Gibbs measures on Brownian paths

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Abstract

This is a summary of results based upon [1, 4] outlining how Gibbs measures can be defined on Brownian paths and what are their most important properties. Such Gibbs measures have a number of applications in Euclidean quantum field theory, statistical mechanics, stochastic (partial) differential equations and other areas.

1 Introduction

Gibbs measures are a fundamental concept and tool originally introduced in classical statistical mechanics, presently reaching far beyond their initial scope. These probability measures were deviced to describe the macroscopic equilibrium states of large systems of microscopically interacting components. In many cases they are defined on a configuration space given by an infinite product E^S of some set E (often called single state space, e.g. $E = \{-1, +1\}$ for Ising spin systems), where S is a suitable topological space (often called index set, e.g. the d-dimensional lattice $\mathbb{Z}^d$). For each compact subset $\Delta \subset S$ and each $\omega \in E^\Delta$, $\bar{\omega} \in E^{S \setminus \Delta}$ a specific conditional probability kernel $\rho_\Delta(\omega|\bar{\omega})$ is given describing the probability of a configuration ω occurring inside Δ when $\bar{\omega}$ is fixed outside Δ. A Gibbs measure is any probability measure ρ on E^S that, when conditioned to $\bar{\omega}$ outside each compact subset Δ, reproduces $\rho_\Delta(\cdot|\bar{\omega})$ for ρ-almost all $\bar{\omega}$. Such a probability measure, if any at all, describes the infinitely large system. Averages with respect to a Gibbs measure of local functions (observables) associated with the system provide the macroscopically measurable values of the system's state. The possible existence of multiple Gibbs measures is of special interest for it means that the system, depending on the values of a control parameter, has different macroscopic states.

Gibbs measures are an interesting object to study also for Brownian paths instead of the simpler spin variables. The reasons for doing this come on the one hand from stochastic dynamics [3], and from quantum field theory (or partial differential equations and related semigroups) on the other [1, 5, 2]. For example, the semigroup generated by a

Schrödinger operator $H = -(1/2)\Delta + V$, with a suitable potential V, can be investigated by using the Feynman-Kac formula, i.e. by running a Brownian motion:

$$(f, e^{-tH} g) = \int_{C(\mathbb{R},\mathbb{R}^d)} \int_{\mathbb{R}^d} f(X_0) g(X_t) e^{-\int_0^t V(X_s) ds} d\mathcal{W}_x(X) dx. \tag{1.1}$$

The formula says that the scalar product (taken on some Hilbert space with suitable f, g) appearing on the left hand side can be computed as an expectation with respect to Brownian motion in the presence of the potential V (on the right hand side X_t denotes a Brownian path evaluated at time t, and $\mathcal{W}_x$ stands for Wiener measure starting from $x \in \mathbb{R}^d$). As it turns out, computing or at least estimating these expectations may be more feasible than doing the same on the left hand side. If H were replaced by a more complicated operator then on the right hand side even a pair interaction potential (or other more complicated expressions) might appear. One example where this actually happens is Nelson's scalar field model investigated by this approach in [5, 2].

The literature on Gibbs measures on path space is relatively little. [8] contains early ideas on $P(\phi)_1$-processes in connection with statistical mechanics. In [7] certain monotonicity, smoothness and convexity conditions are drawn on the external potential so that the use of correlation inequalities becomes possible. There also Gibbs measures with pair potentials are considered, whose existence becomes however a non-trivial issue. The difficulties in setting up Gibbs measures on path space for pair potentials are rooted mainly in the fact that in this case no compactness arguments apply and nothing like superstability properties are around. Below we will report on the progress obtained in [1, 4].

2 Gibbs measure for external potential

First we outline a method how to represent $P(\phi)_1$-processes (i.e., Brownian motion in the presence of an external potential) in terms of Gibbs measures. For details, proofs, and a discussion of the related literature we refer to [1]; the arguments we use are largely based on a spectral theoretic analysis.

Denote $V^+ = \sup\{0, V\}$ and $V^- = \inf\{-V, 0\}$. We consider two classes of external potential $V : \mathbb{R}^d \to \mathbb{R}$:

(V1) V is of Kato-class, i.e. $V^+ \in K_d^{\mathrm{loc}}$ and $V^- \in K_d$, where

$$K_1 = \left\{ V : \sup_{x \in \mathbb{R}} \int_{|x-y| \leq 1} |V(y)| \; dy < \infty \right\},$$

$$K_2 = \left\{ V : \lim_{r \to 0} \sup_{x \in \mathbb{R}^2} \int_{|x-y| \leq r} |V(y)| \log \frac{1}{|x-y|} \; dy = 0 \right\},$$

$$K_d = \left\{ V : \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} |V(y)| \; \frac{1}{|x-y|^{d-2}} \; dy = 0 \right\} \quad \text{if} \quad d \geq 3,$$

and

$$K_d^{\mathrm{loc}} := \left\{ f : f 1_A \in K_d \quad \text{for each compact} \quad A \subset \mathbb{R}^d \right\}. \tag{2.1}$$

(V2) V is bounded from below and continuous, moreover $V(x) = a|x|^{2s} + o(|x|^{2s})$, with some $s > 1$ and $a > 0$.

Examples of Kato-class potentials include smooth functions bounded from below, and also potentials having some local (e.g. Coulomb) singularities. In particular, (V2) is a specific case of (V1). The sets K_d can also be characterized in terms of Wiener integrals.

For V having either of the regularity properties above define the Schrödinger operator H on $L^2(\mathbb{R}^d, dx)$ by

$$H = -\frac{1}{2}\Delta + V(x) \tag{2.2}$$

as a sum of quadratic forms (V is regarded as a multiplication operator). Then $C_0^\infty(\mathbb{R}^d)$ is a form core on which H is essentially self-adjoint and bounded from below. If the bottom of the spectrum E_0 of H is a simple eigenvalue, then the corresponding eigenfunction ψ_0 (ground state) is strictly positive. The semigroup e^{-tH}, $t \geq 0$ exists on $L^2(\mathbb{R}^d, dx)$, and it is an integral operator with positive, continuous, uniformly bounded kernel $G_t(x, y)$. For (V2)-type potentials the semigroup is moreover intrinsically ultracontractive which means the following. Take the probability measure $d\nu = \psi_0^2 dx$ on $\mathbb{R}^d$, and define the isometry $j : L^2(\mathbb{R}^d, d\nu) \to L^2(\mathbb{R}^d, dx)$, $f \mapsto \psi_0 f$. Take now on $L^2(\mathbb{R}^d, d\nu)$ the operator H_ν with $\mathrm{Dom}\, H_\nu = j^{-1}(\mathrm{Dom}\, H)$ defined by

$$H_\nu f = (j^{-1}(H - E_0)j)f = \frac{1}{\psi_0}(H - E_0)(\psi_0 f) = -\frac{1}{2}\Delta f - (\nabla \ln \psi_0, \nabla f)_{\mathbb{R}^d} \tag{2.3}$$

for every $f \in \mathrm{Dom}\, H_\nu$. Then the associated semigroup e^{-tH_ν} exists for all $f \in L^2(\mathbb{R}^d, d\nu)$ and $t \geq 0$. Intrinsic ultracontractivity of e^{-tH} means that e^{-tH_ν} is ultracontractive, i.e. it maps $L^2(\mathbb{R}^d, d\nu)$ into $L^\infty(\mathbb{R}^d, d\nu)$ continuously, or, equivalently, $||e^{-tH_\nu}||_{2,\infty} < \infty$, $\forall t \geq 0$.

Choose H to be a Schrödinger operator such that its ground state ψ_0 exists. Take $t_1 < \ldots < t_n \in \mathbb{R}$, $f_1, \ldots, f_n \in L^2(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$. The *$P(\phi)_1$-process* associated with H is the unique probability measure P on $\mathcal{X}$ defined by

$$\int f_1(X_{t_1}) \ldots f_n(X_{t_n}) dP(X) = (\psi_0 f_1, e^{-(t_2-t_1)H} f_2 \ldots e^{-(t_n-t_{n-1})H} f_n \psi_0)_{L^2(\mathbb{R}^d, dx)}. \tag{2.4}$$

That P is actually a probability measure is a consequence of $e^{-tH}\psi_0 = \psi_0$ and $\|\psi_0\|_{L^2} = 1$.

A $P(\phi)_1$-process is a reversible stationary Markov process with invariant measure $d\nu$ and generator H_ν, and it has almost surely continuous paths. It is moreover the stationary solution of the stochastic differential equation

$$dX_t = (\nabla \ln \psi_0)(X_t)\, dt + dB_t,$$

where B_t denotes Brownian motion on $\mathbb{R}^d$.

In order to give a Gibbsian description to P we need a few further notations. Take $\mathcal{X} = C(\mathbb{R}, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, endowed with the σ-field $\mathcal{A} = \sigma(\pi_t : t \in \mathbb{R})$ generated by the point evaluations $\pi_t : \mathcal{X} \to \mathbb{R}^d$, $X \mapsto \pi_t(X) = X_t$. These will be the configuration space and σ-field for the Gibbs measure, respectively. For $I \subset \mathbb{R}$ we denote by $\mathcal{A}_I$ the σ-field $\sigma(\pi_t : t \in I) \subset \mathcal{A}$, and for $I = [-T, T]$ we will use the notations $\mathcal{X}_T$ respectively $\mathcal{A}_T$.

Denote by $\mathcal{W}_I^{\xi,\eta}$ the Wiener measure conditional on starting in ξ at time T_1 and ending in η at time T_2, with $I = [T_1, T_2]$. Extend this Brownian bridge to a measure on $\mathcal{X}$ by picking any $Y \in \mathcal{X}$ and putting

$$\mathcal{W}_I^Y := \mathcal{W}_I^{Y_{T_1}, Y_{T_2}} \otimes \delta_{I^c}^Y \tag{2.5}$$

where $\delta_{I^c}^Y$ is Dirac measure on $C(I^c, \mathbb{R}^d)$ concentrated on $Y|_{I^c}$. $\mathcal{W}_I^Y$ is a finite measure on $(\mathcal{X}, \mathcal{A})$. We use $\mathcal{W}_I^Y$ further on as reference measure for the Gibbs measure to be constructed.

Take any $A \in \mathcal{A}$ and consider

$$dP_I(A|Y) := \frac{1}{Z_I(Y)} 1_A(X) e^{-\int_I V(X_s)ds}\, d\mathcal{W}_I^Y(X), \tag{2.6}$$

where

$$Z_I(Y) := \int e^{-\int_I V(X_s)ds}\, d\mathcal{W}_I^Y(X) \tag{2.7}$$

is the partition function turning P_I into a probability measure. The $P_I(\cdot|Y)$ are finite time interval Gibbs measures for the potential V and reference measure $\mathcal{W}_I^Y$ under boundary condition Y. The question here to answer is whether also an infinite time interval Gibbs measure exists.

Definition 2.1 *Let $\mathcal{X}^* \subset \mathcal{X}$. A probability measure $\mathcal{P}$ on $(\mathcal{X}, \mathcal{A})$ is called a* Gibbs measure *for the potential V and reference measure $\mathcal{W}_I^Y$ if for every $A \in \mathcal{A}$ and every bounded interval $I \subset \mathbb{R}$ the function $Y \mapsto \mathcal{P}_I(A|Y)$ given by (2.6) is a regular version of the conditional probability $\mathcal{P}(A|\,\mathcal{A}_{I^c})$. A Gibbs measure $\mathcal{P}$ is said to be supported by $\mathcal{X}^*$ if $\mathcal{P}(\mathcal{X}^*) = 1$.*

In the sense of the above definition we then have

Theorem 2.1 *Let P be the $P(\phi)_1$-measure corresponding to the potential V. P is a Gibbs measure for V and $\mathcal{W}_I^Y$.*

A Gibbs measure associated with a $P(\phi)_1$-process need not be unique. An example showing this is the Ornstein-Uhlenbeck process. Take $V_{\mathrm{OU}}(x) = \frac{1}{2}(x^2 - 1)$, $x \in \mathbb{R}$, and consider the corresponding Schrödinger operator H_{OU}. As well-known, its ground state is $\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$, and the $P(\phi)_1$-process corresponding to it is the one-dimensional Ornstein-Uhlenbeck process. Moreover, Mehler's formula gives explicitly the integral kernel of $e^{-tH_{\mathrm{OU}}}$. Fix $\alpha, \beta \in \mathbb{R}$ and define for $s, x \in \mathbb{R}$

$$\psi_s^l(x) := \pi^{-1/4} \exp\left(-\frac{1}{2}(x + \alpha e^{-s})^2\right) \exp\left(\frac{\alpha e^{-s}}{2}\right)^2,$$

$$\psi_s^r(x) := \pi^{-1/4} \exp\left(-\frac{1}{2}(x + \beta e^{s})^2\right) \exp\left(\frac{\beta e^{s}}{2}\right)^2.$$

A simple calculation shows that $e^{-tH_{\mathrm{OU}}}\psi_s^l = \psi_{s+t}^l$, $e^{-tH_{\mathrm{OU}}}\psi_s^r = \psi_{s-t}^r$, and $(\psi_s^l, \psi_s^r) = e^{\alpha\beta/2}$. Thus $\int f_1(X_{t_1})...f_n(X_{t_n}) dP_{\alpha,\beta}(X) = e^{-\alpha\beta/2}(\psi_{t_1}^l f_1, e^{-(t_2-t_1)H_{\mathrm{OU}}} f_2 \ldots e^{-(t_n-t_{n-1})H_{\mathrm{OU}}} f_n \psi_{t_n}^r)$

defines the finite dimensional distributions of a probability measure $P_{\alpha,\beta}$ on $C(\mathbb{R},\mathbb{R})$ of a Gaussian Markov process, which is stationary if and only if $\alpha=\beta=0$. Furthermore, it is easily seen that $P_{\alpha,\beta}$ is a Gibbs measure for every $\alpha,\beta\in\mathbb{R}$. In this case thus uncountably many Gibbs measures exist for the same potential.

A sufficient condition for uniqueness of the Gibbs measure in the sense of Definition 2.1 is provided by the following theorem. Recall that for the Schrödinger operator H the non-negative number $\Lambda := \inf\{\mathrm{Spec}(H)\setminus \inf\mathrm{Spec}(H)\}$ is the spectral gap of H.

Theorem 2.2 *Let H be a Schrödinger operator for a Kato-class potential V such that $\Lambda>0$, and let ψ_0 be its ground state. Put*

$$\mathcal{X}^* := \left\{ X\in\mathcal{X} : \lim_{|t|\to\infty} \frac{e^{-\Lambda|t|}}{\psi_0(X_t)} = 0 \right\}. \tag{2.8}$$

Then the $P(\phi)_1$-measure P corresponding to V is the unique Gibbs measure for V and $\mathcal{W}_I^Y$ supported by $\mathcal{X}^*$.

Restricting to (V2)-type potentials and by making use of the fact that for this class ψ_0 is bounded both from below and above by $C\exp(-\theta|x|^{s+1})$, with suitable constants $C,\theta>0$ for the two bounds respectively, we obtain from Theorem 2.2 that those paths are typical for the $P(\phi)_1$-measure that grow asymptotically like $t^{1/(s+1)}$.

By using the paths taken from $\mathcal{X}^*$ the Gibbs measure P can be constructed by taking limits over the finite time interval Gibbs measures (2.6). The actual support of μ can be shown to lie in $\{X\in\mathcal{X} : \limsup_{|t|\to\infty} g(t)/\psi_0(X_t)=0\}$ with a scale function $g:\mathbb{R}\to\mathbb{R}^+$ such that $\{g(k/2^n)\}_{k\in\mathbb{Z}}$ is summable for every $n\in\mathbb{N}$. For specific choices of V it can be checked that this gives a fairly good approximation of $\mathrm{supp}\,\mu$, in particular covering the results previously known in the literature. Even though $\mathcal{X}^*$ in general does not identify the largest possible set of boundary conditions for which the limiting Gibbs measure is the same, it carries full P-measure and thus it is a large enough subset for this purpose.

3 Gibbs measure for external and pair potential

Next we turn to discussing whether Gibbs measures can be defined also for Brownian motion subjected to both an external and a pair interaction potential. Such a process is in general not Markovian. This case is not accessible for simple spectral analysis and we will develop a cluster expansion. For details and proofs see [4].

We use the same set-up as before. The pair interaction potential is a function $W:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ with the (unessential) symmetry properties $W(\cdot,\cdot,t-s)=W(\cdot,\cdot,|t-s|)$, $W(x,y,\cdot)=W(y,x,\cdot)$, $x,y\in\mathbb{R}^d$, $s,t\in\mathbb{R}$, and satisfying either of the following regularity conditions:

(W1) There is $R>0$ and $\alpha>2$ such that

$$|W(x,y,t-s)| \;\le\; R\,\frac{|x|^2+|y|^2}{1+|t-s|^\alpha} \tag{3.1}$$

for every $x,y\in\mathbb{R}^d$ and $t,s\in\mathbb{R}$.

(W2) There is $R > 0$ and $\alpha > 1$ such that

$$|W(x, y, t - s)| \leq \frac{R}{1 + |t - s|^\alpha} \tag{3.2}$$

for every $s, t \in \mathbb{R}$ and uniformly in $x, y \in \mathbb{R}^d$.

Consider now a $P(\phi)_1$-process with stationary measure $d\nu = \psi_0^2 dx$ and transition probability density

$$g_t(x|y) = \frac{G_t(x, y)}{\psi_0(x)\psi_0(y)e^{-E_0 t}}. \tag{3.3}$$

Denote again the probability distribution of this process by P, and by $P_{[T_1,T_2]}$ its restriction to the field $\mathcal{A}_{[T_1,T_2]}$. We take this as reference measure in constructing the finite time interval Gibbs measures $\mu_{[T_1,T_2]}$ on $\mathcal{X}_{[T_1,T_2]}$ for any $A \in \mathcal{A}$ and boundary condition Y:

$$d\mu_{[T_1,T_2]}(A|Y) = \frac{1}{\mathcal{Z}_{[T_1,T_2]}(Y)} 1_A(X) e^{-\lambda W_{[T_1,T_2]}(X|Y)} dP_{[T_1,T_2]}(X|Y). \tag{3.4}$$

The partition function is

$$\mathcal{Z}_{[T_1,T_2]}(Y) = \int e^{-\lambda W_{[T_1,T_2]}(X|Y)} dP_{[T_1,T_2]}(X|Y). \tag{3.5}$$

Here

$$W_{[T_1,T_2]}(X|Y) = W_{[T_1,T_2]}(X) + W^Y_{[T_1,T_2]}(X) \tag{3.6}$$

is the total energy associated with configuration $X \in \mathcal{X}_{[T_1,T_2]}$ given the boundary configuration $Y = Y^- \cup Y^+$, with $Y^- \in \mathcal{X}_{(-\infty,T_1]}$ resp. $Y^+ \in \mathcal{X}_{[T_2,\infty)}$. Term by term,

$$W_{[T_1,T_2]}(X) = \int_{T_1}^{T_2} \int_{T_1}^{T_2} W(X_t, X_s, s - t) ds dt \tag{3.7}$$

is the "internal energy" associated with the path inside $[T_1, T_2]$, and

$$W^Y_{[T_1,T_2]}(X) = 2 \int_{-\infty}^{T_1} dt \int_{T_1}^{T_2} ds\, W(Y_t^-, X_s, t - s) + 2 \int_{T_1}^{\infty} dt \int_{T_1}^{T_2} ds\, W(Y_t^+, X_s, t - s) \tag{3.8}$$

is the "interaction energy" between X and the boundary path Y. We calibrate the interaction energy such that $W^0_{[T_1,T_2]}(X) = 0$. $P_{[T_1,T_2]}(\,\cdot\,|Y) = P_{[T_1,T_2]}(\,\cdot\,|Y^-_{T_1}, Y^+_{T_2})$ is the conditional distribution of the reference measure for the given boundary condition Y which depends only on the positions attained at T_1 resp. T_2 since P is Markovian. It is easily checked that $\mu_{[T_1,T_2]}(\,\cdot\,|Y)$, with $Y \in C([T_1, T_2]^c, \mathbb{R}^d)$, $-\infty < T_1 < T_2 < \infty$, is a family of finite time interval Gibbs measures. Finally, $\lambda \in \mathbb{R}$ is a parameter which can be interpreted as the strength of the coupling of the pair interaction applied on the Brownian paths.

We are interested whether the sequence of measures $\mu_{[T_1,T_2]}$ has a limit as $T_1 \to -\infty$ and $T_2 \to \infty$. The following notion of convergence will be used. Let generally M be a metric space, and $C(\mathbb{R}, M)$ the space of continuous paths $\{X_t\}$ with values in M. For any interval $I = [T_1, T_2] \subset \mathbb{R}$ let $\mathcal{M}_I \subset \mathcal{M}$ be a sub-σ field of the Borel σ-field $\mathcal{M}$ of M generated by the evaluations $\{X_t : t \in I\}$. The sequence of probability measures $\{\mu_n\}$ on $C(\mathbb{R}, M)$ is said to converge locally weakly to the probability measure μ if for any $-\infty < T_1 < T_2 < \infty$ the restrictions $\mu_n|_{\mathcal{M}_I}$ converge weakly to the measure $\mu_{\mathcal{M}_I}$. We have then the following result.

Theorem 3.1 *Suppose V and W satisfy assumptions (V2), respectively either (W1) or (W2). Take any decreasing sequence $T_1^{(n)}$ and any increasing sequence $T_2^{(n)}$ of real numbers, $T_1^{(n)} < T_2^{(n)}$, such that $T_1^{(n)} \to -\infty$, $T_2^{(n)} \to \infty$, and suppose $0 < |\lambda| \leq \lambda^*$ with λ^* small enough. Then the weak local limit $\lim_{n\to\infty} \mu_{[T_1^{(n)}, T_2^{(n)}]} = \mu$ exists and it is a Gibbs probability measure on $(\mathcal{X}, \mathcal{A})$. Moreover, μ does not depend on the sequences $T_1^{(n)}, T_2^{(n)}$.*

In order to prove existence of the Gibbs measure μ we use a cluster expansion controlled by the small parameter λ. In the next subsection we derive the cluster representation of the partition function (3.5) and outline the main steps of the proof. For details we refer to [4].

3.1 Cluster expansion for the partition function

For simplicity (and without restricting generality) from now on we consider only intervals $[-T, T]$ and the corresponding finite time interval Gibbs measures μ_T. Also, for simplicity we use free boundary conditions, i.e. we choose $Y = 0$ in (3.4).

Take a division of $[-T, T]$ into disjoint intervals $\tau_k = (t_k, t_{k+1})$, $k = 0, ..., N-1$, with $t_0 = -T$ and $t_N = T$, each of length b, i.e. fix $b = 2T/N$; for convenience we choose N to be an even number so that the origin is endpoint to some intervals. We break up a path X into pieces X_{τ_k} by restricting it to τ_k. The total energy contribution of the pair interaction can be written in terms of the sum

$$W_T := \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t) ds dt = \sum_{0 \leq i < j \leq N-1} W_{\tau_i, \tau_j} \tag{3.9}$$

where with the notation $\mathcal{J}_{ij} = \int_{\tau_i} dt \int_{\tau_j} W(X_s, X_t, s-t) ds$ we have

$$W_{\tau_i,\tau_j} = \begin{cases} \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| \geq 2 \\ \frac{1}{2}(\mathcal{J}_{ii} + \mathcal{J}_{jj}) + \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| = 1, \text{ and } i \neq 0,\ j \neq N-1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\, \mathcal{J}_{00} & \text{if } i = 0 \text{ and } j = 1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\, \mathcal{J}_{N-1\ N-1} & \text{if } i = N-1 \text{ and } j = N-2 \end{cases} \tag{3.10}$$

(For keeping the notation simple we do not make explicit the X dependence of these objects.) The following estimate on the double integrals will be useful later on.

Lemma 3.2 *We have*

$$\int_{\tau_i} dt \int_{\tau_j} W(X_t, X_s, s-t)ds \ \leq \ \begin{cases} C_1 b \, \frac{\int_{\tau_i} X^2_{\tau_i}(t)dt + \int_{\tau_j} X^2_{\tau_i}(s)ds}{(|j-i-1|b)^{\alpha}+1} & \text{(W1) case} \\ \frac{C_2 b^2}{(|j-i-1|b)^{\alpha}+1} & \text{(W2) case} \end{cases} \tag{3.11}$$

with some $C_1, C_2 > 0$ in each case respectively.

By using (3.9) we obtain

$$e^{-\lambda W_T} = \prod_{0 \leq i < j \leq N-1} (e^{-\lambda W_{\tau_i,\tau_j}} + 1 - 1) = 1 + \sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i,\tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i,\tau_j}} - 1). \tag{3.12}$$

Here the summation is performed over all nonempty sets of different pairs of intervals, i.e. $\mathcal{R} = \{(\tau_i, \tau_j) : (\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'}) \text{ whenever } (i,j) \neq (i',j')\}$.

In order to keep this and the forthcoming summations in hand we need a few more notations. Two distinct pairs of intervals (τ_i, τ_j) and $(\tau_{i'}, \tau_{j'})$ are called *directly connected* and denoted $(\tau_i, \tau_j) \sim (\tau_{i'}, \tau_{j'})$ if one interval of the pair (τ_i, τ_j) coincides with one interval of the pair $(\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is a collection $\{(\tau_{i_1}, \tau_{j_1}), ..., (\tau_{i_n}, \tau_{j_n})\}$ in which each pair of intervals is connected to another through a sequence of directly connected pairs, i.e., for any $(\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'})$ there exists $\{(\tau_{k_1}, \tau_{l_1}), ..., (\tau_{k_m}, \tau_{l_m})\}$ such that $(\tau_i, \tau_j) \sim (\tau_{k_1}, \tau_{l_1}) \sim ... \sim (\tau_{k_m}, \tau_{l_m}) \sim (\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is called a *contour* and denoted by γ. We denote by $\bar{\gamma}$ the set of all intervals that are elements of the pairs of intervals belonging to a contour γ, and by γ^* the set of time-points of intervals appearing in $\bar{\gamma}$. Clearly, $\mathcal{R}$ can be decomposed into maximal connected components, i.e. contours: $\mathcal{R} = \{\gamma_1, ..., \gamma_r\}$ with $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, $i \neq j$; $i, j = 1, ..., r$.

The sum in (3.12) is then further expanded as

$$\sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i,\tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i,\tau_j}} - 1) = \sum_{r \geq 1} \sum_{\gamma_1, ..., \gamma_r} \prod_{k=1}^{r} \prod_{(\tau_i,\tau_j) \in \gamma_k} (e^{-\lambda W_{\tau_i,\tau_j}} - 1) \tag{3.13}$$

where now summation goes over collections $\{\gamma_1, ..., \gamma_r\}$ of contours such that $\bar{\gamma}_k \cap \bar{\gamma}_{k'} = \emptyset$ unless $k = k'$.

A collection of consecutive intervals $\{\tau_j, \tau_{j+1} ..., \tau_{j+k}\}$, $j \geq 0$, $j + k \leq N - 1$ is called a *chain*. As in the case of contours, $\bar{\varrho}$ and ϱ^* mean the set of intervals belonging to the chain ϱ and the set of time-points in ϱ, respectively. We call two contours γ_1, γ_2 disjoint if they have no intervals in common, i.e. $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$. Two chains ϱ_1, ϱ_2 are called disjoint if they have no common time-points, i.e. $\varrho_1^* \cap \varrho_2^* = \emptyset$. Take now a non-ordered set of disjoint contours and disjoint chains, $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$, with some $r \geq 1$ and $s \geq 0$. Note that such contours and chains may have common time-points. We use the notation $\Gamma^* = (\cup_i \gamma_i^*) \cup (\cup_j \varrho_j^*)$ for the set of all time-points appearing as beginnings or ends of intervals belonging to some contour or chain in Γ. Also, we put $\bar{\Gamma} = (\cup_i \bar{\gamma}_i) \cup (\cup_j \bar{\varrho}_j)$ for the set of intervals appearing in Γ through entering some contours or chains. Denote by $\partial^- \gamma$ resp. $\partial^+ \gamma$ the leftmost resp. rightmost time-points belonging to γ, and the similar objects for ϱ. Γ is called a *cluster* if $\{\gamma_1^*, ..., \gamma_r^*; \varrho_1^*, ..., \varrho_s^*\}$ is a connected collection of sets and for every $\varrho \in \Gamma$ we have that $\partial^- \varrho_l, \partial^+ \varrho_l \in \cup_{j=1}^r \gamma_j^*$. This means that in a cluster chains have no loose ends.

Next we fix the positions of path X at the time-points of the division, i.e we put $X_{t_k} = q_k$, for all $k = 0, ..., N$, with $-T = t_0 < t_1 < ... < t_N = T$. The distribution of path X in interval $[-T, T]$ conditional on the positions attained at the fixed times becomes

$$dP_T(X_{\tau_0}, ..., X_{\tau_{N-1}} | X_{t_0} = q_0, X_{t_1} = q_1, ..., X_{t_N} = q_N) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} | q_k, q_{k+1}). \quad (3.14)$$

We use the shorthand at the right hand side for the corresponding conditional probabilities for easing the notation. Let $p_{t_0,...,t_N}(q_0, ..., q_N)$ be the density with respect to $\prod_{k=0}^N d\nu_k(q_k)$ of the joint distribution of positions of path X recorded at the time-points $t_0, ..., t_N$. Here $d\nu_k$ denotes a copy of $d\nu$ for each $k = 0, ..., N$. By the Markov property it then follows that

$$\begin{aligned} p_{t_0,...,t_N}(q_0, ..., q_N) &= \prod_{k=0}^{N-1} g_b(q_{k+1}|q_k) = \prod_{k=0}^{N-1} (g_b(q_{k+1}|q_k) - 1 + 1) \\ &= 1 + \sum_{\mathcal{M}} \prod_{k:\tau_k \in \mathcal{M}} ((g_b(q_{k+1}|q_k) - 1). \end{aligned}$$

The summation is extended over all nonempty sets $\mathcal{M} = \{\tau_k = (t_k, t_{k+1})\}$ of different pairs of consecutive time-points. In a similar way as before the latter formula can be rewritten in the form

$$\sum_{\mathcal{M}} \prod_{k:\tau_k \in \mathcal{M}} (g_b(q_{k+1}|q_k) - 1) = \sum_{s \geq 1} \sum_{\varrho_1, ..., \varrho_s} \prod_{j=1}^s \prod_{k:\tau_k \in \varrho_j} (g_b(q_{k+1}|q_k) - 1). \quad (3.15)$$

Here $\{\varrho_1, ... \varrho_s\}$ is a collection of connected sets of intervals τ_k such that $\varrho_k^* \cap \varrho_{k'}^* = \emptyset$ unless $k = k'$, i.e. disjoint chains, and this formula explains the way we defined them before.

For every cluster $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$ define the function

$$\kappa_\Gamma = \prod_{l=1}^r \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i \tau_j}} - 1) \prod_{i=1}^s \prod_{k:\tau_k \in \varrho_i} (g_b(q_{k+1}|q_k) - 1). \quad (3.16)$$

Also, introduce the auxiliary probability measure on $\mathcal{X}_T$

$$d\mathcal{P}_T(X) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} | q_k, q_{k+1}) \prod_{k=0}^N d\nu_k(q_k) \quad (3.17)$$

and look at

$$K_\Gamma = \mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma]. \quad (3.18)$$

Note that $\int (g_b(q_{k+1}|q_k) - 1) d\nu(q_{k+1}) = \int (g_b(q_{k+1}|q_k) - 1) d\nu(q_k) = 0$. This is the reason why from a cluster we rule out chains having loose ends; for any such chain $\mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma] = 0$.

By putting (3.13), (3.14), (3.15), (3.16) and (3.18) together we obtain the cluster representation of the partition function $\mathcal{Z}_T$:

Proposition 3.3 *For every $T > 0$*

$$\mathcal{Z}_T = 1 + \sum_{n \geq 1} \sum_{\{\Gamma_1, ..., \Gamma_n\}} \prod_{l=1}^{n} K_{\Gamma_l}. \tag{3.19}$$

Here the summation is performed over non-ordered collections of clusters $\{\Gamma_1, ..., \Gamma_n\} \neq \emptyset$ for which $\Gamma_i^* \cap \Gamma_j^* = \emptyset$ whenever $i \neq j$.

As soon as the the cluster representation for $\mathcal{Z}_T$ is established, the existence of the weak limit measure $\mu := \lim_{T \to \infty} \mu_T$ follows by the cluster estimates below and the general arguments presented in e.g. [6], Chapter 3.

3.2 Cluster estimates

The first crucial estimate for the cluster expansion is given by

Proposition 3.4 *For every cluster Γ we have the bound*

$$|K_\Gamma| \;\leq\; \prod_{\varrho \in \Gamma} (c_1 |\lambda|^{1/3})^{|\bar{\varrho}|} \prod_{\gamma \in \Gamma} \prod_{(\tau_i, \tau_j) \in \gamma} \frac{c_2 |\lambda|^{1/3}}{(|i - j - 1| b)^\delta + 1} \tag{3.20}$$

with $|\bar{\varrho}|$ denoting the number of intervals contained in ϱ, and with constants $c_1, c_2 > 0$ and exponent $\delta > 1$.

In estimate (3.20) the factor accounting for the contribution of chains comes from the uniform upper bound $Ce^{-\Lambda b}$ on $|g_b(q|q') - 1|$ (see second factor in (3.16)), where Λ is the spectral gap for the Schrödinger operator of the underlying $P(\phi)_1$-process, and $C > 0$. This bound, in its turn, is a consequence of the intrinsic ultracontractivity of e^{-tH}, compare Sect. 2. The factor accounting for the contribution of contours comes from an estimate using a generalized variant of the Hölder inequality applied to the products over $e^{-\lambda W_{\tau_i \tau_j}} - 1$ (see first factor in (3.16)), taken together with Lemma 3.2. b is finally chosen in such a combination with λ and Λ that the expression (3.20) results.

The second fundamental estimate leading up to ensuring the convergence of the cluster expansion is

Proposition 3.5 *There is a constant $c > 0$, independent of λ, and a number $0 < \eta(\lambda) < 1$ (with $\eta \to 0$ as $\lambda \to 0$) such that*

$$\sum_{\substack{\Gamma : \Gamma^* \ni 0 \\ |\bar{\Gamma}| = n}} |K_\Gamma| \;\leq\; c\, \eta^n. \tag{3.21}$$

with $|\bar{\Gamma}|$ denoting the number of intervals contained in Γ through some contour or chain.

This estimate follows through a procedure of translating the summation in the left hand side of (3.21) into a combinatorial problem and resumming over (and counting of) first graphs and then trees. The contours are assigned vertices and they are linked into graphs according to the rules connecting them up into clusters.

3.3 Typical path behaviour

An important aspect in understanding the Gibbs measure is what a typical path configuration looks like under it. This is answered by

Theorem 3.6 *With μ-probability 1 we have*

$$|X_t| \leq C\,(\log(|t|+1))^{1/(s+1)} + Q(X) \tag{3.22}$$

with a suitable number $C > 0$ and a functional $Q(X)$, independent of t.

The strategy of proving Theorem 3.6 goes by boosting the typical behaviour of the reference process to the level of the Gibbs measure.

Lemma 3.7 *Take an arbitrary $a > 0$. Then there exist $C > 0$ and $\theta > 0$ such that*

$$P\left(\{X \in \mathcal{X} : \max_{0\leq t\leq 1} |X_t| \geq a\}\right) \leq C\, e^{-\theta a^{s+1}}. \tag{3.23}$$

This can be proven by using Varadhan's Lemma taken together with the upper bound $\exp(-\theta|x|^{s+1})$ for ψ_0 (the ground state of the Schrödinger operator generating the reference process). Then Theorem 3.6 comes about by

Lemma 3.8 *Suppose there exist some numbers $C, \theta > 0$ such that for any $a > 0$*

$$P\left(\{X \in \mathcal{X} : \max_{0\leq t\leq 1} |X_t| \geq a\}\right) \leq C\, e^{-\theta a^{s+1}}. \tag{3.24}$$

Then there exist $C' > 0$ and $\theta' > 0$ such that for any $a > 0$

$$\mu\left(\{X \in \mathcal{X} : \max_{0\leq t\leq 1} |X_t| \geq a\}\right) \leq C' e^{-\theta' a^{s+1}}. \tag{3.25}$$

The proof of this lemma requires once again the use of cluster expansion.

It may actually be conjectured that a result stronger than Theorem 3.6 can be obtained from the following fact proven in [1] on the reference measure. The set of paths satisfying

$$\liminf_{|t|\to\infty} |t|^{\delta}\psi_0(X_t) > 0, \quad \forall \delta > 1 \tag{3.26}$$

has P-measure 1. Thus we expect that a similar property holds also for μ.

3.4 Uniqueness of the Gibbs measure for bounded pair potentials

Uniqueness (in DLR-sense) means that for any increasing sequence or real numbers $\{T_n\}$, $\lim_{n\to\infty} T_n = \infty$, and any corresponding sequence of boundary conditions $\{Y_n\} \subset \mathcal{X}^*$, $\lim_{n\to\infty} \mathbb{E}_{\mu_{T_n}}[F_B|Y_n] = \mathbb{E}_\mu[F_B]$, for every bounded $B \subset \mathbb{R}$ and bounded local function F_B (i.e., measurable with respect to $\mathcal{A}_B$). Here $\mathcal{X}^*$ is the set given by Theorem 2.2.

Theorem 3.9 *Suppose V is of class (V2) and W satisfies (W2). Then we have the following cases:*

1. *If $\alpha > 2$, then whenever the Gibbs measure μ exists, it is unique in DLR sense.*
2. *If $\alpha > 1$, then for sufficiently small $|\lambda|$ the limiting Gibbs measure μ is unique in DLR sense whenever the reference measure is unique.*

If $\alpha > 2$, $|W_T^Y(X)|$ (given by (3.8)) is uniformly bounded in T, and paths X and Y. This implies (since the index set of Gibbs measures is $\mathbb{R}$, i.e. "one dimensional") that only one Gibbs measure can exist. This argument requires no restriction on the values of λ. For $\alpha > 2$ this uniform boundedness does not hold any longer and we once again recourse to cluster expansion.

3.5 Some further properties of the Gibbs measure for bounded pair potentials

Here we list some additional properties of the Gibbs measure for (W2)-type pair potentials useful in various contexts. This case in particular covers Nelson's scalar field model which is discussed in detail in [5, 2].

Theorem 3.10 *Let μ be a Gibbs measure on $(\mathcal{X}, \mathcal{A})$ for V satisfying assumption (V2) and W satisfying (W2). Then the following hold:*

1. [Invariance properties] *μ is invariant with respect to time shift and time reflection:*
$$\mu \circ \tau_t = \mu, \quad \forall t \in \mathbb{R}, \quad \textit{where} \quad (\tau_s X)_t = X_{s+t},$$
$$\mu \circ \vartheta = \mu, \quad \textit{where} \quad (\vartheta X)_t = X_{-t}.$$

2. [Single time distributions] *The distributions φ_T under μ_T of positions q at time $t = 0$ are absolutely continuous with respect to ν, i.e. there exist $C_1, C_2 \in \mathbb{R}$, independent of T and q such that*
$$C_1 \;\leq\; \frac{d\varphi_T}{d\nu}(q) \;\leq\; C_2 \tag{3.27}$$
for every $q \in \mathbb{R}^d$ and $T > 0$. Moreover $\lim_{T\to\infty}(d\varphi_T/d\nu)(q) = (d\varphi/d\nu)(q)$ exists pointwise.

3. [Single time conditional distributions] *The conditional distributions $\mu_T(\,\cdot\,|X_0 = q)$ converge locally weakly to $\mu(\,\cdot\,|X_0 = q)$, for all $q \in \mathbb{R}^d$.*

4. [Mixing properties] *For any bounded functions F, G on $\mathbb{R}^d$ we have on the covariance $\operatorname{cov}_\mu(F_s; G_t) := \mathbb{E}_\mu[F_s G_t] - \mathbb{E}_\mu[F_s]\,\mathbb{E}_\mu[G_t]$ the estimate*
$$|\operatorname{cov}_\mu\,(F_s; G_t)| \;\leq\; \text{const}\; \frac{\sup|F_s|\,\sup|G_t|}{1 + |t-s|^\beta} \tag{3.28}$$
where $\beta > 0$, $F_s := F(X_s)$, $G_t := G(X_t)$, and the constant prefactor is independent of s, t and F, G.

The invariance properties are inherited trivially from the underlying $P(\phi)_1$-process. The proof of the remaining properties makes extensive use of cluster expansion. Not surprisingly, the power law fall-off of correlations follows the power law character of the potentials' decay.

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