ε–Entropy Estimates for Driven Parabolic Equations

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Abstract

We consider parabolic evolution equations on unbounded domains driven by additive coupling to an independent dynamical system. We define the attractor of the combined system and estimate its Kolmogorov ε–entropy (the coarse-grained spatial density of active modes with mesh size ε). The total ε–entropy is at most $\mathcal{O}(\log \varepsilon^{-1})$ larger then the ε–entropy of the driving system. An example of a system whose ε–entropy is strictly larger than that of the driving system is constructed. Remarks on the behaviour of the entropy under spatial scaling are made.

1. Introduction

We investigate the properties of attractors of dissipative systems defined on unbounded domains and driven by an external input of energy. We model them by equations of the following type:

$$\partial_t u = \nu\Delta u + \mathcal{N}(u)u + F\ , \qquad u(x,t) \in \mathbf{C}\ . \tag{1.1}$$

For simplicity we restrict ourselves to one-dimensional systems. Since we consider unbounded domains (without assumptions of decay at infinity, see [BV] for a discussion of these issues), we might expect the dimension of the attractor to be infinite. Nevertheless, as was shown by Vishik and Chepyzhov [VC] for bounded systems and by Collet and Eckmann [CE1–CE3] for infinite systems, the notion of ε–entropy is well-defined and gives a convenient alternative to the dimension (see [KT] for applications to function spaces).

Loosely speaking the ε–entropy is obtained by first fixing a coarse-graining size ε (points on the attractor belonging to the same ball of diameter ε are identified) and then computing the spatial density of these points (number of points in the attractor of the system restricted to a volume V, divided by V). The density of dimension would be obtained by dividing this entropy by $\log\varepsilon^{-1}$ and taking the limit $\varepsilon \to 0$. If we think of the dimension of the attractor as the number of active modes in the system, then the ε–entropy would be *the spatial density of active modes that can be resolved with a precision ε.* It is also related to the topological entropy [CE2,CE3] a classical concept for finite-dimensional dynamical systems. Remark that the two limits involved ($V \to \infty$ and $\varepsilon \to 0$) do not commute and the (maybe unintuitive) order in which we take them ($V \to \infty$ first, $\varepsilon \to 0$ then) is the only way of getting a sensible answer.

The definition of the attractor needs some care, see Eq.(2.8), and takes the form of a fibre bundle. In Theorem 2.5, the ε–entropy is shown to be bounded above by the ε–entropy generated by the driving system (*i.e.* the ε–entropy of the base) plus a term of order at most $\log\varepsilon^{-1}$ (the fibre entropy). We construct an example of a driven system whose ε–entropy is bounded from below by the ε–entropy of the driving system plus a positive number (Theorem 3.2).

One of the physical motivations for studying driven systems is that the driving force defines a natural length scale in the system: the distance over which F makes a significant oscillation. A few simple consequences of that are derived in Section 4.

On a heuristic level, our results show that the coupling of a dissipative system to a smooth force can create additional active modes in the dynamics, but never more than a finite number of modes per unit volume. We can define the *integral* and *dissipative* scales of the system (see [F] for an expository account of these notions in the context of fluid dynamics). When the separation of these two scales tends to infinity, the number of active modes on the dissipative scale becomes independent of the driving system (and stays bounded) whereas the number of active modes at intermediate scales diverges (Eq.(4.3)). The intermediate scales are those which are simultaneously infinitely far from both integral and dissipative scales.

The proof of the upper bound uses a decomposition of the heat kernel into a low frequency part and a high frequency part. The latter is exponentially decaying hence “irrelevant” for large times and the former generates entire analytic functions. For such functions, the Cartwright interpolation formula makes the counting of different functions easy. The lower bound is built on a decaying equation ($\mathcal{N}(0) = -1$ in Eq.(1.1)) plus a forcing F whose restriction to a bounded interval I is almost periodic and any ε–almost period $T(\varepsilon, I)$ satisfies some extensive property with respect to the size of I. This example only produces more entropy than the noise itself, but not a more dimensions, although we believe that there exists systems for which the entropy generated by the dynamics beyond the entropy of the driving system is of order $\log \varepsilon^{-1}$.

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2. Upper Bound on the ε–Entropy

In this section, we first describe the general hypotheses and notations used throughout the paper. We define the attractor in Eq.(2.8). The main result of the section is Theorem 2.5. Its proof is given later in Section 6.

The equations we consider are of the form:

$$\partial_t u(x,t) \;=\; (1+\mathrm{i}\alpha)\partial_x^2 u(x,t) + u(x,t)\mathcal{N}\big(|u(x,t)|^2\big) + F(x,t)\;, \tag{2.1}$$

with $u : \mathbf{R}\times\mathbf{R}^+ \to \mathbf{C}$ and $\alpha \in \mathbf{R}$. The general assumptions on the nonlinearity and the external forcing are:

Assumption 2.1. *The function $\mathcal{N} \in \mathcal{C}^\infty$ satisfies: there are real $q > 0$ and Q_1, Q_2 such that*

$$\mathrm{Re}\,\big(\mathcal{N}(y)\big) \;\le\; Q_1 - y^q\;, \qquad |\mathcal{N}(y)| + y|\mathcal{N}'(y)| \;\le\; Q_2(1+y^2)\;.$$

The smooth function $F : \mathbf{R}\times\mathbf{R}^+ \to \mathbf{C}$ is uniformly bounded by $|F(x,t)| \le \rho$.

A typical example of a nonlinearity is the usual Ginzburg–Landau term $\mathcal{N}(y) = 1-(1+\mathrm{i}\beta)y$ with $\beta \in \mathbf{R}$. Throughout the paper we will use the following notations: $f_t(x) \equiv f(x,t)$; $|z|$, $\overline{z}$ denote respectively the modulus and the conjugate of the complex number z. The norm on $\mathrm{L}^p(\mathbf{R})$ (with Lebesgue measure) is denoted by $\|\cdot\|_p$. In particular $\|f\|_\infty$ is the essential supremum of $|f|$. We also write $[x]$ for the integer part of the positive real x (*i.e.* $[x] = \max\{n \in \mathbf{N} \,:\, n \le x\}$).

We will often use the following two families of cutoff functions,

$$\varphi_y(x) \;=\; \frac{1}{\pi\cosh(x-y)}\;, \qquad \xi_y^{(n)}(x) \;=\; \frac{Q^{(n)}}{(1+(x-y)^2)^{n/2}}\;. \tag{2.2}$$

They satisfy

$$e^{-|x-y|} \;\le\; \pi\varphi_y(x) \;\le\; 2e^{-|x-y|}\;, \qquad \xi_y^{(n)}(x) \;\le\; Q^{(n)}|x-y|^{-n}\;,\quad \forall x,y\in\mathbf{R}\;, \tag{2.3}$$

$$|\partial_x\varphi_y(x)| \;\le\; |\varphi_y(x)|\;, \qquad |\partial_x\xi_y^{(n)}(x)| \;\le\; n|\xi_y^{(n)}(x)|\;, \tag{2.4}$$

$$\int_{-\infty}^{\infty}\varphi_y(x)\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty}\xi_y^{(n)}(x)\,\mathrm{d}x \;=\; 1\;,$$

where the last equation defines $Q^{(n)}$ for $n > 1$.

We consider the evolution defined by Eq.(2.1) in the Banach space H^1_φ which is the completion of $\mathcal{C}^\infty_\mathrm{b}$ (bounded smooth functions) in the norm

$$\|f\|^2_{\mathrm{H}^1_\varphi} = \sup_{y\in\mathbf{R}} \int \varphi_y |f|^2 + \sup_{y\in\mathbf{R}} \int \varphi_y |f'|^2 \,. \tag{2.5}$$

The topology of L^∞ will also be used on that space and we recall the embedding of H^1_φ into L^∞ which follows from the Sobolev inequality:

$$\|f\|^2_\infty \le \frac{\pi}{4}(1+\sqrt{5})\|f\|^2_{\mathrm{H}^1_\varphi} \,. \tag{2.6}$$

Inequality (2.6) follows from writing $|f(x)|^2 = \pi\varphi_x(x)|f(x)|^2$ as the integral of its derivative and then using Eq.(2.4) and Cauchy–Schwarz inequality.

The following proposition is an adaptation of results in [C1,GV,M]:

Proposition 2.2. *There exists a number* $R_0 = R_0(\alpha, q, Q_1, Q_2, \rho)$ *such that for all solutions* u_t *of Eq.(2.1) with* $\|u_0\|_{\mathrm{H}^1_\varphi} < \infty$*, the following holds:*

$$\limsup_{t\to\infty} \|u_t\|_{\mathrm{H}^1_\varphi} \le R_0 \,.$$

Proof. See Section 7.

We introduce a set Σ of complex functions, which defines a family of possible forces $F(x,t)$ yielding a family of semi-flows generated by Eq.(2.1):

Assumption 2.3. *Let* $\Sigma \subset \{f \in \mathrm{L}^\infty \ : \ \|f\|_\infty \le \rho\}$ *satisfy*

$$\begin{gathered} T_y\Sigma = \Sigma \,, \quad \text{where } \bigl(T_y f\bigr)(x) = f(x+y) \,,\ y \in \mathbf{R} \,, \\ \exists\, C_\Sigma(k) \,, k \in \mathbf{N} \ : \ \forall\, f \in \Sigma \,, |f|_{\mathcal{C}^k} \le C_\Sigma(k) \,. \end{gathered}$$

Let the time evolution on Σ *be defined by a flow* $\vartheta^t : \Sigma \to \Sigma$, $t \in \mathbf{R}$, *satisfying*

$$\begin{gathered} \vartheta^t(T_y f) = T_y \vartheta^t(f) \ \text{ for all } y \in \mathbf{R} \,, \\ \|\vartheta^\varepsilon(f) - f\|_\infty \to 0 \ \text{ when } \varepsilon \to 0 \,, \end{gathered}$$

there are $n > 2$, $\delta > 0$, $\varkappa \ge 0$ *such that for all* $y \in \mathbf{R}$ *and* $t > 0$, *the following holds:*

$$\min\Bigl(\|\sqrt{\xi^{(n)}_y}(\vartheta^t(f) - \vartheta^t(g))\|_2 \,, \, \delta \Bigr) \le e^{\varkappa t} \|\sqrt{\xi^{(n)}_y}(f-g)\|_2 \,.$$

Moreover, Σ *is* ϑ^t*–invariant:*

$$\vartheta^t(\Sigma) = \Sigma \,.$$

We denote by Φ_f^t the semi-flow in H^1_φ generated by Eq.(2.1) where $F(x,t) = \vartheta^t(f)(x)$ with $f \in \Sigma$. Using terminology from random dynamical systems (see [A]), we introduce the skew product

$$\begin{aligned} S^t \,:\, \mathrm{H}^1_\varphi \times \Sigma &\to \mathrm{H}^1_\varphi \times \Sigma \\ (u,f) &\mapsto (\Phi_f^t(u), \vartheta^t(f))\,. \end{aligned} \tag{2.7}$$

In this setup, the attractor for Eq.(2.7) is the set

$$\mathcal{A} = \bigcap_{T>0} \overline{\bigcup_{t>T} S^t(B_R \times \Sigma)}^{\|\cdot\|_{\mathrm{H}^1_\varphi}}\,, \tag{2.8}$$

where B_R is the ball of radius R and centre 0 in H^1_φ and $\overline{\mathcal{X}}^{\|\cdot\|_{\mathrm{H}^1_\varphi}}$ is the closure of $\mathcal{X}$ in the norm of H^1_φ. For sufficiently large R, Proposition 2.2 implies that $\mathcal{A}$ is independent of R, hence we do not specify that variable. The set $\mathcal{A}$ satisfies:

i It is invariant under the actions of S^t and $T_y \otimes T_y$ for all $y \in \mathbf{R}$ and $t \in \mathbf{R}^+$.
ii It is compact in $\mathrm{L}^\infty(K) \times \mathrm{L}^\infty(K)$ for bounded $K \subset \mathbf{R}$ (see also [MS]).
iii It attracts some neighbourhood $\mathcal{U}$: $\mathcal{A} \subset \mathcal{U} \times \Sigma \subset \mathrm{H}^1_\varphi \times \Sigma$.

We next define the Kolmogorov–Tikhomirov ε–entropy [KT].

Definition 2.4. *Let $\mathcal{W}$ be a translation invariant subset of $\mathrm{L}^\infty \times \mathrm{L}^\infty$. Let*

$$\mathcal{W}_L = \Big\{ (f|_{[-L,L]}, g|_{[-L,L]}) \,:\, (f,g) \in \mathcal{W} \Big\}\,,$$

where $f|_I$ means the restriction of f to $I \subset \mathbf{R}$. Let $N(\varepsilon, \mathcal{W}_L)$ be the minimal number of balls of radius $\varepsilon > 0$ (in $\mathrm{L}^\infty \times \mathrm{L}^\infty$) needed to cover $\mathcal{W}_L$. The ε–entropy $H_\varepsilon(\mathcal{W})$ of the set $\mathcal{W}$ is defined by

$$H_\varepsilon(\mathcal{W}) = \lim_{L\to\infty} \frac{\log N(\varepsilon, \mathcal{W}_L)}{2L}\,. \tag{2.9}$$

Remark 1. In the case of a subset $\mathcal{V}$ of L^∞, we simply define $H_\varepsilon(\mathcal{V}) = H_\varepsilon(\mathcal{V} \times \{0\})$ (see, *e.g.*, Theorem 2.5 below, where $H_\varepsilon(\Sigma)$ enters).

Remark 2. The existence of the limit in Eq.(2.9) will not be discussed here since it follows from the same sub-additivity argument as in [CE1]. The quantity $\lim_{\varepsilon\to 0} H_\varepsilon(\mathcal{W})/\log(1/\varepsilon)$ can be interpreted as the dimension per unit length of the set $\mathcal{W}$. The order in which the double limit is taken is important, see [CE1,CE2] for a discussion.

We now come to the main result of the section, which is an infinite volume version of [VC], Theorem 3.1:

Theorem 2.5. *There exists $C = C(R_0, \rho, \varkappa)$ such that for all sufficiently small $\varepsilon > 0$, the following holds*

$$0 \le H_\varepsilon(\mathcal{A}) - H_\varepsilon(\Sigma) \le C \log \varepsilon^{-1}\,.$$

Proof. See Section 6.

Remark. We sketch here a relation between this result and the theory of finite dimensional random dynamical systems: the quantity

$$H_\varepsilon(\mathcal{A}\,|\,\Sigma) \;=\; H_\varepsilon(\mathcal{A}) - H_\varepsilon(\Sigma)$$

is usually called the conditional entropy (see, *e.g.,* [KI]). It is interpreted as the entropy created by the flow beyond that created by the forcing alone. Authors often consider systems driven by a noise which is a random process with probability distribution given by a measure **P** on some space Ω. One can define the fibre entropy $H_\varepsilon(\mathcal{A}_{\{f\}})$ (the ε–entropy of the attractor $\mathcal{A}_{\{f\}}$ for a single realisation f of the noise) which is, in reasonable situations, **P**–almost surely independent of f and equal to the conditional entropy [B2]. We conjecture that generically, the fibre entropy of dynamical systems like Eq.(2.1) is $C\log\varepsilon^{-1}$ for some C depending on the equation.

3. Lower Bound on the ε–Entropy for Decaying Equations

In this section we restrict Eq.(2.1) to the case $\rho = \delta \ll 1$ and $\mathcal{N}(x) = -1 + \mathcal{N}_1(x)$. Namely we consider solutions $u_t(x) = u(x,t)$ of

$$\partial_t u \;=\; (1+\mathrm{i}\alpha)\partial_x^2 u - u + \mathcal{N}_1(|u|^2)u + \delta F\;. \tag{3.1}$$

We strengthen Assumption 2.1 and Assumption 2.3 with the additional

Assumption 3.1. *We have* $3\alpha^2 < 1$ *and there are constants* $Q_3 > 0$, $Q_4 > 0$ *such that*

$$\sup_{x,t} \delta|F(x,t)| \;=\; \delta \;\ll\; 1\;,$$
$$|\mathcal{N}_1(x)| + x|\mathcal{N}_1'(x)| \;\le\; Q_3 x \;\text{ for } x \text{ small enough}\;.$$

Furthermore, the forcing term $F(x,t)$ *satisfies:*

1) *For all interval* $I \subset \mathbf{R}$, $|I| < \infty$, *the functions* $t \mapsto F(x,t)$, $x \in I$, *are almost periodic. Namely for all* $\varepsilon > 0$, *there is a* $\Lambda(\varepsilon,|I|) < \infty$ *such that for all* $t \ge 0$, *there is a number* $T(\varepsilon, I) \in [t, t+\Lambda(\varepsilon,|I|)]$ *for which the following holds:*

$$\sup_{s\in\mathbf{R}}\sup_{x\in I}|F(x,s+T(\varepsilon,I)) - F(x,s)| \;\le\; \varepsilon\;.$$

2) *There are positive* x^* *and* B *such that the following holds: there are disjoint intervals* I_n, $n \in \mathbf{Z}$, *with* $x^* < |I_n| < Bx^*$ *and* $2\delta_{k\ell}x^* \le \mathrm{dist}(I_k, I_\ell) \le 2Bx^*$ *such that for all* $n \in \mathbf{N}$ *and all sufficiently large* $T > 0$, *there is a* $t_n > 0$ *such that*

$$\inf_{x\in I_n}\Big|\int_0^T e^{-s}\big(F(x,T+t_n-s) - F(x,T-s)\big)\,\mathrm{d}s\Big| \;\ge\; \tfrac{1}{2}\;.$$

3) *For all* $N \in \mathbf{N}$, *all bounded intervals* $I_1,\dots,I_N$ *with* $\mathrm{dist}(I_k,I_\ell) \ge 2\delta_{k\ell}x^*$ *and all* $t_k > 0$, $k = 1,\dots,N$, *there is a* $\tau > 0$ *such that*

$$\sup_t \max_k \sup_{x\in I_k} |F(x,\tau+t) - F(x,t_k+t)| \;\le\; C^{-1}\varepsilon\;.$$

Item 1) in Assumption 3.1 is the definition of almost periodic functions, item 2) requires that the amplitude of the oscillations of F are of order 1 and item 3) asserts that frequencies of oscillation at different locations on **R** are in some sense independent of each other.

We next provide two examples of functions F that, although not written in the form required by Assumption 2.3, do satisfy Assumption 3.1. The first one is inspired by [C2]:

$$F_1(x,t) \;=\; \prod_{k\in\mathbf{Z}} \operatorname{sign}(-k)\tanh\bigl(x - z_k(t) - 4kx^*\bigr)\;,$$

where $z_k(t) = x^*\cos(\omega_k t + \varphi_k)$ with $\{\omega_k\}_{k\in\mathbf{Z}}$ independent over the rationals. The phases φ_k are arbitrary real numbers. Remark that F_1 is a well-defined bounded analytic function.

The second example is very similar: let $f_0(x)$ be a $\mathcal{C}^\infty$ function supported by $[-x^*, x^*]$, $|f_0(x)| \geq 3/4$ for $|x| \leq x^*$. We define

$$F_2(x,t) \;=\; \sum_{k=-\infty}^{\infty} \cos(\omega_k t + \varphi_k) f_0(x + 4x^*k)\;,$$

where again the frequencies are rationally independent.

The next theorem shows that the ε–entropy is strictly larger than the entropy of the forcing alone, although Eq.(2.1) with $F = 0$ would give $H_\varepsilon = 0$:

Theorem 3.2. *Let $\mathcal{A}$ be defined by Eq.(2.8) in the case of Eq.(3.1) with Assumption 3.1 true. If x^* is sufficiently large, there is a $C = C(\delta, x^*, B, \alpha) > 0$ such that for all $\varepsilon > 0$, the following holds:*

$$H_\varepsilon(\mathcal{A}) - H_\varepsilon(\Sigma) \;\geq\; C\;.$$

By the definition of the attractor (Eq.(2.8)) and the definition of the ε–entropy (Definition 2.4, that uses the product topology) we see that

$$H_\varepsilon(\mathcal{A}) \;\geq\; H_\varepsilon(\Sigma) + \inf_{T>0, f\in\Sigma} H_\varepsilon\bigl(\bigcup_{t>T} \Phi^t_f(0)\bigr)\;, \tag{3.2}$$

where the initial condition is taken to be 0 for simplicity. The proof of Theorem 3.2, given in Section 5, follows from estimating this last term.

4. Scaling Properties of the ε–Entropy

We introduce the equivalent of a "Reynolds number" $Re = \Lambda/\nu \gg 1$ where Λ is the "integral scale" and ν is the square root of the diffusion constant ("viscosity"), defined as follows: let $\mathcal{A}(\Lambda,\nu)$ be the attractor (in the sense of Eq.(2.8)) of the semi-flows generated by the equations

$$\partial_t u \;=\; \nu^2\partial_x^2 u + u(x,t)\mathcal{N}\big(|u(x,t)|^2\big) + F_\Lambda \,. \tag{4.1}$$

The scale Λ is introduced through the forcing $F_\Lambda(x,t) = \vartheta^t\big(f_\Lambda\big)(x)$, $f_\Lambda \in \Sigma(\Lambda)$, where

$$\Sigma(\Lambda) = \big\{f_\Lambda : x \mapsto f(x/\Lambda)\big\}_{f\in\Sigma(1)} \,, \tag{4.2}$$

and $\Sigma(1)$ fulfils Assumption 2.3. From Definition 2.4, we easily obtain the relation $H_\varepsilon(\Sigma(\Lambda)) = \Lambda^{-1}H_\varepsilon(\Sigma(1))$.

If we set $v(x,t) = u(\nu x,t)$, then v satisfies an equation of the form Eq.(2.1) with a forcing $f_{\Lambda/\nu} \in \Sigma(\Lambda/\nu)$. We believe that $\mathcal{N}$ and $\Sigma(1)$ can be chosen such that

$$H_\varepsilon(\mathcal{A}(\Lambda/\nu,1)) \;=\; C\log\varepsilon^{-1} + \frac{\nu}{\Lambda}H_\varepsilon(\Sigma(1)) \,.$$

In that eventuality, rescaling back to u, we would get

$$H_\varepsilon(\mathcal{A}(\Lambda,\nu)) \;=\; \frac{C}{\nu}\log\varepsilon^{-1} + \frac{1}{\Lambda}H_\varepsilon(\Sigma(1)) \,. \tag{4.3}$$

We interpret this result as follows: since

$$\dim(\Lambda,\nu) \;=\; \lim_{\varepsilon\to 0}\frac{H_\varepsilon(\mathcal{A}(\Lambda,\nu))}{\log\varepsilon^{-1}}$$

can be viewed as the dimension per unit length of $\mathcal{A}(\Lambda,\nu)$, we see that in the large Reynolds number regime ($Re \to \infty$), the dimension $\dim(\Lambda,\nu)$ measured in the inertial range ($\Lambda \to \infty$ and $\nu \to 0$) diverges like ν^{-1} whereas the relative "excess" entropy tends to 1:

$$\lim_{Re\to\infty}\frac{H_\varepsilon(\mathcal{A}(\Lambda,\nu)) - H_\varepsilon(\Sigma(\Lambda))}{H_\varepsilon(\mathcal{A}(\Lambda,\nu))} \;=\; 1 \,.$$

Eq.(4.3) actually interpolates between the convergence at rate $1/Re$ to an entropy of the order of $\log(1/\varepsilon)$ (independent of Σ) on the dissipative scale ($\nu = 1$, $\Lambda \to \infty$) and the divergence at rate Re when the entropy is measured on the integral scale ($\nu \to 0$, $\Lambda = 1$). The first asymptotic is consistent with the entropy of the unforced equation [CE1] and the second is consistent with the fact that the dimension of the attractor is infinite (hence the entropy measured on a very large scale is very large).

We next show how this result can be connected with related works by other authors. The first example comes from Kuksin [K1,K2]. He defines the following quantity, called the length scale exponent of the function $f_\nu \in \mathcal{C}^\infty([0,1])$ depending on the parameter $\nu > 0$:

$$\gamma(f_\nu) \;=\; \liminf_{m\to\infty}\liminf_{\nu\to 0}\frac{\log\|f_\nu\|_{\mathrm{H}^m}^{1/m}}{\log\nu^{-1}} \,.$$

Kuksin shows that if u_ν is a solution of (an equation similar to) Eq.(4.1) and F_Λ is "typical" in some sense, then $\gamma(u_\nu) > \delta > 0$. An upper bound is also proved in [K2] (this follows essentially from the smoothness of the solutions to such equations). These bounds mean that the "typical" solutions are concentrated on Fourier modes of frequency $k \in [0, \ell^{-1}] \equiv [0, \nu^{-\delta}]$. Hence the dimension of the set of typical solutions is at least $\nu^{-\delta}$, which is similar to what Eq.(4.3) states.

Such oscillation properties of solutions of Eq.(4.1) have already been observed by many authors, by means of numerical and analytical studies, see [BCD,BGO,LD,LO]. These authors consider the following type of equation:

$$\partial_t u \;=\; (1+\mathrm{i}\alpha)\Delta u + Ru - (1+\mathrm{i}\beta)|u|^{2q}u\;, \tag{4.4}$$

with $\alpha, \beta \in \mathbf{R}$, $R > 0$ and $q \in \mathbf{N}$. The complex field u depends on the variables $x \in [0,1]^d$ and $t \in \mathbf{R}^+$. They show bounds on the (Lyapunov–)dimension of the attractor that behave with R like

$$\dim_{\mathrm{L}}(\mathcal{A}) \;\sim\; R^{d/2}\;,$$

as well as upper bounds on the m^{th} derivative of u of the following kind: there is a $\gamma' > 0$ for which

$$\limsup_{t\to\infty} \|u_t\|_{\mathrm{H}^m} \;\le\; R^{\gamma' m}\;.$$

The exact bounds actually depend on the region in the α–β plane. This is a crucial point in these authors' approach to the problem of turbulence in the Ginzburg–Landau equation. They identify a region of "soft" turbulence where the bound does not depend on the parameters and a region of "hard" turbulence, where the bounds deteriorate as $\alpha\beta \to +\infty$. These claims are supported by theorems [BCD,BGO] and numerical studies [LD]. Without discussing the α–β dependence problem, we show that our results have some connection with these statements. We remark that Eq.(4.4) can be written as

$$\partial_\tau v \;=\; R^{-1}(1+\mathrm{i}\alpha)\Delta v + v - (1+\mathrm{i}\beta)|v|^{2q}v\;,$$

with $v = R^{-1/2}u$ and $\tau = R^{-1}t$. This change of scales shows that the large R limit is indeed a small viscosity limit ($\nu \to 0$). Moreover, the boundary conditions are an alternative way of fixing an integral scale, less "smooth" than the introduction of a forcing. In (4.4), the integral scale is 1. Thus Eq.(4.4) is roughly a rescaled form of Eq.(4.1) with $\nu = R^{-1/2}$ and $\Lambda = 1$. Eq.(4.3) would explain both the behaviour of the dimension ($\sim R^{1/2} = \nu^{-1}$) and the behaviour of the norms, since in the variable $y = x/\sqrt{R}$ we obtain (by an easy extension of Proposition 2.2) that all derivatives $|\partial_y^n v|$ are bounded independently of R, hence $|\partial_x^n u| \sim R^{(1+n)/2}$.

5. Proof of Theorem 3.2

The strategy is to estimate the second term of Eq.(3.2) from below. We show that the solutions of the linearised equation oscillate sufficiently often (in space and in time) and that the nonlinear equation follows that movement close enough.

We first show that the solutions of Eq.(3.1) are small (of order δ) for all times if the initial condition is small (for example 0). We define the function $\zeta(x,t) = \zeta_t(x)$ by

$$\zeta_t(x) \;=\; \delta \int_0^t \bigl(e^{-(t-s)} g_{t-s} \star F_s\bigr)(x)\, \mathrm{d}s \;, \tag{5.1}$$

where g_t is the heat kernel:

$$\begin{aligned} g_t(x) \;&=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}px - t(1+\mathrm{i}\alpha)p^2}\, \mathrm{d}p \\ &=\; \sqrt{\frac{1+\alpha^2}{4\pi t}}\, \exp\Bigl(\frac{\mathrm{i}}{2}\arctan(\alpha) + \frac{\mathrm{i}\alpha x^2}{4t(1+\alpha^2)}\Bigr) \exp\Bigl(-\frac{x^2}{4t(1+\alpha^2)}\Bigr) \;. \end{aligned} \tag{5.2}$$

More simply, ζ is the solution of the linear equation

$$\partial_t \zeta \;=\; \partial_x^2 \zeta - \zeta + \delta F \;, \tag{5.3}$$

with vanishing initial condition. It follows from Eq.(5.1) and Assumption 3.1 that $|\zeta_t(x)| \le \delta$ for all (x,t). Moreover, the following holds:

Lemma 5.1. *There is a $C > 0$ such that if δ is sufficiently small, $\alpha < 1/\sqrt{3}$, and $\|u_0 - \zeta_0\|_{\mathrm{H}^1_\varphi} \le C\delta^3$, then $\|u_t - \zeta_t\|_{\mathrm{H}^1_\varphi} \le C\delta^3$ for all $t > 0$.*

Proof. See at the end of this section.

By Eq.(5.2), we see that for some sufficiently large A, we have

$$\sup_{t\ge 0}\ \sup_{|x| \ge A \log \varepsilon^{-1}} |g_t(x)| \;\le\; \varepsilon \;.$$

That means that $u(x,t)$ only depends on $F(y,s)$ for $s < t$ and $|x-y| \le A \log \varepsilon^{-1}$ up to an error $\mathcal{O}(\varepsilon)$. This implies that $u(x_1,t)$ oscillate “independently” of $u(x_2,t)$ if $|x_1 - x_2| \ge 2A \log \varepsilon^{-1}$. The following proof is a rigorous implementation of this idea.

In the sequel we assume (sufficiently large) T and L, and a (sufficiently small) ε have been (arbitrarily) fixed. We first show that for all $x_0 \in [-L, L]$, $t \mapsto u(x_0,t)$ is almost periodic. Let $I = [x_0 - A\log\varepsilon^{-1}, x_0 + A\log\varepsilon^{-1}]$ and let $\tau = T(\varepsilon, I)$ be an ε–period for F (see Assumption 3.1). Define

$$v_t(x) \;=\; u(x_0 + x, t+\tau) - u(x_0 + x, t) \;, \quad G(x,t) \;=\; \delta F(x_0 + x, t + \tau) - \delta F(x_0 + x, t) \;,$$

and assume $t > T > B \log \varepsilon^{-1}$. By Eq.(3.1), we have (using Eq.(2.2) and Lemma 5.1),

$$\partial_t \int \varphi_y |v_t|^2 \leq -(2+2\delta^2) \int \varphi_y |v_t|^2 + \delta \int \varphi_y |G_t|^2 \leq -\int \varphi_y |v_t|^2 + C\varepsilon^2 \,, \tag{5.4}$$

thus

$$\begin{aligned} |v_t(0)| &= \left| e^{-t}\big(g_t \star v_0\big)(0) + \int_0^t e^{-s}\big(g_s \star |\tilde{\mathcal{N}}_{t-s} v_{t-s} + G_{t-s}\big)(0)\,\mathrm{d}s \right| \\ &\leq Ce^{-t} + C\delta^2 \int_0^t e^{-s}\big(|g_s| \star |v_{t-s}|\big)(0)\,\mathrm{d}s + C\varepsilon \\ &\leq C\varepsilon + C\delta^2 \int_0^t e^{-s} \|g_s/\sqrt{\varphi_0}\|_2 \big(e^{-\frac{1}{2}(t-s)} + \varepsilon^2\big) \\ &\leq C\varepsilon \,, \end{aligned} \tag{5.5}$$

where we have assumed that δ was small enough and A, B where large enough. Hence any ε–period for F on I is an ε–period for $t \mapsto u(x_0, t)$. Moreover, $u(x_0, t)$ oscillates with amplitude $\mathcal{O}(\delta^2)$, for $\zeta(x_0, T)$ does (and Lemma 5.1 holds): let x_0, $t^* > T$ and τ be such that

$$\Big| \int_0^{t^*} e^{-s}\big(F_{t^*+\tau-s}(x) - F_{t^*+s}(x)\big)\,\mathrm{d}s \Big| \geq \tfrac{1}{2} \,,$$

for all x with $|x - x_0| \leq x^*$. Assume that $|\zeta_{t^*+\tau}(x) - \zeta_{t^*}(x)| < \frac{1}{4}\delta$ for all such x. Then

$$\begin{aligned} |\zeta_{t^*+\tau}(x) - \zeta_{t^*}(x)| &= \left| \int_0^{t^*+\tau} e^{-s}\big((1+\mathrm{i}\alpha)\partial_x^2 \zeta_{t^*+\tau-s}(x) + \delta F_{t^*+\tau-s}(x)\big)\,\mathrm{d}s \right. \\ &\qquad \left. - \int_0^{t^*} e^{-s}\big((1+\mathrm{i}\alpha)\partial_x^2 \zeta_{t^*-s}(x) + \delta F_{t^*-s}(x)\big)\,\mathrm{d}s \right| \\ &\geq -Ce^{-t^*} + \delta \left| \int_0^{t^*} e^{-s}\big(F_{t^*+\tau-s}(x) - F_{t^*+s}(x)\big)\,\mathrm{d}s \right| \\ &\qquad - C \int_0^{t^*} e^{-s} \left|\partial_x^2\big(\zeta_{t^*+\tau-s}(x) - \zeta_{t^*-s}(x)\big)\right| \\ &\geq \tfrac{1}{2}\delta - \varepsilon - C \int_0^{t^*} e^{-s} \left|\partial_x^2\big(\zeta_{t^*+\tau-s}(x) - \zeta_{t^*-s}(x)\big)\right| \,. \end{aligned}$$

This implies that $|\partial_x^2(\zeta_{t^*+\tau-s}(x) - \zeta_{t^*-s}(x))|$ has a maximum of order δ somewhere in $|x - x_0| \leq x^*$ and $s < t^*$ hence $|\zeta_{t^*+\tau-s}(x) - \zeta_{t^*-s}(x)| \geq C\delta^2$ for some x, s nearby.

We have thus shown that for all $\{x_{-M}, \ldots, x_M\} \subset [-L, L]$ with $M = [L/(2A \log \varepsilon^{-1})] - 1$ and $|x_k - x_j| \geq \delta_{kj} A \log \varepsilon^{-1}$, there is a bounded function u^* such that for all $\{a_{-M}, \ldots, a_M\} \in [-C\delta^2, C\delta^2]$ there are $t_{-M}, \ldots, t_M$ for which $u(x_k, t_k) = u^*(x_k) + a_k$. Moreover, by Assumption 3.1 there is a τ such that

$$\sup_t \max_k \sup_{|x| \leq A \log \varepsilon^{-1}} |F(x_k + x, \tau + t) - F(x_k + x, t_k + t)| \leq C^{-1}\varepsilon \,.$$

By repeating the calculations made above (Eq.(5.4) and Eq.(5.5)) we obtain that there is a $t^* > T$ such that

$$\max_{k=-M,\dots,M} |u(x_k,t^*) - a_k - u^*(x_k)| \le \varepsilon\ .$$

This proves that $\left(\bigcup_{t>T} \Phi_f^t(0)\right)_L$ contains an ε–distinguishable set $\mathcal{Y}$ with

$$\mathrm{card}(\mathcal{Y}) \ge \left(\frac{2\delta^2}{\varepsilon}\right)^{2M}$$

($\mathcal{Y} = \{x_1,\dots,x_n\} \subset \mathcal{X}$ is called an ε–distinguishable set in $\mathcal{X}$ if $\mathrm{dist}(x_j,x_k) > \varepsilon$ for all $j \neq k$, see [KT]). Hence

$$H_\varepsilon\big(\bigcup_{t>T} \Phi_f^t(0)\big) \ge \frac{\log((2\delta^2\varepsilon^{-1})^{L/(A\log\varepsilon^{-1})-2})}{2L} \ge \frac{1}{16A}\ .$$

With the proof of Lemma 5.1 below, the proof of Theorem 3.2 is complete. □

Proof of Lemma 5.1. Let $\gamma = \sqrt{1+\alpha^2}$, $y \in \mathbf{R}$, $A > 0$, $\eta > 0$. Define $v_t = u_t - \zeta_t$. Using the cutoff function φ_y defined by Eq.(2.2), we have

$$\begin{aligned}
&\tfrac{1}{2}\partial_t \int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) \\
&= -\int \varphi_y\Big(|v_t''|^2 + A|v_t|^2 + (A-1)|v_t'|^2\Big) - \mathrm{Re}\int \varphi_y'(1+\mathrm{i}\alpha)\Big(A\overline{v}_t v_t' + \overline{v}_t' v_t''\Big) \\
&\quad + \mathrm{Re}\int \varphi_y\Big(A\overline{v}_t \mathcal{N}_1(|u_t|^2)u_t + \overline{v}_t'\partial_x\big(\mathcal{N}_1(|u_t|^2)u_t\big)\Big) \\
&\le -\int \varphi_y\left(\Big(|v_t''|^2 - \gamma|v_t''||v_t'| + \tfrac{1}{4}\gamma^2|v_t'|^2\Big) + \Big((A - 1 - \frac{\gamma^2}{4})|v_t'|^2 - A\gamma|v_t'||v_t| + A|v_t|^2\Big)\right) \\
&\quad + \mathrm{Re}\int \varphi_y\Big(A\overline{v}_t \mathcal{N}_1(|u_t|^2)u_t + \overline{v}_t'\partial_x\big(\mathcal{N}_1(|u_t|^2)u_t\big)\Big) \\
&\le -\eta\int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) - \int \varphi_y\Big\{(A - 1 - \tfrac{3}{4}\gamma^2 - \eta)|v_t'|^2 + A(1 - \eta - \tfrac{1}{2}A)|v_t|^2\Big\} \\
&\quad + \mathrm{Re}\int \varphi_y\Big(A\overline{v}_t \mathcal{N}_1(|u_t|^2)u_t + \overline{v}_t'\partial_x\big(\mathcal{N}_1(|u_t|^2)u_t\big)\Big)\ .
\end{aligned}$$

If we choose $A = (11 + 3\alpha^2)/6$ and $\eta = (1 - 3\alpha^2)/12$ ($\eta > 0$ by Assumption 3.1), then the term in curly brackets vanishes. We obtain

$$\begin{aligned}
\partial_t \int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) &\le -2\eta\int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) + \eta\int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) \\
&\quad + \int \varphi_y \eta^{-1}\big(AQ_3^2|u_t|^6 + 2Q_3^2|u_t|^4|u_t'|^2\big) \\
&\le \big(4\eta^{-1}\|u_t\|_\infty^4 Q_3^2 - \eta\big)\int \varphi_y\big(A|v_t|^2 + |v_t'|^2\big) \\
&\quad + 4\eta^{-1}\|u_t\|_\infty^4 Q_3^2 \int \varphi_y\big(A|\zeta_t|^2 + |\zeta_t'|^2\big)\ .
\end{aligned} \tag{5.6}$$

Let $\mathcal{P} = \bigl\{f \in \mathrm{L}^\infty : \|f\|_\infty^4 \le \frac{1}{8}\eta^2 Q_3^{-2}\bigr\}$. If $\|v_t\|_{\mathrm{H}^1_\varphi} = \|u_t - \zeta_t\|_{\mathrm{H}^1_\varphi} \le \delta$, then by Eq.(2.6) we have

$$\|u_t\|_\infty^2 \,\le\, 2\|u_t - \zeta_t\|_\infty^2 + 2\rho^2\delta^2 \,\le\, \frac{\pi}{2}(1+\sqrt{5})\|u_t - \zeta_t\|_{\mathrm{H}^1_\varphi}^2 + 2\rho^2\delta^2 \,\le\, C\delta^2 \;,$$

hence there is a $K = K(Q_2, Q_3, \alpha, \rho)$ such that $u_t \in \mathcal{P}$ for any $\delta < K$. In that case, Eq.(5.6) implies

$$\partial_t \int \varphi_y\bigl(A|v_t|^2 + |v_t'|^2\bigr) \,\le\, -\tfrac{1}{2}\eta \int \varphi_y\bigl(A|v_t|^2 + |v_t'|^2\bigr) + C\delta^4 \int \varphi_y\bigl(A|\zeta_t|^2 + |\zeta_t'|^2\bigr) \;,$$

namely the ball $\{f : \|f - \zeta_t\|_{\mathrm{H}^1_\varphi} \le C\delta^3\}$ is absorbing in $\mathcal{P}$. □

6. Proof of Theorem 2.5

Let $N(\varepsilon, (\mathcal{A})_L)$ be the number of balls of radius at most ε needed to cover $(\mathcal{A})_L$. We have

Proposition 6.1. *There are $A, C > 0$ such that for all $L > 0$ and sufficiently small $\varepsilon > 0$, one has*

$$\frac{N(\varepsilon, (S^1(\mathcal{A}))_L)}{N(\varepsilon, (\vartheta^1(\Sigma))_L)} \,\le\, \frac{N(2\varepsilon, (\mathcal{A})_{L+C})}{N(2\varepsilon, \Sigma_{L+C})} A^{L+1/\varepsilon} \;. \tag{6.1}$$

We continue the proof of Theorem 2.5. We know from Proposition 2.2 that there is an $R \le \rho + R_0$ such that $N(R, (\mathcal{A})_L) = 1$ and $N(R, \Sigma_L) = 1$. Let T be the smallest integer larger than $(\log 2)^{-1}\log(R/\varepsilon)$. By iterating T times the bound (6.1), we obtain

$$\frac{N(\varepsilon, (S^T(\mathcal{A}))_L)}{N(\varepsilon, (\vartheta^T(\Sigma))_L)} \,\le\, A^{L\sum_{n=1,\ldots,T}(1+nC/(\varepsilon L))} \;.$$

Since $N(\varepsilon, (S^T(\mathcal{A}))_L) = N(\varepsilon, (\mathcal{A})_L)$ and $N(\varepsilon, (\vartheta^T(\Sigma))_L) = N(\varepsilon, \Sigma_L)$, we find

$$\lim_{L\to\infty} \frac{\log N(\varepsilon, (\mathcal{A})_L)}{2L} \,\le\, \tfrac{1}{2}T\log A + \lim_{L\to\infty}\frac{\log N(\varepsilon, \Sigma_L)}{2L} \;.$$

By Definition 2.4, that is

$$H_\varepsilon(\mathcal{A}) \,\le\, C(R, A)\log\varepsilon^{-1} + H_\varepsilon(\Sigma) \;.$$

We next prove the lower bound. Fix L, ε and let $\{f_1, \ldots, f_N\}$ be an ε–net for Σ_L (*i.e.* a set $\mathcal{X} \subset \Sigma_L$ such that $\mathrm{dist}(\mathcal{X}, \Sigma_L) \le \varepsilon$, see [KT]) with $N = N(\varepsilon, \Sigma_L) \approx \exp(2LH_\varepsilon(\Sigma))$. By the definition Eq.(2.8) of the attractor, one has

$$\mathcal{A} \ni x_j \,\equiv\, \lim_{t\to\infty}\bigl(\Phi^t_{\vartheta^{-t}f_j}(0), f_j\bigr) \;, \quad j \,=\, 1, \ldots, N \;.$$

By our choice of the product topology in Definition 2.4, $\{x_1, \ldots, x_N\}$ is an ε–net for $\mathcal{A}$. □

Proof of Proposition 6.1. Let u and v be two orbits of Eq.(2.1) with initial conditions u_0 and v_0 such that (u_0, f_1) and (v_0, f_2) belong to $\mathcal{A}$ for some f_1, f_2 in Σ. The difference $w = u - v$ satisfies the equation

$$\partial_t w \;=\; (1+\mathrm{i}\alpha)\partial_x^2 w + G^{(1)} w + G^{(2)} \overline{w} + G^{(3)} \;, \tag{6.2}$$

where (with the shorthands $u = u(x,t)$, $v = v(x,t)$)

$$\begin{aligned} G^{(1)}(x,t) \;&=\; \tfrac{1}{2}\Big(\mathcal{N}(|u|^2) + \mathcal{N}(|v|^2) + (|u|^2 + |v|^2) \int_0^1 \mathcal{N}'\big(t|u|^2 + (1-t)|v|^2\big)\big)\,\mathrm{d}t \Big) \;, \\ G^{(2)}(x,t) \;&=\; uv \int_0^1 \mathcal{N}'\big(t|u|^2 + (1-t)|v|^2\big)\,\mathrm{d}t \;, \\ G^{(3)}(x,t) \;&=\; \vartheta^t\big(f_1\big)(x) - \vartheta^t\big(f_2\big)(x) \;, \qquad \text{for some } f_1, f_2 \in \Sigma \;. \end{aligned}$$

Let $\chi(x)$ be a smooth and monotone function satisfying $\chi(x) = 1$ if $x \le 1$ and $\chi(x) = 0$ if $x \ge 2$. We define two convolution operators by their kernels:

$$K_t^{(1)}(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}px - t(1+\mathrm{i}\alpha)p^2} \chi(|p/p^*|)\,\mathrm{d}p \;, \tag{6.3}$$

$$K_t^{(2)}(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}px - t(1+\mathrm{i}\alpha)p^2} \big(1 - \chi(|p/p^*|)\big)\,\mathrm{d}p \;, \tag{6.4}$$

where $p^* > 4$ is a sufficiently large real number. We write solutions $w_t(x)$ of (6.2) as

$$w_t(x) \;=\; w_t^{(1)}(x) + w_t^{(2)}(x) \;, \tag{6.5}$$

$$w_t^{(1)}(x) \;=\; \big(K_t^{(1)} \star w_0\big)(x) + \int_0^t \big(K_{t-s}^{(1)} \star (G_s^{(1)} w_s + G_s^{(2)} \overline{w}_s + G_s^{(3)})\big)(x)\,\mathrm{d}s \;, \tag{6.6}$$

$$w_t^{(2)}(x) \;=\; \big(K_t^{(2)} \star w_0\big)(x) + \int_0^t \big(K_{t-s}^{(2)} \star (G_s^{(1)} w_s + G_s^{(2)} \overline{w}_s + G_s^{(3)})\big)(x)\,\mathrm{d}s \;. \tag{6.7}$$

The convolution operators $K_t^{(1)}$ and $K_t^{(2)}$ have some regularity and decay properties that we next describe: let the Bernstein class $B_{R,k}$ be the following set of functions:

$$B_{R,k} \;\equiv\; \Big\{ f \in \mathrm{L}^\infty \;:\; f \;\text{ extends to an entire function}\,,\; |f(z)| \;\le\; Re^{k|\mathrm{Im}\, z|} \Big\} \;. \tag{6.8}$$

We have

Lemma 6.2. *For all $p^* > 4$, $t > \frac{1}{2}$, $f \in \mathrm{L}^\infty$, $K_t^{(1)} \star f$ is in $B_{R,2p^*}$ with $R \le 2C_0 \|f\|_\infty$. Moreover, for all $n \in \mathbf{N}$, there is a $C_n > 0$ such that*

$$\begin{aligned} |K_t^{(1)}(x)| \;&\le\; \frac{C_n}{\sqrt{t}} (1 + x^2/t)^{-n} \;. \\ |K_t^{(2)}(x)| \;&\le\; \frac{C_n}{\sqrt{t}} e^{-(p^*)^2 t/2} (1 + x^2/t)^{-n} \;. \end{aligned}$$

We next prove a bound on the short-time growth of w_t in a local norm:

Lemma 6.3. *Let $n > 2$, $y \in \mathbf{R}$. Let $\xi_y^{(n)}$ be given by Eq.(2.2). There is an $M > 0$ such that for all $t > 0$, the following holds*

$$\|\sqrt{\xi_y^{(n)}}w_t\|_2 \;\le\; e^{Mt}\|\sqrt{\xi_y^{(n)}}w_0\|_2 + \int_0^t \bigl(e^{M(t-s)}\|\sqrt{\xi_y^{(n)}}G_s^{(3)}\|_2\bigr)\,\mathrm{d}s\;.$$

The proofs of Lemma 6.2 and Lemma 6.3 are given at the end this section. We next complete the proof of Proposition 6.1. Remark that by Proposition 2.2, we have the following uniform bound for all x, t:

$$\max\bigl(|G_t^{(1)}(x)|, |G_t^{(2)}(x)|\bigr) \;\le\; 1 + 2\sqrt{1+\beta^2}R_0^2\;. \tag{6.9}$$

Let $|f|_d \equiv \sup_{|x|\le d}|f(x)|$. Assume $f_1, f_2 \in \Sigma$ are such that

$$\sup_{|y|\le L}\|\sqrt{\xi_y^{(n)}}(f_1 - f_2)\|_2 \;=\; \sup_{|y|\le L}\|\sqrt{\xi_y^{(n)}}G_0^{(3)}\|_2 \;\le\; 2\varepsilon\;.$$

Then by Assumption 2.3,

$$\sup_{|y|\le L}\|\sqrt{\xi_y^{(n)}}G_t^{(3)}\|_2 \;\le\; 2e^{\varkappa t}\varepsilon \tag{6.10}$$

if $t < 2$. Assume $|w_0|_{L+C(\varepsilon)} \le 2\varepsilon$. If $C(\varepsilon) > ((C_n + Q^{(n)})/\varepsilon)^{1/n}$, Lemma 6.3 and Eq.(6.10) imply

$$\sup_{|y|\le L}\|\sqrt{\xi_y^{(n)}}w_t\|_2 \;\le\; 2\varepsilon e^{Mt}\bigl(1 + e^{\varkappa t}\bigr)$$

for all $t < 2$. We next use that Lemma 6.2 implies $|K_1^{(1)} \star w_0|_L \le C\varepsilon$ and $|K_1^{(2)} \star w_0|_L \le 2C_n e^{-(p^*)^2/2}\varepsilon$. Inserting those bounds into (6.6) and (6.7) gives (by Cauchy–Schwarz inequality)

$$\begin{aligned}|w_1^{(1)}|_L \;&\le\; |K_1^{(1)} \star w_0|_L + C\sup_{|y|\le L}\int_0^1 \|K_{1-s}^{(1)}/\sqrt{\xi_0^{(n)}}\|_2\bigl(\|\sqrt{\xi_y^{(n)}}w_s\|_2 + \|\sqrt{\xi_y^{(n)}}G_s^{(3)}\|_2\bigr)\\ &\le\; C\varepsilon + 2C\varepsilon\int_0^1 \frac{C_n}{\sqrt{1-s}}e^{(M+\varkappa)s}\,\mathrm{d}s\\ &\le\; A\varepsilon\;,\end{aligned} \tag{6.11}$$

where A depends on n but not on p^*, and

$$\begin{aligned}|w_1^{(2)}|_L \;&\le\; |K_1^{(2)} \star w_0|_L + C\sup_{|y|\le L}\int_0^1 \|K_{1-s}^{(2)}/\sqrt{\xi_0^{(n)}}\|_2\bigl(\|\sqrt{\xi_y^{(n)}}w_s\|_2 + \|\sqrt{\xi_y^{(n)}}G_s^{(3)}\|_2\bigr)\\ &\le\; e^{-(p^*)^2/2}\varepsilon + 2C\varepsilon\int_0^1 \frac{C_n e^{-(p^*)^2(1-s)/2}}{\sqrt{1-s}}e^{(M+\varkappa)s}\,\mathrm{d}s\\ &\le\; B(p^*)\varepsilon\;,\end{aligned} \tag{6.12}$$

where $B(p^*) \to 0$ as $p^* \to \infty$. We choose p^* so large that $B(p^*) < \frac{1}{2}$.

Assume next that we have a covering of $(\mathcal{A})_{L+C(\varepsilon)}$ by open sets of radius at most 2ε. If (u, f_1) and (v, f_2) belong to one of those sets, then $w_0 = u - v$ and $G_0^{(3)} = f_1 - f_2$ satisfy the hypotheses stated above that led to inequalities (6.11) and (6.12). We next use a result of Cartwright (see [KT], Eq.(191)): for all f in the Bernstein class $B_{R,2p^*}$ (see Eq.(6.8)), the following identity holds:

$$f(x) = \frac{\sin(8p^*x)}{32(p^*)^2} \sum_{n=-\infty}^{\infty} (-1)^n f(x_n) \frac{\sin(4p^*(x-x_n))}{(x-x_n)^2} , \tag{6.13}$$

where $x_n = \frac{n\pi}{8p^*}$. Let f, g be in $B_{R,2p^*}$. A simple application of Eq.(6.13) shows that

$$|f-g|_L \le C \sup_{|n| \le [8p^*L/\pi] + 8Cp^*/(\varepsilon\pi)} |f(x_n) - g(x_n)| + \tfrac{1}{4}\varepsilon .$$

Hence, among all the functions in $B_{R,2p^*}$ that are bounded by $A\varepsilon$ in $[-L, L]$ (by Eq.(6.11), $w_1^{(1)}$ is such a function), at most $(4A)^{Cp^*L}(4R/\varepsilon)^{Cp^*/\varepsilon}$ of them differ pairwise by more than $\varepsilon/2$ in $[-L, L]$. By taking a ball of radius ε around each of them, and repeating the operation for each element of the original covering of radius 2ε, we get a covering of $(S^1(\mathcal{A}))_L$ by open sets of radius ε. The number of elements in this covering is at most

$$(4A)^{Cp^*L}(4R/\varepsilon)^{Cp^*/\varepsilon} \frac{N(2\varepsilon, (\mathcal{A})_{L+C})}{N(2\varepsilon, \Sigma_{L+C})} N(\varepsilon, (\vartheta^1(\Sigma))_L) .$$

The proof of Proposition 6.1 is complete. ☐

Proofs of Lemma 6.2 and Lemma 6.3. We prove the bound on $K_t^{(1)}$, the one on $K_t^{(2)}$ is similar. We make the change of variable $y = x/\sqrt{t}$ and $k = \sqrt{t}p$ in Eq.(6.3):

$$\begin{aligned}
\sqrt{t}\left(\frac{x}{\sqrt{t}}\right)^n K_t^{(1)}\left(\frac{x}{\sqrt{t}}\right) &= \sqrt{t} y^n K_t^{(1)}(y) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} y^n e^{iky} e^{-(1+i\alpha)k^2} \chi(|k/\sqrt{t}p^*|) \, dk , \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iky} (-i\partial_k)^n \left(e^{-(1+i\alpha)k^2} \chi(|k/\sqrt{t}p^*|) \right) dk ,
\end{aligned}$$

which is bounded because χ is $\mathcal{C}^\infty$ and $\exp(-k^2)$ decays faster than any power of k. The analyticity follows from the Paley–Wiener Theorem using that χ has compact support, hence for complex z, $|\exp(ipz)\chi(|p/p^*|)| \le \exp(2p^*|\mathrm{Im}\, z|)$. The decay of $K_t^{(2)}$ follows from the fact that $1 - \chi(|p/p^*|)$ has support on $\{p > p^*\} = \{k > \sqrt{t}p^*\}$ where $\exp(-k^2) \le \exp(-(p^*)^2t/2)\exp(-k^2/2)$ and we still get a convergent integral.

We next prove Lemma 6.3. We compute a differential inequality for the weighted norm (see Section 7), using the bound $\|(\partial_x \xi_y^{(n)})/\xi_y^{(n)}\|_\infty = n$:

$$\partial_t \int \xi_y^{(n)} |w_t|^2 \le \left(\|G_t^{(1)}\|_\infty + \|G_t^{(2)}\|_\infty + \tfrac{1}{4}n^2(1+\alpha^2) \right) \int \xi_y^{(n)} |w_t|^2 + \int \xi_y^{(n)} |G_t^{(3)}|^2 ,$$

hence using Eq.(6.9),

$$\|\sqrt{\xi_y^{(n)}}w_t\|_2 \leq e^{Mt}\|\sqrt{\xi_y^{(n)}}w_0\|_2 + \int_0^t \left(e^{M(t-s)}\|\sqrt{\xi_y^{(n)}}G_s^{(3)}\|_2\right) \mathrm{d}s .$$

□

7. Proof of Proposition 2.2

We give here a proof of Proposition 2.2 assuming existence, uniqueness, and regularity of solutions over compact intervals of time, which are standard results for smooth enough F (see [P]). The proof is essentially identical to the one given by Mielke [M].

We derive y–independent differential inequalities for $\int \varphi_y|u_t|^2$ and $\int \varphi_y|u_t'|^2$, we integrate them (using Gronwall-like estimates). We first compute the time derivative of the first of these two integrals:

$$\begin{aligned}\partial_t \int \varphi_y|u_t|^2 &= -2\int \varphi_y|u_t'|^2 - 2\mathrm{Re}\int \varphi_y'(1+\mathrm{i}\alpha)\overline{u}_t u_t' \\ &\quad + 2\int \varphi_y|u_t|^2 \mathrm{Re}\,\mathcal{N}(|u_t|^2) + 2\int \varphi_y \mathrm{Re}\,(\overline{u}_t F_t) \\ &\leq \int \varphi_y|u_t|^2\left(\tfrac{3}{2}+\tfrac{1}{2}\alpha^2+2Q_1-2|u_t|^{2q}\right) + \int \varphi_y|F_t|^2 \\ &\leq A - \int \varphi_y|u_t|^2 ,\end{aligned}$$

where $A = A(\rho, \alpha^2, Q_1, q)$. By simple integration we get

$$\int \varphi_y|u_t|^2 \leq \max\left\{A\,,\, \left(\int \varphi_y|u_0|^2 - A\right)e^{-t} + A\right\} . \tag{7.1}$$

We next estimate the time derivative of the second integral described above:

$$\begin{aligned}\partial_t \int \varphi_y|u_t'|^2 &= -2\int \varphi_y|u_t''|^2 - 2\mathrm{Re}\int \varphi_y'(1+\mathrm{i}\alpha)\overline{u}_t' u_t'' + 2\int \varphi_y \mathrm{Re}\,(\overline{u}_t' F_t') \\ &\quad + 2\mathrm{Re}\int \varphi_y\left(|u_t'|^2\mathcal{N}(|u_t|^2) + \overline{u}_t' u\mathcal{N}'(|u_t|^2)\partial_x|u_t|^2\right) \\ &\leq -\int \varphi_y|u_t''|^2 + (2+\alpha^2+2Q_1)\int \varphi_y|u_t'|^2 + \int \varphi_y|F_t|^2 \\ &\quad + 2Q_2\int \varphi_y|u_t'|^2|u_t|^4 .\end{aligned} \tag{7.2}$$

The last term can be bounded by some variant of the Gagliardo–Nirenberg inequalities, see [BGO,M], we also use that for all $\lambda > 0$ one has

$$-\int \varphi_y|f''|^2 \leq -\lambda\int \varphi_y|f'|^2 + \lambda(1+\lambda)\int \varphi_y|f|^2 .$$

It follows that Eq.(7.2) can be estimated by

$$\begin{aligned}\partial_t \int \varphi_y |u_t'|^2 &\leq -\tfrac{3}{4}\int \varphi_y |u_t''|^2 + C_1 + C_2 \int \varphi_y |u_t'|^2 + C_3 \Big(\sup_y \int \varphi_y |u_t|^2\Big)^5 \\ &\leq -\int \varphi_y |u_t'|^2 + A + B\Big(\sup_y \int \varphi_y |u_t|^2\Big)^5 ,\end{aligned}$$

where C_1, C_2, C_3, A, and B are constants independent of u, t, and y. The claim is proved by inserting Eq.(7.1) and by integration.

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