ON THE SCHRÖDINGER OPERATOR IN $\mathbb{R}^2$ WITH AN AHARONOV-BOHM MAGNETIC FIELD

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ABSTRACT. It is proved that the form domain of the magnetic Schrödinger operator S_A in $L^2(\mathbb{R}^2)$ with an Aharonov-Bohm magnetic field is continuously embedded in $L^\infty(\mathbb{R}^+,\ rdr) \otimes L^2(\mathbb{S}^1)$. An implication of this is that, when $V \in L^1(\mathbb{R}^+,\ rdr) \otimes L^\infty(\mathbb{S}^1)$, S_A and $S_A + V$ have the same form domain and coincident essential spectrum, namely $[0, \infty)$

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1 INTRODUCTION

The spectral analysis of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^2)$ is strongly influenced by three related facts:

(i) the Sobolev space $\mathrm{H}^1(\mathbb{R}^2)$ is not continuously embedded in $L^\infty(\mathbb{R}^2)$;

(ii) the Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x} \ \leq\ \text{const}\cdot \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^2\, d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \tag{1}$$

does not hold for $n = 2$;

(iii) there is no Cwikel-Lieb-Rosenblum (CLR) inequality of the form

$$N(\Delta + V) \ \leq\ \text{const}\cdot \int_{\mathbb{R}^2} V_-(\mathbf{x})\, d\mathbf{x}, \quad V_- = \min(V, 0), \tag{2}$$

where $N(\Delta + V)$ denotes the number of negative eigenvalues of $\Delta + V$.

In [1] Laptev and Weidl obtained the elegant result that (ii) does hold when $n = 2$ if the gradient ∇ is replaced by the "magnetic" gradient $\vec{\mathbf{p}} + \vec{A}$, where $\vec{\mathbf{p}} = \frac{1}{i}\nabla$ and $curl\,\vec{A}$ is, in particular, a magnetic field of Aharonov-Bohm type. To be specific, they prove, inter alia, the following [1, Theorem 3]: In polar coordinates (r, θ), let

$$\vec{A}(r,\theta) \;=\; \frac{\Psi(\theta)}{r}\,\bigl(\sin\theta, -\cos\theta\bigr), \tag{3}$$

where $\Psi \in L^\infty(\mathbb{S}^1)$, so that $B = curl\,\vec{A} = 0$ for $r \neq 0$, and $\vec{A}\cdot\vec{x} = 0$. Then, for all $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$,

$$\int\limits_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\,d\mathbf{x} \;\le\; C \int\limits_{\mathbb{R}^2} |(\vec{\mathbf{p}} + \vec{A})u(\mathbf{x})|^2\,d\mathbf{x}, \tag{4}$$

with the sharp constant $C = \bigl(\min\limits_{k\in\mathbb{Z}} |k - \tilde{\Psi}|\bigr)^{-1}$, $\quad \tilde{\Psi} := \frac{1}{2\pi}\int\limits_0^{2\pi} \Psi(\theta)\,d\theta.$

An important consequence is that the Schrödinger operator $S_A = (\vec{\mathbf{p}} + \vec{A})^2$ does not have zero modes in $L^2(\mathbb{R}^2)$, i.e. zero is not an eigenvalue.

In this note we prove that the space $\mathrm{H}^1_A = \bigl\{u \;:\; u,\ (\vec{\mathbf{p}} + \vec{A})u \;\in\; L^2(\mathbb{R}^2)\bigr\}$ with norm

$$\|u\|_{\mathrm{H}^1_A} \;=\; \bigl(\|(\vec{\mathbf{p}} + \vec{A})u\|^2 + \|u\|^2\bigr)^{1/2}$$

is continuously embedded in $L^\infty(\mathbb{R}^+;\ L^2(\mathbb{S}^1);\ rdr)$; thus the space of radial functions in H^1_A is indeed continuously embedded in $L^\infty(\mathbb{R}^2)$. We use this result to prove that for $V \in L^1(\mathbb{R}^+;\ L^\infty(\mathbb{S}^1);\ rdr)$, $S_A + V$ is defined as a form sum and has the same essential spectrum as S_A, which is $[0, \infty)$.

2 AN EMBEDDING THEOREM

The magnetic Schrödinger operator S_A is defined to be the non-negative self-adjoint operator in $L^2(\mathbb{R}^2)$ associated with the closable, non-negative sesquilinear form

$$\mathfrak{s}_A[\varphi] \;=\; \int\limits_{\mathbb{R}^2} |(\vec{\mathbf{p}} + \vec{A})\varphi|^2\,d\mathbf{x}, \quad \varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}). \tag{5}$$

It has form domain $\mathcal{Q}(S_A)$ given by the completion of $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ with respect to $\bigl(\mathfrak{s}_A[\cdot\,] + \|\cdot\|^2\bigr)^{1/2}$, where $\|\cdot\|$ is the $L^2(\mathbb{R}^2)$ norm. We shall use the standard notation H^1_A for $\mathcal{Q}(S_A)$; note that $\mathrm{H}^1_A = \mathcal{D}(S_A^{1/2})$ with the graph norm of $S_A^{1/2}$.

Define

$$X \;=\; \left\{u \;:\; u(r,\cdot\,) \in L^2(\mathbb{S}^1) \;\text{ for }\; r \in \mathbb{R}^+ \;\text{ and }\; \int\limits_0^{2\pi} |u(\cdot\,,\theta)|^2\,d\theta \in L^\infty(\mathbb{R}^+)\right\} \tag{6}$$

with norm

$$\|u\|_X \;=\; \operatorname{ess\,sup}_{r>0}\left\{\left(\int\limits_0^{2\pi} |u(r,\theta)|^2\,d\theta\right)\right\}^{\frac{1}{2}}. \tag{7}$$

In other words, $X = L^\infty(\mathbb{R}^+;\ L^2(\mathbb{S}^1);\ rdr) = L^\infty(\mathbb{R}^+)\otimes L^2(\mathbb{S}^1)$.

THEOREM 1. *For all $u \in \mathrm{H}^1_A$ and $\vec{A}$ defined by (3),*

$$\|u\|_X^2 \ \le\ C\|(\vec{\mathbf{p}}+\vec{A})u\|^2, \tag{8}$$

where $C = \big(\min_{k\in\mathbb{Z}} |k-\tilde{\Psi}|\big)^{-1}$ is sharp.

Proof. Following [1, §1], we observe that S_A is the operator

$$S_A \ =\ -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right)$$

where the operator $K_\theta = i\frac{\partial}{\partial\theta} + \Psi(\theta)$ with domain $\mathrm{H}^1(\mathbb{S}^1)$ in $L^2(\mathbb{S}^1)$ has eigenvalues $\lambda_k = k+\tilde{\Psi}$, $k\in\mathbb{Z}$ and eigenfunctions

$$\varphi_k(\theta) \ =\ \frac{1}{\sqrt{2\pi}} e^{-i\left[\theta(k+\tilde{\Psi}) - \int_0^\theta \Psi(\eta)\ d\eta\right]}.$$

The sequence $\{\varphi_k(\theta)\}$ is an orthonormal basis of $L^2(\mathbb{S}^1)$. Any $u\in L^2(\mathbb{S}^1)$ can therefore be written

$$u(r,\theta) \ =\ \sum_{k\in\mathbb{Z}} u_k(r)\varphi_k(\theta),$$

where $u_k(r) = \int_0^{2\pi} u(r,\theta)\varphi_k(\theta)\ d\theta$, and, for any $u\in H^1_A$,

$$\mathfrak{s}_A[u] \ =\ \sum_{k\in\mathbb{Z}}\left\{\int_0^\infty \left(|u_k'(r)|^2 + \frac{\lambda_k^2}{r^2}|u_k(r)|^2\right)\ rdr\right\}. \tag{9}$$

For any $t\in(0,\infty)$

$$\begin{aligned}
|u_k(t)|^2 \ &=\ 2\ \mathrm{Re}\int_0^t \bar{u}_k(r)u_k'(r)\ dr \\
&\le\ 2\left(\int_0^t |u_k'(r)|^2\ rdr\right)^{\frac12}\left(\int_0^t |u_k(r)|^2\ \frac{dr}{r}\right)^{\frac12} \\
&\le\ \frac{2}{|\lambda_k|}\left(\int_0^t |u_k'(r)|^2\ rdr\right)^{\frac12}\left(\lambda_k^2\int_0^t |u_k(r)|^2\ \frac{dr}{r}\right)^{\frac12} \\
&\le\ \frac{1}{|\lambda_k|}\left\{\int_0^\infty \left(|u_k'(r)|^2 + \frac{\lambda_k^2}{r^2}|u_k(r)|^2\right)\ rdr\right\}.
\end{aligned}$$

Hence, on using (9),

$$\begin{aligned}
\int_0^{2\pi} |u(t,\theta)|^2\ d\theta \ &=\ \sum_{k\in\mathbb{Z}} |u_k(t)|^2 \\
&\le\ \frac{1}{\min_{k\in\mathbb{Z}}|\lambda_k|}\|(\vec{\mathbf{p}}+\vec{A})u\|^2,
\end{aligned}$$

whence (8).

3 SPECTRAL CONSEQUENCE

It follows from (8) that S_A has no zero mode, and hence $S_A^{1/2}$ is injective, and has dense domain $\mathcal{D}(S_A^{1/2})$ and range $\mathcal{R}(S_A^{1/2})$ in $L^2(\mathbb{R}^2)$. Let D_A^1 denote the completion of $\mathcal{D}(S_A^{1/2})$ with respect to

$$\|\varphi\|_{D_A^1} := \|S_A^{1/2}\varphi\| = \|(\vec{\mathbf{p}} + \vec{A})\varphi\|. \tag{10}$$

This is not a subspace of $L^2(\mathbb{R}^2)$, but in view of the Laptev-Weidl inequality (4) it is a functional space.

In the next theorem we need the space

$$Y = \big\{u :\ u(r,\cdot) \in L^\infty(\mathbb{S}^1) \text{ for } r \in \mathbb{R}^+ \ \text{ and } \ \underset{\theta\in(0,2\pi)}{ess\ sup}|u(\cdot\,,\theta)| \in L^1(\mathbb{R}^+,\ rdr)\big\}, \tag{11}$$

with norm

$$\|u\|_Y = \int_0^\infty \underset{\theta\in(0,2\pi)}{ess\ sup}|u(r,\theta)|\ rdr, \tag{12}$$

that is

$$Y = L^1(\mathbb{R}^+;\ L^\infty(\mathbb{S}^1);\ rdr) = L^1(\mathbb{R}^+,\ rdr) \otimes L^\infty(\mathbb{S}^1).$$

THEOREM 2. *Let $V \in Y$. Then V is form bounded relative to S_A with S_A-bound zero, and hence $S_A + V$ is defined as a form sum with form domain $\mathcal{Q}(S_A) = \mathrm{H}_A^1$. Moreover, the essential spectra of $S_A + V$ and S_A coincide.*

Proof. We shall prove that $S_A^{-1/2}VS_A^{-1/2}$ is compact in $L^2(\mathbb{R}^2)$. It is sufficient to prove that $P \equiv S_A^{-1/2}|V|S_A^{-1/2}$ is compact and hence that $T = |V|^{1/2}S_A^{-1/2}$ is compact, since $P = T^*T$.

Given $\varepsilon > 0$, choose $W \in C_0^\infty(\mathbb{R}^+; L^\infty(\mathbb{S}^1))$ with support in $\Omega_\varepsilon = B(0,1/\varepsilon) \setminus B(0,\varepsilon)$ and such that $\|W\|_{L^\infty(\mathbb{R}^2)} \le k_\varepsilon$ for some constant $k_\varepsilon > 0$ and $\|V - W\|_Y < \varepsilon$. Let $\varphi_n \rightharpoonup 0$ in $L^2(\mathbb{R}^2)$. Then with $\psi_n = S_A^{-1/2}\varphi_n,\ \psi_n \rightharpoonup 0$ in D_A^1 and

$$\begin{aligned}
\|T\varphi_n\|^2 &= \||V|^{\frac{1}{2}}\psi_n\|^2 \le \||W|^{\frac{1}{2}}\psi_n\|^2 + \||V-W|^{\frac{1}{2}}\psi_n\|^2 \\
&\le k_\varepsilon \int_{\Omega_\varepsilon} |\psi_n|^2\, d\mathbf{x} + \|V - W\|_Y \|\psi_n\|_X^2 \\
&\le k_\varepsilon \int_{\Omega_\varepsilon} |\psi_n|^2\, d\mathbf{x} + C\varepsilon\|S_A^{\frac{1}{2}}\psi_n\|^2 = I_1 + I_2
\end{aligned} \tag{13}$$

say, by (8). On allowing $n \to \infty$, $I_1 \to 0$ since $|\vec{A}|$ is bounded above and away from 0 on Ω_ε and $\mathrm{H}_A^1(\Omega_\varepsilon)$ is compactly embedded in $L^2(\Omega_\varepsilon)$. It follows that $T\varphi_n \to 0$ and the theorem is proved.

COROLLARY *Let $V \in Y$. Then the essential spectrum of S_A and $S_A + V$ is* $[0, \infty)$.

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