infty(\Omega)} = O([H_{C_3}(\kappa) - \sigma]^{1/2}\kappa^{-1/2}).$$

Moreover, if $\mathcal{N}(\partial\Omega)$ consists of isolated points, then :

$$\frac{\|\psi\|_{L^2(\Omega)}}{\|\psi\|_{L^\infty(\Omega)}} = O(\kappa^{-3/4}).$$

For a related discussion see Remark 2.6.

Remark 1.7. Theorems 1.2 and 1.3 are consequences of Theorem 6.1, and Theorem 1.1 is a consequence of Propositions 2.3 and 5.1. We will see that the proofs involve complicated estimates. We have to mention that one difficulty of the problem comes from the fact that the bottom β_0 of the spectrum of the eigenvalue problem (2.6) on the half plane, which is the limit equation of the linearization of the Ginzburg-Landau system (2.2) at the trivial solution $(0, \mathbf{F})$, is not isolated. In fact, the spectrum of (2.6) fills the interval $[\beta_0, \infty)$.

Remark 1.8. Combining the arguments in this paper together with the results in [HMor] (Section 9), we can deal with non-homogeneous magnetic fields. We can improve the estimates of $\sigma^*(\kappa, H_0)$ (see [LP4]) with better error terms, and prove the localizations of the order parameters. Since the discussions will be parallel, we omit the details.

Remark 1.9. The problem in 3-dimension is less understood. It was proved in [LP5] that, for a bounded 3-dimensional sample with smooth surface, under a homogeneous magnetic field, superconductivity nucleates first at a portion of surface which is tangential to the applied field. If the tangential portion is a 1- or 2-dimensional submanifold of the surface, it would be interesting to see what replace the curvature in this case.

The outline of this paper is the following. In Section 2, we improve the lower bound estimate for H_{C_3} by getting an error term in $O(\kappa^{-1/2})$. In Section 3, we establish a uniform estimate for the lowest eigenvalue of a Schrödinger operator with a magnetic field. Section 4 is devoted to the elliptic estimates on the minimal solutions of the Ginzburg-Landau system. An upper bound estimate for H_{C_3} is proved in Section 5, using results from Sections 3 and 4. Combining the estimates in Sections 2 and 5 we prove Theorem 1.1. The location of the order parameters when the applied field is close to H_{C_3} is discussed in Section 6 and their exponential decay within boundary is established using Agmon's idea [A] (see also [H] for a presentation of the corresponding Helffer-Sjöstrand techniques in the semi-classical context).

§2. Lower Bound Estimate of H_{C_3} Revisited

In this section we give a lower bound estimate of H_{C_3} with an improved error term.

As mentioned in Section 1, we only consider throughout this paper homogeneous applied fields, that is satisfying $\mathcal{H} \equiv \sigma \mathbf{e}_3$, where σ is a constant, see (1.2). We shall treat σ as a large parameter. In order to make our discussion clear, we set

$$\mathcal{A} = \sigma \mathbf{A}, \qquad \varepsilon = \frac{1}{\sqrt{\kappa\sigma}}.$$

It is more convenient to consider a rescaled Ginzburg-Landau functional defined on $\mathcal{W}$ as follows :

$$\mathcal{J}(\psi, \mathbf{A}) = \int_\Omega \{|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi|^2 + \frac{1}{\varepsilon^4}|\text{curl } \mathbf{A} - 1|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}dx, \tag{2.1}$$

where

$$\mathcal{W} = W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{R}^2).$$

Here $W^{1,2}(\Omega, \mathbb{C})$ is the Sobolev space of all complex-valued functions, and $W^{1,2}(\Omega, \mathbb{R}^2)$ is the Sobolev space of all vector-valued functions. Set

$$E(\kappa, \varepsilon) = \inf_{(\psi, \mathbf{A}) \in \mathcal{W}} \mathcal{J}(\psi, \mathbf{A}),$$

Let $(\psi, \mathbf{A})$ be a minimizer of the functional $\mathcal{J}$. Then, $(\psi, \mathbf{A})$ satisfies the Euler equation (see (1.1))

$$\begin{cases} -\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi = \kappa^2(1 - |\psi|^2)\psi, & \\ \text{curl}^2(\mathbf{A} - \mathbf{F}) = -i\varepsilon^2 \Im(\overline{\psi}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi), & \text{in } \Omega, \\ (\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\psi) \cdot \nu = 0, & \\ \text{curl } (\mathbf{A} - \mathbf{F}) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\mathbf{F}$ is the unique vector field satisfying (1.3). We recall that the minimizer is called a *minimal solution* of (2.2). Note that (2.2) has two parameters κ and ε. However, for simplicity we will denote a minimal solution by $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$. Due to the gauge invariance of the Ginzburg-Landau system (2.2), we may always assume that

$$\operatorname{div} \mathbf{A}^\varepsilon = 0 \quad \text{in } \Omega, \qquad \mathbf{A}^\varepsilon \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Denote by $\mu = \mu(b\mathbf{A})$ the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -\nabla^2_{b\mathbf{A}}\phi = \mu\phi & \text{in } \Omega, \\ (\nabla_{b\mathbf{A}}\phi)\cdot\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where b is a real number. The following lemma was given in [LP4] (Lemma 2.1).

Lemma 2.1. *Let $\mathcal{J}$ be the functional defined in (2.1).*

(1) *If $\mu(\frac{1}{\varepsilon^2}\mathbf{F}) < \kappa^2$, then $\mathcal{J}$ has a non-trivial minimizer.*

(2) *If $\mathcal{J}$ has a non-trivial minimizer $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$, then $\mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon) < \kappa^2$.*

The proof of Lemma 2.1 is obvious. Note that $(0, \mathbf{F})$ is a trivial solution of the Ginzburg-Landau system with energy $\mathcal{J}(0, \mathbf{F}) = 0$. If $E(\kappa, \varepsilon) < 0$, then $\mathcal{J}$ has a non-trivial minimizer.

However, Lemma 2.1 is our starting point to estimate H_{C_3}, see [LP4]. In this section, as in [LP4], we shall use Lemma 2.1 to obtain the lower bound estimate of H_{C_3} with an improved control of the error term.

Let us first recall some facts about an eigenvalue variation problem. For every fixed z, let $\beta(z)$ denote the first eigenvalue of the following self-adjoint eigenvalue problem in $L^2(\mathbb{R}_+)$:

$$\begin{cases} -u'' + (z+t)^2 u = \beta(z)u & \text{for } t > 0, \\ u'(0) = 0. \end{cases} \tag{2.5}$$

Lemma 2.2. *There is a unique z_0, $z_0 < 0$, such that*

$$\beta_0 \equiv \inf_{z\in\mathbb{R}^1} \beta(z) = \beta(z_0) = z_0^2.$$

Moreover, $0.5 < \beta_0 < 0.76$.

Lemma 2.2 was proved by Dauge-Helffer in [DH], see also [BH]. For a different proof see [LP2]. For the estimate $0.5 < \beta_0 < 0.76$ see [LP4] (Proposition 2.4). Here we use the notations from [LP2,4]. In the following we denote by $u(t)$ a unique positive eigenfunction of (2.5) for $z = z_0$ and $\beta = \beta_0$ (we may choose u such that $u(0) = 1$). Recall that u has the following property :

$$\int_0^{+\infty} (z_0 + t)u^2(t)dt = 0.$$

Lemma 2.2 is useful in the classification of bounded solutions associated with the bottom of the spectrum of the Neumann realization of the Schrödinger operator $-\nabla_\omega^2$ in the half plane $\mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$:

$$\begin{cases} -\nabla_\omega^2 \phi = \beta_0 \phi & \text{in } \mathbb{R}^2_+, \\ \nabla_\omega \phi \cdot \nu = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.6}$$

where $\omega(x) = \frac{1}{2}(-x_2, x_1)$. It was proved in [LP2] that, the bottom of the spectrum is the number β_0 given in Lemma 2.2, and the *bounded* solutions of (2.6) associated with β_0 are given by $\phi = c\exp(\dfrac{ix_1x_2}{2} + iz_0x_1)u(x_2)$.

In the following, we set

$$C_1 = \frac{u^2(0)}{3\|u\|^2_{L^2(\mathbb{R}_+)}}. \tag{2.7}$$

It is clear from this formula that $C_1 > 0$.

Proposition 2.3 (Lower bound estimate for H_{C_3}). *Assume that Ω is a bounded 2-dimensional domain of class C^4. For κ large, we have*

$$H_{C_3}(\kappa) \geq \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + O(\kappa^{-1/2}), \tag{2.8}$$

where C_1 is the positive constant defined in (2.7).

Proof. Here we need the upper bound of the lowest eigenvalue $\mu(\frac{1}{\varepsilon^2}\mathbf{F})$ of the Schrödinger operator with constant magnetic field : curl $\mathbf{F} \equiv 1$, with an error term in $O(\varepsilon^{-1/2})$, which was obtained by Helffer-Morame in [HMor] (Proposition 10.7) :

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F}) \leq \frac{1}{\varepsilon^2}\{\beta_0 - C_1\kappa_{max}\varepsilon + M\varepsilon^{3/2}\}. \tag{2.9}$$

Using the argument of the appendix in [LP4], (2.8) follows from (2.9). For reader's convenience we include a proof below, which also gives an upper bound of the energy.

For M introduced in (2.9), we choose L such that $L\beta_0^{5/4} > M$. We shall show that, when κ is large, (2.2) has a non-trivial minimal solution for any number σ satisfying

$$0 < \sigma < \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - L\kappa^{-1/2}.$$

From the previous lower bound estimate obtained for $H_{C_3}(\kappa)$ in [LP4], we know already that (2.2) has non-trivial minimal solutions for $\sigma \leq \kappa/\beta_0$ when κ is large. Hence we only need to consider σ's greater than κ/β_0. In the following we choose σ such that

$$\frac{\kappa}{\beta_0} < \sigma < \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - L\kappa^{-1/2}. \tag{2.10}$$

An easy computation gives :

$$\beta_0 > \frac{\kappa}{\sigma} > \beta_0 - \frac{C_1\beta_0^{1/2}}{\kappa}\kappa_{max} + L\beta_0^2\kappa^{-3/2} + O(\kappa^{-2}). \tag{2.11}$$

Recall that $\varepsilon = (\sigma\kappa)^{-1/2}$. From (2.11) we see that

$$\frac{\beta_0^{1/2}}{\kappa} > \ \varepsilon \ > \frac{\beta_0^{1/2}}{\kappa} - \frac{C_1}{2}\kappa_{max}\kappa^{-2} + \frac{L\beta_0^{3/2}}{2}\kappa^{-5/2} + O(\kappa^{-3}).$$

Hence we have

$$\begin{aligned} \frac{\varepsilon}{\beta_0^{1/2}} < \frac{1}{\kappa} \ &< \frac{\varepsilon}{\beta_0^{1/2}} + \frac{C_1}{2\beta_0^{3/2}}\kappa_{max}\varepsilon^2 + O(\varepsilon^{5/2}), \\ \beta_0 > \frac{\kappa}{\sigma} \ &> \beta_0 - C_1\kappa_{max}\varepsilon + L\beta_0^{5/4}\varepsilon^{3/2} + O(\varepsilon^2). \end{aligned} \tag{2.12}$$

Let us write

$$t(\varepsilon) = \varepsilon^2(\kappa^2 - \mu(\frac{1}{\varepsilon^2}\mathbf{F})). \tag{2.13}$$

Using (2.9), (2.11) and (2.12), we have, as $\varepsilon \to 0$,

$$t(\varepsilon) = \varepsilon^2(\kappa^2 - \mu(\frac{1}{\varepsilon^2}\mathbf{F})) \geq (L\beta_0^{5/4} - M)\varepsilon^{3/2} + O(\varepsilon^2) > 0.$$

Let ϕ_ε be an eigenfunction of $-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{F}}$ in Ω associated with the first eigenvalue $\mu(\frac{1}{\varepsilon^2}\mathbf{F})$ of the Neumann realization of $-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{F}}$ and choose $(\lambda\phi_\varepsilon, \mathbf{F})$ as a test function, where λ is a free parameter. We have

$$\begin{aligned} \mathcal{J}(\lambda\phi_\varepsilon, \mathbf{F}) &= \lambda^2 \int_\Omega \{|\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon|^2 - \kappa^2|\phi_\varepsilon|^2 + \frac{\lambda^2\kappa^2}{2}|\phi_\varepsilon|^4\}dx \\ &= -\frac{\lambda^2 t(\varepsilon)}{\varepsilon^2}\|\phi_\varepsilon\|^2_{L^2(\Omega)} + \frac{\lambda^4\kappa^2}{2}\|\phi_\varepsilon\|^4_{L^4(\Omega)}. \end{aligned}$$

Choose

$$\lambda^2 = \frac{t(\varepsilon)\|\phi_\varepsilon\|^2_{L^2(\Omega)}}{\varepsilon^2\kappa^2\|\phi_\varepsilon\|^4_{L^4(\Omega)}} = \frac{t(\varepsilon)\sigma}{\kappa}\frac{\|\phi_\varepsilon\|^2_{L^2(\Omega)}}{\|\phi_\varepsilon\|^4_{L^4(\Omega)}}.$$

Then

$$\mathcal{J}(\lambda\phi_\varepsilon, \mathbf{F}) = -\frac{t(\varepsilon)^2\|\phi_\varepsilon\|^4_{L^2(\Omega)}}{2\varepsilon^4\kappa^2\|\phi_\varepsilon\|^4_{L^4(\Omega)}} = -\frac{t(\varepsilon)^2\sigma^2}{2}\frac{\|\phi_\varepsilon\|^4_{L^2(\Omega)}}{\|\phi_\varepsilon\|^4_{L^4(\Omega)}} < 0.$$

This last inequality shows that, for σ satisfying (2.10), the Ginzburg-Landau system (2.2) has a non-trivial minimal solution $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$. Thus (2.8) holds. □

Remark 2.4. Let ϕ_ε and λ be the same as in the proof of Proposition 2.3. Note that $(\lambda\phi_\varepsilon, \mathbf{F})$ is not a solution of (2.2) (a proof is given below for reader's convenience). From the last part of the proof of Proposition 2.3, we have :

$$\mathcal{J}(\psi^\varepsilon, \mathbf{A}^\varepsilon) < \mathcal{J}(\lambda\phi_\varepsilon, \mathbf{F}).$$

Hence

$$\mathcal{J}(\psi^\varepsilon, \mathbf{A}^\varepsilon) < -\frac{t(\varepsilon)^2\sigma^2}{2}\frac{\|\phi_\varepsilon\|^4_{L^2(\Omega)}}{\|\phi_\varepsilon\|^4_{L^4(\Omega)}}. \tag{2.14}$$

Now we show that, for any $\lambda \neq 0$, $(\lambda\phi_\varepsilon, \mathbf{F})$ is not a solution of (2.2). Otherwise, suppose that $(\lambda\phi_\varepsilon, \mathbf{F})$ were a solution of (2.2). We have

$$\begin{cases} -\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon = \kappa^2(1-\lambda^2|\phi_\varepsilon|^2)\phi_\varepsilon, & \\ \Im(\overline{\phi_\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon) = 0 & \text{in } \Omega, \\ (\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2$'$}$$

Recall that ϕ_ε satisfies

$$\begin{cases} -\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon = \mu(\frac{1}{\varepsilon^2}\mathbf{F})\phi_\varepsilon & \text{in } \Omega, \\ (\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Comparing the first equation of (2.2$'$) with (2.15) we see that

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F})\phi_\varepsilon = \kappa^2(1-\lambda^2|\phi_\varepsilon|^2)\phi_\varepsilon.$$

Since $\phi_\varepsilon \not\equiv 0$, we have :

$$|\phi_\varepsilon(x)|^2 \equiv \frac{\kappa^2 - \mu(\frac{1}{\varepsilon^2}\mathbf{F})}{\kappa^2\lambda^2}.$$

Therefore $|\phi_\varepsilon(x)| \equiv f_\varepsilon$, a constant. Let D be a simply-connected subdomain of Ω. In D we can write $\phi_\varepsilon(x) = f_\varepsilon e^{i\chi_\varepsilon(x)}$, where $\chi_\varepsilon(x)$ is a real-valued C^2 function. Thus

$$\Im(\overline{\phi_\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}\phi_\varepsilon) = \Im[f_\varepsilon e^{-i\chi_\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{F}}(f_\varepsilon e^{i\chi_\varepsilon})] = f_\varepsilon^2(\nabla\chi_\varepsilon - \frac{1}{\varepsilon^2}\mathbf{F}).$$

Hence $\mathbf{F} = \varepsilon^2\nabla\chi_\varepsilon$ and curl $\mathbf{F} \equiv 0$ in D. Repeating this argument on any simply-connected subdomain of Ω, we find curl $\mathbf{F} \equiv 0$ in Ω, which contradicts (1.3). This contradiction shows that $(\lambda\phi_\varepsilon, \mathbf{F})$ is not a solution of (2.2). □

Remark 2.5. Bernoff-Sternberg [BS] gave a formal expansion of $H_{C_3}(\kappa)$.

It was proved in [LP4] that, if Ω is a bounded domain in $\mathbb{R}^2$ of class C^5 and if $\mathbf{F} \in C^5(\Omega)$ satisfying curl $\mathbf{F} \equiv 1$, then, as $\varepsilon \to 0$, we have :

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F}) \le \frac{1}{\varepsilon^2}\{\beta_0 - C_1\kappa_{max}\varepsilon + O(\varepsilon^{8/7})\}.$$

From this estimate Lu-Pan derived a lower bound estimate of H_{C_3} with error term in $O(\kappa^{-1/7})$.

In [DFS] del Pino-Felmer-Sternberg proved, for a $C^{3,\alpha}$ domain, an upper bound of the lowest eigenvalue with a remainder in $o(\frac{1}{\varepsilon})$, namely

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F}) \le \frac{1}{\varepsilon^2}\{\beta_0 - C_1\kappa_{max}\varepsilon + o(\varepsilon)\},$$

which yields a lower bound estimate for $H_{C_3}(\kappa)$ with error term $o(1)$. They also pointed out that, under the stronger assumption that there is a non-degenerate maximum point $z(s^*)$ of the curvature, they could prove an upper bound of the lowest eigenvalue in the form

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F}) \le \frac{1}{\varepsilon^2}\{\beta_0 - C_1\kappa_{max}\varepsilon + \frac{\beta_0^{1/4}}{\sqrt{6}}\sqrt{-\kappa_r''(s^*)}\varepsilon^{3/2} + o(\varepsilon^{3/2})\}.$$

We should also mention that the expression of C_1 in [LP4] is more complicated (and equivalent). Helffer-Morame in [HMor] used $2M_3$ instead of C_1. In this paper we adopt the definition (2.7) for C_1, as in [BS], which shows more explicitly the positivity of C_1.

Remark 2.6. The energy estimate given in (2.14) involves the term $\|\phi_\varepsilon\|^4_{L^2(\Omega)}/\|\phi_\varepsilon\|^4_{L^4(\Omega)}$ which, we believe, is of order $O(\varepsilon^{3/2})$ if $\mathcal{N}(\partial\Omega)$ consists of a finite number of points.

We explain the guess heuristically. For simplicity, let us consider the normalized eigenfunction ϕ_ε, namely, $\|\phi_\varepsilon\|_{L^\infty(\Omega)} = 1$. From Propositions 6.6, 10.6 and 10.7 in [HMor] we believe (and will show later) that the eigenfunction ϕ_ε concentrates on the set $\mathcal{N}(\partial\Omega)$ of the maximum points of the curvature, and decays exponentially away from $\mathcal{N}(\partial\Omega)$, with order $\exp(-\frac{\alpha_1}{\varepsilon}\,\mathrm{dist}(x,\partial\Omega))$ in the normal direction, and with order $\exp(-\frac{\alpha_2}{\sqrt{\varepsilon}}d_t(x,\mathcal{N}(\partial\Omega)))$ in the tangential direction.

Now suppose that $\mathcal{N}(\partial\Omega)$ consists of a finite number of points. Near each point $x_j \in \mathcal{N}(\partial\Omega)$, we can introduce new variables $y = (y_1, y_2)$, where y_1 is the tangential variable and y_2 is the normal variable, to straighten a portion of the boundary around x_j (In fact, y_1 is a translation of $s(x)$, and $y_2 = t(x)$). In the new variables y, after making gauge transformations, the rescaled function of ϕ_ε converges to a bounded solution $c_j\phi_0(y)$ of (2.6) as $\varepsilon \to 0$, where $|c_j| \le 1$, $\phi_0(y) = \exp(iz_0y_1)u(y_2)$ and u is chosen such that $u(0) = 1$ (recall that $\|\phi_\varepsilon\|_{L^\infty(\Omega)} = 1$). So, near x_j, ϕ_ε can be approximated locally by a function in the form $e^{i\chi_j}\phi_0(\frac{y}{\varepsilon})w_j(\frac{y_1}{\sqrt{\varepsilon}})$, here χ_j and w_j are smooth real functions, and $|w_j(y)| \le 1$. Hence

$$\int_{\Omega\cap B_\delta(x_j)} |\phi_\varepsilon|^2 dx \simeq \hat{c}_j\varepsilon^{3/2}.$$

As mentioned above that ϕ_ε concentrates on $\mathcal{N}(\partial\Omega)$. The above computation can be carried out at each of the points $x_1, \cdots, x_n$ in $\mathcal{N}(\partial\Omega)$. Hence

$$\int_\Omega |\phi_\varepsilon|^2 dx \simeq \varepsilon^{3/2}\sum_{j=1}^n \hat{c}_j, \qquad \sum_{j=1}^n \hat{c}_j^2 > 0.$$

Similarly we have

$$\int_\Omega |\phi_\varepsilon|^4 dx \simeq \varepsilon^{3/2}\sum_{j=1}^n \tilde{c}_j, \qquad \sum_{j=1}^n \tilde{c}_j^2 > 0.$$

Therefore

$$\frac{\|\phi_\varepsilon\|^4_{L^2(\Omega)}}{\|\phi_\varepsilon\|^4_{L^4(\Omega)}} \simeq c\varepsilon^{3/2}, \qquad \text{where} \quad c = \frac{[\sum_{j=1}^n \hat{c}_j]^2}{\sum_{j=1}^n \tilde{c}_j}.$$

Remark 2.7. Instead of using the eigenfunction ϕ_ε, we can also choose the test function used in [HMor] (proof of Proposition 10.7) to get a more explicit energy estimate.

§3. Uniform Estimates for the Lowest Eigenvalue

In this section we assume that Ω is a bounded simply-connected domain in $\mathbb{R}^2$ of class C^4. We shall establish a uniform estimate for the lowest eigenvalue $\mu_\varepsilon \equiv \mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon)$ of the Neumann realization of the Schrödinger operator $-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}$ with magnetic potential $\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon$, where $\mathbf{A}^\varepsilon$ satisfies the following conditions

$$\begin{aligned} &\|\mathbf{A}^\varepsilon\|_{C^2(\overline{\Omega})} \le C, \\ &\|\text{curl } \mathbf{A}^\varepsilon - 1\|_{C^1(\overline{\Omega})} = O(\varepsilon^{1/3}), \quad \text{as } \varepsilon \to 0, \\ &\text{curl } \mathbf{A}^\varepsilon = 1, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

We intend to apply the results of this section to the solution $\mathbf{A}^\varepsilon$ of (2.2) in order to prove Theorem 1. However, the results in this section are valid for any family of vector fields satisfying (3.1).

Theorem 3.1. *Under Assumption (3.1), there exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,*

$$\mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon) \ge \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3}) \ , \tag{3.2}$$

where C_1 is given in (2.7).

The proof of Theorem 3.1 will be carried out following the lines of [HMor], with a careful control on the error terms. In this section we always fix a positive number $\delta_0 < \varepsilon(\Omega)$, and let $t(x) = \text{dist}(x, \partial\Omega)$. By the gauge-invariance of the eigenvalue problem and Condition (3.1), we can choose $\mathbf{A}^\varepsilon$ in such a way that, in the neighborhood Ω_{δ_0}, $\mathbf{A}^\varepsilon = (A_1^\varepsilon, A_2^\varepsilon)$ has the following form :

$$\begin{aligned} &A_1^\varepsilon = -t + \frac{t^2}{2}\kappa_r(s) + t^2 b_\varepsilon(s,t), \\ &A_2^\varepsilon = 0, \\ &b_\varepsilon(s,t) = O(\varepsilon^{1/3}). \end{aligned} \tag{3.3}$$

Let $\{\chi_\gamma(x)\}_\Gamma$ be a partition of unity of $\mathbb{R}^2$ introduced in [HMor] ((9.10)-(9.14)) such that $\Gamma = \mathbb{Z}^2$,

$$\begin{aligned} &\chi_\gamma \in C^\infty(\mathbb{R}^2, \mathbb{R}), \\ &\text{spt } (\chi_\gamma) \subset \gamma + [-1,1]^2, \qquad \text{for any } \gamma \in \Gamma, \\ &\sum_{\gamma\in\Gamma} \chi_\gamma(x)^2 = 1, \qquad \sum_{\gamma\in\Gamma} |\nabla\chi_\gamma(x)|^2 < C. \end{aligned}$$

If $\tau(\varepsilon)$ is a function of ε such that $0 < \tau(\varepsilon) < \delta_0$ for all ε small, then we set

$$\chi_{\gamma,\tau(\varepsilon)}(x) = \chi_\gamma(\frac{x}{\tau(\varepsilon)}), \qquad \text{for all } \gamma \in \Gamma.$$

Thus we have a new partition of unity such that

$$\begin{aligned}&\text{spt } (\chi_{\gamma,\tau(\varepsilon)}) \subset \tau(\varepsilon)\gamma + [-\tau(\varepsilon), \tau(\varepsilon)]^2,\\ &\sum_{\gamma\in\Gamma} \chi_{\gamma,\tau(\varepsilon)}(x)^2 = 1, \qquad\qquad \sum_{\gamma\in\Gamma} |\nabla\chi_{\gamma,\tau(\varepsilon)}(x)|^2 < \frac{C}{\tau(\varepsilon)^2}.\end{aligned}$$

Let us introduce :

$$\begin{aligned}\Gamma_{\tau(\varepsilon)}(\Omega) &= \{\gamma \in \Gamma : \text{ spt } (\chi_{\gamma,\tau(\varepsilon)}) \cap \Omega \neq \emptyset\},\\ \Gamma^0_{\tau(\varepsilon)}(\Omega) &= \{\gamma \in \Gamma_{\tau(\varepsilon)}(\Omega) : \text{dist } (\text{spt } (\chi_{\gamma,\tau(\varepsilon)}), \partial\Omega) > \tau(\varepsilon)\},\\ \Gamma^1_{\tau(\varepsilon)}(\Omega) &= \{\gamma \in \Gamma_{\tau(\varepsilon)}(\Omega) : \text{dist } (\text{spt } (\chi_{\gamma,\tau(\varepsilon)}), \partial\Omega) \leq \tau(\varepsilon)\},\\ \Gamma'_{\tau(\varepsilon)}(\Omega) &= \{\gamma \in \Gamma^1_{\tau(\varepsilon)}(\Omega) : \text{ spt } (\chi_{\gamma,\tau(\varepsilon)}) \cap \overline{\Omega} \subset \Omega_{\delta_0}\}.\end{aligned}$$

In the following we denote by φ_ε an eigenfunction of the Neumann Schrödinger operator $-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}$ associated with the lowest eigenvalue μ_ε. We shall recall the estimates for eigenfunctions. Let us begin with the weighted L^2 estimates obtained in [HMor] ((6.25) and (6.26)) :

Lemma 3.2 (Weighted L^2 estimates). *Under Assumption (3.1), there exists positive constants α, ε_1 and C such that, for all $0 < \varepsilon \leq \varepsilon_1$, we have :*

$$\int_\Omega \exp(\frac{\alpha t(x)}{\varepsilon})\{|\varphi_\varepsilon|^2 dx + \varepsilon^2|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\varphi_\varepsilon|^2\}dx \leq C\int_\Omega |\varphi_\varepsilon|^2 dx. \tag{3.4}$$

Propositions 9.6 and 10.2 in [HMor] remain true for a family $\{\mathbf{A}^\varepsilon\}$ satisfying Assumption (3.1).

Lemma 3.3. *Under Assumption (3.1), for any non-negative integer k, there exist positive numbers $\varepsilon_3(k)$ and C_k such that, for any $0 < \varepsilon < \varepsilon_3(k)$, we have :*

$$\begin{aligned}&\int_\Omega t(x)^k|\varphi_\varepsilon(x)|^2 dx \leq C_k\varepsilon^k\|\varphi_\varepsilon\|^2_{L^2(\Omega)},\\ &\int_\Omega t(x)^k|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\varphi_\varepsilon(x)|^2 dx \leq C_k\varepsilon^{k-2}\|\varphi_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Lemma 3.4. *Under Assumption (3.1), we choose $\tau(\varepsilon) = \varepsilon^\rho$ with $0 < \rho \leq \frac{1}{2}$. Then, for any non-negative integer k, there exist positive numbers $\varepsilon_4(k)$ and $C(k)$ such that, for any $0 < \varepsilon < \varepsilon_4(k)$ and $\gamma \in \Gamma'_{\tau(\varepsilon)}(\Omega)$, there exist sequences $\{a_k(\gamma, \varepsilon)\}_\gamma$ and $\{b_k(\gamma, \varepsilon)\}_\gamma$ satisfying*

$$\sum_\gamma a_k(\gamma, \varepsilon)^2 \leq C(k), \qquad\qquad \sum_\gamma b_k(\gamma, \varepsilon)^2 \leq C(k),$$

such that :

$$\int_\Omega t(x)^k |\chi_{\gamma,\tau(\varepsilon)}\varphi_\varepsilon(x)|^2 dx \le a_k(\gamma,\varepsilon)^2 \varepsilon^k \|\varphi_\varepsilon\|^2_{L^2(\Omega)}, \tag{3.5}$$

$$\int_\Omega t(x)^k |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}(\chi_{\gamma,\tau(\varepsilon)}\varphi_\varepsilon(x))|^2 dx \le b_k(\gamma,\varepsilon)^2 \varepsilon^{k-2} \|\varphi_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.6}$$

Remark 3.5. The exponential decay in the normal direction is well-known in the case where the magnetic field is independent of the parameter ε. Under Assumption (3.1), the uniformly exponential decay can also be established, as in [DFS] (Theorem 4.3) and [HMor] (Theorem 6.3) :

Under Assumption (3.1), there exist $\varepsilon_2 > 0$ and, for any multi-index α with $|\alpha| \le 2$, positive constants $c_1(\alpha)$ and $c_2(\alpha)$ such that, for any $0 < \varepsilon < \varepsilon_2$, we have :

$$|D^\alpha \varphi_\varepsilon(x)| \le \frac{c_1(\alpha)}{\varepsilon^{|\alpha|}} \exp(-\frac{c_2(\alpha)}{\varepsilon} \text{ dist } (x, \partial\Omega)) \|\varphi_\varepsilon\|_{L^\infty(\Omega)}, \qquad \text{for all } x \in \Omega. \tag{3.7}$$

Proof of Theorem 3.1.

In the proof C denotes a generic constant which varies from line to line, but is independent of ε and γ. As above, we denote by φ_ε an eigenfunction associated with the lowest eigenvalue μ_ε. Write

$$Q_{\varepsilon^2,\mathbf{A}}[\phi] = \int_\Omega |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}}\phi|^2 dx.$$

Then

$$\mu_\varepsilon = \frac{Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\varphi_\varepsilon]}{\|\varphi_\varepsilon\|^2_{L^2(\Omega)}}.$$

So, a lower bound of $Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\varphi_\varepsilon]$ yields a lower bound for μ_ε. However, we will not compute directly the value of $Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\varphi_\varepsilon]$. Recall that the eigenfunctions concentrate in a small neighborhood of the boundary. Also note that, for any smooth function η, we have :

$$\int_\Omega |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}(\eta\varphi_\varepsilon)|^2 dx = \Re \int_\Omega (-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\varphi_\varepsilon)\overline{(\eta^2\varphi_\varepsilon)} dx + \int_\Omega |\varphi_\varepsilon \nabla\eta|^2 dx, \tag{3.8}$$

hence

$$\mu_\varepsilon \|\varphi_\varepsilon \eta\|^2_{L^2(\Omega)} = Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\varphi_\varepsilon\eta] - \int_\Omega |\varphi_\varepsilon\nabla\eta|^2 dx. \tag{3.9}$$

Therefore, we shall choose a suitable cut-off function η carefully, and estimate each term in the right hand of (3.9). We shall carry out a careful analysis on the behavior of φ_ε near the boundary, and the following notations are needed.

Let $\{\eta_{0,\tau(\varepsilon)}, \eta_{1,\tau(\varepsilon)}\}$ be the partition of unity on $\mathbb{R}$ introduced in [HMor] ((9.22) and (9.23)) such that

$$\begin{aligned}
&\eta^2_{0,\tau(\varepsilon)}(t) + \eta^2_{1,\tau(\varepsilon)} = 1,\\
&|\eta'_{j,\tau(\varepsilon)}(t)| \le \frac{C}{\tau(\varepsilon)}, \qquad \text{for } j = 0, 1,\\
&\text{spt } (\eta_{0,\tau(\varepsilon)}) \subset [\frac{\tau(\varepsilon)}{20}, +\infty),\\
&\text{spt } (\eta_{1,\tau(\varepsilon)}) \subset (-\infty, \frac{\tau(\varepsilon)}{10}].
\end{aligned}$$

Let us choose $\tau(\varepsilon) = \varepsilon^\rho$, where $\frac{1}{3} \leq \rho \leq \frac{1}{2}$. We will fix ρ later. For each $\gamma \in \Gamma^1_{\tau(\varepsilon)}(\Omega)$, we can choose $x_\gamma \in \partial\Omega$, such that, for any $x \in$ spt $(\chi_{\gamma,\varepsilon^{1/3}})$

$$|s - s_\gamma| \leq \frac{3}{2}\varepsilon^{1/3}. \tag{3.10}$$

Set

$$\begin{aligned}
&K(\gamma,\varepsilon) = (s_\gamma - 2\varepsilon^{1/3}, s_\gamma + 2\varepsilon^{1/3}) \times (0, \varepsilon^\rho),\\
&K_\gamma = \kappa_r(s_\gamma),\\
&a_\gamma(t) = 1 - K_\gamma t.
\end{aligned}$$

Let us define

$$\begin{aligned}
&w_\varepsilon(x) = \eta_{1,\tau(\varepsilon)}(t(x))\varphi_\varepsilon(x),\\
&w_{\gamma,\varepsilon}(x) = \chi_{\gamma,\varepsilon^{1/3}}\eta_{1,\tau(\varepsilon)}(t(x))\varphi_\varepsilon(x).
\end{aligned} \tag{3.11}$$

Using (3.9) with $\eta = \eta_{1,\tau(\varepsilon)}$, we have :

$$\mu_\varepsilon \|w_\varepsilon\|^2_{L^2(\Omega)} = Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon] - \int_\Omega |\varphi_\varepsilon \nabla \eta_{1,\tau(\varepsilon)}|^2 dx. \tag{3.12}$$

We shall estimate each term in the right hand of (3.12).

Step 1. We look for a lower bound of $Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon]$.

As in [HMor] ((11.3) and (11.4)), we get

$$Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon] \geq \sum_{\gamma\in\Gamma^1_{\tau(\varepsilon)}(\Omega)} Q_{\varepsilon,\mathbf{A}^\varepsilon}[w_{\gamma,\varepsilon}] - C\varepsilon^{-2/3}\int_\Omega |w_\varepsilon|^2 dx$$

and

$$\begin{aligned}
Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_{\gamma,\varepsilon}] \geq &\int_{K(\gamma,\varepsilon)} a_\gamma(t)\{(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A^\varepsilon_1)w_{\gamma,\varepsilon}|^2 + |\partial_t w_{\gamma,\varepsilon}|^2\}dsdt\\
&-C\int_\Omega \{\varepsilon^{1/3} t |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon} w_{\gamma,\varepsilon}|^2 + \varepsilon^{-11/3}(t^3+t^4)|w_{\gamma,\varepsilon}|^2\}dx.
\end{aligned}$$

From these inequalities, Lemmas 3.4 and 3.5, we get

$$\begin{aligned}
Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon] \geq &\sum_{\gamma\in\Gamma^1_{\tau(\varepsilon)}(\Omega)} \int_{K(\gamma,\varepsilon)} a_\gamma(t)\{(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A^\varepsilon_1)w_{\gamma,\varepsilon}|^2 + |\partial_t w_{\gamma,\varepsilon}|^2\}dsdt\\
&- C\varepsilon^{-2/3}\|w_\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.13}$$

In order to estimate the sum in the right hand of (3.13), we write

$$\begin{aligned}
&A^\varepsilon_1 = A_\gamma(t) + A_{\gamma,\varepsilon},\\
&A_\gamma(t) = -t + \frac{t^2}{2}K_\gamma,\\
&A_{\gamma,\varepsilon} = A^\varepsilon_1 - A_\gamma = \frac{t^2}{2}(\kappa_r(s) - K_\gamma) + t^2 b_\varepsilon(s,t).
\end{aligned} \tag{3.14}$$

We have

$$\begin{aligned}
&\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A_1^\varepsilon)w_{\gamma,\varepsilon}|^2 dsdt\\
=&\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A_\gamma)w_{\gamma,\varepsilon} - \frac{i}{\varepsilon^2}A_{\gamma,\varepsilon}w_{\gamma,\varepsilon}|^2 dsdt\\
=&\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A_\gamma)w_{\gamma,\varepsilon}|^2 dsdt\\
&\quad+\frac{2}{\varepsilon^2}\Im\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)A_{\gamma,\varepsilon}\overline{w_{\gamma,\varepsilon}}(\partial_s - \frac{i}{\varepsilon^2}A_\gamma)w_{\gamma,\varepsilon} dsdt\\
&\quad+\frac{1}{\varepsilon^4}\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|A_{\gamma,\varepsilon}w_{\gamma,\varepsilon}|^2 dsdt.
\end{aligned}$$

Hence

$$Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon] \geq P(\varepsilon) + \frac{2q(\varepsilon)}{\varepsilon^2}, \tag{3.15}$$

where

$$\begin{aligned}
P(\varepsilon) &= \sum_{\gamma\in\Gamma^1_{\tau(\varepsilon)}(\Omega)} \int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|(\partial_s - \frac{i}{\varepsilon^2}A_\gamma)w_{\gamma,\varepsilon}|^2 dsdt,\\
q(\varepsilon) &= \sum_{\gamma\in\Gamma^1_{\tau(\varepsilon)}(\Omega)} q(\gamma,\varepsilon), \qquad\qquad (3.16)\\
q(\gamma,\varepsilon) &= \Im\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)A_{\gamma,\varepsilon}\overline{w_{\gamma,\varepsilon}}(\partial_s - \frac{i}{\varepsilon^2}A_\gamma)w_{\gamma,\varepsilon} dsdt.
\end{aligned}$$

We need a lower bound of each term in the right hand of (3.15). A lower bound of $P(\varepsilon)$ can be obtained by using the argument in [HMor], proof of Theorem 11.1, with the choice $\tau(\varepsilon) = \varepsilon^{5/12}$:

$$P(\varepsilon) \geq \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3})\|w_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.17}$$

In the following, we choose $\tau(\varepsilon) = \varepsilon^{5/12}$ and derive an estimate for $2q(\varepsilon)/\varepsilon^2$. Since $A_\gamma = A_1^\varepsilon - A_{\gamma,\varepsilon}$, we can write

$$\partial_s - \frac{i}{\varepsilon^2}A_\gamma = \partial_s - \frac{i}{\varepsilon^2}A_1^\varepsilon + \frac{i}{\varepsilon^2}A_{\gamma,\varepsilon}.$$

Thus we write

$$\begin{aligned}
q(\gamma,\varepsilon) &= q_1(\gamma,\varepsilon) + q_2(\gamma,\varepsilon),\\
q_1(\gamma,\varepsilon) &= \Im\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)A_{\gamma,\varepsilon}\overline{w_{\gamma,\varepsilon}}(\partial_s - \frac{i}{\varepsilon^2}A_1^\varepsilon)w_{\gamma,\varepsilon} dsdt, \qquad (3.18)\\
q_2(\gamma,\varepsilon) &= \frac{1}{\varepsilon^2}\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)|A_{\gamma,\varepsilon}w_{\gamma,\varepsilon}|^2 dsdt.
\end{aligned}$$

We first estimate $q_1(\gamma,\varepsilon)$. Noting that $\eta_{1,\tau(\varepsilon)}$ depends only on t, using (3.11) we have

$$q_1(\gamma,\varepsilon)=\Im\int_{K(\gamma,\varepsilon)} a_\gamma(t)(1+2K_\gamma t)\eta^2_{1,\tau(\varepsilon)}(t)A_{\gamma,\varepsilon}\overline{\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon}(\partial_s-\frac{i}{\varepsilon^2}A_1^\varepsilon)(\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon)dsdt,$$

so

$$\begin{aligned}|q_1(\gamma,\varepsilon)| &\le C\int_{K(\gamma,\varepsilon)}\eta^2_{1,\tau(\varepsilon)}(t)|A_{\gamma,\varepsilon}\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon||(\partial_s-\frac{i}{\varepsilon^2}A_1^\varepsilon)(\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon)|dsdt\\ &\le C\left(\int_{K(\gamma,\varepsilon)}\eta^2_{1,\tau(\varepsilon)}(t)|A_{\gamma,\varepsilon}|^2|\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon|^2dsdt\right)^{1/2}\\ &\quad\times\left(\int_{K(\gamma,\varepsilon)}\eta^2_{1,\tau(\varepsilon)}(t)|(\partial_s-\frac{i}{\varepsilon^2}A_1^\varepsilon)(\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon)|^2dsdt\right)^{1/2}.\end{aligned}$$

From (3.3), (3.10) and (3.14) we have

$$|A_{\gamma,\varepsilon}|\le t^2(\frac{1}{2}|\kappa_r(s)-\kappa_r(s_\gamma)|+|b_\varepsilon(s,t)|)=O(\varepsilon^{1/3})t^2,\qquad \text{on } K(\gamma,\varepsilon).$$

So we use (3.7) with $\alpha=0$, (3.5) and (3.6), in order to obtain :

$$\begin{aligned}&\int_{K(\gamma,\varepsilon)}\eta^2_{1,\tau(\varepsilon)}(t)|A_{\gamma,\varepsilon}|^2|\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon|^2dsdt\le C\varepsilon^{14/3}a_4(\gamma,\varepsilon)^2\|w_\varepsilon\|^2_{L^2(\Omega)},\\ &\int_{K(\gamma,\varepsilon)}\eta^2_{1,\tau(\varepsilon)}(t)|(\partial_s-\frac{i}{\varepsilon^2}A_1^\varepsilon)(\chi_{\gamma,\varepsilon^{1/3}}\varphi_\varepsilon)|^2dsdt\le C\varepsilon^{-2}b_0(\gamma,\varepsilon)^2\|w_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence we have the following estimate for $q_1(\gamma,\varepsilon)$:

$$|q_1(\gamma,\varepsilon)|\le C\varepsilon^{4/3}a_4(\gamma,\varepsilon)b_0(\gamma,\varepsilon)\|w_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.19}$$

Similarly, we obtain an estimate for $q_2(\gamma,\varepsilon)$:

$$|q_2(\gamma,\varepsilon)|\le C\varepsilon^{8/3}a_4(\gamma,\varepsilon)^2\|w_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.20}$$

Combining (3.16) with (3.18)-(3.20), we get an estimate for $q(\gamma,\varepsilon)$:

$$|q(\gamma,\varepsilon)|\le C\varepsilon^{4/3}[a_4(\gamma,\varepsilon)b_0(\gamma,\varepsilon)+\varepsilon^{4/3}a_4(\gamma,\varepsilon)^2]\|w_\varepsilon\|^2_{L^2(\Omega)}.$$

Therefore

$$\frac{2}{\varepsilon^2}|q(\varepsilon)|\le C\varepsilon^{-2/3}\|w_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.21}$$

Combining (3.15), (3.17) and (3.21), we get a lower bound for $Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon]$:

$$Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon]\ge\frac{1}{\varepsilon^2}(\beta_0-C_1\kappa_{max}\varepsilon-C\varepsilon^{4/3})\|w_\varepsilon\|^2_{L^2(\Omega)}. \tag{3.22}$$

Step 2. We estimate the second term in the right hand of (3.12).

Recall that $\eta_{1,\tau(\varepsilon)} = 1$ if $t \le \tau(\varepsilon)/20$ and $\eta_{1,\tau(\varepsilon)} = 0$ if $t \ge \tau(\varepsilon)/10$. Hence we can find two positive constants $m_1 < m_2$, both independent of ε, such that the support of $|\nabla \eta_{1,\tau(\varepsilon)}|$ is contained in a set $\{x \in \Omega : m_1\tau(\varepsilon) \le t(x) \le m_2\tau(\varepsilon)\}$ for all small ε. Thus

$$\begin{aligned}\int_\Omega |\varphi_\varepsilon \nabla \eta_{1,\tau(\varepsilon)}|^2 dx &\le \frac{c_1}{\tau(\varepsilon)^2}\int_{\{m_1\tau(\varepsilon)\le t(x)\le m_2\tau(\varepsilon)\}} |\varphi_\varepsilon|^2 dx \\ \le \frac{c_2}{\tau(\varepsilon)^4}\int_{\{m_1\tau(\varepsilon)\le t(x)\le m_2\tau(\varepsilon)\}} & t(x)^2|\varphi_\varepsilon|^2 dx \le \frac{c_3\varepsilon^2}{\tau(\varepsilon)^4}\|\varphi_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.23}$$

Here we have used the first inequality in Lemma 3.3 with $k = 2$.

Step 3. Now we can derive a lower bound for the eigenvalue μ_ε.

As in Step 2, we choose $\tau(\varepsilon) = \varepsilon^{5/12}$. From (3.10) we see that $\|w_\varepsilon\|_{L^2(\Omega)} \le \|\varphi_\varepsilon\|_{L^2(\Omega)}$. Using (3.12), (3.22) and (3.23) we have :

$$\begin{aligned}\mu_\varepsilon\|w_\varepsilon\|^2_{L^2(\Omega)} &= Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[w_\varepsilon] - \int_\Omega |\varphi_\varepsilon \nabla\eta_{1,\tau(\varepsilon)}|^2 dx \\ &\ge \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3})\|w_\varepsilon\|^2_{L^2(\Omega)} - \frac{C\varepsilon^2}{\tau(\varepsilon)^4}\|\varphi_\varepsilon\|^2_{L^2(\Omega)} \\ &\ge \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3} - C\varepsilon^{7/3})\|w_\varepsilon\|^2_{L^2(\Omega)},\end{aligned}$$

which yields a lower bound for μ_ε :

$$\mu_\varepsilon \ge \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3}).$$

Now (3.2) is proved. □

Remark 3.6. (1). Lemmas 3.2-3.5 remain true under a weaker condition

$$\begin{aligned}&\|\mathbf{A}^\varepsilon\|_{C^2(\overline{\Omega})} \le C, \\ &\|\text{curl } \mathbf{A}^\varepsilon - 1\|_{C^1(\overline{\Omega})} = o(1), \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{3.1$'$}$$

(2). We can show that, under Condition (3.1), the eigenfunction φ_ε decays exponentially inside the boundary and away from the set of the maximum points of the curvature. Since the proof is similar to the nonlinear case which will be discussed in Section 6, we omit it here.

From the proof above we see that the conclusion of [HMor] (Proposition 10.5) remains true.

Proposition 3.7. *Under Assumption (3.1), there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$ *and* $\phi \in W^{1,2}(\Omega)$*, we have :*

$$Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\phi] \ge \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(x)|\phi|^2 dx, \tag{3.24}$$

where

$$W_\varepsilon(x) = \begin{cases} 1 - C\varepsilon^{1/3} & \textit{if } \textit{dist }(x,\partial\Omega) > 2\varepsilon^{1/3}, \\ \beta_0 - C_1\kappa_r(s)\varepsilon - C\varepsilon^{4/3} & \textit{if } \textit{dist }(x,\partial\Omega) \le 2\varepsilon^{1/3}. \end{cases} \tag{3.25}$$

§4. ELLIPTIC ESTIMATES

In this section we shall derive elliptic estimates for the minimal solutions of the Ginzburg-Landau system (2.2) when the applied field is close to H_{C_3}, namely when we assume

$$(\frac{1}{\beta_0} + o(1))\kappa \leq \sigma < H_{C_3}(\kappa). \tag{4.1}$$

As before we use the notation : $\varepsilon = \dfrac{1}{\sqrt{\kappa\sigma}}$. Using Conclusion (1) of Theorem A, we see that Condition (4.1) implies

$$\varepsilon^2\kappa^2 = \beta_0 + o(1), \qquad \text{as } \varepsilon \to 0. \tag{4.2}$$

In the following, we denote by :

$(\psi^\varepsilon, \mathbf{A}^\varepsilon)$: a non-trivial minimal solution of the Ginzburg-Landau system (2.2),
x^ε : a maximum point of $|\psi^\varepsilon(x)|$,

and we define λ_ε by

$$\lambda_\varepsilon = \|\psi^\varepsilon\|_{L^\infty(\Omega)} = |\psi^\varepsilon(x^\varepsilon)|.$$

We always fix the gauge and assume that $(\psi^\varepsilon, A^\varepsilon)$ satisfies (2.3).

Proposition 4.1. *Assume that Condition (4.1) holds. Then we have :*

$$\textit{dist }(x^\varepsilon, \partial\Omega) = o(\varepsilon) \qquad \textit{and} \qquad \|\psi^\varepsilon\|_{L^\infty(\Omega)} = o(1) \qquad \textit{as } \varepsilon \to 0.$$

For the proof, see [LP4] (Theorem 4.1 and the last paragraph of the proof of Lemma 4.5).

Proposition 4.2. *Under Condition (4.1), there are positive constants C and ε_0 such that, for all $0 < \varepsilon \leq \varepsilon_0$, we have :*

(1) $\|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{C(\overline{\Omega})} \leq \dfrac{C}{\varepsilon}\|\psi^\varepsilon\|_{L^\infty(\Omega)}$;

(2) $\|\textit{curl } \mathbf{A}^\varepsilon - 1\|_{C^1(\overline{\Omega})} \leq C\varepsilon\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}$;

(3) $\|\textit{curl } \mathbf{A}^\varepsilon - 1\|_{C^2(\overline{\Omega})} \leq C\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}$.

(4) *For any multi-index α with $|\alpha| \leq 2$, there exist positive constants $c_1(\alpha)$, $c_2(\alpha)$ such that, for all $0 < \varepsilon < \varepsilon_0$, we have :*

$$|D^\alpha\psi^\varepsilon(x)| \leq \frac{c_1(\alpha)}{\varepsilon^{|\alpha|}}\exp\left(-\frac{c_2(\alpha)}{\varepsilon}\ \textit{dist }(x, \partial\Omega)\right)\|\psi^\varepsilon\|_{L^\infty(\Omega)}, \qquad \textit{for all } x \in \Omega.$$

Proof. Step 1. We prove (1) by contradiction.

Choose $P^\varepsilon \in \overline{\Omega}$ such that

$$|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon(P^\varepsilon)| = \|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{L^\infty(\Omega)}.$$

Suppose Conclusion (1) were false. Then there exists a sequence $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} \frac{\varepsilon_j}{\lambda_{\varepsilon_j}} \|\nabla_{\frac{1}{\varepsilon_j^2}\mathbf{A}^{\varepsilon_j}} \psi^{\varepsilon_j}\|_{L^\infty(\Omega)} = \infty. \tag{4.3}$$

We shall find a contradiction. Passing to a subsequence if necessary, we deal with two cases, namely, Case 1 where P^{ε_j} stays away from the boundary for all j, and Case 2 where P^{ε_j} is close to the boundary as $j \to \infty$. In the following, for the simplicity of notations, we denote ε_j by ε, and consider a "sequence" $\{\varepsilon\}$.

Case 1. $\lim\limits_{\varepsilon\to 0} \dfrac{1}{\varepsilon}$ dist $(P^\varepsilon, \partial\Omega) = \infty$.

We shall show that, in this case, as $\varepsilon \to 0$,

$$\|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon} \psi^\varepsilon\|_{L^\infty(\Omega)} = o(\frac{\lambda_\varepsilon}{\varepsilon}). \tag{4.4}$$

Define the rescaled functions as follows :

$$\begin{aligned} \mathbf{A}_\varepsilon(y) &= \frac{1}{\varepsilon}[\mathbf{A}^\varepsilon(P^\varepsilon + \varepsilon y) - \mathbf{A}^\varepsilon(P^\varepsilon)], \\ \phi_\varepsilon(y) &= \frac{1}{\lambda_\varepsilon} \exp(-\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(P^\varepsilon)\cdot y)\psi^\varepsilon(P^\varepsilon + \varepsilon y). \end{aligned}$$

From (2.2), ϕ_ε satisfies

$$\begin{cases} -\nabla^2_{\mathbf{A}_\varepsilon}\phi_\varepsilon = \varepsilon^2\kappa^2(1 - \lambda_\varepsilon^2|\phi_\varepsilon|^2)\phi_\varepsilon, \\ \text{curl}^2\mathbf{A}_\varepsilon = \varepsilon^2\lambda_\varepsilon^2 \Im(\overline{\phi_\varepsilon}\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon), \qquad \text{in } (\Omega - P^\varepsilon)/\varepsilon. \end{cases} \tag{4.5}$$

Recall that $\varepsilon^2\kappa^2 \to \beta_0 < 1$ as $\varepsilon \to 0$. Using (4.2) and applying the argument in [LP4] (Lemma 4.2), we can show that, for any $R > 0$, there exists $\varepsilon_0(R) > 0$ such that, for $0 < \varepsilon < \varepsilon_0(R)$,

$$\|\mathbf{A}_\varepsilon\|_{C^{2,\alpha}(B_R)} \le C(R).$$

We also obtain (see [LP4], Lemma 4.4) : $\phi_\varepsilon \to 0$ in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Therefore,

$$\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon \to 0 \qquad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^2), \quad \text{as } \varepsilon \to 0.$$

Here we mean by convergence in $C^{k,\alpha}_{loc}$, that for any compact subset $K \subset \mathbb{R}^2$, there exists $\varepsilon_K > 0$ such that we have convergence of ϕ_ε (resp. $\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon$) in $C^{k,\alpha}(K)$ as $\varepsilon \to 0$ ($0 < \varepsilon < \varepsilon_K$). Hence, as $\varepsilon \to 0$,

$$\|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{L^\infty(\Omega)} = |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon(P^\varepsilon)| = \frac{\lambda_\varepsilon}{\varepsilon}|\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon(0)| = o(\frac{\lambda_\varepsilon}{\varepsilon}).$$

So (4.4) holds.

Case 2. dist $(P^\varepsilon, \partial\Omega) \le C\varepsilon$ as $\varepsilon \to 0$.

We shall show that, in this case, as $\varepsilon \to 0$, we have :

$$\|\nabla_{\frac{1}{\varepsilon^2}A^\varepsilon}\psi^\varepsilon\|_{L^\infty(\Omega)} = O(\frac{\lambda_\varepsilon}{\varepsilon}). \tag{4.6}$$

Choose $Q^\varepsilon \in \partial\Omega$ such that $|P^\varepsilon - Q^\varepsilon| = \text{ dist } (P^\varepsilon, \partial\Omega)$. Let $\mathcal{F}_\varepsilon$ be the diffeomorphism which straightens a portion of the boundary around Q^ε as defined in [LP4] (Section 2) with $\mathcal{F}_\varepsilon(0) = Q^\varepsilon$. Let us define $y^\varepsilon = \mathcal{F}_\varepsilon^{-1}(x^\varepsilon)$ and $z^\varepsilon = \frac{y^\varepsilon}{\varepsilon}$. Note that $\{z^\varepsilon\}$ is bounded as $\varepsilon \to 0$. Without loss of generality, we may assume $\lim_{\varepsilon\to 0} z^\varepsilon = z^0$. Note that, although $\mathcal{F}_\varepsilon$ depends on ε, its domain contains a ball R_{R_0} with R_0 independent of ε, and both $\mathcal{F}_\varepsilon$ and $\det D\mathcal{F}_\varepsilon$ are uniformly smooth on this ball. For simplicity, we write now $\mathcal{F} = \mathcal{F}_\varepsilon$ and $g = \det D\mathcal{F}_\varepsilon$. Let $\tilde{\psi}^\varepsilon(y) = \psi^\varepsilon(\mathcal{F}(y))$ and let

$$\mathbf{a}^\varepsilon = [g\mathbf{A}^\varepsilon \cdot \mathbf{e}_1]\mathbf{e}_1 + [\mathbf{A}^\varepsilon \cdot \mathbf{e}_2]\mathbf{e}_2$$

be the vector field associated with $\mathbf{A}^\varepsilon$, see [LP4] ((2.22) and (2.23)). Note that $\mathbf{a}^\varepsilon(0) = \mathbf{A}^\varepsilon(Q^\varepsilon)$, $\mathbf{a}^\varepsilon(0)\cdot\mathbf{e}_2 = 0$. Next we define the following rescaled functions

$$\begin{aligned}\chi_\varepsilon &= \frac{1}{\varepsilon} y_1 \mathbf{A}^\varepsilon(Q^\varepsilon)\cdot\mathbf{e}_1,\\ \tilde{\phi}_\varepsilon(y) &= \frac{1}{\lambda_\varepsilon}\exp(-i\chi_\varepsilon)\tilde{\psi}^\varepsilon(\varepsilon y),\\ \mathbf{a}_\varepsilon(y) &= \frac{1}{\varepsilon}[\mathbf{a}^\varepsilon(\varepsilon y) - \mathbf{a}^\varepsilon(0)],\\ g_\varepsilon(y) &= g(\varepsilon y).\end{aligned}$$

Using (2.2) and the gauge invariance property, we get the equation for $\tilde{\phi}_\varepsilon$:

$$\begin{cases} -\Delta(g_\varepsilon)_{\mathbf{a}_\varepsilon}\tilde{\phi}_\varepsilon = \varepsilon^2\kappa^2(1-\lambda_\varepsilon^2|\tilde{\phi}_\varepsilon|^2)\tilde{\phi}_\varepsilon & \text{in } B^+_{R_0/\varepsilon},\\ \partial_2\tilde{\phi}_\varepsilon = 0 & \text{on } \Gamma_{R_0/\varepsilon}.\end{cases}$$

As in [LP4] (Section 4), we use (4.2) to show that, there is a subsequence (still denoted by $\tilde{\phi}_\varepsilon$) such that $\tilde{\phi}_\varepsilon \to \tilde{\phi}$ and $\mathbf{a}_\varepsilon \to \omega(y)$ in $C^{2+\alpha}_{loc}(\mathbb{R}^2_+)$ as $\varepsilon \to 0$, and $\tilde{\phi}$ is a bounded solution of (2.6). Therefore, as $\varepsilon \to 0$,

$$D(g_\varepsilon)_{\mathbf{a}_\varepsilon}\tilde{\phi}_\varepsilon \to \nabla_\omega\tilde{\phi} \quad \text{in } C^{1+\alpha}_{loc}(\mathbb{R}^2_+).$$

Especially, $D(g_\varepsilon)_{\mathbf{a}_\varepsilon}\tilde{\phi}_\varepsilon(0) \to \nabla_\omega\tilde{\phi}(0)$, so $\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon(0) \to \nabla_\omega\tilde{\phi}(0)$. Hence, as $\varepsilon \to 0$,

$$\|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{L^\infty(\Omega)} = |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon(P^\varepsilon)| = \frac{\lambda_\varepsilon}{\varepsilon}|\nabla_{\mathbf{A}_\varepsilon}\phi_\varepsilon(0)| = O(\frac{\lambda_\varepsilon}{\varepsilon}).$$

So (4.6) is true.

Combining (4.4) and (4.6) we get a contradiction to (4.3). So Conclusion (1) is proved.

Step 2. We prove (2).

From the second equation of (2.2) and using Conclusion (1) we have :

$$\begin{aligned}
&\|\nabla[\mathrm{curl}(\mathbf{A}^\varepsilon-\mathbf{F})]\|_{L^\infty(\Omega)}=\|\mathrm{curl}^2(\mathbf{A}^\varepsilon-\mathbf{F})\|_{L^\infty(\Omega)}\\
=&\varepsilon^2\|\Im(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)}\le\varepsilon^2\|\psi^\varepsilon\|_{L^\infty(\Omega)}\|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon\|_{L^\infty(\Omega)}\\
\le&\varepsilon^2\|\psi^\varepsilon\|_{L^\infty(\Omega)}\cdot\frac{C}{\varepsilon}\|\psi^\varepsilon\|_{L^\infty(\Omega)}=C\varepsilon\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}.
\end{aligned}$$

Recall that $\mathrm{curl}(\mathbf{A}^\varepsilon-\mathbf{F})=0$ on $\partial\Omega$. The above inequalities imply that

$$\|\mathrm{curl}(\mathbf{A}^\varepsilon-\mathbf{F})\|_{L^\infty(\Omega)}\le C'\|\nabla[\mathrm{curl}(\mathbf{A}^\varepsilon-\mathbf{F})]\|_{L^\infty(\Omega)}\le CC'\varepsilon\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}.$$

Therefore

$$\|\mathrm{curl}(\mathbf{A}^\varepsilon-\mathbf{F})\|_{C^1(\overline{\Omega})}\le C''\varepsilon\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}.$$

Hence Conclusion (2) holds (replacing C by a larger number).

Step 3. We prove (3).

Claim 1: Under Condition (4.1), there exist positive constants M and ε_0 such that, for all $0<\varepsilon<\varepsilon_0$, we have :

$$\|\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)}\le\frac{C}{\varepsilon^2}\|\psi^\varepsilon\|^2_{L^\infty(\Omega)}. \tag{4.7}$$

The proof of Claim 1 is similar to the proof of Conclusion (1). Choose $\hat{P}^\varepsilon\in\overline{\Omega}$ such that

$$|[\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)](\hat{P}^\varepsilon)|=\|\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)}.$$

Define the rescaled functions as follows :

$$\begin{aligned}
\hat{\mathbf{A}}_\varepsilon(y)&=\frac{1}{\varepsilon}[\mathbf{A}^\varepsilon(\hat{P}^\varepsilon+\varepsilon y)-\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)],\\
\hat{\phi}_\varepsilon(y)&=\frac{1}{\lambda_\varepsilon}\exp(-\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\cdot y)\psi^\varepsilon(\hat{P}^\varepsilon+\varepsilon y).
\end{aligned}$$

$\hat{\phi}_\varepsilon$ satisfies an equation similar to (4.5) in $\Omega(\varepsilon)=(\Omega-\hat{P}^\varepsilon)/\varepsilon$. As in Step 1 we can show that, there exist $M>0$ and $\varepsilon_0>0$ such that, for any $0<\varepsilon<\varepsilon_0$, we have :

$$\|\nabla(\overline{\hat{\phi}_\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)\|_{L^\infty(\Omega(\varepsilon))}\le M.$$

We compute

$$\begin{aligned}
&\psi^\varepsilon(\hat{P}^\varepsilon+\varepsilon y)=\lambda_\varepsilon\hat{\phi}_\varepsilon(y)\exp(\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\cdot y),\\
&(\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)(\hat{P}^\varepsilon+\varepsilon y)=\frac{\lambda_\varepsilon}{\varepsilon}\exp(\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\cdot y)(\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)(y),\\
&(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)(\hat{P}^\varepsilon+\varepsilon y)=\frac{\lambda_\varepsilon^2}{\varepsilon}(\overline{\hat{\phi}_\varepsilon}\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)(y),\\
&\frac{\partial}{\partial x_j}(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)(\hat{P}^\varepsilon+\varepsilon y)=\frac{\lambda_\varepsilon^2}{\varepsilon^2}\frac{\partial}{\partial y_j}(\overline{\hat{\phi}_\varepsilon}\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)(y).
\end{aligned}$$

So

$$\begin{aligned}&\|\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)} = |[\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)](\hat{P}^\varepsilon)|\\ =&\frac{\lambda_\varepsilon^2}{\varepsilon^2}|\nabla(\overline{\hat{\phi}_\varepsilon}\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)(y)| \le \frac{M\lambda_\varepsilon^2}{\varepsilon^2}.\end{aligned}$$

Claim 1 follows.

Note that

$$\mathrm{curl}^2(\mathbf{A}^\varepsilon - \mathbf{F}) = (\partial_2[\mathrm{curl}(\mathbf{A}^\varepsilon - \mathbf{F})], -\partial_1[\mathrm{curl}(\mathbf{A}^\varepsilon - \mathbf{F})]).$$

Hence, using Claim 1 and the second equation of (2.2), we have :

$$\begin{aligned}&\|\partial_{ij}[\mathrm{curl}(\mathbf{A}^\varepsilon - \mathbf{F})]\|_{L^\infty(\Omega)} \le \|\partial_j[\mathrm{curl}^2(\mathbf{A}^\varepsilon - \mathbf{F})]\|_{L^\infty(\Omega)}\\ =&\varepsilon^2\|\partial_j\Im(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)} = \varepsilon^2\|\Im\partial_j(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)}\\ \le&\varepsilon^2\|\partial_j(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)} \le M\lambda_\varepsilon^2.\end{aligned}$$

Combining this and Conclusion (2) we obtain Conclusion (3).

Step 4. Conclusion (4) can be proved as in [DFS] (Theorem 4.3) and [HMor] (Theorem 6.3). In fact, Conclusion (2) implies that, as $\varepsilon \to 0$, $\|\mathrm{curl}\ \mathbf{A}^\varepsilon - 1\|_{C(\overline{\Omega})} = o(1)$. Then applying the arguments in [DFS] or [HMor] to the equation for ψ^ε one obtains Conclusion (4).

Now, Proposition 4.2 is completed. $\square$

Remark 4.3. (1). From Step 1 of the proof of Proposition 4.2 we see that, for any sequence $\rho_\varepsilon \to \infty$,

$$\lim_{\varepsilon\to 0}\ \max_{\mathrm{dist}\ (x,\partial\Omega)\ge\varepsilon\rho_\varepsilon} \frac{\varepsilon}{\lambda_\varepsilon}|\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon(x)| = 0.$$

(2). Inequality (4.7) says that $|\nabla(\overline{\psi^\varepsilon}\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)|$ is of order $O(\lambda_\varepsilon^2/\varepsilon^2)$. However, we can not conclude that $|\nabla(\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)|$ is also of order $O(\lambda_\varepsilon/\varepsilon^2)$. In fact, from the computations in Step 3 of the proof of Proposition 4.2 we see that

$$\begin{aligned}&[\partial_j(\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)](\hat{P}^\varepsilon + \varepsilon y) = \frac{i\lambda_\varepsilon}{\varepsilon^3}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\exp(\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\cdot y)(\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)(y)\\ &\quad + \frac{\lambda_\varepsilon}{\varepsilon^2}\exp(\frac{i}{\varepsilon}\mathbf{A}^\varepsilon(\hat{P}^\varepsilon)\cdot y)[\partial_j(\nabla_{\hat{\mathbf{A}}_\varepsilon}\hat{\phi}_\varepsilon)](y).\end{aligned}$$

Thus, as $\varepsilon \to 0$, we have :

$$\|\nabla(\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\psi^\varepsilon)\|_{L^\infty(\Omega)} = O(\frac{\lambda_\varepsilon}{\varepsilon^3}).$$

The following Theorem will be used in Section 5 to establish the lower bound of $H_{C_3}(\kappa)$.

Theorem 4.4. *Assume that Condition (4.1) holds and that* $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$ *is a minimal solution of (2.2). Then there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$*, the lowest eigenvalue* $\mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon)$ *of the Neumann Schrödinger operator* $-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}$ *satisfies the following inequality :*

$$\frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3}) \le \mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon) \le \kappa^2. \tag{4.8}$$

Here C_1 *is the number given in (2.7).*

Proof. Since $\sigma < H_{C_3}(\kappa)$, (2.2) has a non-trivial minimal solution $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$. From Lemma 2.1 (2) we have

$$\mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon) < \kappa^2$$

for all $\varepsilon > 0$.

From Proposition 4.2 we see that the family $\{\mathbf{A}^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ satisfies Condition (3.1). Using Theorem 3.1 we conclude that, the lowest eigenvalue $\mu_\varepsilon = \mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon)$ satisfies

$$\mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon) \ge \frac{1}{\varepsilon^2}(\beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3})$$

for all $0 < \varepsilon < \varepsilon_0$. Hence (4.8) holds. □

Next we give L^2 estimates for $|\psi^\varepsilon|$, which will be used in Section 6. The conclusions indicate that, if the applied field is close to the upper critical field in the sense of Condition (4.1), then the order parameter concentrates in a thin layer around the boundary $\partial\Omega$.

Lemma 4.5. *Assume that Condition (4.1) holds and* $\mathbf{F}$ *satisfies (1.3). Let* $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$ *be a minimal solution of (2.2). There exist positive constants* M*,* c *and* ε_0 *such that, for all* $0 < \varepsilon \le \varepsilon_0$*, we have :*

$$\int_\Omega |\psi^\varepsilon|^2 dx \le M \int_{\{\text{dist }(x,\partial\Omega)\le c\varepsilon\}} |\psi^\varepsilon|^2 dx. \tag{4.9}$$

Proof. Note that the weighted L^2 estimate (3.4) is valid for ψ^ε. Especially we have

$$\int_{\{\text{dist }(x,\partial\Omega)\ge c\varepsilon\}} |\psi^\varepsilon|^2 dx \le Ce^{-\alpha c} \int_\Omega |\psi^\varepsilon|^2 dx.$$

Hence

$$(1 - Ce^{-\alpha c}) \int_{\{\text{dist }(x,\partial\Omega)\ge c\varepsilon\}} |\psi^\varepsilon|^2 dx \le Ce^{-\alpha c} \int_{\{\text{dist }(x,\partial\Omega)< c\varepsilon\}} |\psi^\varepsilon|^2 dx.$$

Choose $c > 0$ such that $1 - Ce^{-\alpha c} \ge 1/2$. From the above inequality we find :

$$\int_\Omega |\psi^\varepsilon|^2 dx \le \frac{1}{(1 - Ce^{-\alpha c})} \int_{\{\text{dist }(x,\partial\Omega)< c\varepsilon\}} |\psi^\varepsilon|^2 dx.$$

So (4.9) holds with $M = 1/(1 - Ce^{-\alpha c})$. □

§5. UPPER BOUND ESTIMATES FOR $H_{C_3}(\kappa)$

In this section we keep the notations of Section 1.

Proposition 5.1. *Assume that Ω is a bounded simply-connected 2-dimensional domain of class C^4. Then we have, for κ large,*

$$H_{C_3}(\kappa) \le \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + O(\kappa^{-1/3}). \tag{5.1}$$

Proof. We shall prove the following conclusion : For any σ satisfying $0 < \sigma < H_{C_3}(\kappa)$, we have

$$\sigma < \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + O(\kappa^{-1/3}). \tag{5.2}$$

(5.1) follows from this conclusion.

From the lower bound of H_{C_3}, we only need to consider such σ's that

$$\frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - \frac{L}{\kappa^{1/3}} < \sigma < H_{C_3}(\kappa), \tag{5.3}$$

with $L > 0$ large. For σ satisfying (5.3), Condition (4.1) holds. Ginzburg-Landau system (2.2) has a non-trivial minimal solution $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$. From Theorem 4.4, for all $0 < \varepsilon < \varepsilon_0$, the lowest eigenvalue $\mu_\varepsilon = \mu(\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon)$ satisfies (4.8). From (4.8) and (5.3) we have :

$$\frac{\kappa}{\sigma} = \varepsilon^2\kappa^2 > \beta_0 - C_1\kappa_{max}\varepsilon - C\varepsilon^{4/3}. \tag{5.4}$$

Inequality (5.4) implies that

$$\frac{1}{\sigma} > \frac{\beta_0}{\kappa} - \frac{C_1}{\sigma^{1/2}\kappa^{3/2}}\kappa_{max} - \frac{C}{\sigma^{2/3}\kappa^{5/3}}, \tag{5.5}$$

and

$$\frac{1}{\sigma} = \frac{\beta}{\kappa} - O(\frac{1}{\kappa^{3/2}}). \tag{5.6}$$

Plugging (5.6) into the right side of (5.5), we get

$$\frac{1}{\sigma} > \frac{\beta_0}{\kappa} - \frac{C_1\sqrt{\beta_0}}{\kappa^2}\kappa_{max} - O(\frac{1}{\kappa^{7/3}}).$$

Thus σ must satisfy (5.2). So (5.1) holds. □

Proof of Theorem 1.1. Theorem 1.1 follows from Propositions 2.3 and 5.1. □

§6. Localization of Order Parameters

In this section we assume that Ω is a bounded simply-connected 2-dimensional domain of class C^4 and discuss the concentration phenomena of order parameters when the applied field is close to $H_{C_3}(\kappa)$ and κ is large. We keep the notation $\varepsilon = 1/\sqrt{\sigma\kappa}$ and denote by $(\psi^\varepsilon, \mathbf{A}^\varepsilon)$ a minimal solution of (2.2). Let :

$$\lambda_\varepsilon = \|\psi^\varepsilon\|_{L^\infty(\Omega)}, \qquad \varphi^\varepsilon = \frac{\psi^\varepsilon}{\lambda_\varepsilon}.$$

Recall that we have defined the tangential distance $d_t(x(s,t), \mathcal{N}(\partial\Omega))$ and have introduced the notations Ω_δ and $\mathcal{N}(\partial\Omega)$ (see Section 1).

Theorem 6.1. *Let*

$$\rho = \frac{\beta_0^{3/2}}{C_1}[H_{C_3}(\kappa) - \sigma].$$

(1) *Assume that* $0 < \rho \le L_1\kappa^{-a}$, *where* $0 < a \le \frac{1}{3}$ *and* L_1 *is a positive constant. Then there exist positive constants* α_1, c_1, l_1, ε_1 *and* M_1 *such that, for all* $0 < \varepsilon < \varepsilon_1$, *we have :*

$$\int_\Omega |\psi^\varepsilon|^2 \exp(\frac{\alpha_1}{\sqrt{\varepsilon}}[\kappa_{max} - \kappa_r(s) - l_1\varepsilon^a])dx \le M_1 \int_{\Omega_{c_1\varepsilon}} |\psi^\varepsilon|^2 dx. \tag{6.1}$$

(2). *Assume that* $L_2\kappa^{-1/3} \le \rho = o(1)$, *where* L_2 *is a positive constant. Then there exist positive constants* α_2, c_2, l_2, ε_2 *and* M_2 *such that, for all* $0 < \varepsilon < \varepsilon_2$, *we have :*

$$\int_\Omega |\psi^\varepsilon|^2 \exp(\frac{\alpha_2}{\sqrt{\varepsilon}}[\kappa_{max} - \kappa_r(s) - l_2\rho])dx \le M_2 \int_{\Omega_{c_2\varepsilon}} |\psi^\varepsilon|^2 dx. \tag{6.2}$$

(3) *Assume that* Ω *is not a ball, and let* $\varepsilon_0 < \rho < \kappa_{max} - \kappa_{min}$. *Then, there exist positive constants* α_3, c_3, M_3 *and* ε_3 *such that, for all* $0 < \varepsilon < \varepsilon_3$, *we have :*

$$\int_\Omega |\psi^\varepsilon|^2 \exp(\frac{\alpha_3}{\sqrt{\varepsilon}}[\kappa_{max} - \kappa_r(s) - \rho]_+^{3/2})dx \le \frac{M_3}{\varepsilon^{1/3}} \int_{\Omega_{c_3\varepsilon}} |\psi^\varepsilon|^2 dx. \tag{6.3}$$

In the proof of Theorem 6.1 we need the following notations. For $0 < \delta < \delta_0$, $\mu > 0$, we define :

$$\Gamma(\delta,\mu) = \{x \in \overline{\Omega} : \text{ dist } (x, \partial\Omega) < \delta, \ \kappa_r(s) > \kappa_{max} - \mu\},$$
$$\gamma(z) = \sup\{\text{dist } (x(s), \mathcal{N}(\partial\Omega)) : x(s) \in \partial\Omega, \ \kappa_r(s) \ge \kappa_{max} - z\}, \qquad \text{for } z > 0.$$

If Ω is not a ball and if $\kappa_{min} \le \beta \le \kappa_{max}$, we define :

$$\Gamma_\beta(\delta,\mu) = \{x \in \overline{\Omega} : \text{ dist } (x, \partial\Omega) < \delta, \ \kappa_r(s) > \beta - \mu\}.$$

Lemma 6.2. *Assume that Ω is a bounded domain of class C^4 in $\mathbb{R}^2$ and $\phi_0(s) = \kappa_{max} - \kappa_r(s)$. Then we have the following conclusions.*

1. *There exists a positive constant C such that $|\phi_0'(s)|^2 \leq C\phi_0(s)$.*
2. *Assume that $\mathcal{N}(\partial\Omega)$ consists of non-degenerate maximum points of the curvature. Then there exists a constant $\gamma_0 > 0$ such that $\gamma(z) \leq \gamma_0\sqrt{z}$ for $0 < z < z^*$.*

Proof. Conclusion (1) is a standard result. We prove Conclusion (2). Assume that $\mathcal{N}(\partial\Omega)$ consists of non-degenerate maximum points $x(s_1), \cdots, x(s_n)$ of the curvature. Then we can write

$$\phi_0(s) = \rho_0(s)(s - s_1)^2 \cdots (s - s_n)^2,$$

where $\rho_0(s)$ is a positive continuous function. Hence, we can find a positive constant $c > 0$ such that $\phi_0(s) \geq c d_t(x(s), \mathcal{N}(\partial\Omega))^2$. So the conclusion is true. ☐

Proof of Theorem 6.1. *Step 1.* If $0 < H_{C_3}(\kappa) - \sigma < C$ for large κ, then the number $t(\varepsilon)$ given in (2.13) satisfies

$$\frac{t(\varepsilon)}{\varepsilon} = \beta_0^{3/2}[H_{C_3}(\kappa) - \sigma] + O(\varepsilon^{1/3}), \qquad \text{for small } \varepsilon. \tag{6.4}$$

This follows from (1.5) and the estimate

$$\mu(\frac{1}{\varepsilon^2}\mathbf{F}) = \frac{1}{\varepsilon^2}[\beta_0 - C_1\kappa_{max}\varepsilon + O(\varepsilon^{4/3})] \tag{6.5}$$

obtained in [HMor] (Theorem 11.1). In fact, since $\sigma = H_{C_3}(\kappa) + O(1) = \frac{\kappa}{\beta_0} + O(1)$ and $\varepsilon = 1/\sqrt{\sigma\kappa}$, we have $\varepsilon\sigma = \frac{1}{\sqrt{\beta_0}} + O(\varepsilon)$. Using (6.5) and (1.5) we get

$$\begin{aligned}\frac{t(\varepsilon)}{\varepsilon} &= \frac{1}{\varepsilon}[\varepsilon^2\kappa^2 - \varepsilon^2\mu(\frac{1}{\varepsilon^2}\mathbf{F})] = \frac{\beta_0}{\varepsilon\sigma}[\frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0}\kappa_{max}\varepsilon\sigma - \sigma] + O(\varepsilon^{1/3}) \\ &= \frac{\beta_0}{\varepsilon\sigma}[\frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - \sigma] + O(\varepsilon^{1/3}) = \beta_0^{3/2}[H_{C_3}(\kappa) - \sigma] + O(\varepsilon^{1/3}).\end{aligned}$$

Step 2. We prove an integral inequality.

Let χ be a C^1 function. As in (3.8) we have :

$$\begin{aligned}Q_{\varepsilon^2,\mathbf{A}^\varepsilon}[\chi\varphi^\varepsilon] &\equiv \int_\Omega |\nabla_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}(\chi\varphi^\varepsilon)|^2 dx = \Re\int_\Omega(-\nabla^2_{\frac{1}{\varepsilon^2}\mathbf{A}^\varepsilon}\varphi^\varepsilon)\overline{(\chi^2\varphi^\varepsilon)}dx + \int_\Omega |\varphi^\varepsilon\nabla\chi|^2 dx \\ &= \kappa^2\int_\Omega(1 - \lambda_\varepsilon^2|\varphi^\varepsilon|^2)|\chi\varphi^\varepsilon|^2 dx + \int_\Omega |\varphi^\varepsilon\nabla\chi|^2 dx.\end{aligned}$$

Using this and Proposition 3.7 we get :

$$\int_\Omega W_\varepsilon(x)\chi^2|\varphi^\varepsilon|^2 dx \leq \int_\Omega \{\varepsilon^2\kappa^2(1 - \lambda_\varepsilon^2|\varphi^\varepsilon|^2)|\chi\varphi^\varepsilon|^2 + \varepsilon^2|\varphi^\varepsilon\nabla\chi|^2 dx.$$

Let $\chi = \exp(\frac{\alpha\phi}{\sqrt{\varepsilon}})$, where α and ϕ are to be chosen later. We get :

$$\int_\Omega |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})\{W_\varepsilon(x) - \varepsilon^2\kappa^2 + \lambda_\varepsilon^2\varepsilon^2\kappa^2|\varphi^\varepsilon|^2 - \alpha^2\varepsilon|\nabla\phi|^2\}dx \le 0. \tag{6.6}$$

The term involving $|\varphi^\varepsilon|^4$ in the left hand side of the above inequality has a positive sign, and will be neglected in the following (so the nonlinear effect is neglected, although this term could be useful to estimate the order of λ_ε as $\varepsilon \to 0$).

In terms of the number $t(\varepsilon)$ defined in (2.13), we derive from (6.6) the following inequality :

$$\int_\Omega |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})\{[W_\varepsilon(x) - \varepsilon^2\mu(\frac{1}{\varepsilon^2}\mathbf{F})] - t(\varepsilon) - \alpha^2\varepsilon|\nabla\phi|^2\}dx \le 0.$$

Now, in $\Omega_{2\varepsilon^{1/3}}$, using the upper bound (2.9) of $\mu(\frac{1}{\varepsilon^2}\mathbf{F})$, the lower bound (3.25) of $W_\varepsilon(x)$, we have :

$$W_\varepsilon(x) - \mu(\frac{1}{\varepsilon^2}\mathbf{F}) \ge C_1\varepsilon[\kappa_{max} - \kappa_r(s)] - C'\varepsilon^{4/3},$$

where C' is a positive constant independent of ε. In $\Omega \setminus \Omega_{2\varepsilon^{1/3}}$ we have, for some constant $m > (1-\beta_0)/2$,

$$W_\varepsilon(x) - \varepsilon^2\mu(\frac{1}{\varepsilon^2}\mathbf{F}) - t(\varepsilon) - \alpha^2\varepsilon|\nabla\phi|^2 \ge 1 - \beta_0 + o(1) \ge m.$$

Therefore, we have :

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})\{C_1[\kappa_{max} - \kappa_r(s)] - \frac{t(\varepsilon)}{\varepsilon} - C'\varepsilon^{1/3} - \alpha^2|\nabla\phi|^2\}dx \\ &+ \frac{m}{\varepsilon}\int_{\{\text{dist }(x,\partial\Omega)\ge 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})dx \le 0.\end{aligned} \tag{6.7}$$

Step 3. We consider Case (1).

Set $\phi(x) = \phi_0(x) = \phi_0(s,t) = \phi_0(s) = \kappa_{max} - \kappa_r(s)$. From Conclusion (1) of Lemma 6.2, we have $|\nabla\phi_0(x)|^2 \le C\phi_0(x)$. Choose α such that $0 < \alpha \le \sqrt{\frac{C_1}{2C}}$. From (6.7) we have :

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))\{\frac{C_1}{2}\phi_0(x) - \frac{t(\varepsilon)}{\varepsilon} - C'\varepsilon^{1/3}\}dx \\ &+ \frac{m}{\varepsilon}\int_{\{\text{dist }(x,\partial\Omega)\ge 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \le 0.\end{aligned} \tag{6.8}$$

Suppose that $0 < \rho \le L_1'\varepsilon^{1/3}$. Using (6.4) we find $t(\varepsilon) = O(\varepsilon^{1+a})$. From this and (6.8) we have :

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))\{\frac{C_1}{2}\phi_0(x) - C^*\varepsilon^a\}dx \\ &+ \frac{m}{\varepsilon}\int_{\{\text{dist }(x,\partial\Omega)\ge 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \le 0,\end{aligned} \tag{6.9}$$

where C^* is a constant independent of ε.

In the following, we choose $l_1 \geq \frac{4C^*}{C_1}$. Note that, if $x = x(s,t) \in \Omega_{2\varepsilon^{1/3}} \setminus \Gamma(2\varepsilon^{1/3}, l_1\varepsilon^a)$, then $\phi_0(x) = \phi_0(x(s,t)) = \phi_0(s) \geq l_1\varepsilon^a \geq \frac{4C^*}{C_1}\varepsilon^a$. Therefore we have :

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}\setminus\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))\{\frac{C_1}{2}\phi_0(x) - C^*\varepsilon^a\}dx\\ &\geq C^*\varepsilon^a \int_{\Omega_{2\varepsilon^{1/3}}\setminus\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx.\end{aligned}$$

From this and (6.9) we get :

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}\setminus\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx\\ &\quad + \frac{m}{C^*\varepsilon^{1+a}}\int_{\{\text{dist }(x,\partial\Omega)\geq 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx\\ &\leq \int_{\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx.\end{aligned}$$

Hence, for all $\varepsilon > 0$ small, we have :

$$\int_{\Omega\setminus\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \leq \int_{\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx.$$

Using this and Lemma 4.5 we get :

$$\begin{aligned}&\int_\Omega |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \leq 2\int_{\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx\\ \leq& 2\exp(\frac{2\alpha l_1\varepsilon^a}{\sqrt{\varepsilon}})\int_{\Gamma(2\varepsilon^{1/3},l_1\varepsilon^a)} |\varphi^\varepsilon|^2dx \leq 2M\exp(\frac{2\alpha l_1\varepsilon^a}{\sqrt{\varepsilon}})\int_{\Omega_{c\varepsilon}} |\varphi^\varepsilon|^2dx.\end{aligned}$$

From this we get (6.1) with $\alpha_1 = 2\alpha$, $c_1 = c$ and $M_1 = 2M$.

Step 4. We consider Case (2). Assume $L_2'\varepsilon^{1/3} \ll \rho(\varepsilon) = o(1)$ as $\varepsilon \to 0$. From (6.4) we have :

$$a_1\varepsilon^{1/3} \ll \frac{t(\varepsilon)}{\varepsilon} = C_1\rho + O(\varepsilon^{1/3}) = o(1).$$

Choose $\phi(x) = \phi_0(x) = \kappa_{max} - \kappa_r(s)$ and α as in Case 1. Then (6.8) also holds, and

$$\frac{C_1}{2}\phi_0(x) - \frac{t(\varepsilon)}{\varepsilon} - C'\varepsilon^{1/3} = \frac{C_1}{2}\phi_0(x) - C_1\rho - O(\varepsilon^{1/3}).$$

Choose $l_2 \geq 4$. If $x = x(s,t) \in \Omega_{2\varepsilon^{1/3}} \setminus \Gamma(2\varepsilon^{1/3}, l_2\rho)$, then $\phi_0(x) \geq l_2\rho$, and, for all $\varepsilon > 0$ small,

$$\frac{C_1}{2}\phi_0(x) - \frac{t(\varepsilon)}{\varepsilon} - C'\varepsilon^{1/3} \geq (\frac{C_1 l_2}{2} - C_1)\rho - O(\varepsilon^{1/3}) \geq C_1\rho.$$

So, for small $\varepsilon > 0$, we have :

$$\int_{\Omega_{2\varepsilon^{1/3}}\setminus\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx + \frac{m}{C_1\rho\varepsilon}\int_{\{\text{dist }(x,\partial\Omega)\geq 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx$$
$$\leq \int_{\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx.$$

Hence

$$\int_{\Omega\setminus\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \leq \int_{\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx.$$

Thus

$$\begin{aligned}\int_{\Omega} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx &\leq 2\int_{\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}\phi_0(x))dx \\ \leq& 2\exp(\frac{2\alpha l_2\rho}{\sqrt{\varepsilon}})\int_{\Gamma(2\varepsilon^{1/3},l_2\rho)} |\varphi^\varepsilon|^2 dx \leq 2M\exp(\frac{2\alpha l_2\rho}{\sqrt{\varepsilon}})\int_{\Omega_{c\varepsilon}} |\varphi^\varepsilon|^2 dx.\end{aligned}$$

From this inequality we get (6.4) with $\alpha_2 = 2\alpha$, $c_2 = c$ and $M_2 = 2M$.

Step 5. We consider Case (3). Assume that Ω is not a ball, and $\varepsilon_0 < \rho < \kappa_{max} - \kappa_{min}$. We introduce $\beta = \kappa_{max} - \rho$. Then $\kappa_{min} < \beta < \kappa_{max}$. From (6.4) we have $t(\varepsilon) = C_1(\kappa_{max} - \beta)\varepsilon + O(\varepsilon^{4/3})$. Define

$$\phi(x) = [\beta - \kappa_r(s)]_+^{3/2} = \begin{cases} [\beta - \kappa_r(s)]^{3/2} & \text{if } \kappa_r(s) \leq \beta, \\ 0 & \text{if } \kappa_t(s) > \beta. \end{cases}$$

Choose $\alpha > 0$ small such that $\frac{9}{4}\alpha^2\|\kappa_r'\|^2_{L^\infty} < \frac{C_1}{2}$. For all $x \in \Omega_{2\varepsilon^{1/3}}$ we have :

$$\begin{aligned} &C_1[\kappa_{max} - \kappa_r(s)] - \frac{t(\varepsilon)}{\varepsilon} - \alpha^2|\nabla\phi|^2 \\ =&C_1[\kappa_{max} - \kappa_r(s)] - C_1[\kappa_{max} - \beta] - \frac{9}{4}\alpha^2|\kappa_r'(s)|^2[\beta - \kappa_r(s)]_+ + O(\varepsilon^{1/3}) \\ \geq&C_1[\beta - \kappa_r(s)] - \frac{C_1}{2}[\beta - \kappa_r(s)]_+ + O(\varepsilon^{1/3}).\end{aligned}$$

From this and (6.7) we have, for some constant $C'' > 0$:

$$\begin{aligned}&\int_{\Omega_{2\varepsilon^{1/3}}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})\{2[\beta - \kappa_r(s)] - [\beta - \kappa_r(s)]_+ - C''\varepsilon^{1/3}\}dx \\ &+ \frac{2m}{C_1\varepsilon}\int_{\{\text{dist }(x,\partial\Omega)\geq 2\varepsilon^{1/3}\}} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha\phi}{\sqrt{\varepsilon}})dx \leq 0.\end{aligned}$$

Choose $l_3 \geq 2C''$. On the set $\Gamma_\beta(2\varepsilon^{1/3}, l_3\varepsilon^{1/3})$ we have $[\beta - \kappa_r(s)]_+ \leq l_3\varepsilon^{1/3}$. We can find $M' > 0$ such that, for $\varepsilon > 0$ small :

$$\begin{aligned}&(l_3 - C'')\varepsilon^{1/3}\int_{\Omega\setminus\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}[\beta - \kappa_r(s)]_+^{3/2})dx \\ &\leq \int_{\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})} |\varphi^\varepsilon|^2 \exp(\frac{2\alpha}{\sqrt{\varepsilon}}[\beta - \kappa_r(s)]_+^{3/2})\{[\beta - \kappa_r(s)]_+ - 2[\beta - \kappa_r(s)] + C''\varepsilon^{1/3}\}dx \\ &\leq M'\int_{\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})} |\varphi^\varepsilon|^2 dx.\end{aligned}$$

Hence

$$\int_{\Omega\setminus\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})}|\varphi^\varepsilon|^2\exp(\frac{2\alpha}{\sqrt{\varepsilon}}[\beta-\kappa_r(s)]_+^{3/2})dx\le\frac{M'}{(l_3-C'')\varepsilon^{1/3}}\int_{\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})}|\varphi^\varepsilon|^2dx.$$

Let $M''=M'/(l_3-C'')+1$. For $\varepsilon>0$ small, we use the above inequality and Lemma 4.5 to get :

$$\int_\Omega|\varphi^\varepsilon|^2\exp(\frac{2\alpha}{\sqrt{\varepsilon}}[\beta-\kappa_r(s)]_+^{3/2})dx\le\frac{M''}{\varepsilon^{1/3}}\int_{\Gamma_\beta(2\varepsilon^{1/3},l_3\varepsilon^{1/3})}|\varphi^\varepsilon|^2dx\le\frac{MM''}{\varepsilon^{1/3}}\int_{\Omega_{c\varepsilon}}|\varphi^\varepsilon|^2dx.$$

Hence (6.3) holds with $\alpha_3=2\alpha$, $c_3=c$ and $M_3=MM''$.

Now Theorem 6.1 is proved. □

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of Theorem 6.1. □

Proof of Theorem 1.3. We prove Conclusion (1). We keep the notations in Theorem 6.1.

Assume $0<H_{C_3}(\kappa)-\sigma\le L_1\kappa^{-1/3}$. From Theorem 6.1 we see that, there exists $\varepsilon_0>0$ such that (6.1) holds with $a=1/3$ for all $0<\varepsilon<\varepsilon_0$, namely,

$$\int_\Omega|\psi^\varepsilon|^2\exp(\frac{\alpha_1}{\sqrt{\varepsilon}}[\kappa_{max}-\kappa_r(s)])dx\le M_1\exp(\frac{\alpha_1 l_1}{\varepsilon^{1/6}})\int_{\Omega_{c_1\varepsilon}}|\psi^\varepsilon|^2dx.$$

Hence

$$\int_{\Omega_{\delta_0}\setminus\Gamma(\delta_0,2l_1\varepsilon^{1/3})}|\psi^\varepsilon|^2dx\le M_1\exp(-\frac{\alpha_1 l_1}{\varepsilon^{1/6}})\int_{\Omega_{c_1\varepsilon}}|\psi^\varepsilon|^2dx.\tag{6.10}$$

By the assumption, $\mathcal{N}(\partial\Omega)$ consists of non-degenerate maximum points of the curvature. From Lemma 6.2, Conclusion (2), we have $\gamma(z)\le\gamma_0\sqrt{z}$ for all $0<z<z^*$. So $\gamma(2l_1\varepsilon^{1/3})\le\gamma_0\sqrt{2l_1}\varepsilon^{1/6}$. Let $\ell=\gamma_0\sqrt{2l_1}$. If $d_t(x,\mathcal{N}(\partial\Omega))\ge\ell\varepsilon^{1/6}$ then $\kappa_{max}-\kappa_r(s(x))\ge 2l_1\varepsilon^{1/3}$. From (6.10) we get :

$$\int_{\{x\in\overline{\Omega}:\ \mathrm{dist}(x,\partial\Omega)<\delta_0,\ d_t(x,\mathcal{N}(\partial\Omega))\ge\ell\varepsilon^{1/6}\}}|\psi^\varepsilon|^2dx\le M_1\exp(-\frac{\alpha_1 l_1}{\varepsilon^{1/6}})\|\psi^\varepsilon\|^2_{L^2(\Omega)}.\tag{6.11}$$

Recall that (3.4) holds for ψ^ε. So we have, for some positive constants α and C,

$$\int_{\{r\varepsilon^{5/6}\le\ \mathrm{dist}\ (x,\partial\Omega)\le\delta_0\}}|\psi^\varepsilon|^2dx\le C\exp(-\frac{\alpha r}{\varepsilon^{1/6}})\int_\Omega|\psi^\varepsilon|^2dx.\tag{6.12}$$

Choose $r=\alpha_1 l_1/\alpha$. From (6.11) and (6.12) we have, for all small ε :

$$\int_{\overline{\Omega}\setminus\{x\in\overline{\Omega}:\ \mathrm{dist}\ (x,\partial\Omega)<r\varepsilon^{5/6},\ d_t(x,\mathcal{N}(\partial\Omega))<\ell\varepsilon^{1/6}\}}|\psi^\varepsilon|^2dx\le B_1\exp(-\frac{\alpha_1 l_1}{\varepsilon^{1/6}})\|\psi^\varepsilon\|^2_{L^2(\Omega)},$$

where $B_1=M_1+C$. Choose positive constants ℓ_1, r_1, a_1 and ε_1' such that, for $0<\varepsilon<\varepsilon_1'$,

$$\ell\varepsilon^{1/6}\le\ell_1\kappa^{-1/6},\qquad r\varepsilon^{5/6}\le r_1\kappa^{-5/6},\qquad\frac{\alpha_1 l_1}{\varepsilon^{1/6}}\ge a_1\kappa^{1/6}.$$

If κ is large (so ε is small), we have :

$$\int_{\overline{\Omega}\setminus\{x\in\overline{\Omega}:\ \mathrm{dist}\ (x,\partial\Omega)<r_1\kappa^{-5/6},\ d_t(x,\mathcal{N}(\partial\Omega))<\ell_1\kappa^{-1/6}\}} |\psi^\varepsilon|^2 dx \le B_1 \exp(-a_1\kappa^{1/6})\|\psi^\varepsilon\|^2_{L^2(\Omega)}.$$

So (1.8) is proved.

Conclusion (2) can be proved by a similar argument. □

Remark 6.3. It is interesting to estimate $\|\psi^\varepsilon\|^4_{L^2(\Omega)}/\|\psi^\varepsilon\|^4_{L^4(\Omega)}$ which, we believe, is proportional to $\|\psi^\varepsilon\|^2_{L^2(\Omega)}$ up to a higher order term, as $\varepsilon \to 0$. We may use (6.6) to obtain an estimate for $\|\psi^\varepsilon\|_{L^\infty(\Omega)}$ in terms of $\|\psi^\varepsilon\|^4_{L^2(\Omega)}/\|\psi^\varepsilon\|^4_{L^4(\Omega)}$.

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