The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field

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Abstract

We prove that Nelson's massless field model is infrared divergent in three dimensions. In particular, the Nelson Hamiltonian has no ground state in Fock space and thus it is not unitarily equivalent with the Hamiltonian obtained from the Euclidean quantization. In contrast, for dimensions higher than three the Nelson Hamiltonian has a unique ground state in Fock space and the two Hamiltonians are unitarily equivalent. We also show that the Euclidean Hamiltonian has no spectral gap.

KEYWORDS: Nelson's scalar field model, infrared divergence, ground state, Gibbs measure

1 Introduction

A quantum particle coupled to a free Bose field is described by the Hamiltonian

$$\begin{aligned} H_{\mathrm{gN}} &= \left(-\frac{1}{2}\Delta + V(q)\right) \otimes 1 + 1 \otimes \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk \\ &+ \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} \frac{1}{(2\pi)^{d/2}} \left(\hat{\varrho}(k) e^{ik\cdot q} \otimes a(k) + \hat{\varrho}(-k) e^{-ik\cdot q} \otimes a^*(k)\right) dk. \end{aligned} \tag{1.1}$$

In the first term we recognize the Schrödinger operator $H_{\mathrm{p}} = -(1/2)\Delta + V$ for the quantum particle. V is assumed to be a confining potential which grows at infinity, in particular H_{p} has a purely discrete spectrum. The second term is the energy of the Bose field with dispersion relation $\omega(k) \geq 0$. As a shorthand we set $H_{\mathrm{f}} = \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk$. $a^*(k), a(k)$ are the Bose creation and annihilation operators which satisfy the canonical commutation relations (CCR) $[a(k), a^*(k')] = \delta(k-k')$, $[a(k), a(k')] = 0 = [a^*(k), a^*(k')]$. The last term in (1.1) describes a translation invariant interaction between the particle and the field, which is linear in the field operators. ϱ is the charge distribution and mollifies the coupling between particle and field. We assume ϱ to be smooth, radial and of rapid decay. $\hat{\varrho}$ denotes the Fourier transform of ϱ. We use the normalization $\int_{\mathbb{R}^d} \varrho(x) dx = e \geq 0$; e calibrates then the strength of the interaction. H_{gN} acts on $L^2(\mathbb{R}^d, dq) \otimes \mathcal{F}_{\mathrm{sym}}$ with $\mathcal{F}_{\mathrm{sym}}$ the symmetric Fock space over $L^2(\mathbb{R}^d, dk)$. If $\int_{\mathbb{R}^d} |\hat{\varrho}(k)|^2 (\omega(k)^{-1} + \omega(k)^{-2}) dk < \infty$, then H_{gN} is bounded from below and the interaction is infinitesimally H_{f}-bounded. Therefore H_{gN} is a self-adjoint operator with domain $D(H_{\mathrm{p}} \otimes 1) \cap D(1 \otimes H_{\mathrm{f}})$. ϱ introduces an ultraviolet cut-off, which may be removed by letting $\varrho(x) \to e\delta(x)$ or, equivalently, $\hat{\varrho}(k) \to (2\pi)^{-d/2} e$. In his famous paper [17] Nelson proved that in the case $\omega(k) = \sqrt{k^2 + m_{\mathrm{b}}^2}$, $m_{\mathrm{b}} > 0$, at the expense of renormalizing the energy, H_{gN} stays well-defined in the point charge limit. Here we will keep the ultraviolet cut-off and want to study the infrared behaviour, which is of interest only for massless bosons, $m_{\mathrm{b}} = 0$, i.e.

$$\omega(k) = |k|. \tag{1.2}$$

We call H_{gN} with the dispersion relation (1.2) the massless Nelson model and denote the corresponding Hamiltonian by H_{N}. For $d = 1, 2$ the Nelson Hamiltonian is not bounded from below, thus we require $d \geq 3$. In fact, we will mostly concentrate on dimension $d = 3$ for which, as proved below, the massless Nelson model is infrared divergent.

Before stating our specific project let us explain in general terms the phenomenon of infrared divergence in the context of Hamiltonians of the type (1.1). It is convenient to generalize somewhat and start from

$$H_{\mathrm{G}} = K \otimes 1 + 1 \otimes H_{\mathrm{f}} + \int_{\mathbb{R}^d} (v(k) \otimes a^*(k) + v(k)^* \otimes a(k)) dk. \tag{1.3}$$

K is a self-adjoint operator acting on the particle space $\mathcal{K}$ and it is assumed to have a purely discrete spectrum. $k \mapsto v(k)$ is a function with values in $\mathcal{K}$ such that H_{G} is bounded from below. Clearly, H_{gN} is the special case where $\mathcal{K} = L^2(\mathbb{R}^d, dq)$, $K = H_{\mathrm{p}}$,

$v(k) = (2\pi)^{-d/2} e^{-ik\cdot q} \hat{\varrho}(-k)/(2\omega(k))^{1/2}$. The Hamiltonian H_{G} acts on $\mathcal{K}\otimes\mathcal{F}_{\mathrm{sym}}$. We say that H_{G} is *infrared divergent* if it has no ground state in $\mathcal{K}\otimes\mathcal{F}_{\mathrm{sym}}$. The physical ground state has then an infinite number of very low energy bosons, which forces a representation of the CCR different from the Fock representation. H_{G} should be regarded as a formal object only and one has first to construct the Hilbert space containing the physical ground state, and the corresponding Hamiltonian.

Ideally, one would like to decide infrared divergence directly in terms of K and v in (1.3). At present, this issue is only partially understood. Gérard [9] proves that if, beside some conditions ensuring the self-adjointness of H,

$$\int_{\mathbb{R}^d} \omega(k)^{-2} \|v(k)(K+1)^{-1}\|^2 dk \ < \ \infty \tag{1.4}$$

holds, then H_{G} has a ground state in $\mathcal{K}\otimes\mathcal{F}_{\mathrm{sym}}$. In the particular case of (1.1), if

$$\int_{\mathbb{R}^d} |\hat{\varrho}(k)|^2 \left(\omega(k) + \omega(k)^{-2} + \omega(k)^{-3}\right) dk \ < \ \infty, \tag{1.5}$$

then H_{gN} has a unique ground state in $L^2(\mathbb{R}^d, dx)\otimes\mathcal{F}_{\mathrm{sym}}$, [24]. Thus (1.5) has to be violated for H_{gN} to be infrared divergent. We note that for the massless Nelson model in three dimensions indeed

$$\int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{|k|^3} dk \ = \ \infty. \tag{1.6}$$

On the other hand, by (1.5) the massless Nelson model is infrared convergent for $d \geq 4$.

The integrals in (1.4), resp. (1.5), becoming infinite is only a necessary condition for infrared divergence to occur. To conjecture sufficient conditions we have to rely more closely on model systems. The best understood model is the spin-boson Hamiltonian where $\mathcal{K} = \mathbb{C}^2$, $K = \sigma_x$, $v(k) = \lambda(k)\sigma_z$, and σ_x, σ_z being the Pauli matrices. Some indication comes from the generalized spin-boson models studied in [1], where $v(k) = \sum_{j=1}^{J} B_j \lambda_j(k)$ with $B_j = B_j^*$ acting on $\mathcal{K}$. From these examples we infer that, in addition to (1.4) turning infinite, $v(k)$ must have non-vanishing ground state expectation for small k. If this happens, the model is divergent for *all* couplings $e \neq 0$. If, however, $v(0)$ has zero ground state expectation then there may occur cancellations making the model stay infrared convergent in spite of the integral in (1.4) being infinite. This is the case for the spin-boson Hamiltonian with $v(k) = \alpha\lambda(k)(\sigma_z + m)$ and $\int_{\mathbb{R}^d} |\lambda|^2/\omega^2 dk = \infty$. If $m \neq 0$, the model is infrared divergent for all $\alpha \neq 0$. If $m = 0$, cancellations appear and for sufficiently small α the model is infrared convergent while for large α it is divergent [24]. The same phenomenon occurs in the dipole approximation of the massless Nelson model in $d = 1$ where $\mathcal{K} = L^2(\mathbb{R}, dq)$, $K = H_{\mathrm{p}}$, $v(k) = q\cdot\hat{\varrho}(k)$ [19]. If $V(q) = V(-q)$, infrared divergence depends on how strong the coupling is. Otherwise the model is infrared divergent for all couplings $e \neq 0$. In the three dimensional Nelson Hamiltonian we have

$$v(k) = \frac{e}{\sqrt{2|k|}} \ > \ 0 \tag{1.7}$$

for small k. Thus we expect, and will in fact prove, that H_{N} is infrared divergent for all couplings $e \neq 0$.

Fröhlich [8] considers the massless Nelson model for $V=0$ at fixed total momentum p in which case the Hamiltonian reads

$$H(p)=\frac{1}{2}\left(p-\int_{\mathbb{R}^3} ka^*(k)a(k)dk\right)^2+H_{\mathrm{f}}+\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\frac{1}{\sqrt{2|k|}}(\hat{\varrho}(k)a(k)+\hat{\varrho}(-k)a^*(k))dk \tag{1.8}$$

acting on $\mathcal{F}_{\mathrm{sym}}$. With infrared cut-off and for sufficiently small $|p|$, $H(p)$ has a unique ground state in $\mathcal{F}_{\mathrm{sym}}$. Removing the infrared cut-off yields a physical ground state outside Fock space and a non-Fock representation of CCR. [8] develops the C^*-algebraic approach. In contrast, in our work we follow the route of constructive quantum field theory by introducing a suitable probability measure on the space of classical paths. In terms of this path measure we can define both the physical ground state and the Hamiltonian governing the excitations relative to the ground state. While the explicit construction will take the rest of the paper, we first want to explain some of the particular features of the path measure for H_{N}.

In the Euclidean picture we have the trajectories $t\mapsto q_t$ of the particle, resp. $t\mapsto \xi_t(x)$, $x\in\mathbb{R}^d$, of the field. To attach to them a statistical weight we define the formal Euclidean action by

$$S_{\mathrm{p}}(\{q_t\})=\int_{\mathbb{R}}\left(\frac{1}{2}\dot{q}_t^2+V(q_t)\right)dt \tag{1.9}$$

for the particle, and

$$S_{\mathrm{f}}(\{\xi_t\})=\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{1}{2}\left((\partial_t\xi_t(x))^2+(\nabla\xi_t(x))^2\right)dxdt \tag{1.10}$$

for the field. From (1.1) we infer that the interaction restricted to the time interval $[-T,T]$ is given by

$$S_{\mathrm{int},T}(\{q_t,\xi_t\})=\int_{-T}^{T}\int_{\mathbb{R}^d}\varrho(x-q_t)\xi(t,x)dxdt=\int_{-T}^{T}(\xi_t*\varrho)(q_t)dt \tag{1.11}$$

with $*$ denoting convolution. Thus, heuristically, the Euclidean path measure is

$$\frac{1}{Z}\prod_{t\in\mathbb{R}}dq_t e^{-S_{\mathrm{p}}(\{q_t\})}\prod_{(t,x)\in\mathbb{R}^{d+1}}d\xi_t(x)\ e^{-S_{\mathrm{f}}(\{\xi_t\})}\ e^{-S_{\mathrm{int},T}(\{q_t,\xi_t\})}. \tag{1.12}$$

The first factor above provides an a priori weight for the paths q_t. The corresponding path measure is denoted by $\mathcal{N}^0$, compare with Section 2.1 below. The second factor provides an a priori weight for the paths ξ_t. Clearly, it is a Gaussian measure which we denote by $\mathcal{G}$, see Section 2.2 below. In the spirit of the Feynman-Kac formula we define the semigroup

$$(F,e^{-tH}G)=\mathbb{E}_{\mathcal{N}^0\times\mathcal{G}}[F(q_0,\xi_0)G(q_t,\xi_t)e^{-\int_0^t(\xi_s*\varrho)(q_s)ds}] \tag{1.13}$$

$t\geq 0$. Then e^{-tH} is a contractive semigroup on $L^2(\mathbb{R}^d,d\mathsf{N}^0)\otimes L^2(\mathcal{S}',d\mathsf{G})$, where $d\mathsf{N}^0$ and $d\mathsf{G}$ are the single time distributions under $\mathcal{N}^0$ and $\mathcal{G}$, respectively. e^{-tH} is unitarily equivalent with $e^{-tH_{\mathrm{N}}}$, $t\geq 0$, on $L^2(\mathbb{R}^d,dq)\otimes\mathcal{F}_{\mathrm{sym}}$.

To construct the physical ground state and its associated Hilbert space we have to give sense to the path measure (1.12) in the infinite time limit $T \to \infty$. With respect to the limit measure (whenever it exists), the process $t \mapsto q_t, \xi_t(x)$ jointly is a time reversible stationary Markov process. Therefore this stochastic process has a canonically defined semigroup T_t, $t \geq 0$, self-adjoint on the Hilbert space weighted by the time $t = 0$ stationary measure. Thus T_t has a self-adjoint generator so that $T_t = e^{-tH_{\rm euc}}$, $t \geq 0$, which by definition is the Euclidean Hamiltonian $H_{\rm euc}$. Clearly, by construction, $H_{\rm euc} \geq 0$ and $H_{\rm euc}1 = 0$, i.e. $H_{\rm euc}$ has an eigenvalue at the bottom of its spectrum.

If $H_{\rm N}$ is infrared convergent, then $H_{\rm euc}$ and H are unitarily equivalent and one may work with either of them on an equal footing. However, in the infrared divergent case they are not unitarily equivalent. Since $H_{\rm euc}$ accomodates the ground state, it should be regarded as the proper Hamiltonian of the system.

A short outline of the paper is the following. We recall in Section 2 the basic processes leading up to Section 3 where the definition of the joint particle-field path measure appearing in (1.12) is given. Based on the cluster expansion in [13], under suitable assumptions on V and for sufficiently small e, we establish the existence of the infinite time measure. Thereby $H_{\rm euc}$ is well defined. In Section 4 we characterize infrared divergence in terms of the path measure for the interacting system and prove that the massless Nelson model in $d = 3$ is infrared divergent. In Section 5 we discuss the infrared convergent case appearing for $d \geq 4$. Finally, in Section 6 we show that $H_{\rm euc}$ has no spectral gap.

We remark that the quantized Maxwell field is less infrared divergent since by minimal coupling the Bose field couples to the momentum and not to the position as in (1.1). As proved in [3, 11], the Pauli-Fierz operator has a ground state in Fock space also in three dimensions, which suggests that in the case of the Maxwell field $H_{\rm euc}$ and H are unitarily equivalent.

2 Path measures for the particle and the field

2.1 Euclidean quantization of the particle

On the Hilbert space $L^2(\mathbb{R}^d, dq)$ we define the particle Hamiltonian

$$H_{\rm p} = -\frac{1}{2}\Delta + V(q). \tag{2.1}$$

The external potential $V : \mathbb{R}^d \to \mathbb{R}$ is a multiplication operator. We consider two classes of potentials.

(P1) V is bounded from below, continuous, and having the asymptotics

$$V(q) = C|q|^{2\alpha} + o(|q|^{2\alpha}) \tag{2.2}$$

for large $|q|$, with some constant $C > 0$ and exponent $\alpha > 1$.

(P2) V is of Kato-class bounded from below as

$$V(q) \ \geq\ C|q|^{\alpha} \tag{2.3}$$

with some constant $C > 0$ and exponent $\alpha > 0$.

For details on the Kato class we refer to [5]. In either case above $H_{\rm p}$ has discrete spectrum and a unique, strictly positive ground state ψ_0 lying at its bottom $E_{\rm p}$, i.e. $H_{\rm p}\psi_0 = E_{\rm p}\psi_0$. Associated with $H_{\rm p}$ we consider the stationary $P(\phi)_1$-process q_t with time $t = 0$ distribution (invariant measure)

$$d\mathsf{N}^0 = \psi_0^2(q)dq, \tag{2.4}$$

as defined through the stochastic differential equation

$$dq_t = \nabla \ln \psi_0(q_t)\ dt + dB_t \tag{2.5}$$

where dB_t denotes Brownian motion. The path measure of this process will be denoted by $\mathcal{N}^0$; for more details see [5] and references therein. (Here and henceforth we use calligraphic letters for path measures and straight characters for time $t = 0$ distributions.)

Lemma 2.1 *$\mathcal{N}^0$-almost all paths $Q = \{q_t\}$ are continuous and for any $T > 0$ they satisfy*

$$|q_t - q_s| \ \leq\ C|t-s|^{1/8} \quad \forall s,t \in [-T,T], \tag{2.6}$$

where $C = C(Q,T) > 0$ is independent of s, t. Moreover, $\mathcal{N}^0$-almost surely

$$|q_t| \ \leq\ C_1(\ln(|t|+1))^{1/(\alpha+1)} + C_2, \quad \forall t \in \mathbb{R}, \tag{2.7}$$

where $C_1 > 0$ is a constant dependent only on the dimension of the space, $C_2 = C_2(Q)$, and α is the exponent appearing in (P1), resp. (P2).

Proof: See [5]. □

The process q_t is time-reversible. Its stochastic semigroup is generated by

$$H_{\rm p}^{\rm euc} f = -\frac{1}{2}\Delta f - \nabla \ln \psi_0 \cdot \nabla f. \tag{2.8}$$

$H_{\rm p}^{\rm euc}$ is self-adjoint on the Hilbert space $L^2(\mathbb{R}^d, d\mathsf{N}^0)$. The map

$$f \mapsto f/\psi_0 \tag{2.9}$$

from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, d\mathsf{N}^0)$ is unitary and transforms $H_{\rm p} - E_{\rm p}$ into $H_{\rm p}^{\rm euc}$.

2.2 Euclidean quantization of the free field

We introduce the massless free field in $d+1$ dimensions, $d \geq 3$. For any $f \in \mathcal{S}(\mathbb{R}^{d+1})$, i.e. the Schwartz space over $\mathbb{R}^{d+1}$, we denote

$$\xi(f) = \int_{\mathbb{R}^{d+1}} f(t,x)\xi_t(x)dtdx. \tag{2.10}$$

According to the action (1.10) the quantum field $\{\xi(f), f\in\mathcal{S}(\mathbb{R}^{d+1})\}$ is described by the generalized Gaussian measure $\mathcal{G}$ on $\mathcal{S}'(\mathbb{R}^{d+1})$ with zero mean and covariance

$$\begin{aligned}\operatorname{cov}_{\mathcal{G}}(\xi(f_1),\xi(f_2)) &= \mathbb{E}_{\mathcal{G}}[\xi(f_1)\xi(f_2)]\\ &= \frac{1}{2}\left((-\partial_t^2-\Delta)^{-1}f_1,f_2\right)\\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\hat{f}_1(k_0,k)\hat{f}_2^*(k_0,k)}{k_0^2+k^2}dkdk_0. \end{aligned}\tag{2.11}$$

Here $\hat{f}_1,\hat{f}_2\in\mathcal{S}(\mathbb{R}^{d+1})$ denote the Fourier transforms of $f_1,f_2\in\mathcal{S}(\mathbb{R}^{d+1})$. The scalar product is taken in the space $L^2(\mathbb{R}^{d+1},dtdx)$. $\mathcal{G}$ is a probability measure on the space $\mathcal{S}'(\mathbb{R}^{d+1})$ taken together with its associated Borel σ-field [7, 15].

The particle trajectories are a stochastic process q_t taking values in $\mathbb{R}^d$. Following this example, it will be convenient to regard the field configurations ξ_t as a stochastic process taking values in a suitable Hilbert space to be specified below. To carry out such a construction first note that the covariance (2.11) can be extended to test functions of the form

$$f_t^{\varphi}(x',t')=\varphi(x')\delta(t'-t)\tag{2.12}$$

with $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Thus the random process

$$\xi_t(\varphi)=\xi(f_t^{\varphi}),\quad \varphi\in\mathcal{S}(\mathbb{R}^d),\ t\in\mathbb{R},\tag{2.13}$$

is well-defined. From (2.11) and (2.13) it follows that

$$\mathbb{E}_{\mathcal{G}}[\xi_s(\varphi_1)\xi_t(\varphi_2)]=\frac{1}{4}\int_{\mathbb{R}^d}\frac{\hat{\varphi}_1(k)\hat{\varphi}_2^*(k)}{|k|}\;e^{-|k||s-t|}dk.\tag{2.14}$$

Clearly, $\xi_t(\varphi)$ is a stationary Gaussian process. Its invariant ($t=0$) measure G defined on $\mathcal{S}'(\mathbb{R}^d)$ is again Gaussian with mean 0 and covariance given by (2.14) at $s=t$. Moreover, the process $\xi_t(\varphi)$ is time-reversible [10].

Lemma 2.2 *$\{\xi_t, t\in\mathbb{R}\}$ is a Markov process.*

Proof: See [7, 16]. □

Next we construct a Hilbert space $\mathcal{B}_D$ such that $\xi_t\in\mathcal{B}_D$ and $t\mapsto\xi_t$ is norm-continuous with probability 1. Let D be the positive self-adjoint operator in the space $L^2(\mathbb{R}^d,dx)$ given by a jointly continuous symmetric kernel

$$(D\hat{\xi})(k)=\int_{\mathbb{R}^d}D(k,k')\hat{\xi}(k')dk'\tag{2.15}$$

with $\ker D=\{0\}$. We introduce the real Hilbert space $\mathcal{B}_D\subset\mathcal{S}'(\mathbb{R}^d)$ with norm

$$\|\xi\|^2_{\mathcal{B}_D}=\int_{\mathbb{R}^d\times\mathbb{R}^d}D(k_1,k_2)\hat{\xi}(k_1)\hat{\xi}^*(k_2)dk_1dk_2\tag{2.16}$$

and $\hat{\xi}^*(k) = \hat{\xi}(-k)$. Clearly,

$$\mathbb{E}_{\mathcal{G}}\left[\|\xi_t\|^2_{\mathcal{B}_D}\right] = \mathbb{E}_{\mathsf{G}}\left[\|\xi_0\|^2_{\mathcal{B}_D}\right] = \frac{1}{4}\int_{\mathbb{R}^d} \frac{D(k,k)}{|k|} dk. \tag{2.17}$$

Hence, if

$$\int_{\mathbb{R}^d} \frac{D(k,k)}{|k|} dk < \infty, \tag{2.18}$$

the measure G is concentrated on the space $\mathcal{B}_D$ and the random process ξ_t takes its values from this set; for more details see [10]. By an easy calculation we obtain that

$$\begin{aligned}\mathbb{E}_{\mathcal{G}}(\|\xi_t - \xi_s\|^4_{\mathcal{B}_D}) = \frac{1}{4}\left(\int_{\mathbb{R}^d} \frac{D(k,k)}{|k|}(1 - e^{-|k||t-s|})dk\right)^2 + \\ \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{D(k_1,k_2)^2}{|k_1||k_2|}(1 - e^{-|k_1||s-t|})(1 - e^{-|k_2||s-t|})dk_1 dk_2.\end{aligned} \tag{2.19}$$

Thus if in addition to (2.18) we assume that

$$\int_{\mathbb{R}^d} D(k,k)dk < \infty \quad \text{and} \quad \int\int D(k_1,k_2)^2 dk_1 dk_2 < \infty, \tag{2.20}$$

then, using that $|k|^{-1}(1 - e^{-|k||s-t|}) \leq |s-t|$, we obtain

$$\mathbb{E}_{\mathcal{G}}\left[\|\xi_t - \xi_s\|^4\right] \leq C\,|t-s|^2. \tag{2.21}$$

From (2.21) and Kolmogorov's criterion of continuity of random processes (see [10, 21]) we conclude

Lemma 2.3 *Under conditions (2.18) and (2.20) almost all paths of the process ξ_t are continuous in the metric of $\mathcal{B}_D$. Moreover, for any finite interval $T > 0$ these paths have the property that*

$$\|\xi_s - \xi_t\|_{\mathcal{B}_D} \leq C\,|t-s|^{1/8}, \quad \forall s,t \in [-T,T] \tag{2.22}$$

where $C = C(\{\xi_t\}, T)$ is independent of s,t.

Lemma 2.3 states that with probability 1 the process ξ_t is realized on $C(\mathbb{R}, \mathcal{B}_D)$. We denote the corresponding Gaussian path measure again by $\mathcal{G}$. Since the stationary Markov process ξ_t is reversible, its stochastic semigroup $e^{-tH_{\mathrm{f}}^{\mathrm{euc}}}$ acting on $L^2(\mathcal{B}_D, d\mathsf{G})$ is self-adjoint and $H_{\mathrm{f}}^{\mathrm{euc}}$ is the generator of this semigroup.

Lemma 2.4 *There exists a unitary map $\mathcal{F}_{\mathrm{sym}} \to L^2(\mathcal{B}_D, d\mathsf{G})$ transforming the free field Fock space Hamiltonian H_{f} into $H_{\mathrm{f}}^{\mathrm{euc}}$.*

This map can be constructed by using the Itô-Wick transformation, see [22].

For concreteness we give an explicit form of the kernel $D(k_1,k_2)$ which satisfies (2.18) and (2.20) above. Consider the operator

$$\bar{D} = (-\Delta_k + |k|^2)^{-(d+1)}. \tag{2.23}$$

The eigenvalues of this operator are

$$\lambda_n = \frac{1}{(n_1 + ... + n_d + d/2)^{d+1}}, \tag{2.24}$$

with multi-index $n = (n_1, ..., n_d)$ of integer entries $n_i \geq 0$, $i = 1, ..., d$. Hence $\bar{D}$ is a nuclear and positive operator, and its kernel $\bar{D}(k_1, k_2)$ is jointly continuous [4]. Let $|k|_1$ denote the function

$$|k|_1 = \begin{cases} |k|, & \text{if } |k| < 1, \\ 1, & \text{otherwise.} \end{cases} \tag{2.25}$$

We set

$$D(k_1, k_2) = |k_1|_1 \bar{D}(k_1, k_2)|k_2|_1. \tag{2.26}$$

It is then easily seen that this kernel satisfies conditions (2.18) and (2.20).

For later use we also note that the level sets

$$K_C = \{\xi \in \mathcal{B}_D : (D^{-1/2}\xi, \xi)_{\mathcal{B}_D} \ \leq \ C\} \tag{2.27}$$

are compact in $\mathcal{B}_D$.

3 Path measure for the interacting system

In the previous section we defined the stochastic processes q_t and ξ_t governing the particle, resp. the free field trajectories. The joint field-particle process on the space $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$ is then given by the path measure $\mathcal{P}^0 = \mathcal{N}^0 \times \mathcal{G}$. In order to implement the interaction as in (1.12) we take $\mathcal{P}^0$ as reference process and modify it with a density given by the exponential of S_{int}, compare with (1.11),

$$d\mathcal{P}_T = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt\right) d\mathcal{P}^0, \tag{3.1}$$

where

$$Z_T = \int \exp\left(-\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt\right) d\mathcal{P}^0 \tag{3.2}$$

is the normalizing partition function. Since $\mathcal{G}$ is a Gaussian measure, by an easy computation we obtain

$$Z_T = \int \exp\left(\frac{1}{8}\int_{-T}^{T}\int_{-T}^{T} dt ds \int_{\mathbb{R}^d} \frac{|\hat{\varrho}(k)|^2}{|k|} \cos(k \cdot (q_t - q_s))\ e^{-|k||t-s|} dk\right) d\mathcal{N}^0. \tag{3.3}$$

This integral is well-defined for every $T > 0$.

We will be interested whether the sequence of measures $\mathcal{P}_T$ has a limit as $T \to \infty$. We use the following notion of convergence. Let M be a metric space, and $C(\mathbb{R}, M)$ the space of continuous paths $\{X_t\}$ with values in M. For any interval $I_T = [-T, T] \subset \mathbb{R}$ let $\mathcal{M}_T \subset \mathcal{C}$ be a sub-σ field of the Borel σ-field $\mathcal{C}$ generated by the evaluations $\{X_t : t \in I_T\}$, i.e. $\mathcal{M}_T = \sigma(X_t, t \in I_T)$. We say that a sequence of probability measures $\{\mu_n\}$ on

$C(\mathbb{R}, M)$ converges locally weakly to the probability measure μ if for any $0 \leq T < \infty$ the restrictions $\mu_n|_{\mathcal{M}_T}$ converge weakly to the measure $\mu_{\mathcal{M}_T}$, see [10]. We have then the following result.

Theorem 3.1 *Let $d \geq 3$, $0 < e \leq e^*$, with sufficiently small $e^* > 0$, and V satisfy (P1). Then there exists the local weak limit $\mathcal{P}_T \to \mathcal{P}$ as $T \to \infty$. $\mathcal{P}$ is a probability measure of a stationary reversible Markov process on the path space $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$. Moreover, the stationary distribution P of $\mathcal{P}$ can be obtained as the weak limit $\mathsf{P} = \lim_{T\to\infty} \mathsf{P}_T$, where P_T is the distribution of $\{q_0, \xi_0\}$ under $\mathcal{P}_T$.*

Proof: For any fixed path $Q = \{q_t : t \in \mathbb{R}\} \in C(\mathbb{R}, \mathbb{R}^d)$ consider the conditional distribution

$$\mathcal{P}_T^Q = \mathcal{P}_T(\,\cdot\,|\{q_t\} = Q) \tag{3.4}$$

viewed as a probability measure on $C(\mathbb{R}, \mathcal{B}_D)$. Then the distribution $\mathcal{P}_T$ on $C(\mathbb{R}, \mathbb{R}^d \times \mathcal{B}_D)$ can be written as

$$\mathcal{P}_T(S \times A) = \int_S \mathcal{P}_T^Q(A) d\mathcal{N}_T(Q), \quad \forall T > 0 \tag{3.5}$$

where $A \subset C(\mathbb{R}, \mathcal{B}_D)$ and $S \subset C(\mathbb{R}, \mathbb{R}^d)$, and $\mathcal{N}_T$ is the Q-marginal of $\mathcal{P}_T$. Since the ξ_t process is Gaussian and the coupling is linear in ξ_t, this marginal can be computed explicitly and is given by

$$d\mathcal{N}_T = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T}\int_{-T}^{T} W(q_s - q_t, s-t) ds dt\right) d\mathcal{N}^0, \tag{3.6}$$

where

$$W(q,t) = -\frac{1}{8}\int_{\mathbb{R}^d} \frac{|\hat{\varrho}(k)|^2}{|k|} \cos(k \cdot q)\, e^{-|k||t|} dk. \tag{3.7}$$

$\mathcal{N}_T$ is a Gibbs measure for the finite interval $[-T, T]$. The reversible diffusion process $\mathcal{N}^0$ is the reference measure, and W is the (effective) pair interaction potential. We denote the distribution of q_0 under $\mathcal{N}_T$ by N_T.

We use identity (3.5) to show the existence of the $T \to \infty$ limit of $\mathcal{P}_T$. We will do this by examining separately the limits of $\mathcal{P}_T^Q$ and $\mathcal{N}_T$, starting with the former measure. Since the action $\int_{-T}^{T} (\xi_t * \varrho)(q_t) dt$ is linear in ξ_t, $\mathcal{P}_T^Q$ is a Gaussian measure with covariance given by (2.11) and mean

$$\mathbb{E}_{\mathcal{P}_T^Q}[\xi_t(\varphi)] = \int_{\mathbb{R}^d} \varphi(x) g_T^t(x; Q) dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{3.8}$$

where

$$\begin{aligned} g_T^t(x;Q) &= -\frac{1}{(2\pi)^{d/2}} \int_{-T}^{T} \int_{\mathbb{R}^d} \frac{\hat{\varrho}(k)}{4|k|} \cos(k \cdot (x - q_\tau))\, e^{-|k||t-\tau|} dk d\tau \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ik\cdot x}\, \hat{g}_T^t(k; Q) dk \end{aligned} \tag{3.9}$$

and

$$\hat{g}^t_T(k;Q) = -\frac{\hat{\varrho}(k)}{4|k|}\int_{-T}^{T} e^{-ik\cdot q_\tau} e^{-|k||t-\tau|} d\tau. \tag{3.10}$$

This function has the properties

$$|\hat{g}^t_T(k;Q)| \leq \frac{|\hat{\varrho}(k)|}{2|k|^2} \tag{3.11}$$

$$|\hat{g}^t_T(k;Q) - \hat{g}^s_T(k;Q)| \leq \frac{C|\hat{\varrho}(k)|}{2|k|}\,|t-s| \tag{3.12}$$

for any $s, t \in \mathbb{R}$, $|s-t| < 1$, and with some constant $C > 0$. Furthermore, we have that for any t and Q

$$\lim_{T\to\infty} \hat{g}^t_T(k;Q) = \hat{g}^t(k;Q) \tag{3.13}$$

exists in the norm topology of $\mathcal{B}_D$, uniformly with respect to Q and t in every bounded interval. The limit of (3.9) is given explicitly by

$$g^t(x;Q) = -\frac{1}{(2\pi)^{d/2}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^d} \frac{\hat{\varrho}(k)}{4|k|}\cos(k\cdot(x-q_\tau))\; e^{-|k||t-\tau|} dk d\tau. \tag{3.14}$$

From (3.12) it follows that $\hat{g}^t_T(k;Q)$ as a function of t belongs to $C(\mathbb{R}, \mathcal{B}_D)$ and

$$\sup_t \|\hat{g}^t_T(\,\cdot\,;Q)\|_{\mathcal{B}_D} < \infty \tag{3.15}$$

uniformly in Q and $T \leq \infty$. Moreover, as follows from (3.11), for arbitrary Q, T, and t, $\hat{g}^t_T(\cdot, Q) \in K_C$ for some suitable $C > 0$, with K_C defined by (2.27). Thus by (3.12) and (3.15) the set of restrictions $g^t_T(\cdot, Q)$, $|t| \leq R$ to any interval $[-R, R]$, $\forall Q \in C(\mathbb{R}, \mathbb{R}^d)$, $\forall T \leq \infty$, is a compact set in $C([-R, R], \mathcal{B}_D)$. Hence it follows that the restrictions $\mathcal{P}^Q_T\,|_{\mathcal{M}_R}$ form a weakly compact set of measures.

The characteristic functional of the Gaussian measure $\mathcal{P}^Q_T$ is

$$\begin{aligned}
\chi^Q_T(f) &= \mathbb{E}_{\mathcal{P}^Q_T}[e^{i\xi(f)}] \\
&= \exp\left(-1/8\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^d} \frac{\hat{f}(s,k)\hat{f}^*(t,k)}{|k|} e^{-|k||t-s|} dk\right) \times \\
&\times \exp\left(i\int_{-\infty}^{\infty} dt \int_{\mathbb{R}^d} \hat{g}^t_T(k;Q)\hat{f}(t,k) dk\right).
\end{aligned}$$

From (3.13) it follows that $\chi^Q_T(f) \to \chi^Q(f)$ as $T \to \infty$, where $\chi^Q(f)$ is the characteristic functional of the Gaussian measure $\mathcal{P}^Q$ with covariance (2.14) and mean

$$\mathbb{E}_{\mathcal{P}^Q}[f^\varphi_t] = \int_{\mathbb{R}^d} \varphi(x) g^t(x;Q) dx. \tag{3.16}$$

Thus from the convergence of characteristic functionals χ^Q_T and the weak compactness of the restrictions $\mathcal{P}^Q_T\,|_{\mathcal{M}_R}$ the weak local convergence $\mathcal{P}^Q_T \to \mathcal{P}^Q$ as $T \to \infty$ follows for any $Q \in C(\mathbb{R}, \mathbb{R}^d)$, see [10].

In order to complete our argument we need to control the $\mathcal{N}_T \to \mathcal{N}$ limit. This problem has been investigated in [13] by a cluster expansion technique and some of the results relevant in our context are quoted in Theorem 3.2 below. In particular, the local weak limit $\mathcal{N}_T \to \mathcal{N}$ exists, as stated under (1) in Theorem 3.2. This then completes the proof of the first claim.

Since $\mathcal{P}_T$ has the Markov property and is reversible, the limit distribution $\mathcal{P}$ has the same properties. Stationarity of $\mathcal{P}$ follows by point (2) of Theorem 3.2 and the obvious equality

$$d\mathcal{P}^{\theta_s Q}(\xi_{t-s}) = d\mathcal{P}^{Q}(\xi_t) \tag{3.17}$$

θ_s being the time-shift acting on paths.

The existence of the weak limit $\mathsf{P} = \lim_{T\to\infty} \mathsf{P}_T$ follows by the argument above by borrowing statement (3) of Theorem 3.2 below. This completes the proof of the theorem. □

We conclude this section by stating some facts on $\mathcal{N}$ and related measures. Results on the existence and uniqueness of Gibbs measures relative to Brownian motion are available for some time [19], and a more general framework has been considered more recently in [13]. In the following theorem we list some results taken from [13] to be used below adapted for the case of Nelson's model. For details we refer to this paper, in particular for results on the uniqueness of $\mathcal{N}$ which we do not address here.

Theorem 3.2 *For $d \geq 3$, sufficiently small $e > 0$, and potentials satisfying (P1) the following properties hold:*

1. *There exists the weak local limit $\mathcal{N}_T \to \mathcal{N}$ as $T \to \infty$.*

2. *The measure $\mathcal{N}$ is invariant with respect to time shifts and time reflections.*

3. *The $t = 0$ distributions N_T, resp. N, of the measures $\mathcal{N}_T$, resp. of $\mathcal{N}$, are absolutely continuous with respect to N^0 and*

$$\frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q_0) \to \frac{d\mathsf{N}}{d\mathsf{N}^0}(q_0) \quad as \quad T \to \infty \tag{3.18}$$

pointwise. Moreover, there exists a constant $c > 0$ such that

$$\frac{1}{c} \;\leq\; \frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q_0) \;\leq\; c, \tag{3.19}$$

uniformly in T and q_0.

4. *For N^0-almost all q the conditional distribution $\mathcal{N}_T(\,\cdot\,|q_0 = q)$ converges weakly locally to $\mathcal{N}(\,\cdot\,|q)$.*

5. *$\mathcal{N}$-almost all paths $Q = \{q_t\} \in C(\mathbb{R}, \mathbb{R}^d)$ have the property that for all $t \in \mathbb{R}$*

$$|q_t| \;\leq\; C_1(\ln(|t|+1))^{1/(\alpha+1)} + C_2(Q) \tag{3.20}$$

where $C_1 > 0$ is a constant, and C_2 is a function of the path.

6. For any bounded functions F_1, F_2 on $\mathbb{R}^d$ the following estimate holds on their covariance:

$$\mathbb{E}_{\mathcal{N}}[F_1(q_s)F_2(q_t)] - \mathbb{E}_{\mathcal{N}}[F_1(q_s)]\mathbb{E}_{\mathcal{N}}[F_2(q_t)] \;\leq\; C\,\frac{\sup|F_1|\sup|F_2|}{|s-t|^\gamma+1} \tag{3.21}$$

with some $\gamma > 0$ and a suitable prefactor $C > 0$.

4 The infrared divergent case: $d = 3$

After having introduced the path measures for the interacting system, we turn now to compare the Fock quantization and the Euclidean quantization of the interacting system. The first one is constructed through the Feynman-Kac formula,

$$\mathbb{E}_{\mathcal{P}^0}[F(q_0,\xi_0)G(q_t,\xi_t)e^{-\int_0^t(\xi_s*\varrho)(q_s)ds}] = (F, e^{-tH}G)_{\mathcal{H}^0}, \quad t \geq 0, \tag{4.1}$$

which defines the semigroup $\exp(-tH)$ on $\mathcal{H}^0 = L^2(\mathbb{R}^d \times \mathcal{B}_D, d\mathsf{P}^0)$. Its generator, the Hamiltonian H, is unitarily equivalent with H_{N}. For the Euclidean quantization we start with the semigroup T_t associated with the time reversible Markov stochastic process $\{q_t, \xi_t\}$ distributed according to $\mathcal{P}$ as defined through

$$\mathbb{E}_{\mathcal{P}}[F(q_0,\xi_0)G(q_t,\xi_t)] = (F, T_tG)_{\mathcal{H}}, \quad t \geq 0, \tag{4.2}$$

on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d \times \mathcal{B}_D, d\mathsf{P})$. T_t is a symmetric contractive semigroup. Hence there exists a self-adjoint semibounded operator H_{euc} generating it, i.e. $T_t = \exp(-tH_{\mathrm{euc}})$, which by definition will be viewed as the Hamiltonian of the system obtained in Euclidean quantization. Note that the constant function 1 is the ground state of H_{euc}. Since T_t is positivity improving, this ground state is unique in $\mathcal{H}$.

In this and the following section we investigate whether the Hamiltonians H and H_{euc} are unitarily equivalent. We start by first characterizing infrared divergence in terms of both the approximate ground state and the density of the path measure with respect to the reference measure. We use throughout this paper the following

Definition 4.1 *The Hamiltonian H is called* infrared divergent *if it has no ground state in $\mathcal{H}^0$.*

Let $\{P_\lambda\}$ be the family of spectral projections for the self-adjoint operator H bounded from below, and denote $d\sigma_1(\lambda) = d(1, P_\lambda 1)_{\mathcal{H}^0}$, the spectral measure for $1 \in \mathcal{H}^0$. Then

$$E_0 \;\leq\; E_0' := \inf\operatorname{supp} d\sigma_1(\lambda). \tag{4.3}$$

E_0 is the bottom of the spectrum of H. We also define the approximate ground state

$$\Psi_T = \frac{e^{-T(H-E_0')}1}{\|e^{-T(H-E_0')}1\|}. \tag{4.4}$$

Theorem 4.1 *Suppose V satisfies condition (P1) and the Gibbs measure $\mathcal{N}$ exists. Then the following statements are equivalent:*

(1) H is infrared divergent.

(2) $\lim_{T\to\infty}(1,\Psi_T)=0$.

(3) P is singular with respect to P^0.

Lemma 4.2 *We have the following two cases:*

(i) If $\limsup_{T\to\infty}(1,\Psi_T)>0$, then $\sigma_1(\{E_0'\})>0$ and the limit

$$\lim_{T\to\infty}\Psi_T:=\Psi. \tag{4.5}$$

exists. Moreover, $\Psi>0$, $E_0'=E_0$, and Ψ is the unique ground state of H at eigenvalue E_0.

(ii) If $\limsup_{T\to\infty}(1,\Psi_T)=0$, then $\sigma_1(\{E_0'\})=0$ and

$$\lim_{T\to\infty}\Psi_T=0. \tag{4.6}$$

Moreover, $E_0'=E_0$ and H has no ground state in $\mathcal{H}^0$.

Proof: (i) By the spectral theorem we have

$$(1,\Psi_T)=\frac{\int_{E_0'}^{\infty}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)}{\left(\int_{E_0'}^{\infty}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}}. \tag{4.7}$$

Take some $a>E_0'$. Then

$$\frac{\int_{E_0'}^{\infty}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)}{\left(\int_{E_0'}^{\infty}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}}\leq\frac{\int_{E_0'}^{a}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)+\int_{a}^{\infty}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)}{\left(\int_{E_0'}^{a}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}}. \tag{4.8}$$

Schwartz's inequality gives

$$\int_{E_0'}^{a}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)\leq\left(\int_{E_0'}^{a}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}\left(\sigma_1([E_0',a])\right)^{1/2}. \tag{4.9}$$

Hence (4.8) further becomes

$$\begin{aligned}\frac{\int_{E_0'}^{\infty}e^{-(\lambda-E_0')T}d\sigma_1(\lambda)}{\left(\int_{E_0'}^{\infty}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}}&\leq\left(\sigma_1([E_0',a])\right)^{1/2}+\frac{e^{-(a-E_0')t}}{\left(\int_{E_0'}^{a}e^{-2(\lambda-E_0')T}d\sigma_1(\lambda)\right)^{1/2}}\\&=\left(\sigma_1([E_0',a])\right)^{1/2}+\frac{1}{\left(\int_{E_0'}^{a}e^{-2(\lambda-a)T}d\sigma_1(\lambda)\right)^{1/2}}.\end{aligned}$$

Since $E_0' = \inf \operatorname{supp} d\sigma_1$, the denominator diverges for $T \to \infty$ and

$$0 < \limsup_{T\to\infty}(1, \Psi_T) \le \left(\sigma_1([E_0', a])\right)^{1/2}. \tag{4.10}$$

Taking $a \to E_0'$, this implies $\sigma_1(\{E_0'\}) > 0$. Again, by the spectral theorem the limit (4.5) exists and $\Psi \ge 0$, since e^{-tH} is positivity preserving [3]. Ψ is an eigenfunction of H at eigenvalue E_0'. Since e^{-tH} is positivity improving [3], $\Psi > 0$ and therefore $E_0 = E_0'$ by the uniqueness part of the Perron-Frobenius theorem [20].

(ii) By contraposition take $\sigma_1(\{E_0'\}) > 0$. Then for $T \to \infty$ the numerator of (4.7) converges to $\sigma_1(\{E_0'\})$ and the denominator converges to $\sqrt{\sigma_1(\{E_0'\})}$, which contradicts $\lim_{T\to\infty}(1, \Psi_T) = 0$.

We finally prove that $E_0' = E_0$. We have that

$$\inf \operatorname{supp} d\sigma_1(\lambda) = -\lim_{T\to\infty} \frac{1}{T} \log(1, e^{-TH} 1) = E_0'. \tag{4.11}$$

By an easy computation, for the set of functions

$$0 < c^- \le f \le c^+ < \infty, \tag{4.12}$$

with some constants c^-, c^+, we also have

$$\inf \operatorname{supp} d\sigma_f(\lambda) = -\lim_{T\to\infty} \frac{1}{T} \log(f, e^{-TH} f) = E_0'. \tag{4.13}$$

The linear span of the set of functions defined by (4.12) is dense in $\mathcal{H}$. Indeed, take f as specific combinations of characteristic functions, $f = 1_A + \epsilon 1_{A^c}$ with arbitrary sets $A \subset \mathcal{H}$ and some $\epsilon > 0$; this set is dense in $\mathcal{H}$. If $E_0' > E_0$ then $P_{\{\lambda < E_0'\}}\mathcal{H} \ne \emptyset$, and the linear span of the set (4.12) would not be dense in $\mathcal{H}$. Thus $E_0' = E_0$.

Note that as our proof shows, actually also the limit $\lim_{T\to\infty}(1, \Psi_T)$ exists. $\square$

Lemma 4.3 *We have the following two cases:*

(i) If P *is absolutely continuous with respect to* P^0 *then* $\lim_{T\to\infty}(1, \Psi_T) > 0$ *and*

$$\frac{d\mathsf{P}}{d\mathsf{P}^0} = \lim_{T\to\infty} \frac{d\mathsf{P}_T}{d\mathsf{P}^0} = \lim_{T\to\infty} \Psi_T^2 = \Psi^2. \tag{4.14}$$

(ii) If P *is singular with respect to* P^0, *then* $\lim_{T\to\infty}(1, \Psi_T) = 0$ *and*

$$\lim_{T\to\infty}(1, \Psi_T) = \lim_{T\to\infty} \mathbb{E}_{\mathsf{P}^0}\left[\left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0}\right)^{1/2}\right]. \tag{4.15}$$

Remark: As follows from Lemma 4.2 only two cases can occur: either (i) P is absolutely continuous with respect to P^0, or (ii) P is singular with respect to P^0.

Proof: (i) Note that for any finite $T > 0$ the measure P_T is absolutely continuous with respect to P^0, and

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi) = \frac{Z_T^-(q,\xi)Z_T^+(q,\xi)}{Z_T} \tag{4.16}$$

where

$$Z_T^+(q,\xi) = \mathbb{E}_{\mathcal{P}^0}[e^{-\int_0^T(\xi_t*\varrho)(q_t)dt}|q_0 = q, \xi_0 = \xi]. \tag{4.17}$$

Z_T^- is defined by changing $\int_0^T$ for $\int_{-T}^0$ in the exponent, and obviously $Z_T^- = Z_T^+$. By applying the Feynman-Kac formula we obtain

$$Z_T^+(q,\xi) = (e^{-TH}1)(q,\xi) \tag{4.18}$$

and

$$Z_T = \|e^{-TH}1\|^2 \tag{4.19}$$

in the norm of $L^2(\mathbb{R}^d \times \mathcal{B}_D, d\mathsf{P}^0)$. Thus

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi) = \frac{\big((e^{-TH}1)(q,\xi)\big)^2}{\|e^{-TH}1\|^2}. \tag{4.20}$$

Since we have $\mathsf{P}_T \to \mathsf{P}$ as $T \to \infty$, (4.20) implies (4.14).

(ii) By using Schwartz's inequality and F given by (4.22) below, we get

$$(\Psi_T, 1) = \int \Psi_T d\mathsf{P}^0 = \int \Psi_T F^{1/2} F^{-1/2} d\mathsf{P}^0 \le \left(\int \Psi_T^2 F d\mathsf{P}^0\right)^{1/2} \left(\int F^{-1} d\mathsf{P}^0\right)^{1/2}. \tag{4.21}$$

By Lemma 4.5 below the first integral at the right hand side above converges to zero as $T \to \infty$ and the second integral is bounded, hence the claim follows. □

Proof of Theorem 4.1: We proceed along the scheme (3) ⇒ (2) ⇒ (1) ⇒ (3). (3) ⇒ (2) is given by Lemma 4.2 (ii), (2) ⇒ (1) comes about by Lemma 4.3 (ii).

To show (1) ⇒ (3) suppose that P is not singular with respect to P^0. Then, by the above Remark P must be absolutely continuous with respect to P^0. If, however, $\mathsf{P} \ll \mathsf{P}^0$, then by Lemma 4.3 (ii) we have that there is $l > 0$ such that $\lim_{T\to\infty}(1, \Psi_T) = l$ and $d\mathsf{P}/d\mathsf{P}^0 = \Psi^2$. Moreover, by Lemma 4.2 (i) we have that $\Psi = \lim_{T\to\infty} \Psi_T > 0$ and Ψ is the ground state of H. This is a contradiction proving (1) ⇒ (3). □

Theorem 4.4 *For $d = 3$, (P1)-potentials and $0 < e \le e^*$ H has no ground state in $\mathcal{H}^0$ (and hence H_N has no ground state in $L^2(\mathbb{R}^d, dq) \otimes \mathcal{F}_{\rm sym}$). In particular, H and $H_{\rm euc}$ are not unitarily equivalent.*

We expect that the restriction to weak coupling is of technical nature only.

Lemma 4.5 *For $d = 3$ and $0 < e \le e^*$ P is singular with respect to P^0.*

Proof: Clearly, it suffices to find a function F such that F is almost surely strictly positive with respect to the free measure P^0 and $\mathbb{E}_{\mathsf{P}}[F] = 0$. We choose

$$F(q,\xi) = \exp\left(\int_{\mathbb{R}^3} s(x)\xi(x)dx\right) \tag{4.22}$$

and s satisfying

$$\|s\|^2 := \int_{\mathbb{R}^3} \frac{|\hat{s}(k)|^2}{|k|} dk \;<\; \infty. \tag{4.23}$$

The function F is defined and positive for G-almost all $\xi \in \mathcal{B}_D$ (see [12]). We have the identity

$$\mathbb{E}_{\mathsf{P}_T}[F] = \mathbb{E}_{\mathcal{N}_T}\left[\mathbb{E}_{\mathsf{P}_T^Q}[F]\right] \tag{4.24}$$

According to our standard notation P_T^Q is the time $t=0$ distribution of $\mathcal{P}_T^Q$. P_T^Q is absolutely continuous with respect to G for any Q and its density is given by

$$\frac{d\mathsf{P}_T^Q}{d\mathsf{G}} = \exp\left(\int_{\mathbb{R}^3} \xi(x) m_T(x;Q)dx - 2\int_{\mathbb{R}^3} |\hat{g}_T^0(k;Q)|^2|k|dk\right) \tag{4.25}$$

with

$$m_T(x;Q) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^3} \hat{m}_T(k;Q)e^{ik\cdot x}dk \tag{4.26}$$

and

$$\hat{m}_T(k;Q) = 4|k|\hat{g}_T^0(k;Q) = -\hat{\varrho}(k)\int_{-T}^{T} e^{-ik\cdot q_\tau}e^{-|k||\tau|}d\tau. \tag{4.27}$$

Thus we obtain

$$\mathbb{E}_{\mathsf{P}_T^Q}[F] = \exp\left(\int_{\mathbb{R}^3} \hat{s}(k)\hat{g}_T^0(k;Q)dk + \frac{1}{8}\|s\|^2\right). \tag{4.28}$$

Since $\hat{\varrho}(0) = e > 0$, there exists some $k^* > 0$ such that $\hat{\varrho}(k) > 0$ whenever $0 < |k| < k^*$. We choose then the Fourier transform of s as

$$\hat{s}(k) = \begin{cases} \frac{1}{\hat{\varrho}(k)} \int_{T^*}^{\infty} \frac{e^{-|k||t|}}{\ln t(\ln\ln t)^\zeta}dt, & \text{if } |k| < k^* \\ 0, & \text{otherwise} \end{cases} \tag{4.29}$$

with a suitable constant T^* such that $\ln T^* > 1$, and exponent $0 < \zeta < 1$. First we have to show that (4.23) is satisfied,

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|\hat{s}(k)|^2}{|k|}dk &\leq C\int_0^{k^*} kdk \int_{T^*}^{\infty}\int_{T^*}^{\infty} \frac{e^{-|k|(s+t)}dsdt}{\ln t \ln s(\ln\ln t \ln\ln s)^\zeta} \\
&= C\int_{T^*}^{\infty}\int_{T^*}^{\infty} \frac{1}{(t+s)^2}\frac{dtds}{\ln t\ln s(\ln\ln t)^\zeta(\ln\ln s)^\zeta} \\
&= 2C\int\int_{T^*\le s\le t<\infty} \frac{t/s}{(1+t/s)^2}\frac{dt}{t\ln t(\ln\ln t)^\zeta}\frac{ds}{s\ln s(\ln\ln s)^\zeta} \\
&= 2C\int_0^{\infty}\frac{e^w dw}{(1+e^w)^2}\int_{u^*}^{\infty}\frac{dv}{v(\ln v)^\zeta(v+w)(\ln(v+w))^\zeta} \\
&\leq 2C\int_0^{\infty} e^{-w}dw\int_{u^*}^{\infty}\frac{dv}{v^2(\ln v)^{2\zeta}} < \infty
\end{aligned} \tag{4.30}$$

with $u = \ln t$, $v = \ln s$, $w = u - v$, $u^* = \ln T^*$ above. Now we write

$$\begin{aligned}
\int_{\mathbb{R}^3} \hat{s}(k)\hat{g}^0_T(k;Q)dk &= -\frac{1}{2}\int_{|k|<k^*}\frac{dk}{|k|}\int_{-T}^{T} e^{-i(k,q_s)}e^{-|k||s|}ds\int_{T^*}^{\infty}\frac{e^{-|k||t|}}{\ln t(\ln\ln t)^{\zeta}}dt \\
&= -2\pi\int_{-T}^{T} ds\int_{T^*}^{\infty} dt\int_0^{k^*}\frac{\sin k|q_s|\; e^{-k(|s|+|t|)}}{|q_s|\ln t(\ln\ln t)^{\zeta}}dk \\
&= -4\pi\int_{-T}^{T} ds\int_{T^*}^{\infty} dt\frac{1}{[|q_s|^2+(|s|+|t|)^2]\ln t(\ln\ln t)^{\zeta}} \\
&- 4\pi\int_{-T}^{T} ds\int_{T^*}^{\infty} dt\frac{e^{-|k^*|(|s|+|t|)}\cos(k^*|q_s|)}{[|q_s|^2+(|s|+|t|)^2]\ln t(\ln\ln t)^{\zeta}} \\
&+ 4\pi\int_{-T}^{T} ds\int_{T^*}^{\infty} dt\frac{e^{-|k^*|(|s|+|t|)}\sin(k^*|q_s|/|q_s|)(|t|+|s|)}{[|q_s|^2+(|s|+|t|)^2]\ln t(\ln\ln t)^{\zeta}}. \qquad (4.31)
\end{aligned}$$

The second and third integrals in the latter expression converge for all Q in the $T\to\infty$ limit. Take now the first integral. By (3.20) for any $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that $\mathcal{N}\left(C(\mathbb{R},\mathbb{R}^3)\setminus S_C\right) \le \varepsilon$ where

$$S_C = \{Q \in C(\mathbb{R},\mathbb{R}^3) : |q_t| \;\le\; C_0(\ln(|t|+1))^{1/(\alpha+1)} + C\}. \qquad (4.32)$$

Then for $Q \in S_C$ we estimate

$$\begin{aligned}
&\int_{-T}^{T} ds\int_{T^*}^{\infty}\frac{dt}{(|q_s|^2+(|s|+|t|)^2)\ln t(\ln\ln t)^{\zeta}} \;\ge \\
&\quad\int_{-T}^{T} ds\int_{T^*}^{\infty}\frac{dt}{((C_0(\ln(|s|+1))^{1/(\alpha+1)}+C)^2+(|s|+|t|)^2)\ln t(\ln\ln t)^{\zeta}}
\end{aligned}$$

The right hand side of this last expression goes to infinity as $T\to\infty$. Hence, for any $\delta > 0$ we can find some $\bar{T} = \bar{T}(\delta, C)$ such that

$$\mathbb{E}_{\mathsf{P}^Q_T}[F] \;\le\; \delta, \quad \forall Q \in S_C, \;\; T > \bar{T}. \qquad (4.33)$$

Moreover, it follows from (4.30) and (4.31) that there is a constant B such that $\mathbb{E}_{\mathsf{P}^Q_T}[F] \;\le\; B$ for all Q and T. Hence, by combining (4.28), (4.32) and (4.33), we have

$$\mathbb{E}_{\mathsf{P}_T}[F] = \mathbb{E}_{\mathcal{N}_T}\left[\mathbb{E}_{\mathsf{P}^Q_T}[F]\right] \to 0, \quad \text{as} \quad T\to\infty \qquad (4.34)$$

Consider now the function

$$F_{\bar{C}} = \begin{cases} F(\xi), & \text{if } F(\xi) \le \bar{C} \\ \bar{C}, & \text{otherwise} \end{cases} \qquad (4.35)$$

Clearly, for $T\to\infty$, $\mathbb{E}_{\mathsf{P}_T}[F_{\bar{C}}] \to 0$. On the other hand, $\mathbb{E}_{\mathsf{P}_T}[F_{\bar{C}}] \to \mathbb{E}_{\mathsf{P}}[F_{\bar{C}}]$ with $T\to\infty$, for any $\bar{C}$. This implies then that $\mathbb{E}_{\mathsf{P}}[F_{\bar{C}}] = 0$ for any $\bar{C}$ and hence $\mathbb{E}_{\mathsf{P}}[F] = 0$. This completes the proof Lemma 4.5. $\square$

Proof of Theorem 4.4: It follows straightforward by combining Lemma 4.5 with Theorem 4.1. □

The smallness of the coupling constant was used in (4.32) referring to the typical fluctuations in the Gibbs measure derived in [13]. If we only want to show that H has no ground state in $\mathcal{H}^0$, here is an alternative proof avoiding all restriction to small coupling.

Theorem 4.6 *Suppose $\varrho \geq 0$ and V is of class (P2). Then for $d = 3$ H has no ground state in $\mathcal{H}^0$.*

Proof:

By (4.16), (4.17) and (4.18) we have that

$$(1, \Psi_T) = \frac{\mathbb{E}_{\mathsf{P}^0}[Z_T^+(q,\xi)]}{\sqrt{Z_T}} = \frac{\mathbb{E}_{\mathsf{P}^0}[Z_T^-(q,\xi)]}{\sqrt{Z_T}} \tag{4.36}$$

Furthermore

$$\mathbb{E}_{\mathsf{P}^0}[Z_T^+(q,\xi)] = \mathbb{E}_{\mathsf{N}^0}\left[\mathbb{E}_{\mathcal{N}_+^0}\left[\mathbb{E}_{\mathcal{G}}[e^{-\int_0^T (\xi_t * \varrho)(q_t)dt}]|q_0 = q\right]\right] \tag{4.37}$$

where $\mathcal{N}_+^0$ is the distribution of paths in the forward direction $Q^+ = \{q_t : t \geq 0\}$. A similar expression holds for $\mathbb{E}_{\mathsf{P}^0}[Z_T^+(q,\xi)]$ taken with respect to $\mathcal{N}_-^0$ for paths in the backward direction $Q^- = \{q_t : t \leq 0\}$. Using the Markov property of $\mathcal{N}^0$ and Schwartz inequality, we obtain by (4.36)

$$(1, \Psi_T)^2 \;\leq\; \frac{1}{Z_T}\mathbb{E}_{\mathcal{N}^0}\left[\mathbb{E}_{\mathcal{G}}[e^{-\int_{-T}^0 (\xi_t * \varrho)(q_t)dt}]\, \mathbb{E}_{\mathcal{G}}[e^{-\int_0^T (\xi_t * \varrho)(q_t)dt}]\right]. \tag{4.38}$$

Moreover, since

$$\mathbb{E}_{\mathcal{G}}\left[e^{-\int_0^T (\xi_t * \varrho)(q_t)dt}\right] = e^{-\int_0^T \int_0^T W(q_t - q_s, t-s)dtds}, \tag{4.39}$$

and by a similar expression for the $[-T, 0]$ interval, we get

$$\begin{aligned}
(1, \Psi_T)^2 \;&\leq\; \frac{1}{Z_T}\mathbb{E}_{\mathcal{N}^0}\left[e^{-\int_0^T \int_0^T W(q_t - q_s, t-s)dsdt} e^{-\int_{-T}^0 \int_{-T}^0 W(q_t - q_s, t-s)dsdt}\right] \\
&= \frac{1}{Z_T}\mathbb{E}_{\mathcal{N}^0}\left[e^{-\int_{-T}^T \int_{-T}^T W(q_t - q_s, t-s)dsdt + \int\int_{\Delta_T} W(q_t - q_s, t-s)dtds}\right] \\
&= \frac{1}{Z_T}\mathbb{E}_{\mathcal{N}}\left[e^{\int\int_{\Delta_T} W(q_t - q_s, t-s)dsdt}\right]
\end{aligned} \tag{4.40}$$

Here $\Delta_T = (-T, 0) \times (0, T) \,\cup\, (0, T) \times (-T, 0)$.

In position space the interaction potential reads

$$W(q,t) = -\frac{\pi}{2}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \frac{\varrho(x)\varrho(y)}{(q + x - y)^2 + t^2} \;<\; 0. \tag{4.41}$$

To show that the left hand side of (4.36) converges to zero, we first restrict to the set

$$A_T := \{|q_t| \;\leq\; T^\lambda,\ \forall |t| \leq T\} \tag{4.42}$$

with some $\lambda < 1$. By using the estimate

$$(q_t - q_s + x - y)^2 + (t-s)^2 \;\leq\; 8T^{2\lambda} + 2(x-y)^2 + (t-s)^2, \tag{4.43}$$

we obtain

$$\begin{aligned} &\int\int_{\Delta_T} dtds \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \frac{\varrho(x)\varrho(y)}{(q_t - q_s + x - y)^2 + (t-s)^2} \\ &\geq 2\int_0^T dt \int_0^T ds \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \varrho(x)\varrho(y) \frac{1}{8T^{2\lambda} + 2(x-y)^2 + (t+s)^2} \\ &= \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \varrho(x)\varrho(y) \log \frac{8T^{2\lambda} + 2(x-y)^2 + T^2}{8T^{2\lambda} + 2(x-y)^2}. \end{aligned} \tag{4.44}$$

The right hand side in (4.44) goes to infinity as $T \to \infty$, since $\lambda < 1$. Hence we have shown that

$$\lim_{T\to\infty} \mathbb{E}_{\mathcal{N}_T} \left[\exp\left(\int\int_{\Delta_T} W(q_t - q_s, t-s) dsdt \right) 1_{A_T} \right] = 0. \tag{4.45}$$

We still need to prove that the limit is zero also on the complement of A_T. We use again that $W(q,t) < 0$ and estimate

$$\mathbb{E}_{\mathcal{N}_T} \left[\exp\left(\int\int_{\Delta_T} W(q_t - q_s, t-s) dsdt \right) (1 - 1_{A_T}) \right] \;\leq\; \mathbb{E}_{\mathcal{N}_T} \left[(1 - 1_{A_T}) \right]. \tag{4.46}$$

Writing out the expectation with respect to $\mathcal{N}_T$ and by using Schwartz inequality, we obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{N}_T} \left[(1 - 1_{A_T}) \right] \;&=\; \frac{\mathbb{E}_{\mathcal{N}^0} \left[e^{-\int_{-T}^T \int_{-T}^T W(q_t - q_s, t-s) dtds} (1 - 1_{A_T}) \right]}{\mathbb{E}_{\mathcal{N}^0} \left[e^{-\int_{-T}^T \int_{-T}^T W(q_t - q_s, t-s) dtds} \right]} \\ &\leq\; \frac{\left(\mathbb{E}_{\mathcal{N}^0} \left[e^{-2\int_{-T}^T \int_{-T}^T W(q_t - q_s, t-s) dtds} \right] \right)^{1/2}}{\mathbb{E}_{\mathcal{N}^0} \left[e^{-\int_{-T}^T \int_{-T}^T W(q_t - q_s, t-s) dtds} \right]} \left(\mathbb{E}_{\mathcal{N}^0} \left[(1 - 1_{A_T}) \right] \right)^{1/2} \end{aligned} \tag{4.47}$$

Since

$$0 \;<\; -2\int_{-T}^T \int_{-T}^T W(q_t - q_s, |t-s|) dsdt \;<\; T \int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{|k|^2} dk \tag{4.48}$$

and $\int_{\mathbb{R}^3} dk |\hat{\varrho}|^2/|k|^2 < \infty$, the first factor above is bounded by $\exp(aT)$ with some $a > 0$. To conclude we have to show that this exponential growth is compensated by the second factor. The estimate

$$\mathbb{E}_{\mathcal{N}^0} \left[(1 - 1_{A_T}) \right] = \mathcal{N}^0 \left(\{ q_t : \sup_{t\in[-T,T]} |q_t| \;\geq\; T^\lambda \} \right) \;\leq\; f(T) e^{-cT^{\lambda(\alpha+1)}} \tag{4.49}$$

is a slight modification of Lemma 5.2 in [13] based on Varadhan's lemma in conjunction with the bound $\exp(-c|x|^{\alpha+1})$ on the decay of ψ_0. Here $c > 0$, $f(T)$ is a polynomially growing correction, and $\alpha > 1$ is the exponent appearing in (2.2). By choosing $1/2 < \lambda < 1$, the right hand side of (4.49) goes to zero with $T \to \infty$. This completes the proof. $\square$

We conclude this section by showing that even though for $d = 3$ there is no ground state in Fock space, the particle remains well localized in the physical ground state. (Further properties of the ground state are discussed in [6].) Denote

$$\tilde{\Lambda}_T(q) = \frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q) \tag{4.50}$$

and set

$$\Lambda_T(q) = \tilde{\Lambda}_T(q)\psi_0(q)^2, \tag{4.51}$$

as the particle density with respect to Lebesgue measure in the approximate ground state (ψ_0 is the ground state of the Schrödinger operator (2.1)).

Theorem 4.7 *Suppose the potential V satisfies (P1). Then for sufficiently small couplings $0 < e \leq e^*$ there exist some constants $c_2 > c_1 > 0$ such that the limit $\Lambda(q) = \lim_{T\to\infty} \Lambda_T(q)$ exists and*

$$c_1\psi_0^2(q) \;\leq\; \Lambda(q) \;\leq\; c_2\psi_0^2(q) \tag{4.52}$$

for every $q \in \mathbb{R}^d$.

Proof: The pointwise existence of $\lim_{T\to\infty} d\mathsf{N}_T/d\mathsf{N}^0 = d\mathsf{N}/d\mathsf{N}^0$ (see (3) in Theorem 3.2) and the equality

$$\tilde{\Lambda}(q) := \lim_{T\to\infty} \tilde{\Lambda}_T(q) = \frac{d\mathsf{N}}{d\mathsf{N}^0}(q) \tag{4.53}$$

combined with estimate (3.19) gives the result. $\square$

5 The infrared convergent case: $d \geq 4$

Theorem 5.1 *Suppose $d \geq 4$ and that V satisfies (P1). Then for sufficiently weak couplings $0 < e < e^*$ H has a unique strictly positive ground state Ψ in $\mathcal{H}^0$ at eigenvalue E_0 (and hence H_{N} has a unique ground state in $L^2(\mathbb{R}^d, dq) \otimes \mathcal{F}_{\mathrm{sym}}$). Moreover, there exists a unitary map $\Gamma : \mathcal{H} \to \mathcal{H}^0$ such that $\Gamma^{-1}(H - E_0)\Gamma = H_{\mathrm{euc}}$.*

Proof: Using the representation for $d\mathsf{P}_T^Q/d\mathsf{G}$ given by (4.25) note that there is a constant $C > 0$ such that

$$|\hat{m}_T(k;Q)| \;\leq\; C\frac{|\hat{\varrho}(k)|}{|k|} \tag{5.1}$$

uniformly in Q and T, and hence it follows that the functional $\int_{\mathbb{R}^d} \xi(x) m_T(x;Q)dx$ is well defined for G-almost all $\xi \in \mathcal{B}_D$. Similarly, we have

$$\int_{\mathbb{R}^d} |\hat{g}_T^0(k;Q)|^2 |k| dk \;\leq\; C\int_{\mathbb{R}^d} \frac{|\hat{\varrho}(k)|^2}{|k|^3} dk < \infty \tag{5.2}$$

uniformly in Q and T. Also, the density

$$\frac{d\mathsf{P}^Q}{d\mathsf{G}} = \exp\left(\int_{\mathbb{R}^d}\xi(x)m(x;Q)dx - \frac{1}{4}\int_{\mathbb{R}^d}|\hat{g}^0(k;Q)|^2|k|dk\right) \tag{5.3}$$

exists, where

$$\begin{aligned} m(x;Q) &= \int_{\mathbb{R}^d}\hat{m}(k;Q)e^{ik\cdot x}dk, \\ \hat{m}(k;Q) &= |k|\hat{g}^0(k;Q). \end{aligned}$$

We write now the Radon-Nikodym derivative $d\mathsf{P}_T/d\mathsf{P}^0$ in another way, that is

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi) = \mathbb{E}_{\mathcal{N}_T}\left[\frac{d\mathsf{P}_T^Q}{d\mathsf{G}}(\xi)|q_0 = q\right]\frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q) \tag{5.4}$$

for every $0 < T < \infty$. By Fubini's theorem for any $q \in \mathbb{R}^d$ and $0 < T \leq \infty$ there is a set $\Omega = \Omega(q,T) \subset \mathcal{B}_D$, $\mathsf{P}^0(\Omega) = 1$, such that for $\xi \in \Omega$ the conditional expectation $\mathbb{E}_{\mathcal{N}_T}[d\mathsf{P}_T^Q/d\mathsf{P}^0(\xi)|q_0 = q]$ exists and hence (5.4) is well defined.

Next, from (5.4) we see

$$\mathbb{E}_{\mathsf{P}^0}\left[\left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi)\right)^{1/2}\right] = \int\left(\mathbb{E}_{\mathcal{N}_T}\left[\frac{d\mathsf{P}_T^Q}{d\mathsf{G}}(\xi)|q_0 = q\right]\right)^{1/2}\left(\frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q)\right)^{1/2} d\mathsf{G}d\mathsf{N}^0. \tag{5.5}$$

We apply Jensen's inequality and use (4.25) to obtain

$$\begin{aligned} &\left(\mathbb{E}_{\mathcal{N}_T}\left[\frac{d\mathsf{P}_T^Q}{d\mathsf{P}^0}(\xi)|q_0 = q\right]\right)^{1/2} \geq \\ &\quad \exp\left(\frac{1}{2}\int_{\mathbb{R}^d}\xi(x)\mathbb{E}_{\mathcal{N}_T}\left[m_T(x;Q)|q_0 = q\right]dx - \frac{1}{8}\int_{\mathbb{R}^d}\mathbb{E}_{\mathcal{N}_T}\left[|\hat{g}_T^0(k;Q)|^2 \mid q_0 = q\right]|k|dk\right). \end{aligned}$$

By using once again Jensen's inequality and the fact that the field has zero mean under G, we find

$$\begin{aligned} &\mathbb{E}_{\mathsf{P}^0}\left[\left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi)\right)^{1/2}\right] \geq \\ &\quad \int\exp\left(-\frac{1}{8}\int_{\mathbb{R}^3}\mathbb{E}_{\mathcal{N}_T}\left[|\hat{g}_T^0(k;Q)|^2|q_0 = q\right]|k|dk\right)\left(\frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q)\right)^{1/2} d\mathsf{N}^0(q). \end{aligned} \tag{5.6}$$

Estimate (3.11) gives then

$$\mathbb{E}_{\mathcal{N}_T}\left[|\hat{g}_T^0(k;Q)|^2|q_0 = q\right] \;\leq\; C\frac{|\hat{\varrho}(k)|^2}{|k|^4}. \tag{5.7}$$

Since by (3.19) $d\mathsf{N}_T/d\mathsf{N}^0$ is bounded from below uniformly in T and q, and by using Lemma 4.3 and the Remark following it we arrive at

$$\mathbb{E}_{\mathsf{P}^0}\left[\left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi)\right)^{1/2}\right] \geq C_1 \exp\left(-C_2\int_{\mathbb{R}^d}\frac{|\hat{\varrho}(k)|^2}{|k|^3}dk\right) \geq C_3 > 0. \tag{5.8}$$

By (5.8) it then follows that the left hand side above cannot converge to zero and, as it follows from Lemmas 4.2 and 4.3, there exists a function $\Psi \in \mathcal{H}^0$ such that

$$\lim_{T\to\infty}\frac{d\mathsf{P}_T}{d\mathsf{P}^0} = \Psi^2 = \frac{d\mathsf{P}}{d\mathsf{P}^0}. \tag{5.9}$$

Furthermore we use formulas (5.4) and (5.3) and apply once more Jensen's inequality to obtain

$$\frac{d\mathsf{P}}{d\mathsf{P}^0}(q,\xi) \geq$$
$$\exp\left(\int_{\mathbb{R}^d}\xi(x)\mathbb{E}_{\mathcal{N}}[m(x;Q)|q_0=q]dx - \frac{1}{8}\int_{\mathbb{R}^d}\mathbb{E}_{\mathcal{N}}[|\hat{g}^0(k;Q)|^2|q_0=q]|k|dk\right)\frac{d\mathsf{N}}{d\mathsf{N}^0}(q).$$

Since the function

$$\tilde{m}(x;q) = \mathbb{E}_{\mathcal{N}}[m(x;Q)|q_0=q] \tag{5.10}$$

satisfies the estimate (5.1) it is easy to show that $\exp(\xi(x)\tilde{m}(x;q)) > 0$ for P^0-almost all $(q,\xi) \in \mathbb{R}^d \times \mathcal{B}_D$. By (5.7) and the positivity of $d\mathsf{N}/d\mathsf{N}^0$ we get that $d\mathsf{P}/d\mathsf{P}^0(q,\xi) > 0$ for P^0-almost all (q,ξ). By Lemma 4.3 (i) we have then that the ground state Ψ exists.

We conclude by showing that there is a unitary map $\Gamma : \mathcal{H} \to \mathcal{H}^0$, $F \mapsto F\Psi$ transforming $H - E_0$ into H_{euc}. Indeed, by using (4.1) and (4.2)

$$\begin{aligned}
(F, T_tG)_{\mathcal{H}} &= \mathbb{E}_{\mathcal{P}}[F(q_0,\xi_0)G(q_t,\xi_t)] \\
&= \lim_{T\to\infty}\frac{1}{Z_{2T+t}}\mathbb{E}_{\mathcal{P}^0}[e^{-\int_{-T}^{T+t}(\xi_s*\varrho)(q_s)ds}F(q_0,\xi_0)G(q_t,\xi_t)] \\
&= \lim_{T\to\infty}\frac{(e^{-TH}1, Fe^{-tH}Ge^{-TH}1)_{\mathcal{H}^0}}{(1, e^{-(2T+t)H}1)_{\mathcal{H}^0}} \\
&= \frac{\int\Psi(q_0,\xi_0)F(q_0,\xi_0)[e^{-tH}G\Psi](q_0,\xi_0)d\mathsf{P}^0}{e^{-tE_0}} \\
&= (\Gamma F, e^{-t(H-E_0)}\Gamma G)_{\mathcal{H}^0} \\
&= (F, \Gamma^{-1}e^{-t(H-E_0)}\Gamma G)_{\mathcal{H}}.
\end{aligned}$$

This completes the proof of the theorem. □

The proof above requires a uniform lower bound on $d\mathsf{N}_T/d\mathsf{N}^0$, which is at present available only under the strong assumptions (P1) and $0 < e \leq e^*$. In [24] a uniform lower bound on $(\Psi_T, e^{-\tau H_{\mathrm{P}}} \otimes P_\Omega\Psi_T)$ has been obtained by using, as here, a functional integral representation (P_Ω is the projection onto the Fock vacuum). In our context this technique proves the existence of the ground state of H_{N} for $d \geq 4$ with assumption (P2) on the potential and with no restriction on e.

6 Absence of gap in the spectrum of H_{euc}

Consider the subspace $\hat{\mathcal{H}} \subset \mathcal{H}$ of functions having the property that $\mathbb{E}_{\mathsf{P}}[f] = 0$. Obviously, this subspace is invariant under the semigroup generated by H_{euc}. We denote the restriction of this operator to $\hat{\mathcal{H}}$ by $\hat{H}_{\mathrm{euc}}$.

Theorem 6.1 *For any $d \geq 3$ we have* $\inf \operatorname{Spec} \hat{H}_{euc} = 0$.

Proof: Consider the functional on $\mathcal{H}$

$$F_h(\xi) = \int_{\mathbb{R}^d} \xi(x) h(x) dx := \xi(h), \quad h \in \mathcal{S}(\mathbb{R}^d). \tag{6.1}$$

Clearly, $\xi(h) - \mathbb{E}_{\mathsf{P}}(\xi(h)) := \tilde{F}(h)$ is an element of $\hat{\mathcal{H}}$, and

$$(e^{-\hat{H}_{\mathrm{euc}} t} \tilde{F}_h, \tilde{F}_h)_{\mathcal{H}} = \int_0^\infty e^{-t\lambda} d\sigma_h(\lambda) := G_h(t) \tag{6.2}$$

with

$$d\sigma_h(\lambda) = d(\tilde{F}_h, \hat{P}_\lambda \tilde{F}_h) \tag{6.3}$$

and $\{\hat{P}_\lambda\}$ the spectral family of $\hat{H}_{\mathrm{euc}}$. Lemma 6.2 below implies that $\inf \operatorname{supp} d\sigma_h(\lambda) = 0$, and this on its turn implies $\inf \sup \hat{P}_\lambda = 0$. $\square$

Lemma 6.2 *The following asymptotics holds for $t \to \infty$,*

$$G_h(t) = \frac{C|\hat{h}(0)|^2}{t^{d-1}+1} + o\left(\frac{1}{t^{d-1}+1}\right) \tag{6.4}$$

with some constant $C > 0$.

Proof:

$$\begin{aligned} (e^{-\hat{H}_{\mathrm{euc}} t} \tilde{F}_h, \tilde{F}_h)_{\mathcal{H}} &= \operatorname{cov}_{\mathcal{P}}(\xi_t(h), \xi_0(h)) \\ &= \mathbb{E}_{\mathcal{N}}\left[\mathbb{E}_{\mathcal{P}Q}\left[(\xi_t(h) - \mathbb{E}_{\mathcal{P}Q}[\xi_0(h)])(\xi_0(h) - \mathbb{E}_{Q}[\xi_0(h)])\right]\right] \\ &+ \mathbb{E}_{\mathcal{N}}\left[(\mathbb{E}_{\mathcal{P}Q}[\xi_t(h)] - \mathbb{E}_{\mathcal{P}}[\xi_t(h)])(\mathbb{E}_{\mathsf{P}Q}[\xi_0(h)] - \mathbb{E}_{\mathcal{P}}[\xi_0(h)])\right] \end{aligned}$$

Here we used the identity

$$\begin{aligned} \mathbb{E}_\mu\left[(F - \mathbb{E}_\mu[F])(G - \mathbb{E}_\mu[G])\right] &= \mathbb{E}_\mu\left[(F - \mathbb{E}_\mu[F|\mathcal{F}])(G - \mathbb{E}_\mu[G|\mathcal{F}])\right] \\ &+ \mathbb{E}_\mu\left[(\mathbb{E}_\mu[F|\mathcal{F}] - \mathbb{E}_\mu[F])(\mathbb{E}_\mu[G|\mathcal{F}] - \mathbb{E}_\mu[G])\right] \end{aligned} \tag{6.5}$$

valid for an arbitrary probability measure μ, where $\mathbb{E}_\mu(\,\cdot\,|\mathcal{F})$ is a conditional average with respect to some σ-field $\mathcal{F}$, and F and G are given random variables. Since for any fixed trajectory Q the conditional covariance of $\xi_t(x)$ is the same as before, we get

$$\mathbb{E}_{\mathcal{P}Q}\left[(\xi_t(h) - \mathbb{E}_{\mathcal{P}Q}[\xi_t(h)])(\xi_0(h) - \mathbb{E}_{\mathcal{P}Q}[\xi_0(h)])\right] = \int_{\mathbb{R}^d} \frac{e^{-|k||t|}}{|k|} |\hat{h}(k)|^2 dk. \tag{6.6}$$

Then by choosing a sufficiently rapidly decreasing smooth function h with $\hat{h}(0) = \int h(x)dx \neq 0$, we see that the integral (6.6) behaves asymptotically like const$|\hat{h}(0)|^2/t^{d-1} + o(1/t^{d-1})$.

We turn next to the second term in (6.5). We have

$$\mathbb{E}_{\mathcal{P}^Q}[\xi_t(h)] = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{\mathbb{R}^d} \hat{h}(k)\frac{\hat{\varrho}(k)}{|k|}e^{-ik\cdot q_\tau}e^{-|k||t-\tau|}d\tau dk \tag{6.7}$$

and by stationarity

$$\mathbb{E}_{\mathcal{P}}[\xi_t(h)] = -\frac{1}{4}\int_{\mathbb{R}^d} dk \int_{-\infty}^{\infty} \hat{h}(k)\mathbb{E}_{\mathcal{N}}[e^{-ik\cdot q}]\frac{\hat{\varrho}(k)}{|k|}e^{-|k||t-\tau|}d\tau. \tag{6.8}$$

Hence

$$\mathbb{E}_{\mathcal{P}^Q}[\xi_t(h)] - \mathbb{E}_{\mathcal{P}}[\xi_t(h)] = -\frac{1}{4}\int_{-\infty}^{\infty} d\tau \int_{\mathbb{R}^d} \hat{h}(k)\frac{\hat{\varrho}(k)}{|k|}e^{-|k||t-\tau|}(e^{-ik\cdot q_\tau} - \mathbb{E}_{\mathcal{N}}[e^{-ik\cdot q}])dk. \tag{6.9}$$

Furthermore

$$\begin{aligned}
&(\mathbb{E}_{\mathcal{P}^Q}[\xi_t(h)] - \mathbb{E}_{\mathcal{P}}[\xi_t(h)])\,(\mathbb{E}_{\mathcal{P}^Q}[\xi_0(h)] - \mathbb{E}_{\mathcal{P}}[\xi_0(h)]) = \\
&\quad \frac{1}{16}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_1 d\tau_2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dk_1 dk_2 \hat{h}(k_1)\hat{h}(k_2)\frac{\hat{\varrho}(k_1)\hat{\varrho}(k_2)}{|k_1||k_2|}e^{-|k||t-\tau_1|}e^{-|k_2||\tau_2|} \times \\
&\qquad \times \left(e^{-ik_1\cdot q_{\tau_1}} - \mathbb{E}_{\mathcal{N}}[e^{-ik_1\cdot q}]\right)\left(e^{-ik_2\cdot q_{\tau_2}} - \mathbb{E}_{\mathcal{N}}[e^{-ik_2\cdot q}]\right)
\end{aligned}$$

Making use of (3.21) on the decay of correlations for $\mathcal{N}$ we get that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{N}}\left[(\mathbb{E}_{\mathcal{P}^Q}[\xi_t(h)] - \mathbb{E}_{\mathcal{P}}[\xi_t(h)])(\mathbb{E}_{\mathcal{P}^Q}[\xi_0(h)] - \mathbb{E}_{\mathcal{P}}[\xi_0(h)])\right] \qquad (6.10)\\
&\quad \leq \bar{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_1 d\tau_2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{h}(k_1)\hat{h}(k_2)\frac{\hat{\varrho}(k_1)\hat{\varrho}(k_2)}{|k_1||k_2|}\frac{e^{-|k_1||t-\tau_1|-|k_2||\tau_2|}}{|\tau_1-\tau_2|^\gamma+1}dk_1dk_2 \\
&\quad \leq \bar{C}(\sup|\hat{h}|)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\tau_1 d\tau_2}{(|t-\tau_1|^{d-1}+1)(|\tau_2|^{d-1}+1)(|\tau_1-\tau_2|^\gamma+1)}
\end{aligned}$$

with some $\bar{C} > 0$. Denote

$$L_1(t) = \frac{1}{|t|^{d-1}+1}, \quad L_2(t) = \frac{1}{|t|^\gamma+1} \tag{6.11}$$

The right hand side of (6.10) above is the double convolution $L_1 * L_1 * L_2$. Then we get by Fourier transformation

$$(\widehat{L_1 * L_1 * L_2})(y) = \hat{L}_1(y)^2\hat{L}_2(y), \quad y \in \mathbb{R}. \tag{6.12}$$

By general results on the asymptotics of Fourier transforms [2] we have for small y

$$\begin{aligned}
\hat{L}_1(y) &\sim |y|^{d-2} \\
\hat{L}_2(y) &\sim |y|^{\gamma-1}
\end{aligned}$$

Thus $\hat{L}_1(y)^2\hat{L}_2(y) \sim |y|^{2d+\gamma-5}$. The asymptotics of the integral at the right hand side of (6.10) becomes then $1/t^{2d+\gamma-4}$. Note that for $d \geq 3$ and $\gamma > 0$, $2d+\gamma-4 > d-1$, and thus the second term in (6.5) is $o(1/t^{d-1})$. This completes the proof. $\square$

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