ON THE ZERO MODES OF WEYL-DIRAC OPERATORS AND THEIR MULTIPLICITY

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ABSTRACT. It is proved that the existence of zero modes of Weyl-Dirac operators is a rare phenomenon. An estimate of the multiplicity is given in term of the magnetic potential.

1. INTRODUCTION

In [1] it was proved that the set of magnetic fields which give rise to zero modes of the associated Pauli operators in $\mathbb{R}^3$ is rather sparse. Specifically the following results were proved: let $\mathbb{P}_A$ be the Pauli operator defined as the Friedrichs extension in $[L^2(\mathbb{R}^3)]^2$ of the operator

$$\mathbb{P}_A = \left\{\vec{\sigma}\cdot\left(\frac{1}{i}\vec{\nabla}+\vec{A}\right)\right\}^2 \equiv \sum_{j=1}^{3}\left\{\sigma_j\left(\frac{1}{i}\partial_j + A_j\right)\right\}^2 \qquad (1)$$

on $[C_0^\infty(\mathbb{R}^3)]^2$, where $\vec{A}=(A_1,A_2,A_3)$ is a vector potential and $curl\,\vec{A} =: \vec{B}$ the magnetic field, and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ is the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2)$$

Then, denoting the nullity of $\mathbb{P}_A$, i.e. the dimension of the kernel of $\mathbb{P}_A$, by $nul\,\mathbb{P}_A$,

- for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, $nul\,\mathbb{P}_{tA} = 0$ except for a finite number of values of t in any compact subset of $[0,\infty)$,
- $\{\vec{B} \,:\, nul\,\mathbb{P}_A = 0,\ |\vec{B}| \in L^{3/2}(\mathbb{R}^3)\}$ contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

In this paper we approach the problem through the Weyl-Dirac operator

$$\mathbb{D}_A \;=\; \vec{\sigma}\cdot(\vec{p}+\vec{A}),\quad \vec{p}=\frac{1}{i}\vec{\nabla}. \qquad (3)$$

The analysis has a similar flavour to that in [1] but there are three significant developments: firstly, there is no assumption that $curl\,\vec{A}$ exists and so singular behaviour is allowed; secondly, a result is established for the two-dimensional problem; thirdly, we obtain an upper bound for the number of zero modes in $\mathbb{R}^3$. Similar results to those above for $nul\,\mathbb{P}_A$ are here proved for $nul\,\mathbb{D}_A$, but with the assumption $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$ replaced by $|\vec{A}| \in L^3(\mathbb{R}^3)$. There is a similar result in $\mathbb{R}^2$, but now with $|\vec{A}| \in L^2(\mathbb{R}^2)\cap L^r(\mathbb{R}^2)$ for some $r>2$, the $L^r(\mathbb{R}^2)$ being present only to guarantee that $\mathbb{D}_A$ is defined as an operator sum; note that, when $n=2$,

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there is no dependence on x_3 in (1) and $\vec{\sigma} = (\sigma_1, \sigma_2)$, $\vec{A} = (A_1, A_2)$. The assumption on $|\vec{A}|$ implies that when B is defined (as a scalar field in $\mathbb{R}^2$) and

$$\Phi(r) = \frac{1}{2\pi} \int\limits_{|\mathbf{x}|<r} B(\mathbf{x})\, d\mathbf{x},$$

then

$$\int\limits_0^\infty |\Phi(r)|^2\, dr \le \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} |\vec{A}|^2\, d\mathbf{x}$$

and so the flux $\Phi(\infty) = 0$. The Aharonov-Casher Theorem which assert that

$$nul\, \mathbb{P}_A = \left\{ \frac{1}{2\pi} \left| \int\limits_{\mathbb{R}^2} B(\boldsymbol{x}) d\boldsymbol{x} \right| \right\},$$

where $\{y\}$ denotes the largest integer strictly less than y and $\{0\} = 0$, is therefore not contradicted.

We prove in the final theorem that for $n = 2, 3$ (and indeed the same result holds for any $n \ge 2$)

$$nul\, \mathbb{D}_A \le c \int\limits_{\mathbb{R}^n} |\vec{A}|^n\, d\mathbf{x}, \tag{4}$$

where c is the best constant in the inequality

$$N(|\vec{\mathbf{p}}| - |\vec{A}|) \le c \int\limits_{\mathbb{R}^n} |\vec{A}|^n\, d\mathbf{x} \tag{5}$$

for the number $N(|\vec{\mathbf{p}}| - |\vec{A}|)$ of negative eigenvalues of the operator $|\vec{\mathbf{p}}| - |\vec{A}|$ (see [2]). If $\vec{A}$ is a magnetic potential of the magnetic field $\vec{B} = curl\, \vec{A}$, then $nul\, \mathbb{D}_A = nul\, \mathbb{P}_A$. The nullity is independent of gauge, and if $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, there exist a unique $\vec{A}$ such that $curl\, \vec{A} = \vec{B}$, $div\, \vec{A} = 0$ and

$$\|\vec{A}\|_{L^3(\mathbb{R}^3)} \le C \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)} \tag{6}$$

for some constant C (see [1, Lemma2.2]). Hence, from (4), (see [5]),

$$nul\, \mathbb{D}_A = nul\, \mathbb{P}_A \le \text{const} \int\limits_{\mathbb{R}^3} |\vec{B}|^{3/2}\, d\mathbf{x}. \tag{7}$$

However, this is significantly worse than (4), since $\vec{B}$ may be the *curl* of a highly oscillatory $\vec{A}$ for which $\|\vec{A}\|_{L^3(\mathbb{R}^3)}$ is much smaller than $\|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)}$. A natural question is: given a magnetic field $\vec{B}$, can one determine a gauge $\vec{A}$ which minimizes $\|\vec{A}\|_{L^3(\mathbb{R}^3)}$? We require, for all smoth function f

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int\limits_{\mathbb{R}^3} |\vec{A} + \lambda \vec{\nabla} f|^3\, d\mathbf{x} = 0 \quad \text{at} \quad \lambda = 0$$

and this yields

$$\int\limits_{\mathbb{R}^3} div(|\vec{A}|\vec{A}) f d\mathbf{x} = 0,$$

and hence $div(|\vec{A}|\vec{A}) = 0$.

2. Preliminaries

We define, for $n \geq 2$ and $s = -\frac{1}{2}, \frac{1}{2}, 1$, the spaces

$$D^s(\mathbb{R}^n) \ := \ \left\{u \ : \ |\vec{\mathbf{p}}|^s \hat{u}(\vec{\mathbf{p}}) \ \in \ \left[L^2(\mathbb{R}^n)\right]^2\right\} \tag{8}$$

with $\|u\|_{D^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\vec{\mathbf{p}}|^{2s}|\hat{u}(\vec{\mathbf{p}})|^2 \, d\mathbf{p}$, where $\hat{u}$ denotes the Fourier transform

$$\hat{u}(\vec{\mathbf{p}}) \ = \ \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\vec{\mathbf{x}}\cdot\vec{\mathbf{p}}} u(\vec{\mathbf{x}}) \, d\mathbf{x}$$

and $|\hat{u}(\vec{\mathbf{p}})|$ is the Euclidian norm of $\hat{u}(\vec{\mathbf{p}})$ on $\mathbb{C}^2$. Note that we shall use the standard notation in which $\vec{\mathbf{p}}$ is used to represent both a point in momentum space and the momentum operator $\frac{1}{i}\vec{\nabla}$. The space $\left[L^2(\mathbb{R}^n)\right]^2$ will be denoted by $\mathcal{H}_n$ and its usual inner-product and norm by $(\cdot\,,\cdot)$ and $\|\cdot\|$ respectively.

In view of the way we define the operator in (1) we always take $n = 3$ or $n = 2$. However it will be seen that the analysis holds for a general Dirac operator with magnetic potential in $\left[L^n(\mathbb{R}^n)\right]^n$.

The following well-known facts will be needed:

- There are natural embeddings

$$D^{\frac{1}{2}}(\mathbb{R}^n) \ \hookrightarrow \ \left[L^{\frac{2n}{n-1}}(\mathbb{R}^n)\right]^2, \quad n \geq 2, \tag{9}$$

$$D^1(\mathbb{R}^3) \ \hookrightarrow \ \left[L^6(\mathbb{R}^3)\right]^2, \tag{10}$$

$$\left[\mathrm{H}^1(\mathbb{R}^2)\right]^2 \ \hookrightarrow \ \left[L^q(\mathbb{R}^2)\right]^2, \quad \forall q \in [2, \infty); \tag{11}$$

  see [4].
- The dual $\left(D^{\frac{1}{2}}(\mathbb{R}^n)\right)^*$ of $D^{\frac{1}{2}}(\mathbb{R}^n)$ is $D^{-\frac{1}{2}}(\mathbb{R}^n)$.
- Kato's inequality holds for all $n \geq 2$: for all $u \in D^{\frac{1}{2}}(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} \frac{1}{|\vec{\mathbf{x}}|}|u(\vec{\mathbf{x}})|^2 \, d\mathbf{x} \ \leq \ c_n \int\limits_{\mathbb{R}^n} |\vec{\mathbf{p}}||\hat{u}(\vec{\mathbf{p}})|^2 \, d\mathbf{p} \ = \ c_n \|u\|^2_{D^{\frac{1}{2}}(\mathbb{R}^n)} \tag{12}$$

  where c_n is the best constant.

We shall write $\|\vec{A}\|_{L^n(\mathbb{R}^n)}$ to mean $\| \, |\vec{A}| \, \|_{L^n(\mathbb{R}^n)}$, $|\vec{A}| = (\sum\limits_{j=1}^{n} |A_j|^2)^{1/2}$.

Lemma 1. *Let $|\vec{A}| \in L^n(\mathbb{R}^n)$, $n \geq 2$. Then*

$$\frac{1}{\sqrt{|\vec{\mathbf{p}}|}} \ (\vec{\sigma}\cdot\vec{A}) \ \frac{1}{\sqrt{|\vec{\mathbf{p}}|}} \ : \ \mathcal{H}_n \longrightarrow \mathcal{H}_n \tag{13}$$

is compact. Moreover, for all $\varphi \in D^{\frac{1}{2}}(\mathbb{R}^n)$

$$\|(\vec{\sigma}\cdot\vec{A})\varphi\|_{D^{-\frac{1}{2}}(\mathbb{R}^n)} \ \leq \ \gamma_n^2 \|\vec{A}\|_{L^n(\mathbb{R}^n)} \|\varphi\|_{D^{\frac{1}{2}}(\mathbb{R}^n)}. \tag{14}$$

Proof. Clearly $\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}$ is an isometry from $L^2(\mathbb{R}^n)$ onto $D^{\frac{1}{2}}(\mathbb{R}^n)$ and from $D^{-\frac{1}{2}}(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. Let $\varphi_m \rightharpoonup 0$ in $D^{\frac{1}{2}}(\mathbb{R}^n)$. Then $\|\varphi_m\|_{D^{\frac{1}{2}}(\mathbb{R}^n)} \leq k$ say.

Given $\varepsilon > 0$, set $|\vec{A}| = a_1 + a_2$, where $a_1 \in C_0^\infty(\Omega_\varepsilon)$ for some ball Ω_ε, and $|a_1(\mathbf{x})| \le k_\varepsilon$ say, and $\|a_2\|_{L^n(\mathbb{R}^n)} < \varepsilon$. Then, for $\psi \in D^{\frac{1}{2}}(\mathbb{R}^n)$

$$|\big([\vec{\sigma}\cdot \vec{A}]\varphi_m, \psi\big)| \ \le \ \int\limits_{\mathbb{R}^n} (a_1 + a_2)|\varphi_m||\psi| \ d\mathbf{x} \ := \ I_1 + I_2$$

where, by Kato's inequality (12),

$$\begin{aligned} I_1 \quad &\le \quad k_\varepsilon \int\limits_{\Omega_\varepsilon} \left[\sqrt{|\mathbf{x}|}|\varphi_m(\mathbf{x})|\right]\left[\frac{1}{\sqrt{|\mathbf{x}|}}|\psi(\mathbf{x})|\right] d\mathbf{x} \\ &\le \quad k'_\varepsilon \Big(\int\limits_{\Omega_\varepsilon} |\varphi_m(\mathbf{x})|^2 \ d\mathbf{x}\Big)^{\frac{1}{2}} \Big(\int\limits_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|}|\psi(\mathbf{x})|^2 \ d\mathbf{x}\Big)^{\frac{1}{2}} \\ &\le \quad k''_\varepsilon \Big(\int\limits_{\Omega_\varepsilon} |\varphi_m(\mathbf{x})|^2 \ d\mathbf{x}\Big)^{\frac{1}{2}} \|\psi\|_{D^{\frac{1}{2}}(\mathbb{R}^n)}, \end{aligned}$$

and

$$\begin{aligned} I_2 \quad &\le \quad \|a_2\|_{L^n(\mathbb{R}^n)}\|\varphi_m\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}\|\psi\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)} \\ &\le \quad \varepsilon\gamma_n^2\|\varphi_m\|_{D^{\frac{1}{2}}(\mathbb{R}^n)}\|\varphi\|_{D^{\frac{1}{2}}(\mathbb{R}^n)} \ \le \ \varepsilon\gamma_n^2 k\|\varphi\|_{D^{\frac{1}{2}}(\mathbb{R}^n)}, \end{aligned}$$

where γ_n is the norm of the embedding (9). Hence

$$\|[\vec{\sigma}\cdot \vec{A}]\varphi_m\|_{D^{-\frac{1}{2}}(\mathbb{R}^n)} \ \le \ k''_\varepsilon\|\varphi_m\|_{L^2(\Omega_\varepsilon)} + \varepsilon\gamma_n^2 k.$$

Since $D^{\frac{1}{2}}(\mathbb{R}^n) \hookrightarrow L^2(\Omega_\varepsilon)$ is compact, the compactness of (13) is established. The proof of (14) follows from that for I_2 above.

LEMMA 2. *Suppose that the following conditions are satisfied:*

(i) *when* $n = 3$, $|\vec{A}| \in L^3(\mathbb{R}^3)$,

(ii) *when* $n = 2$, $|\vec{A}| \in L^r(\mathbb{R}^2)$ *for some* $r > 2$.

Then, given any $\varepsilon > 0$ *there exists* $k_\varepsilon > 0$ *such that for all* $\varphi \in D^1(\mathbb{R}^n) \cap \mathcal{H}_n$

$$\|(\vec{\sigma}\cdot \vec{A})\varphi\|^2 \ \le \ \varepsilon^2\|\varphi\|^2_{D^1(\mathbb{R}^n)} + k_\varepsilon\|\varphi\|^2. \tag{15}$$

Proof. Choose $|\vec{A}| = a_1 + a_2$, where $|a_1(\mathbf{x})| \le k_\varepsilon$ and $\|a_2\|_{L^3(\mathbb{R}^3)} < \varepsilon$ in (i) and $\|a_2\|_{L^r(\mathbb{R}^2)} < \varepsilon$ in (ii). Then

$$\|(\vec{\sigma}\cdot \vec{A})\varphi\|^2 \ \le \ \|a_1\varphi\|^2 + \|a_2\varphi\|^2 \le \ k_\varepsilon\|\varphi\|^2 + \varepsilon^2\|\varphi\|^2_{L^t(\mathbb{R}^n)}$$

where $t = 6$ when $n = 3$ and $t = \frac{2r}{r-2} \in (2, \infty)$ when $n = 2$. The lemma follows from the embeddings (10) and (11).

Lemma 2 enables us to define the operator $\mathbb{D}_A$ as the operator sum of $\vec{\sigma}\cdot \vec{\mathbf{p}}$ and $\vec{\sigma}\cdot \vec{A}$ in the two- and three-dimensional cases. Its domain $\mathcal{D}(\mathbb{D}_A)$ is given by

$$\begin{aligned} \mathcal{D}(\mathbb{D}_A) \quad &= \quad \{\varphi \ : \ \varphi, \ \vec{\sigma}\cdot(\vec{\mathbf{p}} + \vec{A})\varphi \ \in \ \mathcal{H}_n\} \\ &= \quad \mathcal{D}(\mathbb{D}_0) \ = \ D^1(\mathbb{R}^n) \cap \mathcal{H}_n. \end{aligned} \tag{16}$$

3. THE MAIN RESULTS

We shall assume hereafter that

$$|\vec{A}| \in \begin{cases} L^3(\mathbb{R}^3), & \text{when } n=3, \\ L^2(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)\ , & \text{for some } r>2, \text{ when } n=2. \end{cases} \tag{17}$$

Lemmas 1 and 2 therefore hold and $\mathbb{D}_A$ is the operator sum

$$\mathbb{D}_A \ = \ \vec{\sigma}\cdot\vec{\mathbf{p}} + \vec{\sigma}\cdot\vec{A}$$

on (16).

Suppose that

$$\mathbb{D}_A\varphi = 0, \quad \varphi \in \mathcal{D}(\mathbb{D}_A) = D^1(\mathbb{R}^n) \cap \mathcal{H}_n. \tag{18}$$

Then $\psi = \sqrt{|\vec{\mathbf{p}}|}\varphi \in D^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathcal{H}_n$, and

$$\|\psi\|^2 \ = \ \int_{\mathbb{R}^n} |\vec{\mathbf{p}}||\hat{\varphi}(\vec{\mathbf{p}})|^2\ d\mathbf{p} \ \le \ \frac{1}{2}(\|\vec{\mathbf{p}}\hat{\varphi}\|^2 + \|\hat{\varphi}\|^2).$$

Hence

$$(E-K)\psi \ = \ 0 \tag{19}$$

where

$$E \ = \ \frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\ (\vec{\sigma}\cdot\vec{\mathbf{p}})\ \frac{1}{\sqrt{|\vec{\mathbf{p}}|}} \ : \ \mathcal{H}_n \longrightarrow \mathcal{H}_n \tag{20}$$

and

$$K \ = \ -\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\ (\vec{\sigma}\cdot\vec{A})\ \frac{1}{\sqrt{|\vec{\mathbf{p}}|}} \ : \ \mathcal{H}_n \longrightarrow \mathcal{H}_n. \tag{21}$$

Furthermore, $E^2 = I$, the identity on $\mathcal{H}_n$, $E^* = E$ and, by Lemma 1, K is compact. Thus, from (19)

$$(I-EK)\psi \ = \ 0. \tag{22}$$

Since $\psi \in D^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathcal{H}_n$, it follows that

$$\begin{aligned} nul\,\mathbb{D}_A \ &\le \ nul(I-EK) \\ &= \ 0 \end{aligned} \tag{23}$$

unless the compact operator EK on $\mathcal{H}_n$ has eigenvalue 1.

THEOREM 1. *Let (17) be satisfied. Then for any $c>0$, $nul\,\mathbb{D}_{tA} = 0$ except at a finite number of values of t in $[0,c]$.*

Proof. This is a consequence of Fredholm theory applied to $F_t = I - EK_t$, where $K_t = -\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\ (\vec{\sigma}\cdot t\vec{A})\ \frac{1}{\sqrt{|\vec{\mathbf{p}}|}}$. Since EK_t is compact on $\mathcal{H}_n$, F_t is Fredholm ([3, Theorem I.3.21]). By [3, Theorem I.3.22] there exists $\delta > 0$ such that for $|t-t'| < \delta$, $nul\,F_{t'} \le nul\,F_t$, and consequently $t \mapsto nul\,F_t$ is upper semi-continuous. Furthermore, by [3, Theorem I.3.25], for small enough δ, $nul\,F_{t'}$ is constant in $\{t' \ : \ 0 < |t-t'| < \delta\}$. For $t < t_0$ say, $nul\,F_t = 0$. It therefore suffices to prove the theorem for $[a,c]$, where $0 < a < t_0$. But this follows from the previous remarks and the compactness of $[a,c]$. An alternative proof can be given by amending the proof of [1, Theorem 4.1] to deal with the non-self-adjoint part EK_t of F_t.

THEOREM 2. *Let* $|\vec{A}| \in L^r(\mathbb{R}^2)$ *for some* $r > 2$ *when* $n = 2$. *Then the set* $\{\vec{A} \ : \ nul\, \mathbb{D}_A = 0 \quad and \quad |\vec{A}| \in L^n(\mathbb{R}^n)\}$ *contains an open dense subset of* $\left[L^n(\mathbb{R}^n)\right]^n$

Proof. We shall prove that with $F_A = I - EK_A$, $K_A = -\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\,(\vec{\sigma}\cdot\vec{A})\,\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}$

$$\{\vec{A} \ : \ nul\, F_A = 0 \ \text{ and } \ |\vec{A}| \in L^n(\mathbb{R}^n)\} \tag{24}$$

is open in $\left[L^n(\mathbb{R}^n)\right]^n$. The theorem will follow from (23) since the density of (24) is implied by Theorem 1.

Given $\varepsilon > 0$, suppose $\|\vec{A} - \vec{A}_0\|_{L^n(\mathbb{R}^n)} < \varepsilon$. Then

$$\|K_A - K_{A_0}\| \ \leq \ \gamma_n^2 \varepsilon$$

by Lemma 1. Let $nul\, \mathbb{D}_{A_0} = 0$. Since EK_{A_0} is compact, this means that 1 is not an eigenvalue and there exists $\delta > 0$ such that

$$\inf_{\|f\|=1} \| \ (I - EK_{A_0})f \ \| \ \geq \delta.$$

Hence, if $\varepsilon < \frac{\delta}{2\gamma_n^2}$

$$\inf_{\|f\|=1} \| \ (I - EK_A)f \ \| \ \geq \frac{\delta}{2}$$

and hence $nul\, F_A = 0$. The theorem therefore follows.

THEOREM 3. *Let (17) be satisfied. Then, for* $n = 2, 3$,

$$nul\, \mathbb{D}_A \ \leq \ c \int_{\mathbb{R}^n} |\vec{A}|^n \ d\mathbf{x}, \tag{25}$$

where c *is the best constant in the Daubechies inequality* [2].

Proof.

We know from (23) that $nul\, \mathbb{D}_A \leq nul(I - EK)$. If $\psi \in \ker(I - EK)$, then, with $\mathcal{S} = EK$, $\|\mathcal{S}\psi\| = \|\psi\|$. Hence, by the min-max theorem

$$nul(I - EK) \ \leq \ \sharp\{\lambda_n(\mathcal{S}) \ : \ \lambda_n(\mathcal{S}) \geq 1\},$$

where $\lambda_n(\mathcal{S})$ denotes the nth singular value of $\mathcal{S}$, i.e. the nth eigenvalue of $|\mathcal{S}| = (\mathcal{S}^*\mathcal{S})^{1/2}$. Furthermore

$$\begin{aligned} \mathcal{S}^*\mathcal{S} \quad &= \quad \left(\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\,(\vec{\sigma}\cdot\vec{A})\,\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\right)^2 \\ &\leq \quad \left(\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\,|\vec{A}|\,\frac{1}{\sqrt{|\vec{\mathbf{p}}|}}\right)^2 \quad =: \ \mathrm{L}^2 \end{aligned}$$

and this gives

$$\begin{aligned} nul(I - EK) \quad &\leq \quad \sharp\{\lambda_n(\mathrm{L}) \ : \ \lambda_n(\mathrm{L}) \geq 1\} \\ &= \quad \sharp\{\text{negative eigenvalues of } |\vec{\mathbf{p}}| - |\vec{A}|\} \end{aligned}$$

by the Birman-Schwinger principle. The theorem now follows from [2].

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