Nonperturbative techniques in the investigation of the spectral properties of many-channel systems $^{(\star)}$

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Abstract. We show that the C^*-algebras generated by the "virtual" hamiltonians of a quantum system (with a qualitatively specified interaction) has in many cases an interesting and nontrivial structure. In particular, the quotient of this algebra with respect to the ideal of compact operators can be explicitly computed. This allows one to determine in a unified way the essential spectrum and to prove the Mourre estimate for large classes of hamiltonians, including: N-Body systems, stratified media, particles subject to Klaus type interactions (widely separated bumps) and other classes of (phase-space) anisotropic hamiltonians. The results presented here are based on a joint work with V. Georgescu.

1 Introduction. Examples of general many-channel systems

We present a couple of quantum systems that have an interesting and rather complex many-channel structure. We will focus on studying the essential spectrum although the algebraic methods introduced in [GI00] fit quite well in the context of the Mourre theory and allow one to investigate refined properties of the spectrum of such systems.

In this section we describe some results that apparently have nothing to do with an algebraic context, and in the rest of the paper we show how they can be obtained by using a unifying C^*-algebraic formalism. We begin with a result which generalizes that of Klaus on "widely separated bumps" types of perturbations. Let $P = -i\nabla$ and Q be the momentum and position observables in $L^2(\mathbb{R}^n)$ and $\mathscr{H}^s = \mathscr{H}^s(\mathbb{R}^n)$ the Sobolev space for $s \in \mathbb{R}$. In a very rough version, we are placed in the following framework: consider a subset L of $\mathbb{R}^n$ which is *rarefied* (or *sparse*) in the sense that it is locally finite and $\lim_{x\in L, |x|\to\infty} \operatorname{dist}(x, L \setminus \{x\}) = \infty$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $C^{-1}|x|^{2s} \le |h(x)| \le C|x|^{2s}$ if $|x| > R$, for some constants $s > 0$, $C > 0$ and $R < \infty$. Denote $H_0 = h(P)$. Let $\{V_l\}_{l\in L}$ be a family of symmetric operators in the space of bounded operators $B(\mathscr{H}^t, \mathscr{H}^{-t})$, for some $t \in [0, s)$, with the property that there is a number $a > 2n$ such that

$$\sup_{l\in L} \|\langle Q\rangle^a V_l\|_{B(\mathscr{H}^t,\mathscr{H}^{-t})} < \infty. \tag{1.1}$$

We give now the result in its first version:

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Theorem 1.1. *Under the above hypotheses* H_0 *is a self-adjoint operator in* $\mathscr{H}$ *with* $D(|H_0|^{1/2}) = \mathscr{H}^s$ *and one has:*
(i) *The series* $\sum_{l\in L} \mathrm{e}^{-ilP} V_l \mathrm{e}^{ilP}$ *converges in the strong topology of* $B(\mathscr{H}^t, \mathscr{H}^{-t})$ *toward a symmetric operator* $V : \mathscr{H}^t \to \mathscr{H}^{-t}$.
(ii) *For each* $\varepsilon > 0$ *there is a constant* $c < \infty$ *such that* $\pm V \leq \varepsilon H_0 + c$. *Thus the form sums* $H = H_0 + V$, $H_l = H_0 + V_l$ *are self-adjoint operators in* $\mathscr{H}$ *with the same form domain as* H_0.
(iii) *The essential spectrum of* H *is:* $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcap_{\substack{F\subset L\\ F\,\mathrm{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}$.

Remarks: (i) We consider in fact an arbitrary abelian locally compact group X instead of $\mathbb{R}^n$. **(ii)** If $s \leq n/2$ and $V_l : \mathbb{R}^n \to \mathbb{R}$ are Borel functions satisfying the condition $\int_{|y-x|<1} |V_l(y)| \cdot |y-x|^{-n+2s-\lambda}\,\mathrm{d}y \leq c\langle x\rangle^{-a}$ $\forall x \in \mathbb{R}^n$ for some constans $c, \lambda > 0$, then the operators V_l of multiplication by the functions V_l satisfy (1.1) for some $t < s$. If $s > n/2$ then the simpler condition $\int_{|y-x|<1} |V_l(y)|\,\mathrm{d}y \leq c\langle x\rangle^{-a}$ suffices. If s is an integer and V_l is a differential operator of order less than $2s$, similar explicit conditions on its coefficients can be stated. **(iii)** The unnatural condition $a > 2n$ may actually be improved to $a > n$ when V_l are local operators. Anyway, the assumption $a > 2n$ may be relaxed in terms of the "degree of rarefaction" of L: the greatest lower bound for a is actually inversely proportional to it. **(iv)** There is no change in our formalism if we replace the Hilbert space $L^2(X)$ of physical states by $L^2(X; \mathbf{E})$ where $\mathbf{E}$ is a finite dimensional Hilbert space. This allows us to treat, for example, Dirac hamiltonians H_0 perturbed by the same class of potentials V.

We give now a second version of the result, involving limits at infinity along ultrafilters. If X is a locally compact space then $\widetilde{X}$ is the set of ultrafilters $\varkappa$ on X finer than the filter of neighbourhoods of infinity; $\widetilde{X}$ should be thought as the "boundary" at infinity of X. Denote $\lim_{l\in\varkappa}$ the limit along the ultrafilter $\varkappa$.

Theorem 1.2. *Under the hypotheses of the preceding theorem, one has:*
(i) *The strong resolvent limit* $\mathrm{s}\text{-}\lim_{l\in\varkappa} \mathrm{e}^{-ilP} H \mathrm{e}^{ilP} = H_\varkappa$ *exists for each* $\varkappa \in \widetilde{L}$.
(ii) *For* $\varkappa \in \widetilde{L}$ *one also has* $H_\varkappa = \mathrm{u}\text{-}\lim_{l\in\varkappa} H_l$ *(limit in the norm resolvent sense).*
(iii) *The essential spectrum of* H *is:* $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{\varkappa\in\widetilde{L}} \boldsymbol{\sigma}(H_\varkappa)$.

Notice that the existence of the limits in (i) and (ii) above and the inclusion $\supset$ in (iii) are not hard to prove. If one prefers to avoid the notion of ultrafilter, the result can be stated (in a less intrinsic form) in terms of usual sequences, as follows:

> *Let* $\mathcal{L}(H)$ *be the set of sequences* $\boldsymbol{l} \equiv \{l_k\}_{k\in\mathbb{N}}$ *of* $L \subset X$ *such that* $\mathrm{s}\text{-}\lim_{k\to\infty} \mathrm{e}^{-il_kP} H \mathrm{e}^{il_kP} =: H_{\boldsymbol{l}}$ *exists. Then:* $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{\boldsymbol{l}\in\mathcal{L}(H)} \boldsymbol{\sigma}(H_{\boldsymbol{l}})$.

As a second example we consider quantum systems with a general configuration space anisotropy:

Theorem 1.3. *Let* $X = \mathbb{R}^n$ *and* H *a self-adjoint operator in* $L^2(X)$ *such that:*

(a) $\lim_{\substack{x\to 0\\ x\in X}} \|(\mathrm{e}^{i\langle x,P\rangle}-1)(H+i)^{-1}\|=0$,
(b) $\lim_{\substack{k\to 0\\ k\in X^*}} \|[\mathrm{e}^{i\langle Q,k\rangle},(H+i)^{-1}]\|=0$,
Then, for each $\varkappa\in\widetilde{X}$ *the limit* $H_\varkappa = \text{s-}\lim_{x\in\varkappa}\mathrm{e}^{i\langle x,P\rangle}H\mathrm{e}^{-i\langle x,P\rangle}$ *exists (in strong resolvent sense), and* $\boldsymbol{\sigma}_{\mathrm{ess}}(H)=\bigcup_{\varkappa\in\widetilde{X}}\boldsymbol{\sigma}(H_\varkappa)$.

We stress that such a pure configuration space description of the essential spectrum is not valid in general (e.g. for systems with full phase-space anisotropy).

2 An algebraic method

Our goal is to show how these results, and many others of this type, involving even much complicated situations, may be treated in a unified manner, by an algebraic approach. The main ideea of this approach is that the admissible hamiltonians corresponding to a physical system subject to a certain type of interactions generate a C^*-algebra $\mathfrak{C}$ (the algebra of energy observables of the system) realized on a Hilbert space $\mathscr{H}$, such that:
(i) $K(\mathscr{H})\subset\mathfrak{C}$
(ii) the quotient C^*-algebra $\mathfrak{C}\,/\,K(\mathscr{H})$ is, in a certain sense, *computable*.
The *consequences for the spectral theory* will concern both the essential spectrum and the Mourre estimate which is the key of a refined investigation of the spectrum (see [ABG96] for further information). Indeeed, if we suppose (for simplicity) $H\in\mathfrak{C}$, and if $\widehat{H}$ denotes its image in $\mathfrak{C}\,/\,K(\mathscr{H})$, these consequences may be stated as:
(i) $\boldsymbol{\sigma}_{\mathrm{ess}}(H)=\boldsymbol{\sigma}(\widehat{H})$
(ii) $\widehat{H}$ satisfies a strict Mourre estimate at some $\lambda\in\mathbb{R}$ if and only if H satisfies a Mourre estimate at λ (modulo some technical conditions).

2.1. We need some definitions. A self-adjoint operator H on $\mathscr{H}$ is *affiliated to a* C^**-algebra* $\mathfrak{C}\subset B(\mathscr{H})$ iff $(H-z)^{-1}\in\mathfrak{C}$ for some $z\in\mathbb{C}\setminus\boldsymbol{\sigma}(H)$; then $\varphi(H)\in\mathfrak{C}$ for all $\varphi\in C_0(\mathbb{R})$ (this will sometimes be denoted by $H\,''\!\in''\,\mathfrak{C}$). The C^**-algebra generated by a family of self-adjoint operators* $\{H\}$ is, by definition, the C^*-algebra generated by the family of resolvents $(H-z)^{-1}$. If $\{\mathfrak{C}_i\}_{i\in I}$ is an arbitrary family of C^*-algebras, the direct product $\prod_{i\in I}\mathfrak{C}_i$ and direct sum $\bigoplus_{i\in I}\mathfrak{C}_i$ are defined as:

$$\prod\nolimits_{i\in I}\mathfrak{C}_i=\{S=(S_i)_{i\in I}\mid S_i\in\mathfrak{C}_i \text{ and } \|S\|:=\sup_{i\in I}\|S_i\|<\infty\}$$
$$\bigoplus\nolimits_{i\in I}\mathfrak{C}_i=\{S=(S_i)_{i\in I}\mid S_i\in\mathfrak{C}_i \text{ and } \|S_i\|\to 0 \text{ as } i\to\infty\}.$$

These are C^*-algebras for the usual operations and $\bigoplus_{i\in I}\mathfrak{C}_i$ is an ideal in $\prod_{i\in I}\mathfrak{C}_i$.

2.2. The notion of *"computable"* for a quotient algebra $\mathfrak{C}\,/\,K(\mathscr{H})$ is crucial for the method we present here. However, we must be aware that its precise definition

depends on the complexity of the problem. Here we describe two situations of increasing complexity, for which the meaning of “computable” is different:
A) Systems as in the case of the usual and generalized N-body problems (the lattice of partitions can be replaced by an arbitrary, even infinite, lattice), or various configuration and phase-space anisotropic systems (e.g. 1-dim operators, multistratified strips, constant magnetic fields). Then the algebra $\mathfrak{C}\big/K(\mathscr{H})$ is “computable” in the following elementary sense: one can find a family $\{\mathfrak{C}_i\}_{i\in I}$ of C^*-algebras (which all have a “much simpler” structure than that of $\mathfrak{C}$) such that:

$$\mathfrak{C}\big/K(\mathscr{H}) \;\hookrightarrow\; \textstyle\prod_{i\in I}\mathfrak{C}_i. \tag{2.1}$$

In other terms, a family of morphisms $\mathcal{P}_i : \mathfrak{C} \to \mathfrak{C}_i$ is given such that $\bigcap_{i\in I}\ker\mathcal{P}_i = K(\mathscr{H})$. If we set $\mathcal{P}[S] = \prod_{i\in I}\mathcal{P}_i[S]$ we shall get a morphism $\mathcal{P} : \mathfrak{C} \to \prod_{i\in I}\mathfrak{C}_i$ with kernel $K(\mathscr{H})$, which induces the required embedding. Now let, for simplicity, H be a bounded admissible hamiltonian and set $H_i = \mathcal{P}_i[H]$. Then, by (2.1), we have an identification $\widehat{H} = \prod_{i\in I}H_i$ and thus:

$$\boldsymbol{\sigma}_{\rm ess}(H) = \overline{\textstyle\bigcup_{i\in I}\boldsymbol{\sigma}(H_i)}. \tag{2.2}$$

This covers the *theorems of HVZ type* known in the usual many-channel situations. We mention that in [DG] is pointed out an abstract class of C^*-algebras graded by infinite semilattices, whose quotient is of the preceding form.
B) A worse situation, which covers systems with infinitely many, widely separated “bumps” cf. [K83]: one can find a family $\{(\mathfrak{C}_i,\mathfrak{J}_i)\}_{i\in I}$ of (“much simpler” than $\mathfrak{C}$) C^*-algebras $\mathfrak{C}_i$ equipped with ideals $\mathfrak{J}_i$ such that there is a natural embedding:

$$\mathfrak{C}\Big/_{K(\mathscr{H})} \;\hookrightarrow\; \textstyle\prod_{i\in I}\mathfrak{C}_i\Big/_{\bigoplus_{i\in I}\mathfrak{J}_i}\,. \tag{2.3}$$

Note that the preceding case (2.1) is obtained when $\mathfrak{J}_i = \{0\}$ for all $i \in I$.
Assume, for example, that each $\mathfrak{C}_i$ is realized on a Hilbert space $\mathscr{H}_i$ and that $\mathfrak{J}_i = K(\mathscr{H}_i)$. We show that if H is a self-adjoint operator affiliated to $\mathfrak{C}$ and $\widehat{H}$ is, in the representation given by the right hand side of (2.3), the quotient of some family $\{H_i\}_{i\in I}$ of operators H_i affiliated to $\mathfrak{C}_i$, then

$$\boldsymbol{\sigma}_{\rm ess}(H) = \textstyle\bigcap_{\substack{F\subset I\\ F\,\text{finite}}}\left\{\left[\bigcup_{i\in F}\boldsymbol{\sigma}_{\rm ess}(H_i)\right] \cup \left[\overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}\right]\right\}. \tag{2.4}$$

Versions of this formula that are more similar to a classic HVZ result can be obtained, as we shall see, by the approach based on ultrafilters.

2.3. We shall describe now some of the ways in which we may construct algebras of energy observables.
A) *Algebras of energy observables constructed as cross products:*
Let $\mathscr{A}$ be a closed $*$-subalgebra of $C^{\rm u}_{\rm b}(\mathbb{R}^n)$ which contains the constants and is stable under translations; put $\mathscr{A}^\infty = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \partial^\alpha\varphi \in \mathscr{A},\ \forall\alpha\}$, let $\Delta = P^2$ be the (positive) Laplace operator and consider self-adjoint operators of the form

$H = \Delta + V$, where V is a first order differential operator with coefficients in $\mathscr{A}^\infty$. We shall denote by X the configuration space $\mathbb{R}^n$ and by X^* its dual. Then:
(i) *the* C^**-algebra of operators on* $L^2(X)$ *generated by the operators* H *is equal to the norm closed subspace generated by operators of the form* $\varphi(Q)\psi(P)$*, where* $\varphi \in \mathscr{A}$ *and* $\psi \in C_0(X)$. This last algebra is denoted by $\llbracket \mathscr{A} \cdot C_0(X^*) \rrbracket$.
(ii) *the algebra* $\llbracket \mathscr{A} \cdot C_0(X^*) \rrbracket$ *is actually the crossed product* $\mathscr{A} \rtimes X$ of $\mathscr{A}$ by the natural action of the translation group on it.
This justifies the interpretation of a general class of crossed products $\mathscr{A} \rtimes X$ as algebras of energy observables. In fact, the class of "algebras of energy observables" is much larger than the class of algebras obtained as cross products, the latter corresponding only to the case of Q-anisotropic or P-anisotropic systems. Other types of algebras of energy observables, which are not crossed products, may be studied by the same method:
B) The simplest type of phase space (or Q-P) anisotropy, where the algebra is (essentially) of the form $\llbracket \mathscr{A} \cdot \mathscr{B} \rrbracket$, where $\mathscr{A}$ and $\mathscr{B}$ are C^*-subalgebras of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $C_{\mathrm{b}}^{\mathrm{u}}(X^*)$ respectively.
C) Full phase space anisotropy: here we consider C^*-algebras naturally associated to the Heisenberg group, which is the simplest type of nilpotent group. The algebra we construct will be called *graded symplectic algebra.*
D) Algebras of creation-anihilation operators on a Fock space. These are generated by hamiltonians of quantum fields with a particle number cut-off. Their quotient with respect to the ideal of compact operators has an especially nice form (see the lecture of V. Georgescu in this volume).

2.4. The *main advantage of crossed product algebras* $\mathscr{A} \rtimes X$ *is that*, due to $\mathbb{K}(X) = \llbracket C_0(X) \cdot C_0(X^*) \rrbracket = C_0(X) \rtimes X$ and to the good functorial properties of cross products, *it suffices to "compute" the quotient at an abelian level*, for example to put in evidence an embedding $\mathscr{A} \,/\, C_0(X) \hookrightarrow \prod_{i\in I}^{X} \mathscr{C}_i$, (where the latter is the greatest subalgebra of $\prod_{i \in I} \mathscr{C}_i$ for which the cross product with X makes sense). Indeed, we first choose a C^*-algebra $\mathscr{A}$, viewed as the algebra of "elementary interactions" (or potentials), such that:

$$\begin{cases} C_0(X) \subset \mathbb{C} + C_0(X) \equiv C_\infty(X) \subset \mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X), \\ \mathscr{A} \ \text{ is stable under translations.} \end{cases} \tag{2.5}$$

If the quotient is computable at the abelian level, as above, then

$$\begin{aligned} [\mathscr{A} \rtimes X] \,/\, \mathbb{K}(X) \ &= \ [\mathscr{A} \rtimes X] \,/\, [C_0(X) \rtimes X] \ \cong \ [\mathscr{A} \,/\, C_0(X)] \ \rtimes X \\ &\hookrightarrow \ \textstyle\prod_{i\in I}^{X} \mathscr{C}_i \ \rtimes X \ \hookrightarrow \ \prod_{i \in I} [\mathscr{C}_i \rtimes X]. \end{aligned}$$

The simplest situation is furnished by a two-body quantum system (or a particle in an external field); then $\mathscr{A} = C_\infty(X)$, $\mathbb{T}(X) \equiv C_\infty(X) \rtimes X$ is the "two-body energy algebra" and we get $\mathbb{T}(X) \,/\, \mathbb{K}(X) = C_0(X^*)$.

2.5. Notice that $C_\infty(X)$ is related to the simplest compactification of X, the one-point (Alexandroff) compactification: $X_\infty \equiv X \cup \{\infty\}$. Indeed, we have a natural identification $C_\infty(X) = C(X_\infty)$ and this is the simplest case of a general phenomenon: the algebras $\mathscr{A}$ with the properties (2.5) are in bjective correspodance with a certain class of compactifications of X. The spectrum $X(\mathscr{A})$ of $\mathscr{A}$ is a compact space and X is homeomorphically and densely embedded in $X(\mathscr{A})$. Also, $\mathscr{A}$ is just the set of functions in $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which have continuous extensions to $X(\mathscr{A})$, i.e. $\mathscr{A} = C(X(\mathscr{A}))$. These compactifications can be described in terms of ultrafilters on X (they are quotients of the Stone-Čech compactification βX). We shall, however, not insist on this point of view, because in non trivial cases the spaces $X(\mathscr{A})$ are rather complicated.

3 "Bumps" Algebras

The class of hamiltonians considered in this section generalizes that of the Schrödinger operators with "widely separated bumps" introduced by M.Klaus in [K83] the quotient algebra being computable in the more general sense we already described in examples of type (2) from §2.2.

3.1. Let X be an abelian locally compact group and $L \subset X$. If $\varphi : X \to \mathbb{C}$, we say that *the limit* L-limφ *exists* iff $\exists\, c \in \mathbb{C}$ such that $\forall\, \varepsilon > 0$ one can find a compact set $\Lambda \subset X$ such that $x \notin L + \Lambda \Longrightarrow |\varphi(x) - c| < \varepsilon$. We denote $c = L\text{-lim}\varphi$. Define:

$$\mathscr{C}_L = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid L\text{-lim}\varphi \text{ exists } \} \quad \text{and} \quad \mathscr{C}_{L,0} = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid L\text{-lim}\varphi = 0\}.$$

Then $\mathscr{C}_L$ and $\mathscr{C}_{L,0}$ are translation invariant C^*-subalgebras of $C_{\mathrm{b}}^{\mathrm{u}}(X)$. One has $C_\infty(X) \subset \mathscr{C}_L$ and $C_0(X) \subset \mathscr{C}_{L,0}$. Also, $\mathscr{C}_{L,0}$ is an ideal of $\mathscr{C}_L$ (and of $C_{\mathrm{b}}^{\mathrm{u}}(X)$) and $\mathscr{C}_L = \mathbb{C} + \mathscr{C}_{L,0}$. We may construct an "energy observables algebra" by taking the cross product with X; as we said before we also have the identities:

$$\mathfrak{C}_L := \mathscr{C}_L \rtimes X = [\![\mathscr{C}_L \cdot C_0(X^*)]\!] \quad \text{and} \quad \mathfrak{C}_{L,0} := \mathscr{C}_{L,0} \rtimes X = [\![\mathscr{C}_{L,0} \cdot C_0(X^*)]\!]. \quad (3.1)$$

The problem of computing the quotients $\mathfrak{C}_L \big/ \mathbb{K}(X)$ and $\mathfrak{C}_{L,0} \big/ \mathbb{K}(X)$ is difficult for general sets L but there are notable particular cases when this is possible, for instance the case of a "widely separated bumps" problem, when L is *rarefied*, i.e. it is locally finite and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$. An explicit description of the quotient algebras will be given below. Notice that we actually can manage sets L that are subject to less restrictive assumptions than that of being rarefied (in the above sense).

3.2. It is possible to give a *new characterization of the algebras* $\mathfrak{C}_L$ *and* $\mathfrak{C}_{L,0}$ *in geometric terms in the phase-space* $\Xi = X \oplus X^*$. We begin by introducing some notations. Let $\{U_x\}_{x \in X}$ and $\{V_k\}_{k \in X^*}$ be the strongly continuous unitary representations of X and X^* in $L^2(X)$ defined by $(U_x f)(y) = f(x + y)$ and $(V_k f)(y) = \mathrm{e}^{i\langle y,k\rangle} f(y)$ respectively. Note that U_x and V_k satisfy the so called canonical commutation relations $U_x V_k = \mathrm{e}^{i\langle x,k\rangle} V_k U_x$. If X is a (real or p-adic) finite dimensional vector space, we have (with the quantum mechanical conventions): $U_x = \mathrm{e}^{i\langle x,P\rangle}$ and $V_k = \mathrm{e}^{i\langle Q,k\rangle}$. If A and T are bounded operators and A is self-adjoint, we shall say that $\|AT^{(*)}\|$ satisfies the property (P) if and only if $\|TA\| + \|AT\|$ verifies (P). Also, $\chi_{L^c_\Lambda}$ will denote the characteristic function of the set $L^c_\Lambda \equiv X \setminus (L + \Lambda)$.

Theorem 3.1. **(1)** *A bounded operator T on $L^2(X)$ belongs to $\mathfrak{C}_{L,0}$ if and only if*
(i) $\lim_{x\to 0} \|(U_x - 1)T^{(*)}\| = 0$,
(ii) $\lim_{k\to 0} \|V_k T V_k^* - T\| = 0$,
(iii) *For each $\varepsilon > 0$ there is a compact set Λ in X such that $\|\chi_{L^c_\Lambda}(Q)T^{(*)}\| < \varepsilon$.*
(2) *$T \in \mathfrak{C}_L$ iff it satisfies* (i), (ii) *and there exists $\widetilde{T} \in C_0(X^*)$ such that for each $\varepsilon > 0$ there is a compact set $\Lambda \in X$ with $\|\chi_{L^c_\Lambda}(Q)\,(T - \widetilde{T}\,)^{(*)}\| < \varepsilon$.*

In order to compute the quotients $\mathfrak{C}_L \,/\, \mathbb{K}(X)$ and $\mathfrak{C}_{L,0} \,/\, \mathbb{K}(X)$ it will be useful to consider the case when for each $l \in L$ is given an algebra $\mathfrak{C}_l \equiv \mathfrak{C}$ independent of $l \in L$. Then we denote $\bigoplus_{l\in L}\mathfrak{C}$ by $\mathfrak{C}^{(L)}$ and one can introduce a new C^*-subalgebra of $\prod_{l\in L}\mathfrak{C}$, namely

$$\mathfrak{C}^{[L]} := \{(S_l)_{l\in L} \in \textstyle\prod_{l\in L}\mathfrak{C} \mid \{S_l \mid l \in L\} \text{ is a relatively compact set in } \mathfrak{C}\}. \tag{3.2}$$

Then $\mathfrak{C}^{(L)}$ is an ideal in $\mathfrak{C}^{[L]}$. If L is equipped with the discrete topology, then

$$\mathfrak{C}^{(L)} = c_0(L;\mathfrak{C}) \cong c_0(L) \otimes \mathfrak{C} \quad \text{and} \quad \mathfrak{C}^{[L]} \cong l^\infty(L) \otimes \mathfrak{C} \subset l^\infty(L;\mathfrak{C}) = \textstyle\prod_{l\in L}\mathfrak{C}.$$

One more object appears naturally: the *L-asymptotic algebra* of $\mathfrak{C}$. This is the quotient algebra: $\mathfrak{C}^{\langle L\rangle} := \mathfrak{C}^{[L]} \,/\, \mathfrak{C}^{(L)}$.

Theorem 3.2. *If L is rarefied, $\mathfrak{C}_L \,/\, \mathbb{K}(X)$ and $\mathfrak{C}_{L,0} \,/\, \mathbb{K}(X)$ are "computable" in the sense that:*
(i) $\mathfrak{C}_{L,0} \,/\, \mathbb{K}(X) \cong \mathbb{K}(X)^{\langle L\rangle}$.
(ii) *One has a natural embedding:* $\mathfrak{C}_L \,/\, \mathbb{K}(X) \;\hookrightarrow\; \mathbb{T}(X)^{[L]} \,/\, \mathbb{K}(X)^{(L)}$.

Proof. We follow the algorithm already described in §2.4. We prove first the theorem at the abelian level, i.e. we show the embeddings:

$$\mathscr{C}_{L,0} \,/\, C_0(X) \cong C_0(X)^{\langle L\rangle} \quad \text{and} \quad \mathscr{C}_L \,/\, C_0(X) \;\hookrightarrow\; C_\infty(X)^{[L]} \,/\, C_0(X)^{(L)}.$$

A precise description of these morphisms is given by the result:

> *There is a unique morphism $\mathscr{C}_L \to C_\infty(X)^{[L]} \,/\, C_0(X)^{(L)}$ such that the image of an element of the form $c + \sum_{l\in M} U_l^* \varphi(Q) U_l$, where $c \in \mathbb{C}$, $M \subset L$, and $\varphi \in C_0(X)$ verifies $\varphi = \chi_\Lambda \varphi$ for some compact set $\Lambda \subset X$, be the quotient of the element $\chi_M \otimes (c + \varphi(Q)) \in C_\infty(X)^{[L]}$ with respect to the ideal $C_0(X)^{(L)}$. The kernel of this morphism is $C_0(X)$ and its restriction to $\mathscr{C}_{L,0}$ induces the canonical isomorphism of $\mathscr{C}_{L,0} \,/\, C_0(X)$ with the L-asymptotic algebra $C_0(X)^{\langle L\rangle}$.*

In the second step of the proof we use the functorial properties of the crossed product in order to obtain the result at the quantized level. ☐

3.3. If H is an observable affiliated to $\mathfrak{C}_L$ and if $\widehat{H}$ is its image through the canonical morphism $\mathfrak{C}_L \to \mathfrak{C}_L \,/\, \mathbb{K}(X)$, then there is a family $(H_l)_{l\in L}$ of observables affiliated to the two-body algebra $\mathbb{T}(X)$ s.t. the quotient of $\prod_{l\in L} H_l$ with respect to the ideal $\mathbb{K}(X)^{(L)}$ be equal to the image of $\widehat{H}$ through the embedding $\mathcal{J}$ below:

$$\begin{tikzcd}
H \text{ ''}\in\text{'' } \mathfrak{C}_L \arrow[d] & \mathbb{T}(X)^{[L]} \text{ ''}\ni\text{'' } (H_l)_{l\in L} \arrow[d] \\
\widehat{H} \text{ ''}\in\text{'' } \mathfrak{C}_L\big/_{\mathbb{K}(X)} \arrow[r, hook, "\mathcal{J}"] & \mathbb{T}(X)^{[L]}\big/_{\mathbb{K}(X)^{(L)}} \text{ ''}\ni\text{'' } \mathcal{J}(\widehat{H}) = \widehat{(H_l)}_{l\in L}
\end{tikzcd}$$

Such a family $(H_l)_{l\in L}$ will be called a *representative* of H. By the discussion above we have $\prod_{l\in L}(H_l - z)^{-1} \in \mathbb{T}(X)^{[L]}$ and the component of $(H_l - z)^{-1}$ in $C_0(X^*)$ is independent of $l \in L$, so $\boldsymbol{\sigma}_{\text{ess}}(H_l)$ is independent of l. Thus the next result is a consequence of (2.4).

Theorem 3.3. *If H is an observable affiliated to $\mathfrak{C}_L$ and $\{H_l\}_{l\in L}$ is a representative of H, then $\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L \\ F\,\text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}$.*

4 The examples revisited

A) Klaus type potentials: The result, in its first version, may be restated as:

Theorem 1.1. BIS: *Under the assumptions and notations of the Theorem 1.1, $H = H_0 + V$ is strictly affiliated to $\mathfrak{C}_L$, $H_l = H_0 + V_l$ is strictly affiliated to $\mathbb{T}(X)$, and the family $\{H_l\}_{l\in L}$ is a representative of H. In particular:*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L \\ F\,\text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}.$$

Since $H = H_0 + V$ is affiliated to $\mathfrak{C}_L$, the second version of the result (Theorem 1.2), based on ultrafilters, follows from:

Theorem 4.1. *For each $S \in \mathfrak{C}_L$ and $\varkappa \in \widetilde{L}$ the limit $\text{s-lim}_{l\in\varkappa} U_l S U_l^* =: S_\varkappa$ exists and belongs to $\mathbb{T}(X) = C_0(X^*) + \mathbb{K}(X)$. The component of $S_\varkappa$ in $C_0(X^*)$ is equal to that of S in $C_0(X^*)$. The map $S \mapsto (S_\varkappa)_{\varkappa\in\widetilde{L}}$ is a morphism of $\mathfrak{C}_L$ into $C(\widetilde{L}\,; \mathbb{T}(X))$ with kernel $\mathbb{K}(X)$. Its range equals the set of $(S_\varkappa)_{\varkappa\in\widetilde{L}}$ such that the component of $S_\varkappa$ in $C_0(X^*)$ is independent of $\varkappa$. Thus: $\mathfrak{C}_L \,/\, \mathbb{K}(X) \hookrightarrow C(\widetilde{L}\,; \mathbb{T}(X))$.*

B) General configuration space anisotropy: Remember the result stated as the Theorem 1.3: apparently it has nothing to do with the preceding algebraic approach. The connection is in fact established via the following theorem.

Theorem 4.2. *Let $X = \mathbb{R}^n$ and H a self-adjoint operator in $L^2(X)$. Then the following assertions are equivalent:*
(i) *H is affiliated to the crossed product algebra $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$*
(ii) *H satisfies the assertions (a) and (b) of the Theorem 1.3, namely*

$$\lim_{\substack{x\to 0\\ x\in X}} \|(\mathrm{e}^{i\langle x,P\rangle}-1)(H+i)^{-1}\| = 0 \quad \textit{and} \quad \lim_{\substack{k\to 0\\ k\in X^*}} \|[\mathrm{e}^{i\langle Q,k\rangle},(H+i)^{-1}]\| = 0.$$

Hence, to prove the Theorem 1.3, it is sufficient to put in evidence the embedding:

$$C_{\mathrm{b}}^{\mathrm{u}}(X)\,/\,C_0(X) \;\hookrightarrow\; C(\widetilde{X}\,;C_{\mathrm{b}}^{\mathrm{u}}(X)).$$

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